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Symmetric generation of M_{22}

Bronson Cade Lim

Symmetric Generation of M_{22}

A Thesis

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in

Mathematics

by

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ABSTRACT

We prove the Mathieu group M_{22} contains two symmetric generating sets with control group $L_3(2)$. The first symmetric generating set consists of order 3 elements while the second consists of involutions. With this knowledge we give two constructions of M_{22}; the first as a homomorphic image of the progenitor $2^{*14} : L_3(2)$ and the second as a homomorphic image of the progenitor $3^{*14} : L_3(2)$. We prove both groups are M_{22} by means of the presentation and the action on the Cayley graph, which is provided via double coset enumeration. The opportunity to present this work as a mathematics thesis gives the author great pleasure. All the work presented is orginal except for the material, for which sources are cited.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 The Classification of the Finite Simple Groups

The Classification Theorem of the Finite Simple Groups (CFSG) is heralded as perhaps the most important result of the 20th Century. The CFSG states that any simple group is one of the following types:

- Cyclic of Prime Order
- Alternating
- Classical
- Exceptional Group of Lie Type
- One of the 26 Sporadic Groups

The CFSG was prematurely announced as completed around 1980; however, some errors were found in proofs. Such problems are a consequence of the CFSG's nature. Michael Ashbacher and Stephen Smith fixed the last known error in 2004 [Asc04]. They presented a 1,200 page long proof.

Richard Brauer pioneered the search for all finite simple groups in 1940 [GLS94]. However, in 1963 Walter Feit and John Griggs Thompson motivated the CFSG in a landmark paper on solvability. In their, at the time, extroardinarily long paper (255 pages), they show every group of odd order is solvable implying every (nonAbelian) simple group is of even order [FT63]. This result is often called the Feit-Thompson theorem or odd order theorem. The CFSG is a collection of large papers like the Feit-Thompson theorem. As such, there is doubt over the CFSG's validity; however, even if the proof is not fully accurate, the general consensus is the CFSG is complete [Wil09].

Recent work with the CFSG has been done by Daniel Gorenstein, Richard Lyons, and Ronald Solomon. Largely motivated by Gorenstein, the so called GLS program was created to write the CFSG clearly in one place; however, this program is ongoing. Of the eleven projected volumes, seven are completed [Wil09], [ALSS11]. It is unfortunate that Gorenstein died in 1992 before he saw his work's completion. The most recent volume, published in 2011, bears the names of Ashbacker and Smith near Lyons and Solomon [ALSS11].

1.2 The Mathieu Groups

The french mathematician Emil Mathieu found 5 highly transitive groups denoted M_{11}, M_{12}, M_{22}, M_{23}, and M_{24}. The small Mathieu groups, M_{11} and M_{12} found in 1861, are sharply 4 and 5-transitive groups, respectively. The only sharply 4-transitive groups are $\mathcal{S}_4$, $\mathcal{S}_5$, $\mathcal{A}_5$, and M_{11}, while the only sharply 5-transitive groups are $\mathcal{S}_5$, $\mathcal{S}_6$, $\mathcal{A}_7$, and M_{12}. The large Mathieu groups, M_{22}, M_{23}, and M_{24} found in 1873, are 3,4, and 5-transitive groups, respectively [Rot95].

We will largely focus on constructing M_{22}. There are many ways M_{22} arises. One involves constructing a transitive extension of $L_3(4)$. Another involves Steiner Systems and their corresponding automorphism groups. It is well known that the automorphism group of the Steiner System of type $\mathcal{S}(3,6,22)$, possesses a simple normal subgroup of index 2 isomorphic to M_{22}. Of course, there is a relationship between the construction of M_{22} given as a transitive extension of $L_3(4)$ and as a subgroup of the automorphism group of $\mathcal{S}(3,6,22)$ [Rot95].

This thesis presents a novel construction. We show M_{22} can be generated two ways: the first by taking 14 elements of order 3 whose set normalizer is $L_3(2)$ and the second by taking 14 elements of order 2 whose set normalizer is the same, $L_3(2)$.

1.3 The History of the Progenitor

A group may be regarded as a collection of objects that interact with each other somehow. It is the study of these interactions that sheds light on the meaning of the group. Given all objects and all interactions the group is known. However, we do not always need this much information to determine the group. We may present the group more ecOnomically. Our studies will be directed towards such a presentation called a progenitor.

The progenitor is a (group) construction developed by Robert T. Curtis in his studies of the Mathieu groups M_{12} and M_{24}. Upon analyzing the structure of these Mathieu groups, Curtis discovered that these groups possess highly symmetric generating sets. Within M_{12} he found 5 generating elements of order 3 whose set normalizer is A_5. In the M_{24} case, he found 7 generating involutions whose set normalizer is $L_2(7)$. We will see that progenitors model this behavior. Curtis constructs these groups and elements explicitly via a special action on conjugacy classes which we investigate in Chapter 6. He later finds them as images of certain progenitors[Cur07].

Chapter 2

Group Theoretic Preliminaries

2.1 Groups

Groups are a natural consequence of studying symmetry in mathematics. We begin with the definition.

Definition 2.1. *A* ***group*** *$(\mathcal{G}, *)$ is a nonempty set $\mathcal{G}$ equipped with an associative binary operation $*$, such that:*

(i) there is an element $e \in \mathcal{G}$ with $e * a = a * e = a$ for all $a \in \mathcal{G}$;

(ii) for every $a \in \mathcal{G}$, there is an element $b \in \mathcal{G}$ with $a * b = e = b * a$.

To avoid cumbersome notation, we omit the group operation $*$ of $\mathcal{G}$, with the understanding that $*$ exists. That is, in place of $(\mathcal{G}, *)$ we write $\mathcal{G}$.

Example 2.2. *Let Ω be a finite set. Define the* ***symmetric group*** *S_Ω to be the set of all bijections of Ω with itself. Then S_Ω is a group under composition of functions.*

Example 2.3. *Let V be a vector space over a field K. Define the* ***general linear group*** *$GL(V)$ to be the set of invertible linear maps from V to itself. Then $GL(V)$ is a group under composition of functions. If a basis for V is specified, then there is a natural correspondence between $GL(V)$ and $GL(n, K)$, the set of $n \times n$ matrices over K with nonzero determinant. It is easy to see that $GL(n, K)$ is a group under matrix multiplication.*

Definition 2.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be groups. A function $f : \mathcal{G} \to \mathcal{H}$ is a* ***homomorphism*** *if, for all $a, b \in \mathcal{G}$,*

$$f(ab) = f(a)f(b).$$

*An **isomorphism** is a homomorphism that is also a bijection. We say that $\mathcal{G}$ is **isomorphic** to $\mathcal{H}$, denoted by $\mathcal{G} \cong \mathcal{H}$, if there exists an isomorphism $f : \mathcal{G} \to \mathcal{H}$.*

It is immediate that the relation $\cong$ is an equivalence relation on the set of all groups.

Example 2.5. *Let $GL(V)$ be as in Example 2.3. Fix a basis β for V. Then the groups $GL(V)$ and $GL(n, K)$ are isomorphic via the map $f : GL(V) \to GL(n, K)$ given by*

$$f(T) = [T]_\beta.$$

Example 2.6. *Let $\Omega = \{a_1, ..., a_n\}$. Then the symmetric group on Ω is the set of bijections of the n elements of Ω. That is, elements of S_Ω permute the n subscripts of the a_i's. Hence, there is an isomorphism between S_Ω and S_n, where S_n is the set of bijections of $\{1, ..., n\}$. For this reason, we say that S_n is the set of permutations of n letters.*

Example 2.7. *Let $\mathcal{G}$ be a group. An isomorphism $f : \mathcal{G} \to \mathcal{G}$ is called an **automorphism** of $\mathcal{G}$. Denote the set of all automorphisms of $\mathcal{G}$ by $Aut(\mathcal{G})$. Then $Aut(\mathcal{G})$ is a group under composition of functions.*

2.2 Group Action

The study of groups and how they interact with various structures is of tremendous importance. In Examples 2.2 and 2.3, we constructed groups in relationship to an underlying structure. The symmetric group S_Ω has a natural way of interacting with the elements of Ω. The general linear group interacts instead with vectors. In this section, we investigate this group action on structures.

Definition 2.8. *[Rot95] If Ω is a set and $\mathcal{G}$ is a group, then Ω is a **$\mathcal{G}$-set** if there is a function $\alpha : \mathcal{G} \times \Omega \to \Omega$ (called an **action**), denoted by $\alpha : (g, a) \mapsto ga$, such that:*

(i) $1a = a$ for all $a \in \Omega$; and

(ii) $g(ha) = (gh)a$ for all $g, h \in \mathcal{G}$ and $a \in \Omega$.

*One also says that $\mathcal{G}$ **acts** on Ω. If $|\Omega| = n$, then n is called the **degree** of the $\mathcal{G}$-set Ω.*

It is customary to omit α, since the action can be written as ga for $g \in \mathcal{G}$ and $a \in \Omega$, with the understanding that α exists.

Example 2.9. *The symmetric group on n letters has an action on the set of n letters given by permutation. We see that both conditions are satisfied in Definition 2.8, the latter following from associativity of functions (permutations).*

The next theorem states that provided we have some $\mathcal{G}$-set Ω, then there is an explicit homomorphism $f : \mathcal{G} \to \mathcal{S}_\Omega$. This will be very useful in the later chapters when we prove simplicity of certain groups.

Theorem 2.10. *[Rot95] If Ω is a $\mathcal{G}$-set, then there is a homomorphism $f : \mathcal{G} \to \mathcal{S}_\Omega$.*

Proof. Since Ω is a $\mathcal{G}$-set, each element $g \in \mathcal{G}$ is a permutation of the elements of Ω, say π_g. Define $f : \mathcal{G} \to \mathcal{S}_\Omega$ by $f(g) = \pi_g$. We see that $f(gh) = \pi_{gh}$. But $\pi_{gh}a = (gh)a = g(ha) = \pi_g(ha) = \pi_g\pi_h a$. Thus, $f(gh) = f(g)f(h)$. □

Definition 2.11. *[Rot95] If Ω is a $\mathcal{G}$-set and $a \in \Omega$, then the **$\mathcal{G}$-orbit** of a is*

$$\mathcal{O}(a) = \{ga : g \in \mathcal{G}\} \subset \Omega.$$

We typically say the orbits of a under $\mathcal{G}$, or simply the orbit of a when no confusion arises, instead of $\mathcal{G}$-orbit.

Definition 2.12. *[Rot95] If Ω is a $\mathcal{G}$-set and $a \in \Omega$, then the **stabilizer** of a, denoted by $\mathcal{G}^a$, is the subgroup*

$$\mathcal{G}^a = \{g \in \mathcal{G} | ga = a\} \leq \mathcal{G}.$$

Theorem 2.13. *[Rot95] If Ω is a $\mathcal{G}$-set and $a \in \Omega$, then*

$$|\mathcal{O}(a)| = [\mathcal{G} : \mathcal{G}^a].$$

Proof. Define a map $f : \mathcal{O}(a) \to \mathcal{G}/\mathcal{G}^a$ by $f(y) = g\mathcal{G}^a$, where $ga = y$. This map is well defined for if $ga = ha$, then $h^{-1}g(a) = a$ and $h^{-1}g \in \mathcal{G}^a$. Thus, $g\mathcal{G}^a = h\mathcal{G}^a$. The function f is injective: for if $f(ga) = f(ha)$, then $g\mathcal{G}^a = h\mathcal{G}^a$ and $h^{-1}g \in \mathcal{G}^a$. We have $h^{-1}ga = a$ and so $ga = ha$. Now f is surjective, for if $g \in \mathcal{G}$, then $f(ga) = g\mathcal{G}^a$. We conclude that f is a bijection and so

$$|\mathcal{O}(a)| = [\mathcal{G} : \mathcal{G}^a].$$

□

Corollary 2.14. *[Rot95] If $a \in \mathcal{G}$, the number of conjugates of a is equal to the index of its centralizer:*

$$|a^{\mathcal{G}}| = [\mathcal{G} : C_{\mathcal{G}}(a)],$$

and this number is a divisor of $|\mathcal{G}|$, when $\mathcal{G}$ is finite.

Proof. The set $\{a^{\mathcal{G}}\}$ of conjugates of a is a $\mathcal{G}$-set, so we may apply the preceeding theorem. Note that $\mathcal{G}_a = C_{\mathcal{G}}(a)$. The result follows. □

2.2.1 Transitivity

Definition 2.15. *[Rot95] Let Ω be a $\mathcal{G}$-set of degree n and let $k \leq n$ be a positive integer. Then Ω is **k-transitive** if, for every pair of $k-$tuples having distinct entries in Ω, say $(a_1, ..., a_k)$ and $(b_1, ..., b_k)$, there is a $g \in G$ with $ga_i = b_i$ for $i = 1, ..., k$*

Example 2.16. *The symmetric group S_n is $n-$transitive for it is the set of all bijections of $\{1, ..., n\}$ with itself.*

Example 2.17. *Any group $\mathcal{G}$ is **transitive** (1-transitive) on itself, where the action is given either by left or right multiplication.*

Theorem 2.18. *[Rot95] If Ω is a transitive $\mathcal{G}$-set of degree n, and if $a \in \Omega$, then*

$$|\mathcal{G}| = n|\mathcal{G}^a|.$$

If Ω is faithful, then $|\mathcal{G}^a|$ is a divisor of $(n-1)!$.

Proof. By Theorem 2.13, for $a \in \Omega$, the orbit of a, $\mathcal{O}(a)$ has size: $|\mathcal{O}(a)| = [\mathcal{G} : \mathcal{G}^a]$. But $\mathcal{G}$ is transitive and so $\mathcal{O}(a) = \Omega$. We have $[\mathcal{G} : \mathcal{G}^a] = n$ as desired. The last claim follows since $\mathcal{G}^{\Omega - \{a\}} \leq S_{n-1}$. □

Theorem 2.19. *[Rot95] If $\mathcal{H} \leq \mathcal{G}$, then $\mathcal{G}$ acts transitively on the set of all right cosets of $\mathcal{H}$.*

Proof. Clearly, $\mathcal{G}$ has a well defined action on the set of right cosets. Suppose there exists $\mathcal{H}g_1, \mathcal{H}g_2 \in \mathcal{G}/\mathcal{H}$. By Cayley's theorem, there exists a $g \in \mathcal{G}$ such that $g_1 g = g_2$. We conclude that $\mathcal{H}g_1 g = \mathcal{H}g_2$ and so the action of $\mathcal{G}$ is transitive. □

2.3 Blocks, Primitivity, and Simplicity Criterion

There are properties of group action that we need to investigate before we can find a simplicity criteria. The action of a group $\mathcal{G}$ on a set Ω tells us information about the structure of the group. We begin this section with block systems, then we will move to primitivity, and finally criterion for determining simplicity.

Definition 2.20. *[Rot95] If Ω is a $\mathcal{G}$-set, then a **block** is a subset B of Ω such that, for each $g \in \mathcal{G}$, either $gB = B$ or $gB \cap B = \emptyset$.*

Example 2.21. *The subsets $\emptyset$, Ω, and the set of one-point sets of a $\mathcal{G}$-set X are called **trivial blocks**. Other blocks are called **nontrivial**.*

Example 2.22. *Let $2^2 = \langle(1,2)(3,4),(1,3)(2,4)\rangle$, then the subsets $B = \{1,3\}$ and $B' = \{2,4\}$ are blocks of $\Omega = \{1,2,3,4\}$.*

Definition 2.23. *[Rot95] A transitive $\mathcal{G}$-set Ω is **primitive** if it contains no nontrivial block; otherwise, it is **imprimitive**.*

Example 2.24. *The symmetric group S_n with natural action on $\Omega = \{1,...,n\}$ is primitive. For if $B = \{i_1,...,i_k\}$ is a nontrivial block, then there is a $j \in \Omega$ such that $j \notin B$. Let $\pi \in S_n$ such that $\pi(i_m) = i_m$ for $m = 1,...,k-1$ and $\pi(i_k) = j$. Then $\pi B \neq B$ and $\pi B \cap B \neq \emptyset$.*

The next result will tell us when the action of $\mathcal{G}$ is primitive. Recall, a subgroup $\mathcal{H}$ of $\mathcal{G}$ is **maximal** if for every $\mathcal{H} \leq \mathcal{K} \leq \mathcal{G}$, we have either $\mathcal{H} = \mathcal{K}$ or $\mathcal{K} = \mathcal{G}$.

Theorem 2.25. *[Wil09] Suppose that the group $\mathcal{G}$ acts transitively on Ω, and let $\mathcal{G}^a$ be the stabilizer of $a \in \Omega$. Then $\mathcal{G}$ is primitive if and only if $\mathcal{G}^a$ is a maximal subgroup of $\mathcal{G}$.*

Proof. Suppose that $\mathcal{G}$ is primitive and $\mathcal{G}^a$ is not maximal. Then there exists a subgroup $\mathcal{K}$ of $\mathcal{G}$ with $\mathcal{G}^a < \mathcal{K} < \mathcal{G}$. Since the cosets of $\mathcal{G}^a$ in $\mathcal{G}$ are in one to one correspondence with the points in Ω, the cosets of $\mathcal{G}^a$ in $\mathcal{K}$ forms a block of Ω.

Conversely, suppose that $\mathcal{G}^a$ is maximal and the action of $\mathcal{G}$ is imprimitive. Then there exists a nontrivial block B of X such that $gB = B$ or $gB \cap B = \emptyset$ for all $g \in \mathcal{G}$. Since $\mathcal{G}$ is transitive, there exists $g' \in \mathcal{G}$ and $b \in B$ such that $g'b = a$. Let $\mathcal{K} = \{g \in \mathcal{G} | gg'B = g'B\}$, K is obviously a subgroup. Furthermore, $\mathcal{G}^a < \mathcal{K} < \mathcal{G}$. The first containment follows from $g' \in \mathcal{K}$ and $g' \notin \mathcal{G}^a$, while the latter follows from B being nontrivial. This contradiction completes the proof. □

We need one more result before we end our discussion on primitivity but first a lemma.

Lemma 2.26. *[Rot95] let $\mathcal{G}$ is a group acting faithfully and primitively on Ω of degree $n \geq 2$. If $\mathcal{H}$ is a normal nontrivial subgroup of $\mathcal{G}$, then Ω is a transitive $\mathcal{H}$-set.*

Proof. If $\mathcal{H}$ is nontrivial, then Ha is a block for all $a \in \Omega$. Since the action of $\mathcal{G}$ is primitive, $\mathcal{H}a = \emptyset$ (plainly impossible), $\mathcal{H}a = \{a\}$, or $\mathcal{H}a = \Omega$. Suppose $\mathcal{H}a = \{a\}$, then we must have $\mathcal{H} \leq \mathcal{G}^a$, the stabilizer of a. But $\mathcal{G}$ is transitive, so there exits $g \in \mathcal{G}$ with $ga = b$. By normality of $\mathcal{H}$, we have that $\mathcal{H} = g\mathcal{H}g^{-1} \leq g\mathcal{G}^a g^{-1} = \mathcal{G}^b$. Hence, $\mathcal{H} \leq \cap_{b\in\Omega}\mathcal{G}^b = 1$. This contradiction shows $\mathcal{H}$ must be transitive. □

We will end this section with a result from Kenkichi Iwasawa, originally proved in 1941 [Iwa41]. Recall, a group is said to be **perfect** if it is equal to its **derived subgroup** (commutator subgroup). That is, $\mathcal{G}$ is perfect if $\mathcal{G} = \mathcal{G}'$, where $\mathcal{G}' = \langle xyx^{-1}y^{-1} | x, y \in \mathcal{G}\rangle$.

Theorem 2.27 (Iwasawa's Lemma). *[Wil09] If $\mathcal{G}$ is a finite perfect group, acting faithfully and primitively on a set Ω, such that the point stabiliser $\mathcal{G}^a$ has a normal Abelian subgroup $\mathcal{A}$ whose conjugates generate $\mathcal{G}$, then $\mathcal{G}$ is simple.*

Proof. Let $\mathcal{H}$ be a normal subgroup of $\mathcal{G}$ with $1 < \mathcal{H} < \mathcal{G}$, then $\mathcal{H}$ is transitive on Ω by Lemma 2.26. By hypothesis, each $g \in \mathcal{G}$ is of the form $g = \prod g_i a_i g_i^{-1}$, where $g_i \in \mathcal{G}$ and $a_i \in \mathcal{A}$. Since $\mathcal{H}$ acts transitively, we have $\mathcal{G} = \mathcal{H}\mathcal{G}^a$. Any element g of $\mathcal{G}$ can be written as $g = hs_a$, where $h \in \mathcal{H}$ and $s_a \in \mathcal{G}^a$. In particular, $g_i = h_i s_i$. Now

$$g = \prod h_i s_i a_i s_i^{-1} h_i^{-1}.$$

But $\mathcal{A}$ is normal in $\mathcal{G}^a$ and so $s_i a_i s_i^{-1} \in \mathcal{A}$, we conclude $g \in \mathcal{H}\mathcal{A}\mathcal{H} \leq \mathcal{H}\mathcal{A}$. This implies that $\mathcal{G} = \mathcal{H}\mathcal{A}$ which gives us

$$\mathcal{G}/\mathcal{H} = \mathcal{H}\mathcal{A}/\mathcal{H} \cong \mathcal{A}/(\mathcal{H} \cap \mathcal{A}).$$

But $\mathcal{A}$ is Abelian, therefore any quotient of $\mathcal{A}$ is Abelian. Thus $\mathcal{G}' \leq \mathcal{H}$. Since $\mathcal{G}$ is perfect, we must have $\mathcal{H} = \mathcal{G}$. We conclude $\mathcal{G}$ is simple. □

2.4 Finitely Presented Groups, Free Products, and the Semidirect Product

Consider the symmetric group $\mathcal{S}_3$ on $\{1,2,3\}$. It is easily seen that $\mathcal{S}_3 = \langle (1,2)(1,2,3) \rangle$. In fact, $\mathcal{S}_n = \langle (1,2),(1,...,n) \rangle$. Suppose we set $x = (1,2)$ and $y = (1,2,3)$. We notice that $xy = (2,3)$ has order 2. We can then say $\mathcal{S}_3$ is generated by x and y subject to the relations $x^2 = 1$, $y^3 = 1$, and $(xy)^2 = 1$. This is called a presentation for $\mathcal{S}_3$. We generalize this idea of presenting groups on generators subject to some relations now.

Definition 2.28. *[Rot95] Let $\mathcal{G}$ be a group generated by $\mathcal{X}$ satisfying a set of relations $\mathcal{R}$. Then a **presentation** for $\mathcal{G}$ is an ordered pair $\langle \mathcal{X} | \mathcal{R} \rangle$.*

In the previous case, we had $\mathcal{X} = \{x, y\}$ and $\mathcal{R} = \{x^2 = 1, y^3 = 1, (xy)^2 = 1\}$. We conclude that $\langle \mathcal{X} | \mathcal{R} \rangle$ is a presentation for $\mathcal{S}_3$ and write $\mathcal{S}_3 = \langle x, y | x^2 = y^3 = (xy)^2 = 1 \rangle$.

Suppose now we have a collection of groups $\{\mathcal{K}_i\}_{i \in I}$, where I is some indexing set. We endow this set with the group operation given by juxtaposition, which we denote by $*$. We will call this the **free product** of the groups $\{\mathcal{K}_i\}_{i \in I}$. If the $\mathcal{K}_i$ are isomorphic, then denote this group $\mathcal{K}^{*I}$, where $\mathcal{K} \cong \mathcal{K}_i$. The next example will make this clear.

Example 2.29. *Let C_2 be the cyclic group of order 2, which we denote by 2. Define $\mathcal{K}_i = 2$ for $i = 1,2,3$. Then the free product of the $\{\mathcal{K}_i\}$ is the group 2^{*3}. Suppose now that $\mathcal{K}_i = \langle t_i \rangle$, then we have*

$$2^{*3} = \langle t_1 \rangle * \langle t_2 \rangle * \langle t_3 \rangle.$$

In terms of presentations, we may write:

$$2^{*3} = \langle t_1, t_2, t_3 | t_1^2 = t_2^2 = t_3^2 = 1 \rangle.$$

Eventually, we will define groups containg groups such as 2^{*3}. We need to introduce another group construction called the semidirect product. The semidirect product is a group constructed by two subgroups in which one subgroup acts on the other subgroup. We follow the standard construction provided by Curtis in [Cur07].

Definition 2.30. *[Cur07] Let $\mathcal{K}$ be a group and $\mathcal{Q} \leq Aut(\mathcal{K})$, be a subgroup of the automorphism group of $\mathcal{K}$. Let $\mathcal{G} = \mathcal{Q} \times \mathcal{K}$ be the Cartesian product. Define a binary* *operation $\circ$ by $(a,x) \circ (b,y) = (ab, x^b y)$, for $a,b \in \mathcal{Q}$ and $x,y \in \mathcal{K}$. We call $\mathcal{G}$ the* ***semidirect product** of $\mathcal{K}$ by $\mathcal{Q}$ and write $\mathcal{G} = \mathcal{K} : \mathcal{Q}$.*

Proposition 2.31. *[Cur07] Let $\mathcal{G}$ be the semidirect product of $\mathcal{K}$ by $\mathcal{Q}$ as in Definition 2.30. Then $\mathcal{G}$ is a group under $\circ$.*

Proof. It is clear the $\mathcal{G}$ is closed. The identity is given by $(1_{\mathcal{Q}}, 1_{\mathcal{K}})$. The inverse of $(a,x) \in \mathcal{G}$ is $(a^{-1}, (x^{-1})^{a^{-1}})$. Finally,

$$[(a,x)(b,y)](c,z) = (ab, x^b y)(c,z) = (abc, x^{bc} y^c z),$$

and

$$(a,x)[(b,y)(c,z)] = (a,x)(bc, y^c z) = (abc, x^{bc} y^c z),$$

for all $a,b,c \in \mathcal{Q}$ and $x,y,z \in \mathcal{K}$. □

It should be noted that the semidirect product $\mathcal{K} : \mathcal{Q}$ is often constructed from the Cartesian product $\mathcal{K} \times \mathcal{Q}$. However, for this thesis the construction given above is more favorable in terms of utility. Futhermore, $\mathcal{K} : \mathcal{Q}$ has two natural subgroups $\tilde{\mathcal{K}} = \{(1_{\mathcal{Q}}, x) | x \in \mathcal{K}\} \cong \mathcal{K}$ and $\tilde{\mathcal{Q}} = \{(a, 1_{\mathcal{K}}) | a \in \mathcal{Q}\} \cong \mathcal{Q}$ such that $\mathcal{K} : \mathcal{Q} = \tilde{\mathcal{Q}}\tilde{\mathcal{K}}$. For this reason, we often ignore the notation (a,x) for an arbitrary element of $\mathcal{K} : \mathcal{Q}$ (this is called the external semidirect product). Instead of (a,x) we write ax where multiplication is given by

$$axby = abx^b y.$$

We will use this internal semidirect product for the duration of the thesis. Before we go further, suppose we set $a = b^{-1}$ and $y = 1_{\mathcal{K}}$. We would arrive at $b^{-1}xb = x^b$, which is conjugation by b. This observation allows us to perform simplifications of elements inside the groups we construct in this manner.

2.5 The Progenitor and Symmetric Presentations

We introduce the progenitor via the following example. Let $\mathcal{G} \cong \mathcal{S}_{n+1}$ and $\mathcal{T} = \{(1,2), ..., (1,n+1)\}$. Then $\bar{\mathcal{T}} = \{\langle(1,i)\rangle\}_{i=2}^{n+1}$. The subgroup $\mathcal{N}_{\mathcal{G}}(\bar{\mathcal{T}})$ can only permute the second entry in $(1,i)$. It follows that $\mathcal{N}_{\mathcal{G}}(\bar{\mathcal{T}}) \cong \mathcal{S}_n$, where $\mathcal{S}_n$ acts on $\{2, ..., n+1\}$. We could venture to say that $\mathcal{S}_{n+1} = \langle \mathcal{S}_n, t \rangle$, with $t = (1,2)$. This motivates the next definition.

Definition 2.32. *[Cur07] Let $\mathcal{G}$ be a group and $\mathcal{T} = \{t_0, t_1, ..., t_n\} \subset \mathcal{G}$, then define $\bar{\mathcal{T}} = \{\mathcal{T}_0, \mathcal{T}_1, ..., \mathcal{T}_n\}$, where $\mathcal{T}_i = \langle t_i \rangle$, the cyclic subgroup generated by t_i. We further define $\mathcal{N} = \mathcal{N}_{\mathcal{G}}(\bar{\mathcal{T}})$, the set normalizer of $\mathcal{T}$ in $\mathcal{G}$. We say that $\mathcal{T}$ is a* ***symmetric generating set*** *for $\mathcal{G}$ if the following conditions hold:*

(i) $\mathcal{G} = \langle \mathcal{T} \rangle$, and

(ii) $\mathcal{N}$ permutes $\bar{\mathcal{T}}$ transitively, not necessarily faithfully.

If $\mathcal{G}$ possesses a symmetric generating set $\mathcal{T}$, then $\mathcal{G}$ is said to be ***symmetrically generated****. We refer to the subgroup $\mathcal{N}$ as the* ***control subgroup*** *and to generators of the free product as the* ***symmetric generators****.*

Considering $\mathcal{S}_{n+1}$, we see that condition (i) holds: for $\mathcal{T}$ is known to generate $\mathcal{G}$. Since $\mathcal{S}_n$ is n-transitive, we have that condition (ii) holds as well. By the preceding definition, $\mathcal{T}$ is a symmetric generating set for $\mathcal{S}_{n+1}$. We can now say $\mathcal{S}_{n+1}$ is symmetrically generated.

Notice, the conjugates of $t = (1,2)$ under $\mathcal{S}_n$ have relations among them. That is, $(1,2)(1,3) = (1,2,3)$ has order 3, etc. Suppose there are no such relations. This results in an infinite free product of cyclic groups together with some permutation group acting via conjugation.

Definition 2.33. *[Cur07] An* ***involutory progenitor*** *is a semidirect product of the following form:*

$$\mathcal{P} \cong 2^{*n} : \mathcal{N} = \{\pi\omega | \pi \in \mathcal{N}, \omega \text{ a reduced word in the } t_i\},$$

where 2^{*n} denotes a free product of n copies of the cyclic group of order 2 generated by involutions t_i for $i = 1, ..., n$; and $\mathcal{N}$ is a transitive permutation group of degree n which acts on the free product by permuting the generators.

We may generalize $\mathcal{S}_{n+1}$ to a progenitor in the following way. Recall, $2^{*n} = \langle t_1 \rangle * \cdots * \langle t_n \rangle$. Define $\mathcal{N} = \mathcal{S}_n$ on $\{1, ..., n\}$. We have arrived at $2^{*n} : \mathcal{S}_n$. Moreover, any transitive permutation group $\mathcal{N}$ on n letters generalizes to a progenitor $2^{*n} : \mathcal{N}$.

Now, if we add relations to $2^{*n} : \mathcal{S}_n$, namely $(t_i t_j)^3 = 1$ for $i, j = 1, ..., n$, then we may obtain some finite group. Appropriate relations would yield $\mathcal{S}_{n+1}$. The process of adding relations is called **factoring by the relations**. If we add a relation, say $(\pi\omega)^a = 1$, then the factored progenitor is denoted:

$$\mathcal{G} \cong \frac{2^{*n} : S_n}{(\pi\omega)^a}.$$

If $\mathcal{G}$ is also finite then we say $\mathcal{G}$ is a **finite homomorphic image** of the progenitor $2^{*n} : \mathcal{N}$. The meaning of homomorphic image is illustrated in the next theorem.

Theorem 2.34. *[Cur07] Let $\mathcal{G}$ be a finite nonAbelian simple group. Then $\mathcal{G}$ is the homomorphic image of the progenitor $2^{*n} : \mathcal{N}$, where $\mathcal{N}$ is a transitive subgroup of the symmetric group $\mathcal{S}_n$.*

Proof. The Feit-Thompson theorem, [FT63], guarantees $\mathcal{G}$ is of even order and hence contains and element of order 2. Furthermore, $\mathcal{G}$ is generated by such elements. If $\mathcal{M}$ is a maximal subgroup of $\mathcal{G}$, there exists an element $x \in \mathcal{G}$ such that $\mathcal{G} = \langle \mathcal{M}, x \rangle$. This follows by maximality of $\mathcal{M}$.

We now show that $\langle x^{\mathcal{M}} \rangle \lhd \mathcal{G}$. Let $m_1, m \in \mathcal{M}$. Then $mx^{m_1}m^{-1} = x^{mm_1} \in \langle x^{\mathcal{M}} \rangle$. Now since $x = x^e$, we have $x, x^{-1} \in \langle x^{\mathcal{M}} \rangle$. Hence for $m \in \mathcal{M}$, we have $x^{-1}x^m x \in \langle x^{\mathcal{M}} \rangle$. Since $\mathcal{G} = \langle \mathcal{M}, x \rangle$, we must have $\langle x^{\mathcal{M}} \rangle \lhd \mathcal{G}$. But we must necessarily have that $\langle x^{\mathcal{M}} \rangle = \mathcal{G}$ since $\langle x^{\mathcal{M}} \rangle \neq 1$ and $\mathcal{G}$ is simple.

Now define $\mathcal{M} = \mathcal{N}$ and let $n = |x^{\mathcal{M}}|$. If we index the set of conjugates $x^{\mathcal{M}}$, we may define a mapping $\phi : 2^{*n} : N \to \mathcal{G}$ by $\phi(t_i) = x_i$ and $\phi(g) = g$ for all $g \in \mathcal{N}$. Furthermore, we have that $\mathcal{M}$ acts faithfully: for if $x_i^m = x_i$ for every element x_i of the generating set, then $m \in \mathcal{Z}(\mathcal{G})$. Since $\mathcal{G}$ is simple, it has trivial center and so $m = 1$. □

The progenitor generalizes to symmetric generators of arbitrary order.

Definition 2.35. *[Cur07] Let $\mathcal{G}$ be a group and $\mathcal{T} = \{t_1, t_2, ..., t_n\}$ be a symmetric generating set for $\mathcal{G}$ with $|t_i| = m$. Then if $\mathcal{N} = \mathcal{N}_{\mathcal{G}}(\bar{\mathcal{T}})$, then we define the **progenitor** to be the semidirect product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_m.*

Note, we may define $\mathcal{N}$ to act in a nonpermuation way on $\bar{\mathcal{T}}$. However, we will not need this for this thesis. We see Theorem 2.34 generalizes to arbitrary groups.

Theorem 2.36. *[Cur07] Let $\mathcal{G}$ be a group and $\mathcal{T} = \{t_1, t_2, ..., t_n\}$, with $|t_i| = m$ for all i, be a symmetric generating set for $\mathcal{G}$. Then $\mathcal{G}$ is a homomorphic image of the progenitor $\mathcal{P} = m^{*n} : \mathcal{N}$, where $\mathcal{N} = \mathcal{N}_{\mathcal{G}}(\bar{\mathcal{T}})$.*

Proof. Define a homomorphism $\phi : \mathcal{P} \to \mathcal{G}$ by $\phi(t_i) = t_i$ and $\phi(m) = m$ for all $t_i \in \mathcal{T}$ and $m \in \mathcal{N}$. □

If $\mathcal{G}$ is a finite homomorphic image of the progenitor $m^{*n} : \mathcal{N}$, then $\phi(\pi\omega) = 1$ for some element $\pi\omega \in m^{*n} : \mathcal{N}$. Associating $\phi(\pi)$ with π and $\phi(\omega)$ with ω, we see that $\phi(\pi\omega) = 1$ if $\omega = \pi^{-1}$. Hence, a finite homomorphic image is factored by elements of $\mathcal{N} \cap 2^{*n}$. The following lemma, which will we refer to as the Famous Lemma (named by John Bray), tells us which relations to factors the progenitor $\mathcal{P}$ by so that we may look for finite homomorphic images.

Lemma 2.37 (The Famous Lemma). *[Cur07]*

$$\mathcal{N} \cap \langle t_i, t_j \rangle \leq C_{\mathcal{N}}(\mathcal{N}_{ij}),$$

where $\mathcal{N}_{ij}$ denotes the stabilizer in $\mathcal{N}$ of the two points i and j.

Proof. If $\pi \in \mathcal{N}$ and $\pi = \omega(t_i, t_j)$ is a word in the t_i, t_j, and if $a \in \mathcal{N}_{ij}$, then

$$\pi^a = \omega(t_i, t_j)^a = \omega(t_i, t_j) = \pi.$$

Since $\pi^a = \pi$, we must have $\pi \in C_{\mathcal{N}}(\mathcal{N}_{ij})$. □

Note, that the lemma generalizes to any number of generators. That is,

$$\mathcal{N} \cap \langle t_1, ..., t_n \rangle \leq C_{\mathcal{N}}(\mathcal{N}_{1...n}),$$

where we use the arbitrary indexing of $\{1, ..., n\}$ on the t_i's to avoid cumbersome notation.

We defined the presentation of a progenitor to be of the form $2^{*n} : \mathcal{N}$, where $\mathcal{N}$ permutes the $t_i's$ transitively. Since $\mathcal{N}$ is transitive on $\{t_1, ..., t_n\}$, we may determine the number of conjugates of t_1 by taking the index of the point stabilizer $\mathcal{N}^1$ in $\mathcal{N}$. In terms of presentations, we see that if $\mathcal{N} = \langle \mathcal{X} | \mathcal{R} \rangle$. then

$$2^{*n} : \mathcal{N} \cong \langle \mathcal{X}, t | \mathcal{R}, t^2, [\mathcal{N}^1, t] \rangle.$$

That is, we define t to commute with the point stabilizer $\mathcal{N}^1$ and so $|t^{\mathcal{N}}| = [\mathcal{N} : \mathcal{N}^1]$ as desired. To illustrate this, the natural progenitor $2^3 : \mathcal{S}_3$ has point stabilizer $\mathcal{S}_2$. Hence, we have the presentation $2^3 : \mathcal{S}_3 \cong \langle x, y, t | x^3, y^2, (xy)^2, t^2, [y, t] \rangle$, where $\mathcal{S}_2 = \langle y \rangle$.

In lieu of the famous lemma, we add the relations $(\pi\omega)^a = 1$ to the presentation of $2^{*n} : \mathcal{N}$, where a is an integer. Of course, we may add more than one such relation. We can then compute which integers a result in finite groups.

2.6 Coset Enumeration

When we have found a finite image $\mathcal{G}$ of a progenitor $\mathcal{P}$, we desire to construct a homomorphism from $\mathcal{G}$ to $\mathcal{S}_n$ for some n. We do this via the action of $\mathcal{G}$ on the cosets of the control group $\mathcal{N}$ of $\mathcal{P}$. To accomplish this, we exploit the properties of $\mathcal{N}$ within $\mathcal{G}$. Since $\mathcal{G}$ is an image of $\mathcal{P}$, any element g of $\mathcal{G}$ can be represented in the form $g = \pi\omega$, where $\pi \in \mathcal{N}$ and ω is a reduced word in the t_i's. Through conjugation of g by $\mathcal{N}$, we arrive at several elements of $\mathcal{G}$ represented by a permutation followed by a word of length the same as ω.

What this means is we can represent a large amount of the single cosets $\mathcal{N}\omega$ via $\mathcal{N}\omega^\pi$ for $\pi \in \mathcal{N}$. Observe that $N\omega\pi = N\pi\pi^{-1}\omega\pi = N\pi\omega^\pi = N\omega^\pi$. That is, we may represent some of the cosets of $\mathcal{N}$ in $\mathcal{G}$ by double cosets $\mathcal{N}\omega\mathcal{N}$. Recall, if $\mathcal{H}$ and $\mathcal{K}$ are subgroups of $\mathcal{G}$, then the $\mathcal{H}-\mathcal{K}$ **double coset** is a subset of $\mathcal{G}$ of the form $\mathcal{H}g\mathcal{K}$, where $g \in \mathcal{G}$, and the $\mathcal{H}-\mathcal{K}$ double cosets partition $\mathcal{G}$. This provides a useful tool for constructing a coset table for $\mathcal{G}$, one which we will use to give a permutation representation of M_{22}. The following lemma allows us to see how many elements are in a given double coset.

Lemma 2.38. *[Cur07] If $\mathcal{H}$ and $\mathcal{K}$ are finite subgroups of the group $\mathcal{G}$ and $x \in \mathcal{G}$, then $|\mathcal{H}x\mathcal{K}| = |\mathcal{H}||\mathcal{K}|/|\mathcal{H}^x \cap \mathcal{K}|$.*

Proof. We will count the right cosets of $\mathcal{H}$ in $\mathcal{H}x\mathcal{K}$. Suppose we have two distinct cosets $\mathcal{H}xk_1$ and $\mathcal{H}xk_2$, where $k_1, k_2 \in \mathcal{K}$, then $\mathcal{H}xk_1k_2^{-1}x^{-1} \neq \mathcal{H}$ implies that $xk_1k_2^{-1}x^{-1} \notin \mathcal{H}$. But this implies that $k_1k_2^{-1} \notin x^{-1}\mathcal{H}x$ and so $k_1k_2^{-1} \notin \mathcal{H}^x \cap \mathcal{K}$. Finall, we have $(\mathcal{H}^x \cap \mathcal{K})k_1 \neq (\mathcal{H}^x \cap \mathcal{K})k_2$. This arguement works in the reverse direction as well. Thus the single cosets of $\mathcal{H}$ in $\mathcal{H}x\mathcal{K}$ are in one to one correspondence with the single cosets of $\mathcal{H}^x \cap \mathcal{K}$ in $\mathcal{K}$. The result follows. □

In the special case that $\mathcal{N} = \mathcal{H} = \mathcal{K}$, then this amounts to determining the number of elements in the double coset $\mathcal{N}\omega\mathcal{N}$ or equivalently the number of single cosets of $\mathcal{N}$ in $\mathcal{N}\omega\mathcal{N}$. Investigating this further, we have that that if π is in the **coset stabilizer** $\mathcal{N}^{(\omega)} = \{\pi \in \mathcal{N} | \mathcal{N}\omega\pi = \mathcal{N}\omega\}$, then $\mathcal{N}\omega\pi\omega^{-1} = \mathcal{N}$. But then $\pi \in \mathcal{N}^\omega$ and $\pi \in \mathcal{N}$ implies that $\pi \in \mathcal{N}^\omega \cap \mathcal{N}$. In fact, $\mathcal{N}^{(\omega)} = \mathcal{N}^\omega \cap \mathcal{N}$. That is, we may determine the number of single cosets of $\mathcal{N}\omega\mathcal{N}$ by computing $|\mathcal{N}|/|\mathcal{N}^{(\omega)}|$.

What have gathered so far is that instead of enumerating the single cosets $\mathcal{N}\omega$, we can enumerate the double cosets $\mathcal{N}\omega\mathcal{N}$ and arrive at a complete list of single cosets.

We will define what is meant by a complete list shortly. But first let us illustrate the process of double coset enumeration with the following example.

Example 2.39. *Consider the symmetric group $\mathcal{S}_3$ on $\{1,2,3\}$. Let $\mathcal{T} = \{(1,2),(1,3)\}$ with $t_1 = (1,2)$ and $t_2 = (1,3)$. Then $\mathcal{N} = \mathcal{N}_{\mathcal{S}_3}(\bar{\mathcal{T}}) = \langle(2,3)\rangle \cong \mathcal{S}_2$. We will begin with the double coset $\mathcal{N}e\mathcal{N}$, which we denote by [*] for brevity. Since $\mathcal{N}e\mathcal{N} = \mathcal{N}$, there is only one element here.*

We will find new single cosets by multiplying by elements of $\mathcal{S}_3$. If we multiply $\mathcal{N}$ by elements of $\mathcal{N}$, then we do not arrive at a new double coset, we do not even arrive at a new single coset! We must then multiply by t_1 and t_2. If we take the ***single coset representative*** *$\mathcal{N}e$ of [*] and multiply by t_1 and t_2, then we get $\mathcal{N}t_1$ and $\mathcal{N}t_2$. But do these belong to the same double coset or are they in distinct double cosets? Consider $\pi = (2,3) \in \mathcal{N}$. Then $\mathcal{N}t_1^{\pi} = \mathcal{N}t_2$ and so these single cosets are in the same double coset. We may denote the double coset $\mathcal{N}t_1\mathcal{N}$ by [1]. We know that $|[1]| = |\mathcal{N}|/|\mathcal{N}^{(\omega)}|$, but $\mathcal{N}^{(\omega)}$ is easily seen to be trivial. But then we have $|[1]| = 2$. Since $|\mathcal{S}_3| = 6$, we have found the double coset decomposition*

$$\mathcal{S} = \mathcal{N} \cup \mathcal{N}t_1\mathcal{N},$$

which amounts to the single coset decomposition

$$\mathcal{S} = \langle(2,3)\rangle \cup \langle(2,3)\rangle t_1 \cup \langle(2,3)\rangle t_2.$$

In the preceding example, we knew that $|\mathcal{S}_3| = 6$ and it was clear when to stop the process. Had we not known when the process stops, we would have to multiply the single coset representative $\mathcal{N}t_1$ of [1] by t_1 and t_2 to look for new single cosets. However, we know that $t_1t_2 = (1,3,2) = (2,3)(1,2) \in \mathcal{N}t_1 \in [1]$. So the process would have stopped regardless. In general, the next lemma provides us with a way to determine when we have stopped.

Lemma 2.40. *[Rot95] Let $\mathcal{G}$ be a finite group, $\mathcal{X}$ a set of generators of $\mathcal{G}$, $\mathcal{H} \leq \mathcal{G}$ a subgroup and $\mathcal{H}\omega_1, ..., \mathcal{H}\omega_n$ some distinct cosets of $\mathcal{H}$. It $\cup_{i=1}^{n}\mathcal{H}\omega_i$ is closed under right multiplication by every $a \in \mathcal{X} \cup \mathcal{X}^{-1}$, then $\mathcal{G} = \cup_{i=1}^{n}\mathcal{H}\omega_i$, and $[\mathcal{G} : \mathcal{H}] = n$ and $\mathcal{G} = n|\mathcal{H}|$.*

Proof. Suppose there is another single coset $\mathcal{H}\omega$. Since $\mathcal{G}$ is transitive on $\mathcal{G}/\mathcal{H}$, there exists a word a on the set $\mathcal{X} \cup \mathcal{X}^{-1}$ such that $\mathcal{H}\omega a = \mathcal{H}\omega_k$ for some $1 \leq k \leq n$. Hence

$\cup_{i=1}^{n}\mathcal{H}\omega_i$ is not closed under multiplication, a contradiction. The later claim follows from Lagrange. □

In terms of progenitors, when multiplication by the symmetric generating set $\{t_i\}$ and the inverses $\{t_i^{-1}\}$ ceases to produce new single cosets we have found a complete list. For completeness, we include the double coset enumeration process beginning with an arbitrary double coset:

(a) Determine the coset stabilizer $\mathcal{N}^{(\omega)}$ of the single coset representative $\mathcal{N}\omega$ of $[\omega]$.

(b) Multiply $\mathcal{N}\omega$ on the right by the orbits of $\mathcal{N}^{(\omega)}$ on $\{t_i\} \cup \{t_i^{-1}\}$.

(c) Determine if there are any new single cosets.

This simple process takes quite a bit of time and we will spend the majority of the construction of M_{22} in this phase.

2.7 Double Coset Enumeration over a Maximal Subgroup

Manual double coset enumeration can get complicated, which results in computations that are quite messy. To remedy this we will take a closer look at this process. Recall, double coset enumeration is a process by which we decompose a group $\mathcal{G}$ into sets of the form $\mathcal{N}\omega\mathcal{N}$, where ω is a word in the t_is. This allows us to enumerate the single cosets and embed $\mathcal{G}$ into $\mathcal{S}_\Omega$, where $\Omega = \{Ng\}$.

If we instead find the single coset decomposition of $\mathcal{G}$ over $\mathcal{M}$, where $\mathcal{N} \leq \mathcal{M} \leq G$, we see the number of elements of $\Omega' = \{\mathcal{M}g\}$ is less than Ω. Hence, the number of double cosets of the form $\mathcal{M}\omega\mathcal{N}$ decrease. We then find the single coset decomposition of $\mathcal{M}$ over $\mathcal{N}$, which is equivalent to finding the double coset decomposition of $\mathcal{M}$ over $\mathcal{N}$. If T is a transversal for $\mathcal{N}$ in $\mathcal{M}$, then

$$\mathcal{M} = \cup_{x\in T}\mathcal{N}x.$$

Similarly if S is a transversal for $\mathcal{M}$ in G, we have:

$$\mathcal{G} = \cup_{y\in S}\mathcal{M}y.$$

Hence,

$$\mathcal{G} = \cup_{y\in S}\mathcal{M}y = \cup_{y\in S}\cup_{x\in T}\mathcal{N}xy = \cup_{x\in T,y\in S}\mathcal{N}xy.$$

But this is exactly what double coset enumeration of $\mathcal{G}$ over $\mathcal{N}$ accomplishes.

Recently (2003), Wiedorn has used this technique in [Wie03] to decompose the symmetric presentation for the smallest Janko group, $\mathcal{J}_1$ given by:

$$\mathcal{J}_1 \cong \frac{2^{*5} : A_5}{(xt)^7},$$

into double cosets of the form $\mathcal{L}\omega\mathcal{A}$, where $\mathcal{L} \cong L_2(11)$ and $\mathcal{A} \cong \mathcal{A}_5$, ω is a word in the t_i's of length at most 6. We illustrate this idea with an example.

2.7.1 Double Coset Enumeration of $\mathcal{S}_5$ over $\mathcal{S}_3$

We consider a known presentation of $\mathcal{S}_5$ [Cur07]. Consider the progenitor $\mathcal{P} = 2^{*4} : \mathcal{A}_4$, with natural action on the t_is factored by the relation $((0,1,2)t_0)^4 = 1$. We have the result:

$$\mathcal{S}_5 \cong \frac{2^{*4} : \mathcal{A}_4}{(0,1,2)t_0^4}.$$

We proceed to manual Double Coset Enumeration of $\mathcal{S}_5$ over the group generated by $\mathcal{S}_4 \cong \langle \mathcal{A}_4, t_0t_1t_0 \rangle$. We will then do manual double coset enumeration of $\mathcal{S}_4$ over $\mathcal{A}_4$.

2.7.2 $\mathcal{S}_5$ over $\mathcal{S}_4$

Let $\mathcal{S}_5 = \mathcal{M}$. We begin with the double coset $\mathcal{M}e\mathcal{N}$. The stabilizer of $\mathcal{M}e$ is easily seen to be all of $\mathcal{N}$, which is transitive on the $\{t_0, t_1, t_2, t_3\}$. Take an element form the orbit, say t_0 and multiply it by the single coset representative $\mathcal{M}e$. This results in a new double coset $\mathcal{M}t_0\mathcal{N}$, which we denote [0].

2.7.3 $\mathcal{M}t_0\mathcal{N}$

We compute the point stabilizer $\mathcal{N}^0$ to be $\mathcal{A}_3$ on $\{1, 2, 3\}$. Since we have no additional relations we conclude that $\mathcal{N}^{(0)} \geq \mathcal{N}^0$. The orbits of $\mathcal{N}^{(0)}$ are $\{\{t_0\}, \{t_1, t_2, t_3\}\}$. Now take an element from each orbit ,say t_0 and t_1, and multiply by the single coset representative $\mathcal{M}t_0$. We get the following:

$$\mathcal{M}t_0t_0 = \mathcal{M} \in [*] \text{ and } \mathcal{M}t_0t_1 = \mathcal{M}t_0 \in [0],$$

the later relation being given by $\mathcal{M}t_0t_1t_0 = \mathcal{M}$ since $t_0t_1t_0 \in \mathcal{M}$.

Since there are no new double cosets, the process ends.

2.7.4 The Cayley Graph of $\mathcal{S}_5$ over $\mathcal{S}_4$

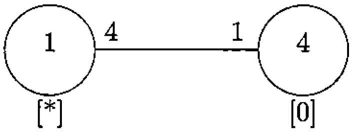

Figure 2.1: The Cayley Graph of $\mathcal{S}_5$ Over $\mathcal{S}_4$

2.7.5 $\mathcal{S}_4$ over $\mathcal{A}_4$

We now perform manual double coset enumeration of $\mathcal{S}_4$ over $\mathcal{A}_4$. Note $[\mathcal{S}_4 : \mathcal{A}_4] = 2$, so we anticipate that there are two double cosets, both consisting of a single element. Now $\mathcal{S}_4 \cong \langle \mathcal{A}_4, t_0t_1t_0 \rangle$, thus the t_is in this case or rather the s_is are the conjugates of $s_0 = t_0t_1t_0$. There are 12 conjugates of s_0 under $\mathcal{A}_4$, we seek to find which are equal.

The relation $t_0t_1 \sim t_0t_2$ grants $t_0t_1t_0 \sim t_0t_2t_0$. Conjugating by $x = (1,2,3)$, we see that $t_0t_2t_0 \sim t_0t_3t_0$. Now we again use the relation $(0,1,2)t_0t_2t_1t_0 = 1$. By multiplying on the left by $t_2t_0(0,2,1)$, we achieve:

$$t_1t_0 = (0,2,1)t_1t_2.$$

Thus the we get the relation $t_0t_1t_0 \sim t_0(0,2,1)t_1t_2 \sim t_2t_1t_2$. We conclude that $t_it_jt_i \sim t_kt_lt_k$ for $i,j,k,l \in \{0,1,2,3\}$. Hence the 12 conjugate of s_0, reduce to 1.

We conclude that there are two double cosets $\mathcal{N}e\mathcal{N}$ and $\mathcal{N}t_0t_1t_0\mathcal{N}$.

2.7.6 The Cayley Graph of $\mathcal{S}_4$ over $\mathcal{A}_4$

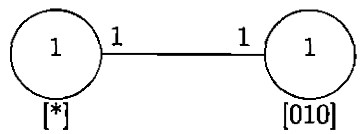

Figure 2.2: The Cayley Graph of $\mathcal{S}_4$ Over $\mathcal{A}_4$

2.7.7 The Single Coset Decomposition of $\mathcal{A}_4$ in $\mathcal{S}_5$

We now replace the group $\mathcal{M}$ with its coset decomposition with respect to $\mathcal{A}_4$. We have that,

$$\mathcal{M} = \mathcal{A}_4 \cup \mathcal{A}_4 t_0 t_1 t_0$$

and

$$\mathcal{G} = \mathcal{M} \cup \mathcal{M}t_0 \cup \mathcal{M}t_1 \cup \mathcal{M}t_2 \cup \mathcal{M}t_3.$$

By substituting in $\mathcal{M}$, we have that the single cosets are:

$$\mathcal{A}_4, \mathcal{A}_4 t_0 t_1 t_0, \mathcal{A}_4 t_0, \mathcal{A}_4 t_0 t_1, \mathcal{A}_4 t_1, \mathcal{A}_4 t_0 t_1 t_0 t_1, \mathcal{A}_4 t_2, \mathcal{A}_4 t_0 t_1 t_0 t_2, \mathcal{A}_4 t_3, \mathcal{A}_4 t_0 t_1 t_0 t_3.$$

But since $t_i t_j t_i \sim t_k t_l t_k$, we may make suitable adjustments to get the list of single cosets:

$$\mathcal{A}_4, \mathcal{A}_4 t_0 t_1 t_0, \mathcal{A}_4 t_0, \mathcal{A}_4 t_0 t_1, \mathcal{A}_4 t_1, \mathcal{A}_4 t_1 t_2, \mathcal{A}_4 t_2, \mathcal{A}_4 t_2 t_1, \mathcal{A}_4 t_3, \mathcal{A}_4 t_3 t_1.$$

Notice, that $[\mathcal{S}_5 : \mathcal{A}_4] = 10$, which is the number of single cosets that we have arrived at. Now by computing the action of G on the set of single cosets, we arrive at a transitive embedding of $\mathcal{S}_5$ into $\mathcal{S}_{10}$.

Chapter 3

Symmetric Generating Sets for M_{22}

Curtis constructs a symmetric generating set for M_{12} and M_{24} in [Cur07]. He constructs the symmetric generating set for M_{12} by looking at the conjugacy class Λ of $a = (1, 2, 3, 4, 5)$ in A_5. From here, he defines a special type of conjugation of elements of order 3 on Λ (see Section 6.1.1). In turn, this defines an element s_1 of S_{12}. If $\hat{a}$ is the image of a given by conjugation on Λ, then he shows $M_{12} = \langle s_1, \hat{a} \rangle$. Furthermore, s_1 has 5 conjugates under $\hat{a}$, say s_1, s_2, s_3, s_4, s_5. He then shows that $M_{12} = \langle s_1, s_2, s_3, s_4, s_5 \rangle = \langle s_1, \hat{a} \rangle$. That is the set $\{s_1, s_2, s_3, s_4, s_5\}$ is a symmetric generating set for M_{12} with control group A_5 [Cur07].

For M_{24}, Curtis considers the group $L_2(7) \cong L_3(2)$ and takes the class Λ of $a = (1, 2, 3, 4, 5, 6, 7)$. Again, he acts on Λ in a particular way by involutions in $L_2(7)$. This will define an element s_1 of S_{24}. If $\hat{a}$ is the image of a given by conjugation on Λ, then he shows that $M_{24} = \langle s_1, \hat{a} \rangle$. Again, he shows s_1 has 7 conjugates under $\hat{a}$, say $s_1, s_2, s_3, s_4, s_5, s_6, s_7$. Moreover, $M_{24} = \langle s_1, s_2, s_3, s_4, s_5, s_6, s_7 \rangle$. Thus the set $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ is a symmetric generating set for M_{24} with control group $L_2(7)$ [Cur07].

It is a tragedy that no such nice way seems to exist for M_{22}. The groups M_{12} and M_{24} are truly exceptional. To find a symmetric generating set for M_{22} we will instead look inside permutation representations of high degree. Let us first show how to construct the smallest of the large Mathieu groups, M_{22}, in a natural way.

3.1 The Mathieu Group M_{22}

Let $\mathcal{G}$ be a k-transitive group. If we stabilize a point, then we are returned a $(k-1)$-transitive group. Transitive extensions arise by beginning with a $(k-1)$-transitive group and finding its corresponding k-transitive group. That is, transitive extensions take a point stabilizer and find the original group (not that it is necessarily unique). The reader is referred to the maximal subgroups of M_{22} in the ATLAS,[CCN+85]. You will find that $L_3(4)$ is maximal in M_{22}. It will be shown that $L_3(4)$ acts doubly transitively on 21 letters, while M_{22} is 3-transitive on 22 letters. We might think that we could begin with $L_3(4)$ and attempt to find M_{22} via a transitive extension. In this section, we will show that M_{22} is a 3-transitive simple group of order 443,520 whose point stabilizer is $L_3(4)$. However, we will not use this information to find symmetric generating sets. The order of $L_3(4)$ is 20,160, which would be quite a large control group. We find in Section 3.2 that we can use $L_3(2)$ as the control group instead.

3.1.1 The Projective Plane $P^2(4)$

As usual, let V be a vector space over a field K and denote the set $[x] = \{y|y = \lambda x, x \in V^\# = V - \{0\}, \lambda \in K\}$ be the **homogeneous coordinates** for x. If V is $(n+1)$-dimensional, denote $P^n(V) = \{[x]|x \in V^\#\}$ be the **projective n-space.** If V is a vector space over $GF(q)$, denote $P^n(V)$ by $P^n(q)$.

Lemma 3.1. *[Rot95] Let V be a $(n+1)$-dimensional vector space over $GF(q)$, then:*

(i) For every $n \geq 0$ and every prime power q,

$$|P^n(q)| = q^n + q^{n-1} + ... + q + 1.$$

(ii) The group $L_{n+1}(q)$ acts doubly transitively on $P^n(q)$.

If $n = 2$ and $q = 4$, then $|P^2(4)| = 21$ and $L_3(4)$ acts doubly transitively on $P^2(4)$.

Proof. *(i)* : Since $V^\#$ has $q^{n+1} - 1$ vectors and $P^n(q)$ partitions these vectors into equivalence classes of $(q-1)$ vectors, we have that

$$|P^n(q)| = (q^{n+1} - 1)/(q-1) = q^n + q^{n-1} + ... + q + 1.$$

Sketch of (ii) : Take two pairs of projective points $([x],[y])$ and $([x'],[y'])$. Extend the linearly independent sets $\{x,y\}$ and $\{x',y'\}$ to bases $\{x,y,z_1,...,z_{n-1}\}$ and $\{x',y',z'_1,...,z'_{n-1}\}$ for V. Then there exists a $g \in GL(V)$ such that $g(x)=x'$, $g(y)=y'$ and $g(z_i)=z'_i$. If $det(g)=\lambda \neq 1$, then define $h \in GL(V)$ by $h(x)=\lambda^{-1}x'$, $h(y)=y'$ and $h(z_i)=z'_i$. Then $h \in L_{n+1}(q)$ and $L_{n+1}(q)$ acts doubly transitively as desired. □

3.1.2 Transitive Extensions

The idea is to extend the doubly transitive group $L_3(4)$ on 21 letters to a 3-transitive group on 22 letters. To do this and to prove simplicity of the constructed groups we need a few lemmas.

Lemma 3.2. *[Rot95] Let Ω be a $\mathcal{G}$-set. If $k \geq 2$, then Ω is k-transitive if and only if, for each $a \in \Omega$, the $\mathcal{G}^a$-set $\Omega - \{a\}$ is $(k-1)$-transitive*

Proof. If Ω is k-transitive, then it is clear that $\Omega - \{a\}$ is $(k-1)$-transitive. Suppose that for each $a \in \Omega$, $\Omega - \{a\}$ is $(k-1)$ transitive. Let $(a_1,...,a_k)$ and $(b_1,...,b_k)$ be $k-tuples$ consisting of distinct elements of Ω. Then there exists $g \in \mathcal{G}^{a_k}$ such that $g(a_1,...,a_k)=(b_1,...,b_{k-1},a_k)$ and $h \in \mathcal{G}^{b_1}$ such that $h(b_1,...,b_{k-1},x_k)=(b_1,...,b_k)$. We conclude that hg is the desired element and Ω is k-transitive. □

Lemma 3.3. *[Rot95] Let Ω be a faithful primitive k-transitive $\mathcal{G}$-set with $\mathcal{G}^a$ a simple group. Then either $\mathcal{G}$ is simple or every nontrivial normal subgroup H of $\mathcal{G}$ is a regular normal subgroup. Furthermore, if $k \geq 3$ and $|\Omega|$ is not a power of 2, then either $\mathcal{G} \cong S_3$ or $\mathcal{G}$ is simple.*

Proof. (*Sketch*) Let $\mathcal{H}$ be a nontrivial normal subgroup of $\mathcal{G}$, then $\mathcal{H}$ is transitive. Let $\mathcal{G}^a$ be a stabilizer of a point $a \in \Omega$. Since $\mathcal{G}^a$ is simple, either $\mathcal{H} \cap \mathcal{G}^a = 1$ or $\mathcal{H} \cap \mathcal{G}^a = \mathcal{G}^a$. If $\mathcal{H} \cap \mathcal{G}^a = 1$ for all $a \in \Omega$, then Ω is regular. If $\mathcal{H} \cap \mathcal{G}^a = \mathcal{G}^a$, then $\mathcal{G}^a \leq \mathcal{H}$. But Ω is primitive and so $\mathcal{G}^a$ is maximal. Thus $\mathcal{H} = \mathcal{G}$ and $\mathcal{G}$ is simple.

Now suppose that $k \geq 3$ and $|\Omega|$ is not a power of 2. If $\mathcal{G}$ is not simple, then any normal subgroup $\mathcal{H}$ is regular. Now let $\mathcal{G}^a$ act on $\mathcal{H}^\#$, $\mathcal{H}^\#{=}H-\{1\}$, by conjugation. For $h \in \mathcal{H}$, the set $\{h,h^{-1}\}$ is easily seen to be a block: for $ghg^{-1}=h$, then $ghg^{-1}=h^{-1}$ or if $ghg^{-1}=h^{-1}$, then $gh^{-1}g^{-1}=h$. Since $k \geq 3$, we have that $\mathcal{H}^\#$ is a doubly transitive $\mathcal{G}^a$-set. But then $\mathcal{H}^\#$ is primitive and so $\mathcal{H}^\# = \{h,h^{-1}\}$ or $\{h,h^{-1}\}=\{h\}$ for all $h \in \mathcal{H}^\#$.

The latter case cannot happen for this would imply that $\mathcal{H}$ is a elementary Abelian 2-group. Since $\mathcal{H}$ is regular, $|\Omega|$ divides $|\mathcal{H}|$. Thus $|\Omega| = 2^m$ for some m. If $\mathcal{H}^{\#} = \{h, h^{-1}\}$ then $|\mathcal{H}| = 3$ and so $\mathcal{H} \cong \mathbb{Z}_3$. Thus $|\Omega| = 3$, for $|\Omega| = |\mathcal{H}|$ and so we must have that $\mathcal{G} \cong \mathcal{S}_3$. □

Lemma 3.4. *[Rot95] If Ω is a doubly transitive $\mathcal{G}$-set and $a \in \Omega$, then $\mathcal{G} = \mathcal{G}^a \cup \mathcal{G}^a g \mathcal{G}^a$ for some $g \notin \mathcal{G}^a$.*

Proof. (*Sketch*) Define a map $f : \{\mathcal{G}^a\text{–}orbits\} \to \{(\mathcal{G}^a\text{–}\mathcal{G}^a)\text{– double cosets}\}$ by $f(\mathcal{G}^a b) = \mathcal{G}^a g \mathcal{G}^a$, where $ga = b$. It can be shown f is bijective. Since $\mathcal{G}$ is doubly transitive, there are only two orbits of $\mathcal{G}^a$ on Ω, $\Omega - \{a\}$ and $\{a\}$. Hence $\mathcal{G} = \mathcal{G}^a e \mathcal{G}^a \cup \mathcal{G}^a g \mathcal{G}^a = \mathcal{G}^a \cup \mathcal{G}^a g \mathcal{G}^a$. □

Definition 3.5. *[Rot95] Let $\mathcal{G}$ be a permutation group on Ω and let $\tilde{\Omega} = \Omega \cup \{\infty\}$, where $\infty \notin \Omega$. A transitive permutation group $\tilde{\mathcal{G}}$ on $\tilde{\Omega}$ is a* ***transitive extension*** *of $\mathcal{G}$ if $\mathcal{G} \leq \tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}^{\infty} = \mathcal{G}$.*

Theorem 3.6. *[Rot95] Let $\mathcal{G}$ be a doubly transitive permutation group on a set X. Suppose there is $a \in \Omega$, $\infty \notin \Omega$, $g \in \mathcal{G}$, and a permutation h of $\tilde{\Omega} = \Omega \cup \{\infty\}$ such that:*

(i) $g \notin \mathcal{G}^a$;

(ii) $h(\infty) \in \Omega$;

(iii) $h^2 \in \mathcal{G}$ and $(gh)^3 \in \mathcal{G}$; and

(iv) $h\mathcal{G}^a h = \mathcal{G}^a$.

Then $\tilde{\mathcal{G}} = \langle \mathcal{G}, h \rangle \leq S_{\tilde{Omega}}$ is a transitive extension of $\mathcal{G}$.

Proof. Let $\tilde{\mathcal{G}} = \langle \mathcal{G}, h \rangle$, then it is clear that $\tilde{\mathcal{G}}$ is transitive on $\tilde{\Omega}$ by condition (ii). We show that $\tilde{\mathcal{G}} = \mathcal{G} \cup \mathcal{G}h\mathcal{G}$, for then $\tilde{\mathcal{G}}^{\infty} = \mathcal{G}$ as desired. Now $\mathcal{G} \cup \mathcal{G}h\mathcal{G} \subseteq \tilde{\mathcal{G}}$ and so if $\mathcal{G} \cup \mathcal{G}h\mathcal{G}$ is a group, then we must have equality. It follows that $\mathcal{G} \cup \mathcal{G}h\mathcal{G}$ is a group if it is closed. We have:

$$\begin{aligned} (\mathcal{G} \cup \mathcal{G}h\mathcal{G})(\mathcal{G} \cup \mathcal{G}h\mathcal{G}) &\subseteq \mathcal{G}\mathcal{G} \cup \mathcal{G}\mathcal{G}h\mathcal{G} \cup \mathcal{G}h\mathcal{G}\mathcal{G} \cup \mathcal{G}h\mathcal{G}\mathcal{G}h\mathcal{G} \\ &\subseteq \mathcal{G} \cup \mathcal{G}h\mathcal{G} \cup \mathcal{G}h\mathcal{G}h\mathcal{G}, \end{aligned}$$

where we have made the identification $\mathcal{G}\mathcal{G} = \mathcal{G}$. Hence, we must have $\mathcal{G}h\mathcal{G}h\mathcal{G} \subseteq \mathcal{G} \cup \mathcal{G}h\mathcal{G}$ or equivalently $h\mathcal{G}h \subseteq \mathcal{G} \cup \mathcal{G}h\mathcal{G}$, since $g_1 h g_2 h g_3 \in \mathcal{G} \cup \mathcal{G}h\mathcal{G}$ if and only if $g_1^{-1} g_1 h g_2 h g_3 g_3^{-1} = h g_2 h \in \mathcal{G} \cup \mathcal{G}h\mathcal{G}$.

Now $\mathcal{G}$ acts doubly transitively on Ω and so $\mathcal{G} = \mathcal{G}^a \cup \mathcal{G}^a g \mathcal{G}^a$, $g \notin \mathcal{G}^a$. By (iii) and (*iv*) there exists $\gamma, \delta \in \mathcal{G}$ such that $h^2 = \gamma$ and $(gh)^3 = \delta$. This implies $h\gamma^{-1} = h^{-1} = \gamma^{-1}h$ and $hgh = g^{-1}h^{-1}g^{-1}\delta$. We compute $h\mathcal{G}h$ now:

$$\begin{aligned}
h\mathcal{G}h &= h(\mathcal{G}^a \cup \mathcal{G}^a g \mathcal{G}^a)h, \\
&= h\mathcal{G}^a h \cup h\mathcal{G}^a g \mathcal{G}^a h, \\
&= h\mathcal{G}^a h \cup (h\mathcal{G}^a h)h^{-1}gh^{-1}(h\mathcal{G}^a h), \\
&= \mathcal{G}^a \cup \mathcal{G}^a h^{-1}gh^{-1}\mathcal{G}^a, \\
&= \mathcal{G}^a \cup \mathcal{G}^a(\gamma^{-1}h)g(h\gamma^{-1})\mathcal{G}^a, \\
&\subset \mathcal{G} \cup \mathcal{G}hgh\mathcal{G}, \\
&= \mathcal{G} \cup \mathcal{G}g^{-1}h^{-1}g^{-1}\delta\mathcal{G}, \\
&= \mathcal{G} \cup \mathcal{G}h^{-1}\mathcal{G}, \\
&= \mathcal{G} \cup \mathcal{G}\gamma^{-1}h\mathcal{G}, \\
&= \mathcal{G} \cup \mathcal{G}h\mathcal{G}.
\end{aligned}$$

We conclude that $\tilde{\mathcal{G}} = \langle \mathcal{G}, h \rangle$ is a transitive extension of $\mathcal{G}$.

□

3.1.3 M_{22} as a Transitive Extension of $L_3(4)$

We now construct M_{22} as a transitive extension of $L_3(4)$ acting on $P^2(4) \cup \{\infty\}$. We then show that M_{22} is a simple group of order 443,520. There is only one simple group of order 443,520, the Mathieu group M_{22} [Par70]. One may also check the ATLAS, [CCN+85].

Theorem 3.7. *[Rot95] There exists a 3-transitive group M_{22} of degree 22 and order* $443,520 = 22 \cdot 21 \cdot 20 \cdot 48 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ *such that the stabilizer of a point is $L_3(4)$.*

Proof. (*Sketch*) We will show that M_{22} is a transitive extension of $L_3(4)$ acting on $\tilde{P^2}(4) = P^2(4) \cup \{\infty\}$. Let $x = [1,0,0] \in P^2(4)$, $g[\lambda, \mu, \nu] = [\mu, \lambda, \nu]$, and $h = (x\ \infty)f$ with $f[\lambda, \mu, \nu] = [\lambda^2 + \mu\nu, \mu^2, \nu^2]$. Note, that f fixes x and so h is well-defined and g does not fix x. It can be shown that $(gh)^3 = 1$ and $h^2 = 1$ as well.

Now take $k \in \mathcal{G}^a$, then we see that $hkh(\infty) = (\infty)$ and $hkh(x) = x$. Hence, we can suppose hkh acts solely on $P^2(4)$ only since $hkh = (x\infty)fk(x\infty)f = fkf$. But $f \in \mathcal{G}^a$ implies that $hkh \in \mathcal{G}^a$ and so $M_{22} = \langle L_3(4), h \rangle$ is a transitive extension of M_{22}.

Now M_{22} is 3-transitive since $L_3(4)$ was doubly transitive. Since $|M_{22}| = |P^2(4) \cup \{\infty\}||\mathcal{G}^\infty|$ and the stabilizer of ∞ is $L_3(4)$ which has order 20,160, we conclude that $|M_{22}| = 22 \cdot 20160 = 22 \cdot 21 \cdot 20 \cdot 48$ as desired. □

Theorem 3.8. *[Rot95] The group M_{22} is simple.*

Proof. Since M_{22} is a faithful 3-transitive group with $\mathcal{G}^\infty = L_3(4)$ and $|\tilde{P^2}(4)| = 22$, we must have that $\mathcal{G}$ is simple. □

One can also construct a transitive extension M_{23} of M_{22}. Similarly, one can construct a transitive extension M_{24} of M_{23}. This results in a simple 4-transitive group and a simple 5-transitive group on 23 and 24 letters, respectively. The groups M_{22}, M_{23}, and M_{24} make up the large Mathieu groups. One can also obtain the small Mathieu groups M_{11} and M_{12} in a similar way, with the exception of the stabilizers. That is, one builds M_{11} as a transitive extension of the non-simple group M_{10}. Then M_{12} is built as a transitive extension of the simple group M_{11} [Rot95].

The interested reader is referred to [DM96] and [Rot95] for a treatment of the Mathieu groups and the nesting property. It is known that all Mathieu groups are subgroups of the largest Mathieu group M_{24}. While this is obvious for M_{22} and M_{23}, it is not clear for the smaller Mathieu groups M_{11} and M_{12}.

3.2 Two Symmetric Generating Sets for M_{22}

The reader is referred to the ATLAS, [CCN+85], for the permutation representations used in this section. We will look inside the group structure of M_{22} for two symmetric generating sets, both with the same control group. We will find that both symmetric generating sets have 14 elements, one consisting of elements of order 3 and the other of order 2.

Theorem 3.9. *There exist a symmetric generating set $\mathcal{T} = \{t_1, ..., t_{14}\}$ of M_{22}, such that $|t_i| = 3$ and $\mathcal{N}_{M_{22}}(\bar{\mathcal{T}}) \cong L_3(2)$, where $\bar{\mathcal{T}} = \{\langle t_1\rangle, ..., \langle t_{14}\rangle\}$.*

Proof. Consider the permutation representation of M_{22} on 176 letters given by the action of M_{22} on the set of cosets of the maximal subgroup $\mathcal{A}_7$. Within the maximal subgroup $2^3 : L_3(2)$ there are three class of subgroups isomorphic to $L_3(2) = \mathcal{N}$. In one of these classes, there exists a point stabilizer isomrphic to $\mathcal{A}_4$ such that the centralizer $\mathcal{C}_{M_{22}}(\mathcal{A}_4) \cong \mathcal{A}_4$. Take an element of order 3 in the centralizer, say $t \in \mathcal{C}_{M_{22}}(\mathcal{A}_4)$. Now we have that $\mathcal{N}^t = \mathcal{C}_{\mathcal{N}}(t) = \mathcal{A}_4$. Since the number of conjugates $|t^{\mathcal{N}}| = |\mathcal{N} : \mathcal{N}^t|$, we have that $|t^N\mathcal{N}| = |\mathcal{N} : \mathcal{A}_4| = 14$. That is, t has 14 conjugates under the action of $\mathcal{N}$. We can label these conjugates as $t_1, ..., t_{14}$ so that the generators x and y of $L_3(2)$ act like $x = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$ and $y = (1,8)(2,13)(3,10)(4,5)(6,9)(11,12)$ on $\{t_1, ..., t_{14}\}$.

Let $\mathcal{H} = \langle t_1, ..., t_{14}\rangle$. It is clear that $L_3(2) \leq \mathcal{N}_{M_{22}}(\mathcal{H})$. Furthermore, $2^3 : L_3(2) \cong \langle L_3(2), t_1\bar{t}_8t_1\rangle$ is also a subgroup of the normalizer $\mathcal{N}_{M_{22}}(\mathcal{H})$. But $2^3 : L_3(2) \neq \mathcal{N}_{M_{22}}(\mathcal{H})$, since $t_1 \notin 2^3 : L_3(2)$. This implies that the maximal subgroup $2^3 : L_3(2)$ is a proper subgroup of $\mathcal{N}_{M_{22}}(\mathcal{H})$. Hence, $\mathcal{N}_{M_{22}}(\mathcal{H}) = M_{22}$, lest we contradict the maximality of $2^3 : L_3(2)$. But then $\mathcal{H} \lhd M_{22}$. Since $\mathcal{H}$ is nontrivial, we conclude $\mathcal{H} = M_{22}$. Moreover, $L_3(2)$ is the normal closure of $\{\langle t_1\rangle, ..., \langle t_{14}\rangle\}$ and acts transitively on the t_i's. □

Corollary 3.10. *M_{22} is a homomorphic image of the progenitor* $3^{*14} : L_3(2)$.

Proof. A symmetric generating set is supplied by Theorem 3.9. □

Theorem 3.11. *There exist a symmetric generating set $\mathcal{T} = \{t_1, ..., t_{14}\}$ of M_{22}, such that $|t_i| = 2$ and $\mathcal{N}_{M_{22}}(\bar{\mathcal{T}}) \cong L_3(2)$, where $\bar{\mathcal{T}} = \{\langle t_1\rangle, ..., \langle t_{14}\rangle\}$.*

Proof. As in Theorem 3.9, there is a copy of $L_3(2)$ within $2^3 : L_3(2)$ such that the stabilizer of a point is $\mathcal{A}_4$ and $\mathcal{C}_{M_{22}}(\mathcal{A}_4) \cong \mathcal{A}_4$. Take an involution with $\mathcal{C}_{M_{22}}(\mathcal{A}_4)$, say t. Then as before there are 14 conjugates of t under $L_3(2)$. We may label the conjugates as $t_1, ..., t_{14}$ so that the generators x and y of $L_3(2)$ act like the permutations $x = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$, $y = (1,12)(2,3)(4,11)(5,8)(6,13)(9,10)$ on the set $\{t_1, ..., t_{14}\}$.

Let $\mathcal{H} = \langle t_1, ..., t_{14}\rangle$. As before, we find that $2^3 : L_3(2) = \langle L_3(2), t_1t_8\rangle$ and $2^3 : L_3(2)$ is properly contained in $\mathcal{N}_{M_{22}}(\mathcal{H})$. Again, we must have that $\mathcal{H}$ is normal in M_{22} and so $\mathcal{H} = M_{22}$. Moreover, $L_3(2)$ is the normal closure of $\{\langle t_1\rangle, ..., \langle t_{14}\rangle\}$ and acts transitively on the t_i's. □

Corollary 3.12. *M_{22} is a homomorphic image of the progenitor* $2^{*14} : L_3(2)$.

Proof. A symmetric generating set is supplied by Theorem 3.11. □

3.3 The Progenitors $3^{*14} : L_3(2)$ and $2^{*14} : L_3(2)$

Since M_{22} is a homomorphic image of the progenitor $3^{*14} : L_3(2)$ and $2^{*14} : L_3(2)$ by Corollary 3.10 and Corollary 3.12, respectively, we are now in a position to find M_{22}. If we have the presentation $N = L_3(2) \cong \langle x, y | x^7, y^2, (xy)^3, (x,y)^4 \rangle$, then we may construct the progenitors via the familiar formula:

$$m^{*14} : L_3(2) = \langle x, y, t | x^7, y^2, (xy)^3, (x,y)^4, t^m, [N^7, t] \rangle.$$

In Theorem 3.9 and Theorem 3.11, $L_3(2)$ acted differently on the t_i's and so N^7 will not be the same in both cases. We present the progenitors now:

$$3^{*14} : L_3(2) = \langle x, y, t | x^7, y^2, (xy)^3, (x,y)^4, t^3, (t^{x^4}, xy), (t, y) \rangle,$$

where $x \sim (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$, $y \sim (1,8)(2,13)(3,10)(4,5)(6,9)(11,12)$ and $t \sim t_7$,

$$2^{*14} : L_3(2) = \langle x, y, t | x^7, y^2, (xy)^3, (x,y)^4, t^3, (t^{x^2}, yx^{-1}), (t, y) \rangle,$$

where $x \sim (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$, $y \sim (1,12)(2,3)(4,11)(5,8)(6,13)(9,10)$ and $t \sim t_7$.

Note, that since x and y act as permutations of the t_i's, we must have that x and y act identically on t_i^{-1}'s. This is apparent since $\pi^{-1} t_i \pi = t_i^{\pi} = t_j$, implies that $\pi^{-1} t_i^{-1} \pi = (t_i^{-1})^{\pi} = t_j^{-1}$, for $\pi \in L_3(2)$. With this in mind, we have written the generators x and y acting on $\{t_1, ..., t_{14}\}$ instead of $\{t_1, ..., t_{14}, \bar{t}_1, ..., \bar{t}_{14}\}$, where $t_i^{-1} = \bar{t}_i$.

We see that in both cases the two point stabilizer $N_{7,1}$ is trivial and so we are free of restrictions in our relations.

Table 3.1: Relations of the Progenitors $m^{*14} : L_3(2)$ That We Are Considering

$2^{*14} : L_3(2)$	$3^{*14} : L_3(2)$
$(xt_7)^a = 1$	$(xt_7)^a = 1$
$(yt_2)^b = 1$	$(x^{-1}t_7)^b = 1$
$(x^{-1}yxyt_2)^c = 1$	$(xyt_7)^c = 1$
$(xyt_7)^d = 1$	$(xyt_7^{-1})^d = 1$
	$(xyt_2)^e = 1$

Chapter 4

M_{22} as a Homomorphic Image of $3^{*14} : L_3(2)$

We will first enumerate the single cosets of the form $\mathcal{M}\omega$, where $L_3(2) \leq \mathcal{M} \leq \mathcal{G}$, ω is a word in the t_i's, and $\mathcal{G}$ is a factored progenitor. This will supply us with an action of $\mathcal{G}$ on the set of single cosets of $\mathcal{M}$ in $\mathcal{G}$. It will follow $\mathcal{G}$ acts faithfully and primitively on these single cosets. Furthermore, the subgroup $\mathcal{M}$ of $\mathcal{G}$ will possess a normal Abelian subgroup whose conjugates generate $\mathcal{G}$. Applying Iwasawa's lemma, we see that $\mathcal{G}$ is simple. Checking the ATLAS, [CCN+85], there is only one simple group of order $|\mathcal{G}|$, which is M_{22}.

Factor the progenitor $3^{*14} : L_3(2)$ by the relations $(xyt)^5, (xyt^{-1})^5, (xyt^{x^2})^5$ to obtain the following homomorphic image:

$$G = \frac{3^{*14} : L_3(2)}{(xyt)^5, (xyt^{-1})^5, (xyt^{x^2})^5}.$$

Set $\pi = xy$, then $\pi = (1, 13, 14)(2, 10, 12)(3, 5, 9)(6, 7, 8)$. The relation $(xyt)^5 = 1$ yields $\pi^2 t_7^{\pi} t_7 t_7^{\pi^2} t_7^{\pi} t_7 = 1$, which is:

$$\pi^2 t_8 t_7 = \bar{t}_7 \bar{t}_8 \bar{t}_6.$$

The relation $(xyt^{-1}) = 1$ yields $\pi^2 \bar{t}_7^{\pi} \bar{t}_7 \bar{t}_7^{\pi^2} \bar{t}_7^{\pi} \bar{t}_7 = 1$, where we have made the identification $t^{-1} = \bar{t}$. We arrive at:

$$\pi^2 \bar{t}_8 \bar{t}_7 = t_7 t_8 t_6.$$

The relation (xyt^{x^2}) yields $\pi^2 t_2^{\pi} t_2 t_2^{\pi^2} t_2^{\pi} t_2 = 1$, which is:

$$\pi^2 t_{10} t_2 = \bar{t}_2 \bar{t}_{10} \bar{t}_{12}$$

Consider the subgroup of $\mathcal{G}$ generated by $\mathcal{N} = L_3(2)$ and $t_1 \bar{t}_8 t_1$, say $\mathcal{M} = \langle \mathcal{N}, t_1 \bar{t}_8 t_1 \rangle$.

4.1 Some Relations

Being a coset enumeration process, double coset enumeration involves knowing relations among the cosets of $\mathcal{M}$ in $\mathcal{G}$. For example, $\mathcal{M} t_7 \bar{t}_{14} t_7 = \mathcal{M}$ which means $\mathcal{M} t_7 t_{14} = \mathcal{M} \bar{t}_7 \bar{t}_{14}$ by right multiplication. Define an equivalence relation $\sim$ on the set of words on $\{t_i, t_i^{-1}\}$ by $\omega \sim \omega'$ if $\mathcal{M}\omega = \mathcal{M}\omega'$. Since the set of single cosets of $\mathcal{M}$ in $\mathcal{G}$ partition $\mathcal{G}$, the relation $\sim$ is a well defined equivalence relation. Note, that any element of $\mathcal{G}$ is of the form $\pi\omega$, where $\pi \in \mathcal{N}$ and ω is a word. Hence, we only require $\sim$ defined on the set of words on $\{t_i, t_i^{-1}\}$: for if $\pi\omega \sim \pi'\omega'$, then $\mathcal{M}\pi\omega = \mathcal{M}\omega$ and $\mathcal{M}\pi'\omega' = \mathcal{M}\omega'$. We conclude $\mathcal{M}\omega = \mathcal{M}\omega'$ and so $\omega \sim \omega'$.

Any relation in the presentation gives a strict equality among the elements. However, the relation $\sim$ gives no such promise. While equality is more desirable, we often can only guarantee $\sim$ holds. Let us now prove some relations.

Since $t \sim t_7$, it is beneficial to write the relations in $t_7 t_i \sim t_j t_k t_l$ form. We begin with $\pi^2 t_8 t_7 = \bar{t}_7 \bar{t}_8 \bar{t}_6$. Conjugating by $(1,14,8,7)(2,6,5,11)(3,10)(4,9,13,12)$, we have the following relation:

$$(1,5,7)(3,6,4)(8,12,14)(10,13,11) t_7 t_1 = \bar{t}_1 \bar{t}_7 \bar{t}_5.$$

We conjugate the relation $\pi^2 t_{10} t_2 = \bar{t}_2 \bar{t}_{10} \bar{t}_{12}$ by $(1,5,10,7,11,13,2)(3,14,4,6,9,8,12)$ and obtain:

$$(1,3,7)(2,5,4)(8,10,14)(9,12,11) t_7 t_1 = \bar{t}_1 \bar{t}_7 \bar{t}_3.$$

We will see later that $t_7 t_1 \sim t_1 t_7$. So relations involving $t_7 t_i$ are useful. Conjugating both of the relations above by $(1,7)(2,12)(4,11)(5,9)(6,13)(8,14)$, yields:

$$(1,7,9)(2,8,14)(3,13,11)(4,10,6) t_1 t_7 = \bar{t}_7 \bar{t}_1 \bar{t}_9,$$

and

$$(1,7,3)(2,4,5)(8,14,10)(9,11,12) t_1 t_7 = \bar{t}_7 \bar{t}_1 \bar{t}_3.$$

Lemma 4.1. $t_7 t_1 \bar{t}_2 \sim \bar{t}_6 t_3 t_7$

Proof. Since $\pi^2 t_8 t_7 = \bar{t}_7 \bar{t}_8 \bar{t}_6$, we have that if $\tau = (1,5,8,12)(2,4,7,10)(3,9,11,14)(6,13)$, then $(\pi^2)^\tau t_{12} t_{10} t_{13} = \bar{t}_{10} \bar{t}_{12}$. Now by the relation $\pi^2 t_{10} t_2 = \bar{t}_2 \bar{t}_{10} \bar{t}_{12}$, we have:

$$\begin{aligned} \pi^2 t_{10} t_2 &= \bar{t}_2 (\pi^2)^\tau t_{12} t_{10} t_{13}, \\ &= (\pi^2)^\tau \bar{t}_4 t_{12} t_{10} t_{13}. \end{aligned}$$

Now conjugate by $(1,5,10,7,11,13,2)(3,14,4,6,9,8,12)$. □

4.2 $\mathcal{M} \cong 2^3 : L_3(2)$

Lemma 4.2. $t_1 \bar{t}_8 t_1$ *has order 2.*

Proof. See Appendix E for code. □

Lemma 4.3. $t_1 \bar{t}_8 t_1$ *has 7 distinct conjugates under* $L_3(2)$.

Proof. The element $t_1 \bar{t}_8 t_1$ is an involution and so $t_1 \bar{t}_8 \bar{t}_1 \bar{t}_8 t_1 = 1$ and $t_8 t_1 t_8 = \bar{t}_1$. Hence $t_1 \bar{t}_8 \bar{t}_1 \bar{t}_8 t_1 \bar{t}_1 t_8 = \bar{t}_1 t_8$ and so $t_8 t_1 \bar{t}_8 \bar{t}_1 = t_8 \bar{t}_1 t_8$. Finally, $\bar{t}_1 t_8 \bar{t}_1 = t_8 \bar{t}_1 t_8$. □

Lemma 4.4. *Let* $t_{1,8} = t_1 \bar{t}_8 t_1$. *Then* $\mathcal{N}$ *acts as* $L_3(2)$ *on* $\{t_{1,8}, \ldots, t_{7,14}\}$.

Proof. Follows from the identification $t_1 \bar{t}_8 t_1 = t_8 \bar{t}_1 t_8$. □

Lemma 4.5. *The group* $\langle t_{1,8}, \ldots, t_{7,14} \rangle$ *is an elementary Abelian 2-group of order* 2^3.

Proof. We compute $t_{1,8} t_{2,9} = t_{4,11} = (t_{4,11})^{-1} = (t_{1,8} t_{2,9})^{-1}$. Hence, every element of $\langle t_{1,8}, \ldots, t_{7,14} \rangle$ is an involution. Now $t_{5,12} = t_{2,9} t_{3,10}$, $t_{6,13} = t_{3,10} t_{4,11} = t_{3,10} t_{1,8} t_{2,9}$, and $t_{7,14} = t_{4,11} t_{2,9} t_{3,10} = t_{1,8} t_{2,9} t_{2,9} t_{3,10} = t_{1,8} t_{3,10}$. Thus $t_{4,11}$, $t_{5,12}$, $t_{6,13}$, and $t_{7,14}$ may be omitted from the generating set. We have $\langle t_{1,8}, t_{2,9}, t_{3,10} \rangle$ is elementary Abelian of order 2^3. □

Theorem 4.6. $\mathcal{M} \cong 2^3 : L_3(2)$.

Proof. Let $\mathcal{H}$ be the elementary Abelian 2-group of order 2^3. Then $\mathcal{N}$ acts as $L_3(2)$ on $\mathcal{H}$ and so

$$\mathcal{M} = \langle \mathcal{N}, \mathcal{H} \rangle = \mathcal{H} : \mathcal{N} \cong 2^3 : L_3(2).$$

□

4.3 Double Coset Enumeration over $2^3 : L_3(2)$

We proceed to do manual double coset enumeration over $\mathcal{M}$. Where we use the notation $[\omega]$ to be the double coset $\mathcal{M}\omega\mathcal{N}$, where ω is a word in the t_i's.

Throughout the process, we will consider orbits on $\{t_1, ..., t_{14}\}$. The orbits on $\{\bar{t}_1, ..., \bar{t}_{14}\}$ will be the same since $\mathcal{N}$ acts the same on the inverses.

4.3.1 $\mathcal{M}e\mathcal{N}$

We begin with the double coset $\mathcal{M}e\mathcal{N}$, which we will denote $[e]$. This coset has one single coset in it, namely $\mathcal{M}$. The single coset stabiliser is then just $\mathcal{M}$, which has two orbits:

$$\mathcal{O} = \{\{1, ...14\}\}.$$

So that we take an element from each orbit say t_7 and $\bar{t}_7$ and multiply the single coset representative $\mathcal{M}$ by each to obtain $\mathcal{M}t_7$ and $\mathcal{M}\bar{t}_7$. We have two new double cosets $\mathcal{M}t_7\mathcal{N}$, denote it $[7]$, and $\mathcal{M}\bar{t}_7\mathcal{N}$, denote it $[\bar{7}]$.

4.3.2 $\mathcal{M}t_7\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7\mathcal{N}$ we find the single coset stabiliser by first computing the point stabiliser $\mathcal{N}^7$. This is found to be

$$\mathcal{N}^7 \geq \langle (1,8)(2,13)(3,10)(4,5)(6,9)(11,12), (1,6,12)(2,11,3)(4,10,9)(5,8,13) \rangle.$$

Since $|\mathcal{N}^{(7)}| \geq 12$, the number of elements in $[7]$ is $168/12 \leq 14$. Furthermore, the orbits of $\mathcal{N}^{(7)}$ on $\{t_1, ..., t_{14}\}$ are:

$$O = \{\{7\}, \{14\}, \{1, ..., 13\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7$ of the double coset $\mathcal{M}t_7\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7t_7 &= \mathcal{M}\bar{t}_7 \in [\bar{7}], \\
\mathcal{M}t_7t_1 &\in [7,1], \\
\mathcal{M}t_7t_{14} &\in [7,14], \\
\mathcal{M}t_7\bar{t}_7 &= \mathcal{M} \in [*], \\
\mathcal{M}t_7\bar{t}_1 &\in [7,\bar{1}], \\
\mathcal{M}t_7\bar{t}_{14} &= N\bar{t}_7 \in [\bar{7}].
\end{aligned}$$

We see that the new double cosets are $[7,1], [7,14], [7,\bar{1}]$.

4.3.3 $\mathcal{M}\bar{t}_7\mathcal{N}$

We have that $\mathcal{N}^{(\bar{7})} \geq \mathcal{N}^{(7)}$ and so the orbits are the same as the previous section. We again take the single coset representative and multiply on the right by an element from each of the six orbits. We have:

$$\begin{aligned}
\mathcal{M}\bar{t}_7t_7 &= \mathcal{M} \in [*], \\
\mathcal{M}\bar{t}_7t_1 &= \mathcal{M}t_3\bar{t}_{11}[7,\bar{1}], \\
\mathcal{M}\bar{t}_7t_{14} &= \mathcal{M}t_7 \in [7], \\
\mathcal{M}\bar{t}_7\bar{t}_7 &= \mathcal{M}t_7 \in [7], \\
\mathcal{M}\bar{t}_7\bar{t}_1 &\in [\bar{7},\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_{14} &= \mathcal{M}\bar{t}_7t_{14}t_{14} = \mathcal{M}t_7t_{14} \in [7,14].
\end{aligned}$$

We see that there is only one new double coset, which is $[\bar{7},\bar{1}]$.

4.3.4 $\mathcal{M}t_7t_1\mathcal{N}$

We have that $N^{7,1}$ is trivial. The relation $\mathcal{M}t_7t_1 = \mathcal{M}t_1t_7$ adds the element $\pi = (1,7)(2,12)(4,11)(5,9)(6,13)(8,14)$ to $\mathcal{N}^{(71)}$. We have $\mathcal{N}^{(7,1)} \geq \langle\pi\rangle$.

Since $|\mathcal{N}^{(7,1)}| \geq 2$, we have that $[7,1]$ contains $|\mathcal{N}|/|\mathcal{N}^{(7,1)}| \leq 84$ single cosets. The orbits of $\mathcal{N}^{(7,1)}$ are

$$\mathcal{O} = \{\{1,7\},\{2,12\},\{3\},\{4,11\},\{5,9\},\{6,13\},\{8,14\},\{10\}\}.$$

Taking an element from each orbit and multipling the single coset representative $\mathcal{M}t_7t_1$ on the right we arrive at:

$$\begin{aligned}
\mathcal{M}t_7t_1t_1 &= \mathcal{M}t_7\bar{t}_1 \in [7,\bar{1}],\\
\mathcal{M}t_7t_1t_2 &= \mathcal{M}\bar{t}_4\bar{t}_3 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7t_1t_3 &= \mathcal{M}\bar{t}_7\bar{t}_1 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7t_1t_4 &= \mathcal{M}t_6t_2 \in [71],\\
\mathcal{M}t_7t_1t_5 &= \mathcal{M}\bar{t}_7\bar{t}_1 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7t_1t_6 &= \mathcal{M}\bar{t}_4\bar{t}_5 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7t_1t_8 &= \mathcal{M}t_2t_1 \in [7,1],\\
\mathcal{M}t_7t_1t_{10} &= \mathcal{M}t_6t_{13} \in [7,14],
\end{aligned}$$

and now by the inverses:

$$\begin{aligned}
\mathcal{M}t_7t_1\bar{t}_1 &= \mathcal{M}t_7 \in [7],\\
\mathcal{M}t_7t_1\bar{t}_2 &= \mathcal{M}t_9\bar{t}_{10} \in [7,\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_3 &\in [7,1,\bar{3}],\\
\mathcal{M}t_7t_1\bar{t}_4 &= \mathcal{M}t_5t_3 \in [7,1],\\
\mathcal{M}t_7t_1\bar{t}_5 &= \mathcal{M}t_3t_4\bar{t}_6 \in [7,1,\bar{3}],\\
\mathcal{M}t_7t_1\bar{t}_6 &= \mathcal{M}t_2\bar{t}_5 \in [7,\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_8 &= \mathcal{M}t_1t_{13} \in [7,1],\\
\mathcal{M}t_7t_1\bar{t}_{10} &= \mathcal{M}\bar{t}_7\bar{t}_1 \in [\bar{7},\bar{1}].
\end{aligned}$$

4.3.5 $\mathcal{M}t_7t_{14}\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_{14}\mathcal{N}$ we find the point stabiliser satisfies $\mathcal{N}^{7,14} = \mathcal{N}^7$. Since $|t_7\bar{t}_{14}t_7| = 2$, we have

$$t_7t_{14}\bar{t}_7\bar{t}_{14} \sim t_7\bar{t}_{14}\bar{t}_{14}\bar{t}_7\bar{t}_{14} = t_7\bar{t}_{14}t_7 \sim e,$$

where $\bar{t}_{14}\bar{t}_7\bar{t}_{14} = t_7$. This relation implies $\mathcal{M}t_7t_{14} = \mathcal{M}t_{14}t_7$ and expands the single coset stabilizer to:

$$\mathcal{N}^{(7,14)} \geq \langle N^{7,14}, (1,9,10,13)(2,3,6,8)(4,12,11,5)(7,14)\rangle$$

Since $|\mathcal{N}^{(7,14)}| \geq 24$, the number of elements in $[7,14]$ is $168/24 \leq 7$. Furthermore, the orbits of $\mathcal{N}^{(7,14)}$ on $\{t_1, ..., t_{14}\}$ are:

$$\mathcal{O} = \{\{7,14\}, \{1,...,13\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_{14}$ of the double coset $\mathcal{M}t_7t_{14}\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7t_{14}t_1 &= \mathcal{M}\bar{t}_5\bar{t}_{13} \in [\bar{7}\bar{1}], \\
\mathcal{M}t_7t_{14}\bar{t}_1 &= \mathcal{M}t_5t_{13} \in [7,1], \\
\mathcal{M}t_7t_{14}t_{14} &= \mathcal{M}\bar{t}_7 \in [\bar{7}], \\
\mathcal{M}t_7t_{14}\bar{t}_{14} &= \mathcal{M}t_7 \in [7],
\end{aligned}$$

4.3.6 $\mathcal{M}t_7\bar{t}_1\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7\bar{t}_1\mathcal{N}$, we find $\mathcal{N}^{7,\bar{1}} = \mathcal{N}^{7,1} = 1$. The relation $\mathcal{M}t_7\bar{t}_1 = \mathcal{M}t_{13}\bar{t}_8$ expands the single coset stabilizer to:

$$\mathcal{N}^{(7,\bar{1})} \geq \langle (1,8)(3,5)(4,11)(6,14)(7,13)(10,12)\rangle.$$

Since $|\mathcal{N}^{(7,\bar{1})}| \geq 2$, the number of elements in $[7,\bar{1}]$ is $168/2 \leq 84$. Furthermore, the orbits of $\mathcal{N}^{(7,\bar{1})}$ are:

$$\mathcal{O} = \{\{1,8\}, \{2\}, \{3,5\}\{4,11\}, \{6,14\}, \{7,13\}, \{9\}, \{10,12\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7\bar{t}_1$ of the double coset $\mathcal{M}t_7\bar{t}_1\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7\bar{t}_1t_1 &= \mathcal{M}t_7 \in [7],\\
\mathcal{M}t_7\bar{t}_1t_2 &= \mathcal{M}t_3t_{14}\bar{t}_8[7,1,\bar{3}],\\
\mathcal{M}t_7\bar{t}_1t_3 &= \mathcal{M}t_5\bar{t}_4 \in [7,\bar{1}],\\
\mathcal{M}t_7\bar{t}_1t_4 &= \mathcal{M}\bar{t}_4\bar{t}_5 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7\bar{t}_1t_6 &= \mathcal{M}t_2t_4 \in [7,1],\\
\mathcal{M}t_7\bar{t}_1t_7 &= \mathcal{M}t_1t_5\bar{t}_6 \in [7,1,\bar{3}],\\
\mathcal{M}t_7\bar{t}_1t_9 &= \mathcal{M}t_3\bar{t}_{11} \in [7,\bar{1}],\\
\mathcal{M}t_7\bar{t}_1t_{10} &= \mathcal{M}t_7t_4 \in [7,1],
\end{aligned}$$

and by the inverses:

$$\begin{aligned}
\mathcal{M}t_7\bar{t}_1\bar{t}_1 &= \mathcal{M}t_7t_1 \in [7,1],\\
\mathcal{M}t_7\bar{t}_1\bar{t}_2 &= \mathcal{M}t_3\bar{t}_{11} \in [7,\bar{1}],\\
\mathcal{M}t_7\bar{t}_1\bar{t}_3 &= \mathcal{M}t_2t_3\bar{t}_5 \in [7,1,\bar{3}],\\
\mathcal{M}t_7\bar{t}_1\bar{t}_4 &= \mathcal{M}\bar{t}_5 \in [\bar{7}],\\
\mathcal{M}t_7\bar{t}_1\bar{t}_6 &= \mathcal{M}\bar{t}_3\bar{t}_1 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7\bar{t}_1\bar{t}_7 &= \mathcal{M}t_5\bar{t}_4 \in [7,\bar{1}],\\
\mathcal{M}t_7\bar{t}_1\bar{t}_9 &= \mathcal{M}t_{10}t_{13}\bar{t}_8 \in [7,1,\bar{3}],\\
\mathcal{M}t_7\bar{t}_1\bar{t}_{10} &= \mathcal{M}\bar{t}_8\bar{t}_9 \in [\bar{7},\bar{1}].
\end{aligned}$$

4.3.7 $\mathcal{M}\bar{t}_7\bar{t}_1\mathcal{N}$

Continuing with the double coset $\mathcal{M}\bar{t}_7\bar{t}_1\mathcal{N}$, we find $\mathcal{N}^{\bar{7}\bar{1}} = \mathcal{N}^{71} = 1$. The relation $\mathcal{M}\bar{t}_7\bar{t}_1 = \mathcal{M}\bar{t}_1\bar{t}_7$ expands the single coset stabilizer to:

$$\mathcal{N}^{(\bar{7},\bar{1})} \geq \langle(1,7)(2,12)(4,11)(5,9)(6,13)(8,14)\rangle.$$

Since $|\mathcal{N}^{(\bar{7},\bar{1})}| \geq 2$, the number of elements in $[\bar{7},\bar{1}]$ is $168/2 \leq 84$. Furthermore, the orbits of $\mathcal{N}^{(\bar{7},\bar{1})}$ are:

$$\mathcal{O} = \{\{1,7\},\{2,12\},\{3\}\{4,11\},\{5,9\},\{6,13\},\{8,14\},\{10\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}\bar{t}_7\bar{t}_1$ of the double coset $\mathcal{M}\bar{t}_7\bar{t}_1\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}\bar{t}_7\bar{t}_1t_1 &= \mathcal{M}\bar{t}_7 \in [\bar{7}], \\
\mathcal{M}\bar{t}_7\bar{t}_1t_2 &= \mathcal{M}t_5\bar{t}_1 \in [7,\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_1t_3 &= \mathcal{M}t_7t_1\bar{t}_3 \in [7,1,\bar{3}], \\
\mathcal{M}\bar{t}_7\bar{t}_1t_4 &= \mathcal{M}\bar{t}_5\bar{t}_3 \in [\bar{7},\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_1t_5 &= \mathcal{M}t_1t_{12}\bar{t}_{13} \in [7,1,\bar{3}], \\
\mathcal{M}\bar{t}_7\bar{t}_1t_6 &= \mathcal{M}t_{13}\bar{t}_{14} \in [7,\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_1t_8 &= \mathcal{M}\bar{t}_1\bar{t}_{13} \in [\bar{7},\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_1t_{10} &= \mathcal{M}t_7t_1 \in [7,1],
\end{aligned}$$

and now by the inverses:

$$\begin{aligned}
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_1 &= \mathcal{M}\bar{t}_7t_1 = \mathcal{M}t_3\bar{t}_{11} \in [7,\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_2 &= \mathcal{M}t_4t_3 \in [7,1], \\
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_3 &= \mathcal{M}t_7t_1 \in [7,1], \\
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_4 &= \mathcal{M}\bar{t}_6\bar{t}_2 \in [\bar{7},\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_5 &= \mathcal{M}t_7t_1 \in [7,1], \\
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_6 &= \mathcal{M}t_4t_5 \in [7,1], \\
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_8 &= \mathcal{M}\bar{t}_2\bar{t}_1 \in [\bar{7},\bar{1}], \\
\mathcal{M}\bar{t}_7\bar{t}_1\bar{t}_{10} &= \mathcal{M}t_6t_{13} \in [7,14],
\end{aligned}$$

4.3.8 $\mathcal{M}t_7t_1\bar{t}_3\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_1\bar{t}_3\mathcal{N}$, we find the point stabilizer to be trivial. The relation $\mathcal{M}t_7t_1\bar{t}_3 = \mathcal{M}t_1t_7\bar{t}_3$ adds the element

$$(1,7)(2,12)(4,11)(5,9)(6,13)(8,14)$$

to the coset stabilizer. The relation $\mathcal{M}t_7t_1\bar{t}_3 = \mathcal{M}t_5t_9\bar{t}_{10}$ adds the element

$$(1,9)(2,8)(3,10)(5,7)(6,13)(12,14)$$

to the coset stabilizer. We conclude

$$\mathcal{N}^{(7,1,\bar{3})} \geq \langle (1,7)(2,12)(4,11)(5,9)(6,13)(8,14), (1,9)(2,8)(3,10)(5,7)(6,13)(12,14) \rangle.$$

Since $|\mathcal{N}^{(7,1,\bar{3})}| \geq 4$, the number of elements in $[7,1,\bar{3}]$ is $168/4 \leq 42$. Furthermore, the orbits of $\mathcal{N}^{(7,1,\bar{3})}$ are:

$$\mathcal{O} = \{\{1,5,7,9\},\{2,8,12,14\},\{3,10\}\{4,11\},\{6,13\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_1\bar{t}_3$ of the double coset $\mathcal{M}\bar{t}_7t_1\bar{t}_3\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7t_1\bar{t}_3t_7 &= \mathcal{M}t_4\bar{t}_6 \in [7,\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_3t_{14} &= \mathcal{M}t_{11}t_1 \in [7,1],\\
\mathcal{M}t_7t_1\bar{t}_3t_3 &= \mathcal{M}t_7t_1 \in [7,1],\\
\mathcal{M}t_7t_1\bar{t}_3t_4 &= \mathcal{M}t_7\bar{t}_6 \in [7,\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_3t_6 &= \mathcal{M}t_9t_5\bar{t}_{10} \in [7,1,\bar{3}]
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}t_7t_1\bar{t}_3\bar{t}_7 &= \mathcal{M}t_7\bar{t}_5 \in [7,\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_3\bar{t}_{14} &= \mathcal{M}\bar{t}_{11}\bar{t}_1 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_3\bar{t}_3 &= \mathcal{M}t_7t_1t_3 = \mathcal{M}\bar{t}_7\bar{t}_1 \in [\bar{7},\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_3\bar{t}_4 &= \mathcal{M}t_8\bar{t}_{10} \in [7,\bar{1}],\\
\mathcal{M}t_7t_1\bar{t}_3\bar{t}_6 &= \mathcal{M}t_1t_7\bar{t}_3 \in [7,1,\bar{3}]
\end{aligned}$$

We see the set of cosets $\mathcal{M}\omega$ is closed under multiplication by $\{t_i\} \cup \{t_i^{-1}\}$. Hence, we have arrived at a full list of single cosets.

4.4 The Cayley Graph of $\mathcal{G}$ Over $\mathcal{M}$

We now represent the process of double coset enumeration as a Cayley graph. The circles represent double cosets and lines represent multiplication by t_i's. The numbers inside of the circles represent the number of single cosets within the double coset, while the numbers on the outside of the circles indicate the number of t_i's going to the next double coset.

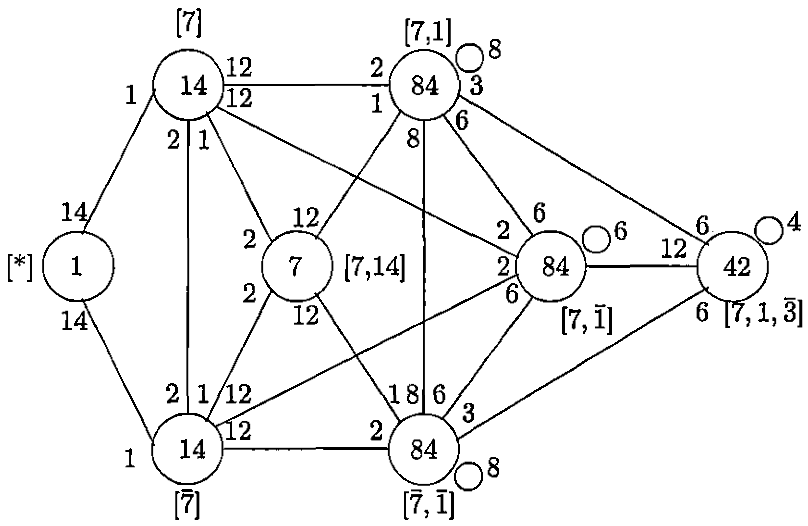

Figure 4.1: The Cayley Graph of $\mathcal{G}$ Over $\mathcal{M}$

4.5 $\mathcal{G} \cong M_{22}$

We will use Iwasawa's Lemma and the transitive action of $\mathcal{G}$ on the set of single cosets $\{\mathcal{M}\omega | \omega \text{ is a word in the } t_i's\}$.

Lemma 4.7. *The order of $\mathcal{G}$ is 443,520. Furthermore, $\mathcal{G}$ acts faithfully on the set $\{\mathcal{M}\omega | \omega$ is a word in the $t_i's\}$.*

Proof. Since $\Omega = \{\mathcal{M}\omega\}$ is a transitive $\mathcal{G}$-set of degree 330, we have:

$$|\mathcal{G}| = 330|\mathcal{G}^1|,$$

where $\mathcal{G}^1$ is the stabilizer of the single coset $\mathcal{M}$. But $\mathcal{M}$ is only stabilized by elements of $\mathcal{M}$. Hence $\mathcal{G}^1 = \mathcal{M}$ and $|\mathcal{G}^1| = |\mathcal{M}| = 1344$. We conclude that $|\mathcal{G}| = 443,520$. Furthermore, we must have that Ω is faithful, lest $|\mathcal{G}| > 443,520$. □

Lemma 4.8. *The group $\mathcal{G}$ acts primitively on $\{\mathcal{M}\omega\}$.*

Proof. Since $\mathcal{G}$ is transitive, if B is a nontrivial block then we may assume that $\mathcal{M} \in B$. If $\mathcal{M}t_i \in B$, then $\mathcal{M}t_i^{\mathcal{N}} \in B$: for $\mathcal{M} \in B$ and $\mathcal{M}\mathcal{N} = \mathcal{M}$ implies $B\mathcal{N} = B$. Similarly, if $\mathcal{M}\bar{t}_i \in B$, then $\mathcal{M}\bar{t}_i^{\mathcal{N}} \in B$. We show that if $\mathcal{M}t_i \in B$, then $B = \{\mathcal{M}\omega\}$. Suppose $\mathcal{M}t_7 \in B$, then $\mathcal{M}t_7\bar{t}_7 = \mathcal{M} \in B$ implies $B\bar{t}_7 = B$. Hence, $B = Bt_7$. Furthermore, $B = B\mathcal{N} = Bt_7\mathcal{N} = B\bar{t}_7\mathcal{N}$ implies that multiplication under t_i's and $\bar{t}_i$'s stabilizes B. But this is exactly coset enumeration, hence $B = \{\mathcal{M}\omega\}$.

Now suppose B is any block not containing an element $\mathcal{M}t_i$ for $i = 1, ..., 14$. By the Cayley graph, we may assume that $B = \{\mathcal{M}, \mathcal{M}t_7t_{14}, ..., \mathcal{M}t_1t_8\}$: for the other double cosets are stabilized by a t_i and so $\mathcal{M}t_i \in B$. But since $|B|$ must divide $|\{\mathcal{M}\omega\}| = 330$, we cannot have $|B| = 8$. We conclude the action is primitive. □

Lemma 4.9. *The group $\mathcal{G}$ is perfect.*

Proof. Since $\mathcal{G} = \langle \mathcal{N}, t \rangle$, we have that $\mathcal{N} \leq \mathcal{G}'$, for $\mathcal{N}$ is simple and therefore perfect. We show that $t \in \mathcal{G}$. We consider the following two commutators: $[t_7t_6, t_6\bar{t}_7]$, $[t_7, t_6]$. Evaluating the first, we have:

$$\begin{aligned}[t_7t_6, t_6\bar{t}_7] &= t_7t_6t_6\bar{t}_7\bar{t}_6\bar{t}_7t_7t_6, \\ &= t_7\bar{t}_6\bar{t}_7\bar{t}_6t_6, \\ &= t_7\bar{t}_6\bar{t}_7.\end{aligned}$$

Evaluating the second:

$$[t_7, t_6] = t_7t_6\bar{t}_7\bar{t}_6,$$

But then $[t_7t_6, t_6\bar{t}_7][t_7, t_6] = \bar{t}_6$. Thus $\mathcal{G}' \geq \langle \mathcal{N}, \bar{t}_6 \rangle = \mathcal{G}$. We conclude that $\mathcal{G}$ is perfect. □

Lemma 4.10. *The point stabilizer $\mathcal{M}$ of $\mathcal{G}$ posseses an Abelian normal subgroup K whose conjugates generate $\mathcal{G}$.*

Proof. Since $\mathcal{M} \cong 2^3 : L_3(2)$, we have the normal Abelian subgroup $\mathcal{K} = \langle t_i\bar{t}_{i+7}t_i \rangle \cong 2^3$, for $i = 1, ..., 7$. Since $t_1\bar{t}_8t_1 \in K$ is an involution, we want $t_8\bar{t}_1 \in K$, for then $t_8\bar{t}_1t_1\bar{t}_8t_1 = t_1 \in K$.

Now $t_8t_1, t_1t_8 \in \mathcal{K}$, for $t_1\bar{t}_8t_1^{t_1} = \bar{t}_8\bar{t}_1$ and $t_1\bar{t}_8t_1^{\bar{t}_1} = \bar{t}_1\bar{t}_8$. Hence, by conjugation we must have $t_it_{i+7}, t_{i+7}t_i \in \mathcal{K}$. Now consider the elements t_1t_8, t_7t_{14}. We have the product:

$$\omega = t_1t_8t_7t_{14} = t_1\pi t_7t_8t_6t_{14} = \pi t_{13}t_7t_8t_6t_{14}.$$

But then $\omega t_{14}t_7 = \pi t_{13}t_7t_8t_6t_7 = \omega'$. Since $t_8t_6t_7 = \pi^2\bar{t}_6\bar{t}_8$, we have $\omega' = t_{13}^{\pi^2}t_7^{\pi^2}\bar{t}_6\bar{t}_8$. This simplifies to $t_1t_6\bar{t}_6\bar{t}_8 = t_1\bar{t}_8$. Finally, $(t_1\bar{t}_8)^{-1} = t_8\bar{t}_1 \in \mathcal{K}$. Hence, $\mathcal{K} = \langle t_1, ..., t_{14}\rangle$.

It remains to be shown that $x, y \in \mathcal{K}$. From the factored relations, we have $xy \in \mathcal{K}$. Moreover, let $xy = \omega$, where ω is either relation. Then $x = \omega y$ and $y = x^{-1}\omega$. Now $x = \omega y = \omega x^{-1}\omega$. But conjugating by x we have $x = x^{-1}\omega\omega^x$. Now $x^2 = \omega\omega^x$. Since $\omega, \omega^x \in \mathcal{K}$, we have $x^2 \in \mathcal{K}$. Of course, we have $x \in \mathcal{K}$ and since $xy = \omega$, $y \in \mathcal{K}$. □

Theorem 4.11. *The group $\mathcal{G}$ is simple. Furthermore, $\mathcal{G} \cong M_{22}$.*

Proof. We have that $\mathcal{G}$ is a perfect group acting faithfully and primitively on $\{\mathcal{M}\omega\}$. The stabilizer of the single coset $\mathcal{M}$ possesses a normal Abelian subgroup $\mathcal{K} \cong 2^3$ whose conjugates generate $\mathcal{G}$. By Iwasawa's lemma, $\mathcal{G}$ is a simple group. But $|\mathcal{G}| = 443,520$ and a quick check in the ATLAS, [CCN+85], and [Par70] shows there is only one simple group of this order, M_{22}. We conclude $\mathcal{G} \cong M_{22}$. □

Chapter 5

M_{22} as a Homomorphic Image of $2^{*14} : L_3(2)$

To prove the result, we adopt the same approach as in Chapter 4. That is, we will find a faithful and primitive action of $\mathcal{G}$ on 330 points that satisfies Iwasawa's lemma. It will then follow $\mathcal{G}$ is isomorphic to M_{22}.

Factor the progenitor $2^{*14} : L_3(2)$ by the relations $(yt^{x^2})^5, (xyt)^{11}, (yt^x t)^3$ to obtain the homomorphic image:

$$\mathcal{G} = \frac{2^3 : L_3(2)}{(yt^{x^2})^5, (xyt)^{11}, (yt^x t)^3}.$$

Now $(yt^{x^2})^5 = 1$ can be written as $1 = (yt_2)^5 = yt_2 t_2^y t_2 t_2^y t_2$, which is the relation:

$$yt_2t_3 = t_2t_3t_2.$$

Let $\pi = xy = (1,3,11)(4,8,10)(5,13,14)(6,7,12)$, then $(xyt)^{11} = 1$ can be written as $1 = (\pi t_7)^{11}$, which yields the following calculation:

$$\begin{aligned} 1 &= (\pi t_7)^{11}, \\ &= \pi^2 t_7^{\pi} t_7 t_7^{\pi^2} t_7^{\pi} t_7 t_7^{\pi^2} t_7^{\pi} t_7 t_7^{\pi^2} t_7^{\pi} t_7, \\ &= \pi t_{12} t_7 t_6 t_{12} t_7 t_6 t_{12} t_7 t_6 t_{12} t_7. \end{aligned}$$

Thus, we have the relation:

$$\pi t_{12} t_7 t_6 t_{12} t_7 = t_7 t_{12} t_6 t_7 t_{12} t_6.$$

Now $(yt^x t)^3 = 1$ can be written as $1 = (yt_1t_7)^3 = yt_1t_7t_1^y t_7^y t_1 t_7$, which is the relation:

$$yt_1t_7t_{12} = t_7t_1t_7.$$

Define the subgroup $\mathcal{M}$ of $\mathcal{G}$ to be the group generated by the control group $\mathcal{N} = L_3(2)$ and $t_7t_{14} = tt^{xyx^2}$. That is,

$$\mathcal{M} = \langle \mathcal{N}, tt^{xyx^2} \rangle.$$

We decompose $\mathcal{G}$ into the double cosets $\mathcal{M}\omega\mathcal{N}$, where ω is a word in the t_i's, via double coset enumeration.

5.1 Some Relations

Lemma 5.1. *$t_it_{i+1} \sim t_it_{i+1}t_i$ for $i = 1, ..., 6$.*

Proof. Consider the relation $t_2t_3 \sim t_2t_3t_2$. By conjugating this relation by powers of x, the result follows immediately. □

Lemma 5.2. *For $j \neq i + 7$, $t_it_j \sim t_it_jt_i$.*

Proof. Apply Lemma 1.1, with $i = 1$ to get $t_1t_2 \sim t_1t_2t_1$. We now compute the stabilizer $\mathcal{N}^0$ of t_0. This is computed to be:

$$\mathcal{N}^0 = \langle (2, 11, 12)(3, 7, 13)(4, 5, 9)(6, 10, 14), (2, 4, 14)(3, 12, 6)(5, 13, 10)(7, 9, 11) \rangle.$$

The orbits of $\mathcal{N}^0$ are $\{\{t_1\}, \{t_8\}, \{t_2, t_3, ..., t_7, t_9, ..., t_{14}\}\}$. Since $\mathcal{N}$ is transitive, we conclude that this holds for all other i. □

Lemma 5.3. *For $i = 1, ..., 6$, $t_it_{i+1} \sim t_it_{i+8}$ and $t_7t_1 \sim t_7t_8$.*

Theorem 5.4. *$t_7t_1t_5 \sim t_8t_7$.*

Proof. Consider the relation $t_1t_7t_{12} \sim t_7t_1t_7 \sim t_7t_1$. Now if we conjugate by

$$x^{-1}yx^3yx^{-2} \sim (1, 7, 8, 14)(2, 3, 11, 6)(4, 13, 9, 10)(5, 12),$$

then we get $t_7t_8t_5 \sim t_8t_7$. But by the previous lemma, $t_7t_8 \sim t_7t_1$. Thus $t_7t_1t_5 \sim t_8t_7$. □

5.2 $\mathcal{M} \cong 2^3 : L_3(2)$

Lemma 5.5. *$t_i t_{i+7}$ has order 2.*

Proof. See Appendix E □

Lemma 5.6. *$t_i t_{i+7}$ has 7 conjugates under the action of $\mathcal{N}$.*

Proof. Since $t_1 t_8$ has inverses $t_1 t_8$ and $t_8 t_1$, by uniqueness, we must have $t_1 t_8 = t_8 t_1$. But $t_1 t_8$ has 14 conjugates under the action of $\mathcal{N}$ and so by above there are only 7 conjugates. □

Lemma 5.7. *Let $s_i = t_i t_{i+7}$ for $i = 1, ..., 7$. Then x acts as $(1,2,3,4,5,6,7)$ and y acts as $(1,5)(2,3)$ on $\{s_1, ..., s_7\}$.*

Proof. Follows from the observation $t_1 t_8 = t_8 t_1$. □

Lemma 5.8. *The group $\langle s_1, .., s_7 \rangle$ is an elementary Abelian 2-group of order 2^3.*

Proof. We compute $s_7 s_1 = s_5$. Hence, $(s_7 s_1)^{-1} = s_5^{-1} = s_5 = s_7 s_1$ and so $s_7 s_1$ is an involution. Moreover, we have $s_i s_j$ must also be an involution. Since $s_7 s_1 = s_5$, we may omit s_5 from the generating set. Similarly, we may also omit s_6, s_4, and s_3: for $s_6 = s_1 s_2$, $s_4 = s_6 s_7 = s_1 s_2 s_7$, $s_3 = s_7 s_1 s_6 = s_7 s_1 s_1 s_2 = s_7 s_2$. We have $\langle s_1, ..., s_7 \rangle = \langle s_7, s_1, s_2 \rangle$. Now we may not omit s_1, s_2, nor s_7 for s_3, s_4, s_5, s_6 rely on them. Hence $\langle s_7, s_1, s_2 \rangle$ is an elementary Abelian 2-group of order 2^3. □

Theorem 5.9. $\mathcal{M} \cong 2^3 : L_3(2)$.

Proof. Let $\mathcal{H} \cong 2^3$ be the elementary Abelian 2-group of order 8 in $\mathcal{M}$. The group $\mathcal{N}$ acts as $L_3(2)$ on $\mathcal{H}$. Hence, $\mathcal{M} = \mathcal{H} : \mathcal{N} \cong 2^3 : L_3(2)$. □

5.3 Double Coset Enumeration Over $2^3 : L_3(2)$

We proceed to do manual double coset enumeration over $\mathcal{M}$. Denote $[\omega]$ to be the double coset $\mathcal{M}\omega\mathcal{N}$, where ω is a word in the t_i's.

5.3.1 $\mathcal{M}e\mathcal{N}$

We begin with the double coset $\mathcal{M}e\mathcal{N}$, denote it $[e]$. This double coset contains only one single coset, namely $\mathcal{N}$. The single coset stabiliser is N, which has one orbit:

$$O = \{\{1, 2, ..., 14\}\}.$$

Take an element from $\mathcal{O}$ say t_7 and multiply the single coset representative $\mathcal{M}$ by it to obtain $\mathcal{M}t_7$. This is a new double coset $\mathcal{M}t_7\mathcal{N}$, denote it [7].

5.3.2 $\mathcal{M}t_7\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7\mathcal{N}$, we find the point stabiliser $\mathcal{N}^7$. This is

$$\mathcal{N}^7 = \langle (1,12)(2,3)(4,11)(5,8)(6,13)(9,10), (1,6,9)(2,8,13)(3,12,11)(4,10,5) \rangle.$$

We have the relation $\mathcal{M}t_7 = \mathcal{M}t_{14}$, and so the element $(2,13)(3,4)(5,12)(6,9)(7,14)(10,11)$ belongs to the coset stabilizer $\mathcal{N}^{(7)}$. We conclude that:

$$\mathcal{N}^{(7)} \geq \langle N^7, (2,13)(3,4)(5,12)(6,9)(7,14)(10,11) \rangle.$$

Since $|\mathcal{N}^{(7)}| \geq 24$, the number of elements in [7] is $168/24 \leq 7$. Furthermore, the orbits of $\mathcal{N}^{(7)}$ on $\{t_1, ..., t_{14}\}$ are:

$$\mathcal{O} = \{\{7, 14\}, \{1, ..., 6, 8, ..., 13\}\}.$$

Take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7$ of the double coset $\mathcal{M}t_7\mathcal{N}$. We have:

$$\begin{aligned} \mathcal{M}t_7t_7 &= \mathcal{M} \in [*], \\ \mathcal{M}t_7t_1 &\in [7,1]. \end{aligned}$$

The single coset $\mathcal{M}t_7t_1$ is new, so we must have a new double coset $\mathcal{M}t_7t_1 = [7, 1]$.

5.3.3 $\mathcal{M}t_7t_1\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_1\mathcal{N}$ we find the single coset stabiliser $\mathcal{N}^{(7,1)}$. The relation $\mathcal{M}t_7t_1 = \mathcal{M}t_{14}t_1$ enlarges the coset stabilizer to

$$\mathcal{N}^{(7,1)} \geq \langle (2,13)(3,4)(5,12)(6,9)(7,14)(10,11) \rangle.$$

Now the relation $\mathcal{M}t_7t_1 = \mathcal{M}t_7t_8$, is stabilized by $(1,8)(2,10)(3,9)(4,6)(5,12)(11,13)$ and so belongs to $\mathcal{N}^{(71)}$. We conclude that

$$\mathcal{N}^{(71)} \geq \langle (2,13)(3,4)(5,12)(6,9)(7,14)(10,11), (1,8)(2,10)(3,9)(4,6)(5,12)(11,13) \rangle.$$

Since $|\mathcal{N}^{(7,1)}| \geq 4$, the number of elements in $[7,1]$ is $168/4 \leq 42$. Furthermore, the orbits of $\mathcal{N}^{(71)}$ on $\{t_1, ..., t_{14}\}$ are:

$$\mathcal{O} = \{\{1,8\},\{5,12\},\{7,14\},\{2,10,11,13\},\{3,4,6,9\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_1\mathcal{N}$ of the double coset $\mathcal{M}t_7t_1\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7t_1t_1 &= \mathcal{M}t_7 \in [7], \\
\mathcal{M}t_7t_1t_2 &\in [7,1,2], \\
\mathcal{M}t_7t_1t_3 &\in [7,1,3], \\
\mathcal{M}t_7t_1t_7 &= \mathcal{M}t_7t_1 \in [7,1], \\
\mathcal{M}t_7t_1t_5 &= \mathcal{M}t_8t_7 \in [7,1]
\end{aligned}$$

The new double cosets have single coset representatives $\mathcal{M}t_7t_1t_2$ and $\mathcal{M}t_7t_1t_3$, we represent them as $[7,1,2]$ and $[7,1,3]$ respectively.

5.3.4 $\mathcal{M}t_7t_1t_2\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_1t_2\mathcal{N}$ we find the single coset stabiliser is trivial. However, the relation $\mathcal{M}t_7t_1t_2 = \mathcal{M}t_5t_1t_9$ will add the element

$$(2,9)(3,11)(4,10)(5,7)(6,13)(12,14)$$

to the coset stabilizer $N^{(7,1,2)}$. We conclude:

$$\mathcal{N}^{(7,1,2)} \geq \langle (2,9)(3,11)(4,10)(5,7)(6,13)(12,14) \rangle.$$

Since $|\mathcal{N}^{(7,1,2)}| \geq 2$, the number of elements in $[7,1,2]$ is $168/2 \leq 84$. Furthermore, the orbits of $\mathcal{N}^{(7,1,2)}$ on $\{t_1, ..., t_{14}\}$ are:

$$\mathcal{O} = \{\{1\}, \{8\}, \{2,9\}, \{3,11\}, \{4,10\}, \{5,7\}, \{6,13\}, \{12,14\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_1t_2$ of the double coset $\mathcal{M}t_7t_1t_2\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7t_1t_2t_1 &= = \mathcal{M}t_{11}t_1t_2 = \mathcal{M}t_4t_1t_2 \in [7,1,3], \\
\mathcal{M}t_7t_1t_2t_2 &= \mathcal{M}t_7t_1 \in [7,1], \\
\mathcal{M}t_7t_1t_2t_3 &\in [7,1,2,3], \\
\mathcal{M}t_7t_1t_2t_4 &\in [7,1,2,4], \\
\mathcal{M}t_7t_1t_2t_5 &= \mathcal{M}t_8t_5t_{11}t_3 \in [7,1,2,3] \\
\mathcal{M}t_7t_1t_2t_6 &= \mathcal{M}t_{12}t_2t_8 \in [7,1,2] \\
\mathcal{M}t_7t_1t_2t_8 &= \mathcal{M}t_7t_1t_2 \in [7,1,2] \\
\mathcal{M}t_7t_1t_2t_{12} &= \mathcal{M}t_7t_3t_{11}t_{13} \in [7,1,2,3]
\end{aligned}$$

The new double cosets are $\mathcal{M}t_7t_1t_2t_3\mathcal{N}$ and $\mathcal{M}t_7t_1t_2t_4\mathcal{N}$, which we represent by $[7,1,2,3]$ and $[7,1,2,4]$ respectively.

5.3.5 $\mathcal{M}t_7t_1t_3\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_1t_3\mathcal{N}$ we find the single coset stabiliser is trivial. However, the relation $\mathcal{M}t_7t_1t_3 = \mathcal{M}t_{12}t_1t_{10}$ will add the element

$$(2,6)(3,10)(4,11)(5,14)(7,12)(9,13)$$

to the coset stabilizer $\mathcal{N}^{(7,1,3)}$. We conclude:

$$\mathcal{N}^{(7,1,3)} \geq \langle (2,6)(3,10)(4,11)(5,14)(7,12)(9,13) \rangle.$$

Since $|\mathcal{N}^{(7,1,3)}| \geq 2$, the number of elements in $[7,1,3]$ is $168/2 \leq 84$. Furthermore, the orbits of $\mathcal{N}^{(7,1,3)}$ on $\{t_1, ..., t_{14}\}$ are:

$$\mathcal{O} = \{\{1\}, \{8\}, \{2,6\}, \{3,10\}, \{4,11\}, \{5,14\}, \{7,12\}, \{9,13\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_1t_3$ of the double coset $\mathcal{M}t_7t_1t_3\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7t_1t_3t_1 &= \mathcal{M}t_{13}t_1t_3 = \mathcal{M}t_2t_1t_{10} \in [7,1,2] \\
\mathcal{M}t_7t_1t_3t_2 &= \mathcal{M}t_2t_6t_3t_7 \in [7,1,2,3] \\
\mathcal{M}t_7t_1t_3t_3 &= \mathcal{M}t_7t_1[7,1] \\
\mathcal{M}t_7t_1t_3t_4 &= \mathcal{M}t_9t_{10}t_1 \in [7,1,3] \\
\mathcal{M}t_7t_1t_3t_5 &= \mathcal{M}t_2t_4t_6t_1 \in [7,1,2,3] \\
\mathcal{M}t_7t_1t_3t_7 &= \mathcal{M}t_4t_{10}t_2t_5 \in [7,1,2,3] \\
\mathcal{M}t_7t_1t_3t_8 &= \mathcal{M}t_7t_1t_3 \in [7,1,3] \\
\mathcal{M}t_7t_1t_3t_9 &\in [7,1,3,9].
\end{aligned}$$

We see that the only new double coset is $\mathcal{M}t_7t_1t_3t_9\mathcal{N}$, which is represented by $[7,1,3,9]$.

5.3.6 $\mathcal{M}t_7t_1t_2t_3\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_1t_2t_3\mathcal{N}$ we find the single coset stabiliser is trivial. However, the relation $\mathcal{M}t_7t_1t_2t_3 = \mathcal{M}t_{14}t_6t_2t_{10}$ will add the element

$$(1,6)(3,10)(4,12)(5,11)(7,14)(8,13)$$

to the coset stabilizer $\mathcal{N}^{(7,1,2,3)}$. We conclude:

$$\mathcal{N}^{(7,1,2,3)} \geq \langle (1,6)(3,10)(4,12)(5,11)(7,14)(8,13) \rangle.$$

Since $|\mathcal{N}^{(7123)}| \geq 2$, the number of elements in $[7,1,2,3]$ is $168/2 \leq 84$. Furthermore, the orbits of $\mathcal{N}^{(7,1,2,3)}$ on $\{t_1, ..., t_{14}\}$ are:

$$O = \{\{2\},\{9\},\{1,6\},\{3,10\},\{4,12\},\{5,11\},\{7,14\},\{8,13\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_1t_2t_3$ of the double coset $\mathcal{M}t_7t_1t_2t_3\mathcal{N}$. We have:

$$\begin{aligned}
\mathcal{M}t_7t_1t_2t_3t_1 &= \mathcal{M}t_1t_{14}t_6 \in [7,1,2]\\
\mathcal{M}t_7t_1t_2t_3t_2 &= \mathcal{M}t_7t_{12}t_2t_3 = \mathcal{M}t_3t_{13}t_2t_7 \in [7,1,2,3]\\
\mathcal{M}t_7t_1t_2t_3t_3 &= \mathcal{M}t_7t_1t_2 \in [7,1,2]\\
\mathcal{M}t_7t_1t_2t_3t_4 &= \mathcal{M}t_3t_{13}t_5 \in [7,1,2]\\
\mathcal{M}t_7t_1t_2t_3t_5 &= \mathcal{M}t_6t_3t_4 \in [7,1,3]\\
\mathcal{M}t_7t_1t_2t_3t_7 &= \mathcal{M}t_{13}t_5t_9 \in [7,1,3]\\
\mathcal{M}t_7t_1t_2t_3t_8 &= \mathcal{M}t_3t_4t_6 \in [7,1,3]\\
\mathcal{M}t_7t_1t_2t_3t_9 &= \mathcal{M}t_7t_1t_2t_3 \in [7,1,2,3]
\end{aligned}$$

5.3.7 $\mathcal{M}t_7t_1t_2t_4\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_1t_2t_4\mathcal{N}$ we find the single coset stabiliser is trivial. However, the relation $\mathcal{M}t_7t_1t_2t_4 = \mathcal{M}t_{10}t_8t_2t_5$ will add the element

$$\pi_1 = (1,8)(3,14)(4,5)(6,13)(7,10)(11,12)$$

to the coset stabilizer $\mathcal{N}^{(7,1,2,4)}$. The relation $\mathcal{M}t_7t_1t_2t_4 = \mathcal{M}t_{13}t_{12}t_2t_7$ will add the element

$$\pi_2 = (1,12,10)(3,8,5)(4,7,13)(6,11,14)$$

to the coset stabilizer $\mathcal{N}^{(7,1,2,4)}$. We conclude:

$$\mathcal{N}^{(7,1,2,4)} \geq \langle \pi_1, \pi_2 \rangle.$$

Since $|\mathcal{N}^{(7124)}| \geq 12$, the number of elements in $[7,1,2,4]$ is $168/12 \leq 14$. Furthermore, the orbits of $\mathcal{N}^{(7124)}$ are:

$$\mathcal{O} = \{\{2\},\{9\},\{1,3,...,8,10,...,14\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_1t_2t_4$ of the double coset $\mathcal{M}t_7t_1t_2t_4N$. We have:

$$\mathcal{M}t_7t_1t_2t_4t_1 = \mathcal{M}t_{11}t_4t_2 \in [7,1,2]$$
$$\mathcal{M}t_7t_1t_2t_4t_2 = \mathcal{M}t_3t_6t_9t_7 \in [7,1,3,9]$$
$$\mathcal{M}t_7t_1t_2t_4t_9 = \mathcal{M}t_7t_1t_2t_4 \in [7,1,2,4].$$

5.3.8 $\mathcal{M}t_7t_1t_3t_9\mathcal{N}$

Continuing with the double coset $\mathcal{M}t_7t_1t_3t_9\mathcal{N}$ we find the single coset stabiliser is trivial. However, the relation $\mathcal{M}t_7t_1t_3t_9 = \mathcal{M}t_{14}t_{11}t_3t_2$ will add the element

$$\pi_1 = (1,11)(2,9)(4,8)(5,6)(7,14)(12,13)$$

to the coset stabilizer $\mathcal{N}^{(7,1,3,9)}$. The relation $\mathcal{M}t_7t_1t_3t_9 = \mathcal{M}t_{12}t_{14}t_3t_6$ will add the element

$$\pi_2 = (1,14,5)(2,13,11)(4,9,6)(7,12,8)$$

to the coset stabilizer $\mathcal{N}^{(7,1,3,9)}$. We conclude:

$$\mathcal{N}^{(7,1,3,9)} \geq \langle \pi_1, \pi_2 \rangle.$$

Since $|\mathcal{N}^{(7,1,3,9)}| \geq 12$, the number of elements in $[7,1,3,9]$ is $168/12 \leq 14$. Furthermore, the orbits of $\mathcal{N}^{(7,1,3,9)}$ are:

$$\mathcal{O} = \{\{3\},\{10\},\{1,2,4,...,9,11,...,14\}\}.$$

We now take an element from each orbit and multiply on the right by the single coset representative $\mathcal{M}t_7t_1t_3t_9$ of the double coset $\mathcal{M}t_7t_1t_3t_9\mathcal{N}$. We have:

$$\mathcal{M}t_7t_1t_3t_9t_1 = \mathcal{M}t_{11}t_2t_3 \in [7,1,3]$$
$$\mathcal{M}t_7t_1t_3t_9t_3 = \mathcal{M}t_7t_8t_3t_9 = \mathcal{M}t_7t_1t_3t_9 \in [7,1,3,9]$$
$$\mathcal{M}t_7t_1t_3t_9t_{10} = \mathcal{M}t_7t_8t_{10}t_6 \in [7,1,2,4].$$

5.4 Cayley Graph of $\mathcal{G}$ Over $2^3 : L_3(2)$

We now represent the process of double coset enumeration as a Cayley graph. The circles represent double cosets and lines represent multiplication by t_i's. The numbers inside of the circles represent the number of single cosets within the double coset, while the numbers on the outside of the circles indicate the number of t_i's going to the next double coset.

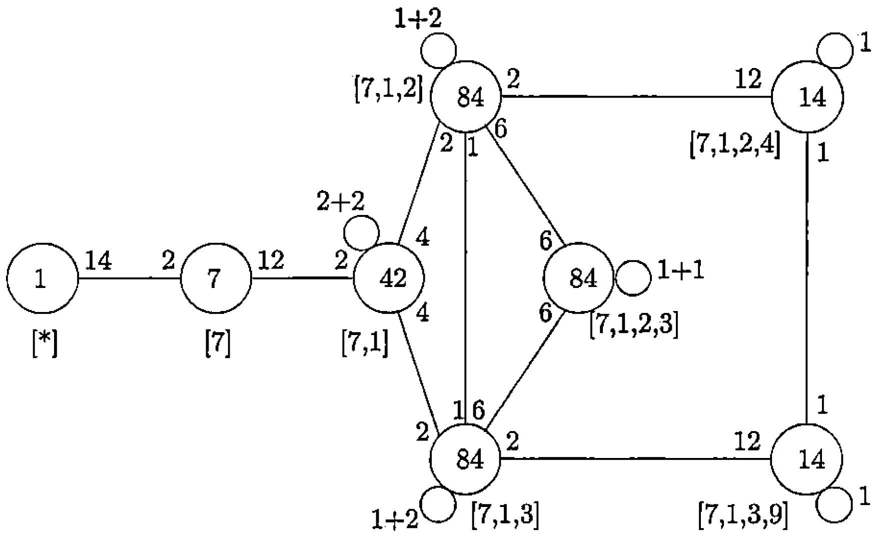

Figure 5.1: The Cayley Graph of $\mathcal{G}$ Over $\mathcal{M}$

5.5 $\mathcal{G} \cong M_{22}$

Again, we use Iwasawa's Lemma and the transitive action of $\mathcal{G}$ on the set of single cosets $\{\mathcal{M}\omega | \omega$ is a word in the $t_i's\}$. It will follow that $\mathcal{G}$ is simple of order 443,520. By, insert paper, there is only one simple group of order 443,520, the Mathieu Group M_{22}. We will conclude that $\mathcal{G} \cong M_{22}$.

Lemma 5.10. *The order of $\mathcal{G}$ is 443,520. Furthermore, $\mathcal{G}$ acts faithfully on $\mathcal{G}/\mathcal{M}$.*

Proof. Since $\{\mathcal{M}\omega\}$ is a transitive $\mathcal{G}$-set of degree 330. Then

$$|\mathcal{G}| = 330|\mathcal{G}^1|,$$

where $\mathcal{G}^1$ is the stabilizer of the single coset $\mathcal{M}$. But $\mathcal{M}$ is only stabilized by elements of $\mathcal{M}$. Hence $\mathcal{G}^1 = \mathcal{M}$ and $|\mathcal{G}^1| = |\mathcal{M}| = 1344$. We conclude that $|\mathcal{G}| = 443,520$. Finally, $\{\mathcal{M}\omega\}$ is faithful, lest $|\mathcal{G}| > 443,520$. □

Lemma 5.11. *The group $\mathcal{G}$ is perfect.*

Proof. We apply Iwasawa's lemma using the transitive action of $\mathcal{G}$ on $\{\mathcal{M}\omega\}$. We first show that $\mathcal{G}$ is perfect. Since $\mathcal{G} = \langle \mathcal{N}, t \rangle$, we have that $\mathcal{N} \leq \mathcal{G}'$, for $\mathcal{N}$ is simple and therefore perfect. We show that $t \in \mathcal{G}$, Consider, the relation:

$$\begin{aligned} \pi^{-1} &= t_{12}t_7t_6t_{12}t_7t_6t_{12}t_7t_6t_{12}t_7, \\ &= [t_{12}t_6, t_6t_7][t_{12}t_6, t_6t_7]t_6. \end{aligned}$$

We see that $t_6 = \pi[t_{12}t_6, t_6t_7]^2 \in \mathcal{G}'$ and so $\mathcal{G} \geq \mathcal{G}' \geq \langle x, y, t_6 \rangle = \mathcal{G}$. We conclude that $\mathcal{G} = \mathcal{G}'$ and $\mathcal{G}$ is perfect. □

Lemma 5.12. *The stabilizer $\mathcal{G}^1 \leq \mathcal{G}$, possesses a normal Abelian subgroup $\mathcal{K}$ whose conjugates generate $\mathcal{G}$.*

Proof. Now $\mathcal{G}^1 = \mathcal{M}$ and $\mathcal{M} \cong 2^3 : L_3(2)$ possesses a normal Abelian subgroup $\mathcal{K} = \langle t_i t_{i+7} \rangle \cong 2^3$. We have $(t_7t_{14})^{t_1} = t_1t_7t_{14}t_1 \in \mathcal{K}$. But then $t_1t_7t_{14}t_1 = y^{x^5}t_1t_7t_1t_{14}t_1$ and so $y^{x^5}t_7 \in \mathcal{K}$. We compute $y^{x^5} = (t_2t_9)^{t_{12}t_{11}}$. This completes the proof. □

Lemma 5.13. *The group $\mathcal{G}$ acts primitively on $\mathcal{G}/\mathcal{M} = \{\mathcal{M}\omega\}$.*

Proof. Since $\mathcal{G}$ is transitive, if B is a nontrivial block then we may assume that $\mathcal{M} \in B$. Now if $\mathcal{M}t_i \in B$, then we must have $B = \{\mathcal{M}\omega\}$: for $\mathcal{M}$ is stabilized by $\mathcal{N}$ and $\mathcal{M}t_1t_1 \in B$ implies $\mathcal{M}t_7t_1 \in B$. Hence, $\mathcal{M}(t_7t_1)^{\mathcal{N}} \in B$. Continuing in this manner, we have B is the complete list and is therefore nontrivial. For any other coset $\mathcal{M}\omega \in B$, we have $\mathcal{M}\omega \in \mathcal{M}\omega\mathcal{N}$ and so there exists a t_i such that $\mathcal{M}\omega t_i \in \mathcal{M}\omega\mathcal{N}$ by the Cayley graph. That is, each single coset representative on two letters or more is stabilized by a t_i. But of course this implies $\mathcal{M}\omega t_i \in B$ and we have $\mathcal{M}t_i \in B$. Hence, B is trivial. □

Theorem 5.14. *The group $\mathcal{G}$ is simple. Moreover, $\mathcal{G} \cong M_{22}$.*

Proof. The group $\mathcal{G}$ acts faithfully and primitively on the set $\{\mathcal{M}\omega\}$. Furthermore, the point stabilizer $\mathcal{M} \cong 2^3 : L_3(2)$ possesses a normal Abelian subgroup $\langle t_1t_8, t_2t_9, t_7t_{14} \rangle$, whose conjugates generate $\mathcal{G}$. By Iwasawa's lemma, we have $\mathcal{G}$ is simple. Since $|\mathcal{G}| = 443,520$, we have $\mathcal{G} \cong M_{22}$ by ATLAS, [CCN+85], and [Par70]: for there is only one simple group of this order. □

Chapter 6

Class Action on Groups

As Curtis describes in his construction of M_{12} and M_{22}, we may build larger groups from smaller ones by action on conjugacy classes [Cur07]. We can enumerate the elements of a conjugacy class then act on it via elements of the group to obtain a homomorphism into a larger permutation group.

Consider $\mathcal{S}_4$. Take the elements of the conjugacy class $\mathcal{C}_x$ of $x = (1, 2, 3, 4)$. We may write them in a table.

Table 6.1: Conjugacy Class of $x = (1, 2, 3, 4)$

1	(1,2,3,4)	2	(1,2,4,3)
3	(1,3,2,4)	4	(1,3,4,2)
5	(1,4,2,3)	6	(1,4,3,2)

Define an element t_0 to act on $\mathcal{C}_x$ in the following manner:

$$t_0 : (1, x, y, z) \mapsto (1, x, y, z)^{(y,z)} = (1, x, z, y).$$

This action is well defined since $(y, z) \in \mathcal{S}_4$ for all $y, z \in \{1, 2, 3, 4\}$. If we compute $t_0 : \mathcal{C}_x \to \mathcal{C}_x$, we obtain the permutation

$$\hat{t}_0 = (1, 2)(3, 4)(5, 6).$$

Now x acts on $\mathcal{C}_x$ by conjugation to yield: $\hat{x} = (2, 5, 4, 3)$. Let $\mathcal{S} = \langle \hat{t}_0, \hat{x} \rangle \cong \mathcal{S}_5$. Notice that $\mathcal{S}$ is transitive on $\{1, 2, 3, 4, 5, 6\}$. Hence, $\mathcal{S}_5$ has a transitive action on 6 points.

We did not have to take the permutation (y, z), instead we could have used (x, z) or (x, y). While (x, y) will yield $\mathcal{S}_5$ in the same way as above, (x, z) will yield 4×2.

This begs the question: What are the other possible groups? First let us make this idea clear.

Let $\mathcal{G}$ be a permutation group acting on $\{1, ..., n\}$ containing an n-cycle x. Let $\mathcal{G}_x$ be the conjugacy class of x. Note, we can represent any element $y \in \mathcal{C}_x$ as $y = (1, y_2, ..., y_n)$. Take $\mathcal{G}^1$, the point stabilizer of $\mathcal{G}$. Let $t \in \mathcal{G}^1$, then $x^t = (1, x_2^t, ..., x_n^t)$, defines a permutation t_0 of the subscripts $\{2, ..., n\}$. Define $\tilde{\mathcal{G}} = \langle \tilde{x}, \tilde{t}_0 \rangle$, then we can view $\tilde{\mathcal{G}}$ as being induced by t and x.

Furthermore, if $\hat{\mathcal{G}}$ has been induced from $\mathcal{G}$, say from t_0 and x. If $|t_0| = m$, then $\hat{\mathcal{G}}$ is a homomorphic image of the progenitor:

$$m^{*n} : \mathcal{G}.$$

6.1 The Alternating Group $\mathcal{A}_n$

Since the method for constructing the class action relies on the ability to fix a point within the conjugacy class, we can apply these methods when n is odd. Since the conjugacy classes of $\mathcal{A}_n$ get large very quickly, we only look at $\mathcal{A}_5$. One can do $\mathcal{A}_7$ in a similar manner; however, the images do not appear to be very interesting.

6.1.1 $\mathcal{A}_5$

Consider $\mathcal{A}_5$ and the class of $x = (1, 2, 3, 4, 5)$. Curtis showed that by taking 3-cycles and x, we obtain M_{12}. We will see what the involutions will give us.

As above, we enumerate the class $\mathcal{C}_x$. We compute the action of the involutions of $\mathcal{A}_5$, the fourgroup, on $\mathcal{C}_x$. That is, define:

$$t_0 : (1, x, y, z, w) \mapsto (1, x, y, z, w)^{(x,w)(y,z)} = (1, w, z, y, x).$$

Table 6.2: Conjugacy Class of $x = (1, 2, 3, 4, 5)$

1	(1, 5, 2, 4, 3)	2	(1, 3, 2, 5, 4)
3	(1, 5, 4, 3, 2)	4	(1, 4, 5, 2, 3)
5	(1, 4, 2, 3, 5)	6	(1, 3, 4, 2, 5)
7	(1, 2, 3, 4, 5)	8	(1, 2, 4, 5, 3)
9	(1, 4, 3, 5, 2)	10	(1, 2, 5, 3, 4)
11	(1, 3, 5, 4, 2)	12	(1, 5, 3, 2, 4)

Again, let $\hat{t}_0$ and $\hat{x}$ be the images of t_0 and x, respectively. Then

$$\tilde{\mathcal{G}} = \langle \hat{x} = (1,11,12,9,2)(4,6,8,5,10), \hat{t}_0 = (1,6)(2,4)(3,7)(5,12)(8,11)(9,10)\rangle.$$

Since $[\hat{x}, \bar{t}_0] = 1$, we must have that $\hat{\mathcal{G}} = 5 \times 2 \cong 10$.

Now consider $t_0 = (x,y)(z,w)$. As above we obtain:

$$\hat{t}_0 = (1,10)(2,7)(3,4)(5,8)(6,9)(11,12).$$

Let $\hat{\mathcal{G}} = \langle \hat{t}_0, \hat{X}\rangle$, then we find that $\hat{\mathcal{G}} = 2{\cdot}(2^4 : \mathcal{S}_6)$. The results are summarized in the following table. We include Curtis's result on M_{12} for completeness.

Table 6.3: Groups Induced from $\mathcal{A}_5$

t_0	Conjugates of $\hat{t}_0$ under $\hat{x}$	$\hat{\mathcal{G}}$
$Id(\mathcal{A}_4)$	1	5
(x,y,z)	5	M_{12}
$(x,y)(z,w)$	5	$2{\cdot}(2^4 : \mathcal{S}_6)$
$(x,w)(y,z)$	1	10

6.2 The Symmetric Group $\mathcal{S}_n$

The symmetric group $\mathcal{S}_n$ is divided into conjugacy classes based upon cycle types. Hence, we may apply the method of construction for all n, both odd and even. We will consider $n = 4$ and $n = 5$.

6.2.1 $\mathcal{S}_4$

In the beginning of this section, we considered the involutions of $\mathcal{S}_4$ to define the action. We may also consider the 3-cycles of $\mathcal{S}_3$ on $\{x,y,z\}$. Let $x = (1,2,3,4)$ and C_x be its conjugacy class. Where $y \in C_x$ can be taken to be $y = (1,x,y,z)$. Let $t_0 = (x,y,z)$. Then we may enumerate the class C_x as before. In this way, we achieve

$$\hat{t}_0 = (1,4,5)(2,6,3).$$

Again, $\hat{x} = (2,5,4,3)$ and so $\hat{\mathcal{G}} = \langle \hat{t}_0, \hat{x}\rangle \cong \mathcal{S}_6$. For completeness, if $t_0 = e$, then $\hat{\mathcal{G}} = \langle \hat{x}\rangle \cong 4$. Hence the possible images of $\mathcal{S}_4$ acting on itself are as follows:

Table 6.4: Groups Induced from $\mathcal{S}_4$

t_0	$\hat{t}_0$	$\hat{\mathcal{G}}$
$Id(\mathcal{S}_3)$	$Id(\mathcal{S}_6)$	4
(x, z)	$(1,6)(2,4)(3,5)$	4×2
(y, z)	$(1,2)(3,4)(5,6)$	$\mathcal{S}_5$
(x, y, z)	$(1,4,5)(2,6,3)$	$\mathcal{S}_6$

6.2.2 $\mathcal{S}_5$

We will take $x = (1,2,3,4,5)$ and enumerate the conjugacy class $\mathcal{C}_x$ of x. Let $(1, x, y, z, w)$ be an arbitrary element of $\mathcal{C}_x$. Then the point stabilizer, $\mathcal{S}_4$, consists of permutations on $\{x, y, z, w\}$. We present the table.

Table 6.5: Groups Induced from $\mathcal{S}_5$

t_0	Conjugates of $\hat{t}_0$ under $\hat{x}$	$\hat{\mathcal{G}}$
$Id(\mathcal{S}_4)$	1	5
(x, y)	5	$(L_2(11) \times L_2(11)) : 2$
(x, z, y)	5	$M_{12} \times M_{12}$
(x, w, z, y)	5	$(\mathcal{A}_{12} \times \mathcal{A}_{12}) : 2$
(y, z)	5	$2{\cdot}(2^4 : \mathcal{S}_6)$
(x, y, w, v)	1	20

6.3 The Linear Group $L_3(2)$

The projective special linear group $L_3(2)$ has a transitive action on both 7 points and 14 points. If we consider the action on 7 points, then we may fix a point in the class of $(1,2,3,4,5,6,7)$. If we consider the action on 14 points, by fixing 1 in $(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$ we also fix 8. We conclude that we may compute the action on classes in this case as well. However, we should not expect to get the same result, since point stabilizers depend on the action. That is, the stabilizer of a point in $L_3(2)$ on 7 letters is $\mathcal{S}_4$, while the stabilizer of a point in $L_3(2)$ on 14 letters is $\mathcal{A}_4$.

6.3.1 $L_3(2)$ on 14 Points

Again, one of the justifications for M_{24} to be considered as a homomorphic image of the progenitor $2^{*7} : L_3(2)$ is due to this class action. Since we have arrived at M_{22} as a homomorphic image of both $3^{*14} : L_3(2)$ and $2^{*14} : L_3(2)$, we may wonder if M_{22} arises in this natural way. It is unfortunate that this is not the case. However, we investigate the other possible induced groups in this section.

As before, begin with $x = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)$. Since x is not transitive on 14 points, we will need another element of $L_3(2)$. Let y=(1, 8)(2, 13)(3, 10)(4, 5)(6, 9)(11, 12). If we consider an arbitrary element

$$(1,x,y,z,u,v,w)(8,x',y',z',u',v',w')$$

of the conjugacy class of x. Then we would ask what permuations of {x, y, z, u, v, w, z', y', z', u', v', w' } are allowed. To answer this, take an element of the point stabilizer $(L_3(2))^1$ and conjugate x by it. For example, the element $t_0 = (2,14,12)(3,6,11)(4,10,13)(5,9,7)$ takes x to $(1,14,6,10,9,11,5)(8,7,13,3,2,4,12)$. If x were to be represented as (1, x, y, z, u, v, w)(8, x', y', z', u', v', w'), then $t_0 = (x,w,v')(y,u',z)(x',w',v)(y',u,z')$ would act as desired.

Enumerate the conjugacy class C_x of x. Since $|C_x| = 24$, we only give the table and present the code in the appendix. We omit duplicate results in the table, unless they arise from elements that have different orders.

Table 6.6: Groups Induced from $L_3(2)$ on 14 Points

t_0	Conjugates of $\hat{t}_0$ under $L_3(2)$	$\hat{\mathcal{G}}$
$Id(L_3(2))$	1	$L_3(2)$
$(x,w,v')(y,u',z)(x',w',v)(y',u,z')$	7	$\mathcal{A}_{24}$
$(x,u,y)(z,v,w)(x',u'y')(z',v',w')$	7	$3.L_3(2)$
$(x,z)(y,y')(u,v')(u',v)(w,w')(x',z')$	7	M_{24}

Notice that M_{24} appears in the list; however, the number of conjugates of $\hat{t}_0$ under $L_3(2)$ is still 7 (but this is not new). In fact, this is true for all of the groups in the table. One may wonder if the number of conjugates is independent of the action.

6.3.2 $L_3(2)$ on 7 Points

Curtis showed that if we fix 1 in the class of $(1,2,3,4,5,6,7)$, and label the positions $(1,x,y,z,u,v,w)$, then we may take any of the nontrivial elements of the elements

$\{(v,z)(x,y),(x,y)(u,w),(u,w)(v,z)\}$ to get M_{24}. We now see what happens to the rest of the elements.

We can figure out how to act on $(1,x,y,z,u,v,w)$ by Section 6.3.1. Enumerate the class C_x of x. Since x is transitive on 7 points, we will only find $\hat{x}$, ignoring the other generator. Note, we required the other before, since we were looking for 14 conjugates and x was not a 14-cycle. We again only present the table and omit duplicate results unless they arise in a different way.

Table 6.7: Groups Induced from $L_3(2)$ on 7 Points

t_0	Conjugates of $\hat{t}_0$ under $\hat{x}$	$\hat{\mathcal{G}}$
$Id(L_3(2))$	1	7
$(x,v)(y,u)$	7	$\mathcal{A}_{24}$
$(x,u,z,v)(y,w)$	7	$\mathcal{A}_{24}$
$(y,w)(u,v)$	7	M_{24}
$(y,u)(v,w)$	7	$(3^7:2^3):14$
$(x,u,y)(z,v,w)$	1	21

Appendix A

Some Images of $m^{*n} : \mathcal{S}_n$

We will consider the progenitors $5^{*3} : \mathcal{S}_3$, $5^{*4} : \mathcal{S}_4$, $7^{*3} : \mathcal{S}_3$, and $7^{*4} : \mathcal{S}_4$. What follows is largely based on computer proofs given by a permutation representation induced by coset action.

Table A.1: Presentations of the Progenitors $m^{*n} : \mathcal{S}_n$ That We Are Considering

(i)	$\langle x, y, t \vert x^3, y^2, (xy)^2, t^5, (t, y)\rangle \cong 5^{*3} : \mathcal{S}_3$
	$x \sim (0, 1, 2)$, $y \sim (1, 2)$, $t \sim t_0$
(ii)	$\langle x, y, t \vert x^3, y^2, (xy)^2, t^7, (t, y)\rangle \cong 7^{*3} : \mathcal{S}_3$
	$x \sim (0, 1, 2)$, $y \sim (1, 2)$, $t \sim t_0$
(iii)	$\langle x, y, t \vert x^4, y^2, (xy)^2, t^5, (t, y)\rangle \cong 5^{*4} : \mathcal{S}_4$
	$x \sim (0, 1, 2, 3)$, $y \sim (1, 2)$, $t \sim t_0$
(iv)	$\langle x, y, t \vert x^4, y^2, (xy)^2, t^7, (t, y)\rangle \cong 7^{*4} : \mathcal{S}_4$
	$x \sim (0, 1, 2, 3)$, $y \sim (1, 2)$, $t \sim t_0$

The lemma says that $\mathcal{S}_3 \cap \langle t_0, t_1\rangle \leq C_{\mathcal{S}_3}((\mathcal{S}_3)^{0,1})$. But any two point stabilizer in $\mathcal{S}_3$ is trivial and so we may take any $\pi \in \mathcal{S}_3$ and any product of t_0 and t_1. Moreover, by taking $\mathcal{S}_4 \cap \langle t_0, t_1, t_2\rangle$, we have that the three point stabilizer is trivial and so we may again taken any $\pi \in S_4$ and any product of t_0, t_1, t_2. We now list the relations and the some images. We have made this distinction between necessary relations and unnecessary relations by putting the value in bold.

Table A.2: Relations of the Progenitors $m^{*n} : \mathcal{S}_n$ That We Are Considering

$5^{*3} : \mathcal{S}_3$	$7^{*3} : \mathcal{S}_3$
$((0,1,2)t_0)^a = 1$	$(0,1,2)t_0)^a = 1$
$((0,1)t_0)^b = 1$	$((0,1)t_0)^b = 1$
$5^{*4} : \mathcal{S}_4$	$7^{*4} : \mathcal{S}_4$
$((0,1,2,3)t_0)^a = 1$	$((0,1,2,3)t_0)^a = 1$
$((1,2)(3,4)t_0)^b = 1$	$((1,2)(3,4)t_0)^b = 1$
$((1,2,3)t_0)^c = 1$	$((1,2,3)t_0)^c = 1$

Table A.3: Some Finite Images of the Progenitor $5^{*3} : \mathcal{S}_3$

Parameters		Order of G	Shape of $\langle t_0, t_1 \rangle$	Shape of $\langle T \rangle$	Shape of G
a	b				
3	10	150	5^2	5^2	$5^2 : \mathcal{S}_3$
5	**6**	124800	$U_3(4)$	$U_3(4)$	$U_3(4) : 2$
6	**5**	95040	$L_2(11)$	G	M_{12}.

Table A.4: Some Finite Images of the Progenitor $7^{*3} : \mathcal{S}_3$

Parameters		Order of G	Shape of $\langle t_0, t_1 \rangle$	Shape of $\langle T \rangle$	Shape of G
a	b				
3	14	294	7^2	7^2	$7^2 : \mathcal{S}_3$
7	4	2184	$L_2(13)$	$L_2(13)$	$L_2(13) : 2$

Table A.5: Some Finite Images of the Progenitor $5^{*4} : \mathcal{S}_3$

Parameters			Order of G	Shape of $\langle t_0, t_1 \rangle$	Shape of $\langle T \rangle$	Shape of G
a	b	c				
3	14	294	7^2	7^2	$7^2 : \mathcal{S}_3$	

A.1 $5^2 : S_3$

Consider the factored progenitor:

$$\mathcal{G} \cong \frac{5^{*3} : S_3}{(xt)^3}.$$

Notice that $(xt)^3 = 1$, implies $t_2 t_1 = \bar{t}_0$. Conjugating by $(1, 2)$, we have that $t_1 t_2 = \bar{t}_0$ and so $[t_1, t_2] = 1$. Also, $[t_0, t_1] = [t_0, t_2] = 1$ as can be seen. Hence, $\langle t_1, t_2 \rangle \cong 5^2$. Since there are no relation involving the control group, we conclude that S_3 has no image in 5^2. This can only mean that it's action is preserved in the image. Hence, $\mathcal{G} \cong 5^2 : S_3$.

Furthermore, we see that $\mathcal{G}$ has three maximal subgroups of index 2,3, and 25, respectively: $\mathcal{H}_1 \cong \langle x, y \rangle$, $\mathcal{H}_2 \cong \langle y, (t^x)^2 \rangle$, and $\mathcal{H}_3 \cong \langle x, (t^x)^2 \rangle$.

Now the group $\mathcal{H}_1$ is of order 6. Our group possess only one subgroup of order 6. This subgroup is isomorphic to S_3. We can also see that the group $\mathcal{H}_2$ is of order 50 and is hence isomorphic to $5^2 : 2$. Finally, the group $\mathcal{H}_3$ is of order 75 and is hence isomorphic to $5^2 : 3$, where $3 \leq S_3$.

A.2 $U_3(4) : 2$

Adding the relations $(xt)^5, (y^{x^2} t)^6$ to the progenitor $5^{*3} : S_3$ we obtain the finite homomorphic image:

$$\mathcal{G} \cong \frac{5^{*3} : S_3}{(xt)^5, (y^{x^2} t)^6}.$$

The composition factors of this group are $U_3(4)$ and 2. Define a subgroup $\mathcal{H}$ of $\mathcal{G}$ to be generated as follows:

$$\mathcal{H} = \langle y, t \rangle.$$

Now define $\phi : G \to S_{65}$ by

$$\begin{aligned}\phi(x) &= (1,2,5)(3,8,6)(4,10,19)(7,14,25)(9,17,15)(11,18,31)(12,22,35)(13,24,21)\\&\quad(16,28,36)(20,26,41)(23,37,50)(27,42,34)(29,44,49)(30,46,54)(32,48,56)\\&\quad(38,45,47)(39,40,52)(51,59,62)(53,60,65)(55,61,58)(57,64,63),\\\phi(y) &= (1,3)(2,6)(4,9)(5,8)(7,13)(10,15)(11,18)(12,16)(14,21)(17,19)(20,27)(22,36)\\&\quad(23,29)(24,25)(26,34)(28,35)(30,32)(33,43)(37,49)(38,45)(39,40)(41,42)\\&\quad(44,50)(46,56)(48,54)(51,53)(55,58)(57,63)(59,65)(60,62),\\\phi(t) &= (1,4,11,21,34)(2,7,15,27,43)(3,9,18,14,26)(5,12,23,38,28)(6,13,10,20,33)\\&\quad(8,16,29,45,35)(17,30,37,44,55)(19,32,49,50,58)(22,24,39,51,60)\\&\quad(25,40,53,62,36)(31,47,52,61,64)(41,48,57,65,46)(42,54,63,59,56).\end{aligned}$$

A.3 M_{12}

Adding the relations $(xt)^6, (y^{x^2}t)^5$ to the progenitor $5^{*3} : S_3$ we obtain the finite homomorphic image:

$$\mathcal{G} \cong \frac{5^{*3}S_3}{(xt)^6, (y^{x^2}t)^5}.$$

Define a subgroup $\mathcal{H}$ of $\mathcal{G}$ by:

$$\mathcal{H} = \langle t, yxt^{-1}x^{-1}, xt^{-1}x^{-1}t^{-1}x, x^{-1}t^2x^{-1}tx^{-1}\rangle$$

We define the map $\phi : \mathcal{G} \to S_{12}$ by computing the action of $\mathcal{G}$ on the set of cosets of $\mathcal{H}$ in $\mathcal{G}$:

$$\begin{aligned}\phi(x) &= (1,2,4)(3,6,5)(7,11,8)(9,10,12)\\\phi(y) &= (1,3)(2,5)(4,6)(7,9)(8,10)(11,12)\\\phi(t) &= (2,6,10,12,7)(4,8,11,9,5).\end{aligned}$$

A.4 S_8

Adding the relations $((x^2y)^2t)^6$, $(xt)^8$, and $((xy)^yt)^7$ to the progenitor $5^{*4} : S_4$ to obtain the finite homomorphic image:

$$\mathcal{G} \cong \frac{5^{*4} : \mathcal{S}_4}{((x^2y)^2t)^6, (xt)^8, ((xy)^yt)^7}.$$

Defina a subgroup $\mathcal{H}$ of $\mathcal{G}$ to be generated as follows:

$$\mathcal{H} = \langle x, y, txt^{-1}, x^t, t^2xt^{-2} \rangle.$$

The action of $\mathcal{G}$ on the set of cosets of $\mathcal{H}$ in $\mathcal{G}$ induces a map $\phi : \mathcal{G} \longrightarrow \mathcal{S}_8$ by

$$\begin{aligned}
\phi(x) &= (5,6,8,7),\\
\phi(y) &= (7,8),\\
\phi(t) &= (1,2,4,5,3).
\end{aligned}$$

We compute the maximal subgroups of $\mathcal{G}$ to be:

$$\begin{aligned}
\mathcal{H}_1 &= \langle x^2t^{-1}x^{-1}t^{-1}x^2t^{-1}xt^{-1}, t^{-1}xt^{-1}x^2t^{-1}xt^2x^{-1}t^{-1} \rangle,\\
\mathcal{H}_2 &= \langle x^2yt^2x^{-1}txt^{-1}x^{-1}t^{-2}, txt^{-1}x^{-1}tx^{-1}tx, y, x^2yx^{-1} \rangle,\\
\mathcal{H}_3 &= \langle x^2yt^{-1}xytx^{-1}, x^2yt^2xyt^{-1}x^{-1}t^{-2}, x^2tx^{-1}tx^{-1}tx^{-1}tx, xyx^2y,\\
&\quad x^2yt^{-2}xt^{-1}xt^{-2}x^{-1}t^{-1}, x^2ytx^{-1}t^2xt^{-2}x^{-1}t^{-1}, x^2yt^{-1}x^{-1}t, txyt^{-1}x^{-1}t^{-1}x^2t^{-2}x \rangle,\\
\mathcal{H}_4 &= \langle y, tx^{-1}txt^{-2}x^{-1}t^{-1}x, t^2x^{-1}t^{-2}x^{-1}yxt^{-2}xt \rangle,\\
\mathcal{H}_5 &= \langle xyx^2y, txt^2x^2txtx^{-1} \rangle,\\
\mathcal{H}_6 &= \langle xyx^2y, x^{-1}yt^{-1}xyt^{-1}xt, x^2yt^2x^{-1}t^{-2}, xtxyt^{-1}x^{-1}, t^2xt^2x^{-1}t,\\
&\quad x^2ytx^{-1}t^2xt^{-2}x^{-1}t^{-1}, xytx^2t^2xt^2x^{-1}, yt^2x^{-1}yt^2x^{-1}t, yt^{-1}x^{-1}t^{-2}xt^2x^{-1}t \rangle,\\
\mathcal{H}_7 &= \langle x^2yt^{-2}xt^{-1}xt^{-2}x^{-1}t^{-1}, xt^2x^{-1}yt^{-1}xyt^{-1}x^{-1} \rangle.
\end{aligned}$$

We compute these groups to be:

$$\begin{aligned}
\mathcal{H}_1 &\cong L_2(7) : 2,\\
\mathcal{H}_2 &\cong \mathcal{S}_5 \times \mathcal{S}_3,\\
\mathcal{H}_3 &\cong (\mathcal{A}_4 : 2^4) : 2,\\
\mathcal{H}_4 &\cong \mathcal{S}_6 \times \mathcal{S}_2,\\
\mathcal{H}_5 &\cong \mathcal{S}_7\\
\mathcal{H}_6 &\cong (2^4 : 3^2) : \mathcal{D}_8,\\
\mathcal{H}_7 &\cong \mathcal{A}_8.
\end{aligned}$$

A.5 $5^3 : \mathcal{S}_4$

Adding the relation $(xt)^4$ to the progenitor $5^{*4} : \mathcal{S}_4$ we obtain the following finite homomorphic image:

$$\mathcal{G} \cong \frac{5^{*4} : \mathcal{S}_4}{(xt)^4}.$$

The relation $(xt)^4 = 1$ yields $t_3t_2t_1t_0 = 1$. Now $[t_0, t_1] = t_0t_1\bar{t}_0\bar{t}_1 = \bar{t}_2\bar{t}_3t_3t_2t_1\bar{t}_1 = 1$. Conjugating $[t_0, t_1]$ by the point stabilizer $\mathcal{S}_3$ on $\{1, 2, 3\}$ shows $[t_0, t_i] = 1$. Similarly, we conjguate by $\mathcal{S}_3$ on $\{0, 2, 3\}$ to obtain $[t_1, t_i] = 1$. Of course, this implies $[t_i, t_j] = 1$. Now the relation $t_3t_2t_1 = \bar{t}_0$ implies $\langle t_1, t_2, t_3, t_0\rangle = \langle t_1, t_2, t_3\rangle$. The group $\langle t_1, t_2, t_3\rangle$ is elementary Abelian of order $5^3 = 125$. Moreover, the action of $\mathcal{S}_4$ is preserved and so we must have:

$$\mathcal{G} \cong 5^3 : \mathcal{S}_4.$$

We see that the maximal subgroups of $\mathcal{G}$ are:

$$\begin{aligned}
\mathcal{H}_1 &= \langle x, x^2yx^{-1}, yx^2y, (x^2y)^2\rangle,\\
\mathcal{H}_2 &= \langle yxt^{-1}xyt^{-1}x, ytx^{-1}t, x^{-1}t^{-2}xt^{-2}, xt^{-2}x^{-1}t^{-2}, x^2t^2xt^{-2}x\rangle,\\
\mathcal{H}_3 &= \langle x, yx^2y, (x^2yt)^2, x^{-1}t^{-2}xt^{-2}, xt^{-2}x^{-1}t^{-2}, x^2t^2xt^{-2}x\rangle,\\
\mathcal{H}_4 &= \langle x^2yx^{-1}, yx^2y, (x^2yt)^2, x^{-1}t^{-2}xt^{-2}, xt^{-2}x^{-1}t^{-2}, x^2t^2xt^{-2}x\rangle.
\end{aligned}$$

We compute these groups to be:

$$\begin{aligned}
\mathcal{H}_1 &\cong \mathcal{S}_4\\
\mathcal{H}_2 &\cong 5^3 : \mathcal{S}_3\\
\mathcal{H}_3 &\cong 5^3 : D_8\\
\mathcal{H}_4 &\cong 5^3 : \mathcal{A}_4.
\end{aligned}$$

A.6 $7^2 : \mathcal{S}_3$

Adding the relation $(xt)^3$ to the progenitor $7^{*3} : \mathcal{S}_3$ we obtain the following finite homomorphic image:

$$\mathcal{G} \cong \frac{7^{*3} : \mathcal{S}_3}{(xt)^3}.$$

As in Appendix A.2, we have the relation $(xt)^3$ implies that $[t_i, t_j] = 1$ for $i, j = 1, 2, 3$. Furthermore, we have $t_2 t_1 = \bar{t}_0$ and so $\langle t_0, t_1, t_2 \rangle = \langle t_1, t_2 \rangle$. Furthermore, $\langle t_1, t_2 \rangle$ is elementary Abelian of order 7^2. Hence, we have:

$$\mathcal{G} \cong 7^2 : \mathcal{S}_3.$$

We compute the maximal subgroups of $\mathcal{G}$ to be:

$$\begin{aligned} \mathcal{H}_1 &= \langle x, y \rangle, \\ \mathcal{H}_2 &= \langle y, (x^{-1}t^{-1}x)^3, xt^{-2}x^{-1}t \rangle, \\ \mathcal{H}_3 &= \langle x, (x^{-1}t^{-1}x)^3, xt^{-2}x^{-1}t \rangle. \end{aligned}$$

We compute these groups to be:

$$\begin{aligned} \mathcal{H}_1 &\cong \mathcal{S}_3, \\ \mathcal{H}_2 &\cong 7^2 : 2, \\ \mathcal{H}_3 &\cong 7^2 : 3. \end{aligned}$$

A.7 $L_2(13) : 2$

Adding the relations $(xt)^7$, $(y^{x^2}t)^4$ to the progenitor $7^{*3} : \mathcal{S}_3$ we obtain the following finite homomorphic image:

$$\mathcal{G} \cong \frac{7^{*3} : \mathcal{S}_3}{(xt)^7, (y^{x^2}t)^4}.$$

Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ generated as follows:

$$\mathcal{H} = \langle x, ytxt^2, yt^2xt \rangle.$$

By computing the action of $\mathcal{G}$ on the set of cosets of $\mathcal{H}$ in $\mathcal{G}$ we may define $\phi : \mathcal{G} \to \mathcal{S}_{14}$ by

$$\begin{aligned} \phi(x) &= (3,7,9)(4,10,11)(5,12,13)(6,8,14), \\ \phi(y) &= (1,2)(3,5)(4,6)(7,13)(8,11)(9,12)(10,14), \\ \phi(t) &= (1,3,8,10,7,12,4)(2,5,11,14,13,9,6). \end{aligned}$$

We compute the maximal subgroups of $\mathcal{G}$ to be:

$$\begin{aligned}
\mathcal{H}_1 &= \langle x^{-1}t^{-1}x^{-1}txt^2x, xt^3xt^2x^{-1}, xyt^{-2}, x^{-1}, t^2, x^{-1}t\rangle,\\
\mathcal{H}_2 &= \langle x^{-1}t^{-2}xtxt, x^t, xyt^{-2}xt^2, xt^{-2}xt^2x\rangle,\\
\mathcal{H}_3 &= \langle yxt^{-1}x^{-1}txt^3, x^{-1}t^2x^{-1}t^{-1}x^{-1}t^2, t^2x^{-1}t^{-1}x^{-1}t^2, tx^{-1}tx^{-1}t^2xt^2\rangle,\\
\mathcal{H}_4 &= \langle xt^{-1}xt^3, yxt^{-1}xt^2x, x^{-1}t^{-1}x^{-1}txt^2x, xt^{-1}xtx^{-1}t^2\rangle,\\
\mathcal{H}_5 &= \langle tx^{-1}tx^{-1}t^{-2}x^{-1}t, t^2x^{-1}t^{-1}x\rangle.
\end{aligned}$$

We compute these groups to be:

$$\begin{aligned}
\mathcal{H}_1 &\cong 7:2^2,\\
\mathcal{H}_2 &\cong 6:2^2,\\
\mathcal{H}_3 &\cong 2^2:L_2(2),\\
\mathcal{H}_4 &\cong 13:12,\\
\mathcal{H}_5 &\cong L_2(13).
\end{aligned}$$

A.8 $7^3 : \mathcal{S}_4$

Adding the relation $(xt)^4$ to the progenitor $7^{*4} : \mathcal{S}_4$ we obtain the following finite homomorphic image:

$$\mathcal{G} \cong \frac{7^{*4} : \mathcal{S}_4}{(xt)^4}.$$

As in Appendix A.5, we have that $(xt)^4 = 1$ implies $[t_i, t_j] = 1$ and $t_3t_2t_1 = \bar{t}_0$, so $\langle t_0, t_1, t_2, t_3\rangle = \langle t_1, t_2, t_3\rangle$. Hence, $\langle t_1, t_2, t_3\rangle$ is elementary Abelian of order 7^3. Therefore:

$$\mathcal{G} \cong 7^3 : \mathcal{S}_4.$$

We compute the maximal subgroups of $\mathcal{G}$ to be:

$$\begin{aligned}
\mathcal{H}_1 &= \langle x, y\rangle,\\
\mathcal{H}_2 &= \langle yxt^{-1}xyt^{-1}x, ytx^{-1}t, x^{-1}t^{-2}xt^{-2}, xt^{-2}x^{-1}t^{-2}, x^2t^2xt^{-2}x\rangle,\\
\mathcal{H}_3 &= \langle x, yx^2y, (x^2yt)^2, x^{-1}t^{-2}xt^{-2}, xt^{-2}x^{-1}t^{-2}, x^2t^2xt^{-2}x\rangle,\\
\mathcal{H}_4 &= \langle x^2yx^{-1}, yx^2y, (x^2yt)^2, x^{-1}t^{-2}xt^{-2}, xt^{-2}x^{-1}t^{-2}, x^2t^2xt^{-2}x\rangle.
\end{aligned}$$

We compute these groups to be:

$$\begin{aligned}\mathcal{H}_1 &\cong \mathcal{S}_4,\\ \mathcal{H}_2 &\cong 7^3 : \mathcal{S}_3,\\ \mathcal{H}_3 &\cong 7^3 : D_8,\\ \mathcal{H}_4 &\cong 7^3 : \mathcal{A}_4.\end{aligned}$$

Appendix B

Some Images of $2^{*14} : L_3(2)$ and $3^{*14} : L_3(2)$

We detailed the construction of $2^{*14} : L_3(2)$ and $3^{*14} : L_3(2)$ in Section 3.3. The presentations are therefore assumed.

B.1 $\mathcal{A}_7$

Adding the relation $(x^{-1}t)^4$ to the progenitor $3^{*14} : L_3(2)$ we obtain the finite homomorphic image:

$$\mathcal{G} \cong \frac{3^{*14} : L_3(2)}{(x^{-1}t)^4}.$$

Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ generated as follows:

$$\mathcal{H} = \langle y, tx^{-1}, t^x, x^{-2}t \rangle$$

This induces a map $\phi : G \to \mathcal{S}_7$ given by

$$\begin{aligned} \phi(x) &= (1,2,3,4,5,7,6), \\ \phi(y) &= (1,2)(4,6), \\ \phi(t) &= (3,5,7). \end{aligned}$$

The group $\langle \phi(x), \phi(y), \phi(t) \rangle$ contains all 3-cycles and even permutations. Hence, $\mathcal{G} \cong \mathcal{A}_7$

B.2 $2^4 : \mathcal{A}_7$

Adding the relations $(xyt)^4, (xyt^{-1})^5$ to the progenitor $3^{*14} : L_3(2)$ we obtain the finite homomorphic image:

$$\mathcal{G} \cong \frac{3^{*14} : L_3(2)}{(xyt)^4, (xyt^{-1})^5}.$$

We compute the composition factors of $\mathcal{G}$ to be $\mathcal{A}_7, 2, 2, 2, 2$. Also, we compute that there is a minimal normal subgroup of order 16. Hence, $\mathcal{G} \cong 2^4 : \mathcal{A}_7$

Define a subgroup $\mathcal{H}$ of $\mathcal{G}$ generated as follows:

$$\mathcal{H} = \langle x, y, txt^{-1}, t^{-1}x^{-2}tx^{-1}t, t^{-1}x^2t^{-1}x^{-3}t, t^{-1}x^2yx^2yx^{-1}t\rangle.$$

This induces a map $\phi : \mathcal{G} \to \mathcal{S}_{16}$ by

$$\begin{aligned}
\phi(x) &= (3,4,6,11,10,9,5)(7,8,15,13,16,12,14),\\
\phi(y) &= (4,7)(5,8)(6,12)(9,10)(11,15)(13,16),\\
\phi(t) &= (1,2,3)(4,8,9)(5,10,7)(6,13,11)(12,16,15).
\end{aligned}$$

The maximal subgroups of $\mathcal{G}$ are:

$$\begin{aligned}
\mathcal{H}_1 &= \langle t, ytxyt^{-1}xt^{-1}x^{-1}t^{-1}\rangle,\\
\mathcal{H}_2 &= \langle xtx^{-1}ytx^2t, x^{-1}yxyt^{-1}x^{-1}t^{-1}x^{-1}t, yx^{-1}t^{-1}xt^{-1}xtx^{-1}t,\\
&\quad xtxtx^{-1}ytx, x^2yx^{-1}t^{-1}xtx^{-1}yx, (xyx^{-1}t)^2, (xt^{-1})^4, txyx^{-1}t^{-1}xyx^{-1}t, (x^{-1}t)^4\rangle,\\
\mathcal{H}_3 &= \langle xt^{-1}xtx^{-3}t^{-1}x^{-1}t^{-1}, t^{-1}xyt^{-1}x^{-1}txt^{-1}x^{-1}, (xyx^{-1}t)^2, (xt^{-1})^4,\\
&\quad txyx^{-1}t^{-1}xyx^{-1}t, (x^{-1}t)^4\rangle,\\
\mathcal{H}_4 &= \langle x^2tx^{-3}yt^{-1}, yt^{-1}xt^{-1}xtxt^{-1}x, (xyx^{-1}t)^2, (xt^{-1})^4,\\
&\quad txyx^{-1}t^{-1}xyx^{-1}t, (x^{-1}t)^4\rangle,\\
\mathcal{H}_5 &= \langle tx^2yt^{-1}xt^{-1}x^{-1}t, xt^{-1}x^{-2}txtx^{-1}t^{-1}x, (xyx^{-1}t)^2, (xt^{-1})^4,\\
&\quad txyx^{-1}t^{-1}xyx^{-1}t, (x^{-1}t)^4\rangle,\\
\mathcal{H}_6 &= \langle x^{-1}t^{-1}x^{-1}t^{-1}x^{-2}ytx^{-1}, t^{-1}xtx^{-1}t^{-1}x^{-1}t^{-1}x^2t, (xyx^{-1}t)^2, (xt^{-1})^4,\\
&\quad txyx^{-1}t^{-1}xyx^{-1}t, (x^{-1}t)^4\rangle.
\end{aligned}$$

We compute these groups to be:

$$\begin{aligned}
\mathcal{H}_1 &\cong \mathcal{A}_7,\\
\mathcal{H}_2 &\cong 2^4 : (\mathcal{A}_4 \times 3 : 2)\\
\mathcal{H}_3 &\cong 2^4 : \mathcal{S}_5,\\
\mathcal{H}_4 &\cong 2^4 : L_2(7)\\
\mathcal{H}_5 &\cong 2^4 : L_2(7),\\
\mathcal{H}_6 &\cong 2^4 : \mathcal{A}_6.
\end{aligned}$$

B.3 $2{\cdot}M_{22}$

Adding the relations $(xt)^5, (x^{-1}t)^5$ to the progenitor $3^{*14} : L_3(2)$ we obtain the finite homomorphic image:

$$\mathcal{G} \cong \frac{3^{*14} : L_3(2)}{(xt)^5, (x^{-1}t)^5}.$$

The composition factors for $\mathcal{G}$ are M_{22} and 2. Where $\mathcal{Z}(\mathcal{G}) \cong 2$. Hence $\mathcal{G}$ is the double cover for M_{22}.

B.4 $3{\cdot}M_{22}$

Adding the relations $(yt^{x^2})^5, (yt^x t)^3$ to the progenitor $2^{*14} : L_3(2)$ we obtain the finite homomorphic image:

$$\mathcal{G} \cong \frac{2^{*14} : L_3(2)}{(yt^{x^2})^5, (yt^x t)^3}$$

We compute the composition factors of $\mathcal{G}$ to be M_{22} and 3. Futhermore, $\mathcal{Z}(\mathcal{G}) = 3$ and we conclude $\mathcal{G} \cong 3{\cdot}M_{22}$, the triple cover for M_{22}.

B.5 $2^3 : L_3(2)$

Adding the relations $(xt)^7$, $(yt^{x^2})^4$, and $(x^{-1}yxyt^{x^2})^4$ to the progenitor $2^{*14} : L_3(2)$ we obtain the finite homomorphic image:

$$\mathcal{G} = \frac{2^{*14} : L_3(2)}{(xt)^7, (yt^{x^2})^4, (x^{-1}yxyt^{x^2})^4}.$$

We consider the relation $(xt)^7 = 1$. This yields $t^{x^6}t^{x^5}t^{x^4}t^{x^3}t^{x^2}t^{x}t = 1$ which may be written as follows:

$$t_7t_6t_5 = t_1t_2t_3t_4.$$

Now consider $(yt^{x^2})^4 = 1$. This yields $t^{x^2y}t^{x^2}t^{x^2y}t^{x^2} = 1$, which may be written as follows:

$$t_3t_2 = t_2t_3.$$

Finally, the relation $(x^{-1}yxyt^{x^2})^4 = 1$. Note that

$$\pi = x^{-1}yxy = (1, 12, 8, 5)(2, 6, 10, 4)(3, 11, 9, 13)(7, 14),$$

and so the relation yields $t^{x^2\pi^3}t^{x^2\pi^2}t^{x^2\pi}t^{x^2} = 1$, which may be written as:

$$t_4t_{10} = t_2t_6.$$

We proceed to show that $\mathcal{G}$ is indeed $2^3 : L_3(2)$.

Lemma B.1. *For $j \neq i + 7$ and $i < 8$, $t_it_j = t_jt_i$.*

Proof. Fix $i = 1$, then the point stabilizer N^1 has three orbits: $\{\{1\}, \{8\}, \{2, ..., 7, 9, ..., 14\}\}$. Hence $t_1t_j = t_jt_1$ for all $j \neq 8$. For arbitrary i, we may conjugate this relation by powers of x. Provided $j \neq i + 7$, equality will hold. This completes the proof. □

In G, we have that $[t_i, t_j] = 1$ most of the time. The one exception occuring when $i \equiv_7 j$.

Lemma B.2. *For $j \neq i + 7$ and $i < 8$, $t_it_j = t_k$ for some k.*

Proof. We consider the relation $t_7t_6t_5 = t_1t_2t_3t_4$ as our playground. Now by relation $t_4t_{10} = t_2t_6$, we get $t_3t_9 = t_1t_5$. Hence, we have the following:

$$\begin{aligned} t_7t_6t_5 &= t_1t_2t_3t_4, \\ t_7t_6t_1t_5 &= t_2t_3t_4, \\ t_7t_6t_3t_9 &= t_2t_4t_3, \\ t_7t_6t_3t_9t_3 &= t_2t_4, \\ t_7t_6t_9 &= t_2t_4. \end{aligned}$$

Again, consider the relation $t_4t_{10} = t_2t_6$. By the lemma we get $t_2t_4 = t_{10}t_6$, and we may now consider:

$$\begin{aligned} t_7t_9t_6 &= t_2t_4, \\ t_7t_9t_6 &= t_{10}t_6, \\ t_7t_9 &= t_{10}. \end{aligned}$$

□

Theorem B.3. *For all i, $t_i = t_{i+7}$.*

Proof. By the previous lemma, we have that $t_7t_9 = t_{10}$. Now by the relation $t_3t_9 = t_1t_5$, we take conjugate by an element of N^9 to get $t_7t_9 = t_{12}t_{13}$. Finally, take the relation $t_7t_9 = t_{10}$ and conjugate by

$$\pi = (2,6)(3,10)(4,11)(5,14)(7,12)(9,13),$$

to get $t_{12}t_{13} = t_3$. From here we have the sequence of equalities

$$t_{10} = t_7t_9 = t_{12}t_{13} = t_3.$$

We find $t_i = t_{i+7}$ by conjugating by various powers of x. □

Theorem B.4. *The group $\langle t_1, ..., t_7\rangle = \langle t_7, t_1, t_2\rangle$ is elementary Abelian of order 2^3.*

Proof. We have $t_7t_2 = t_3$ and so t_3 may be omitted from the generating set. Similarly, $t_4 = t_1t_3 = t_1t_7t_2$, $t_5 = t_2t_4 = t_2t_1t_7t_2 = t_1t_7$, and $t_6 = t_3t_5 = t_7t_2t_1t_7 = t_2t_1$. We may not omit t_7, t_1, or t_2 because these elements are needed to represent t_4. This completes the proof. □

Theorem B.5. *The group $\mathcal{G}$ is isomorphic to $2^3 : L_3(2)$.*

Proof. Since $\mathcal{G}$ is a homomorphic image of the progenitor $2^{*14} : L_3(2)$, we need only remark that $\langle t_7, t_1, t_2\rangle$ is a normal subgroup. □

Appendix C

Class Action Code

```
S14:=Sym(14);
G<x,y>:=sub<S14|(1,2,3,4,5,6,7)(8,9,10,11,12,13,14),
(1,8)(2,13)(3,10)(4,5)(6,9)(11,12)>;
S24:=Sym(24);
C:=Classes(G);
C;

C1:=Setseq(Class(G,G!(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)));
C1;
/* Class of order 7 elements */

/* Conver C to a sequence that way we may define the element sequences*/

T:=[[1,1^(C1[1]),1^((C1[1])^2),1^((C1[1])^3),1^((C1[1])^4),
1^((C1[1])^5),1^((C1[1])^6),8,8^(C1[1]),8^((C1[1])^2),8^((C1[1])^3),
8^((C1[1])^4),8^((C1[1])^5),8^((C1[1])^6)]];
for i in [2..#C1] do T:=T cat
[[1,1^(C1[i]),1^((C1[i])^2),1^((C1[i])^3),1^((C1[i])^4),
1^((C1[i])^5),1^((C1[i])^6),8,8^(C1[i]),8^((C1[i])^2),8^((C1[i])^3),
8^((C1[i])^4),8^((C1[i])^5),8^((C1[i])^6)]]; end for;
T;
/*Defines a sequence of sequences T such that an element (1,2,3,4,5,6,7)
 is represented as [1,2,3,4,5,6,7] */

N:=Stabilizer(G,1);
CN:=Classes(N);
CN;

for n in N do
```

```
nn:=T[1];
zz:=[1,1^(C1[1]*n),1^(C1[1]^2*n),1^(C1[1]^3*n),1^(C1[1]^4*n),
1^(C1[1]^5*n),1^(C1[1]^6*n),8,8^(C1[1]*n),8^(C1[1]^2*n),8^(C1[1]^3*n)
,8^(C1[1]^4*n),8^(C1[1]^5*n),8^(C1[1]^6*n)];

CIT:=[1,2,3,4,5,6,7,8,9,10,12,13,14];
for i in [1..14] do for j in [1..14] do
if nn[i] eq zz[j] then CIT[i]:=j;
end if; end for; end for;
t:=S14!CIT;
t;
/*Stabilize the point 1 and defines a rule,
t, based off of the action of y on C1[1]*/

CIT:=[i: i in [1..#T]];
for k in [1..#T] do h:=T[k];
for j in [1..#T] do
for i in [1..14] do h[i^t]:=T[k][i]; end for;
if h eq T[j] then CIT[k]:=j;
end if; end for; end for;
tt:=S24!CIT;
/* Computes the image of t via placement switching in T*/

CIT:=[i: i in [1..#T]];
for i in [1..#T] do for j in [1..#T] do
if C1[i]^x eq C1[j] then CIT[i]:=j;
end if;
end for; end for;
xx:=S24!CIT;
/*Computes the image of x by conjugation,
note C1 is ordered the same as T by above
definition so the labellings are the same*/

CIT:=[i: i in [1..#T]];
for i in [1..#T] do for j in [1..#T] do
if C1[i]^y eq C1[j] then CIT[i]:=j;
end if;
end for; end for;
yy:=S24!CIT;
/*Computes the image of x by conjugation,
note C1 is ordered the same as T by above definition
so the labellings are the same*/

M:=sub<S24|xx,yy,tt>; H:=sub<M|xx,yy>;
```

```
CompositionFactors(M);  n; #Conjugates(H,tt);
end for;
```

Appendix D

General MAGMA Code

For Chapters 4, 5, and 6 we made use of the computer program MAGMA. In Chapters 4 and 5, MAGMA was able to tell verify our relations. In Chapter 6, we used MAGMA to compute the class action. More details on MAGMA can be found at http://magma.maths.usyd.edu.au/magma/handbook/

We will describe the following commands we used most often.

- $G < x1, ..., xn > := Group < x1, ..., xn | r_1(x_1, ..., x_n), ..., r_m(x_1, ..., x_n) >$; - Defines a finitely presented group on n generators, subject to the relations r_i.
- $SchreierSystem(G, sub < G|Id(G) >)$; - Returns a list of elements of a finitely presented group G.
- $H := sub < G|y1, ..., yn >$; - Defines a subgroup H of G, generated by y1,..,yn.
- $f, G1, k := CosetAction(G, H)$; - Computes the action of G on the cosets of a subgroup H in G, provided the coset table is closed. Defines the image of G by $G1$ with corresponding homomorphism f and kernel k.
- $CompositionFactors(G)$; - Returns the composition factors for a group G.
- $Classes(G)$; - Returns a set of representatives of the conjugacy classes of G, together with the order and length of the class.
- $Class(G, x)$; - Returns the conjugacy class of x in G.
- $Conjugates(H, x)$; - Returns the conjugates of x in H.

- $LowIndexSubgroups(G, < m, n >)$; - Returns a set of subgroups that have index k such that $m \leq k \leq n$.
- for i in [1..k] do r[i]; end for; - Iterates a process $r[i]$ over some indexing set and returns the outputs for each i.
- if S eq T then R; end if; - Returns R if true and returns nothing if false.

Appendix E

MAGMA Code for M_{22} from $3^{*14} : L_3(2)$

```
G<x,y,t>:=Group<x,y,t|x^7,y^2,(x*y)^3,(x,y)^4,t^3,(t^(x^4),x*y),
(t,y),(x*y*(t^-1))^5,(x*y*t^(x^2))^5>;
H1:=sub<G|x,y>;
H2:=sub<G|x,y,t^x*(t^(x*y))^-1*t^x>;
#DoubleCosets(G,H2,H1);

S:=Sym(28);
p:=S!(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)
(15,16,17,18,19,20,21)(22,23,24,25,26,27,28);
q:=S!(1,8)(2,13)(3,10)(4,5)(6,9)(11,12)(15,22)
(16,27)(17,24)(18,19)(20,23)(25,26);
N:=sub<S|p,q>;
#N;

cst := [null : i in [1 .. 2640]] where null is [Integers() | ];
f, G1, k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;

ts:=[Id(G1):i in [1..28]];
ts[7]:=f(t);
ts[1]:=(f(t)^f(x));
ts[2]:=(f(t)^f(x^2));
ts[3]:=(f(t)^f(x^3));
ts[4]:=(f(t)^f(x^4));
ts[5]:=(f(t)^f(x^5));
ts[6]:=(f(t)^f(x^6));
```

```
ts[8]:=ts[1]^f(y);
ts[9]:=ts[6]^f(y);
ts[10]:=ts[3]^f(y);
ts[11]:=ts[10]^f(x);
ts[12]:=ts[11]^f(x);
ts[13]:=ts[12]^f(x);
ts[14]:=ts[13]^f(x);
ts[15]:=ts[1]^-1;
ts[16]:=ts[2]^-1;
ts[17]:=ts[3]^-1;
ts[18]:=ts[4]^-1;
ts[19]:=ts[5]^-1;
ts[20]:=ts[6]^-1;
ts[21]:=ts[7]^-1;
ts[22]:=ts[8]^-1;
ts[23]:=ts[9]^-1;
ts[24]:=ts[10]^-1;
ts[25]:=ts[11]^-1;
ts[26]:=ts[12]^-1;
ts[27]:=ts[13]^-1;
ts[28]:=ts[14]^-1;
N1:=sub<G1|f(x),f(y),ts[1]*ts[22]*ts[1]>;

prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
*/
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;

N7:=Stabiliser(N,7);
S:={[7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7] eq n*(ts[(Rep(SSS[i]))[1]])
then  print Rep(SSS[i]);
end if;
end for;
```

```
end for;

T7:=Transversal(N,N7);
for i in [1..#T7] do
ss:=[7]^T7[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N7);

N21:=Stabiliser(N,21);
S:={[21]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[21] eq n*(ts[(Rep(SSS[i]))[1]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

T21:=Transversal(N,N21);
for i in [1..#T21] do
ss:=[21]^T21[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N21);

N71:=Stabiliser(N7,1);
S:={[7,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1] eq n*(ts[(Rep(SSS[i]))[1]]*ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
```

```
end if;
end for;
end for;

for n in N do if [7,1]^n eq [1,7]
then N71:=sub<N|N71,n>; end if; end for;

T71:=Transversal(N,N71);
for i in [1..#T71] do
ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N71);

N714:=Stabiliser(N7,14);
S:={[7,14]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[14] eq n*(ts[(Rep(SSS[i]))[1]]*ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,14]^n eq [14,7]
then N714:=sub<N|N714,n>; end if; end for;

T714:=Transversal(N,N714);
for i in [1..#T714] do
ss:=[7,14]^T714[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N714);
```

```
N715:=Stabiliser(N7,15);
S:={[7,15]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[15] eq n*(ts[(Rep(SSS[i]))[1]]*ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,15]^n eq [13,22]
then N715:=sub<N|N715,n>; end if; end for;

T715:=Transversal(N,N715);
for i in [1..#T715] do
ss:=[7,15]^T715[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N715);

N2115:=Stabiliser(N7,15);
S:={[21,15]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[21]*ts[15] eq n*(ts[(Rep(SSS[i]))[1]]*ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [21,15]^n eq [15,21]
then N2115:=sub<N|N2115,n>; end if; end for;

T2115:=Transversal(N,N2115);
for i in [1..#T2115] do
ss:=[21,15]^T2115[i];
```

```
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

Orbits(N2115);

N713:=Stabiliser(N71,17);
S:={[7,1,17]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1]*ts[17] eq n*(ts[(Rep(SSS[i]))[1]]*
ts[(Rep(SSS[i]))[2]]*ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,1,17]^n eq [15,21]
then N2115:=sub<N|N2115,n>; end if; end for;

T713:=Transversal(N,N713);
for i in [1..#T713] do
ss:=[7,1,17]^T713[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
```

Appendix F

MAGMA Code for M_{22} from $2^{*14} : L_3(2)$

```
G<x,,y,t>:=Group<x,y,t|x^7,y^2,(x*y)^3,(x,y)^4,t^2,(t,x^2*y*x^-3),
(t,y),(y*t^(x^2))^5,(x*y*t)^11,(y*t^x*t)^3>;
H:=sub<G|x,y>;
Index(G,H);
/*(t,x^2*y*x^-3)=(t^(x^2),y*x^-1)*/
c:=G!t*t^(x^-1*y*x);
H:=sub<G|x,y,c>;
f, G1, k:=CosetAction(G,H);
IN:=sub<G1|f(x),f(y)>;
N1:=sub<G1|f(x),f(y),f(c)>;



S:=Sym(14);
p:=S!(1,2,3,4,5,6,7)(8,9,10,11,12,13,14);
q:=S!(1,12)(2,3)(4,11)(5,8)(6,13)(9,10);
N:=sub<S|p,q>;
#N;

cst := [null : i in [1 .. 2640]] where null is [Integers() | ];

ts:=[Id(G1):i in [1..14]];
ts[7]:=f(t);
ts[1]:=(f(t)^f(x));
ts[2]:=(f(t)^f(x^2));
ts[3]:=(f(t)^f(x^3));
```

```
ts[4]:=(f(t)^f(x^4));
ts[5]:=(f(t)^f(x^5));
ts[6]:=(f(t)^f(x^6));
ts[12]:=ts[1]^f(y);
ts[13]:=ts[12]^f(x);
ts[14]:=ts[13]^f(x);
ts[8]:=ts[14]^f(x);
ts[9]:=ts[8]^f(x);
ts[10]:=ts[9]^f(x);
ts[11]:=ts[10]^f(x);

prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
 */
 v := pt;
for i in I do
 v := v^(Q[i]);
end for;
return v;
end function;

N7:=Stabiliser(N,7);
S:={[7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7] eq n*(ts[(Rep(SSS[i]))[1]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if 7^n eq 14 then N7:=sub<N|N7,n>; end if; end for;


T7:=Transversal(N,N7);
for i in [1..#T7] do
ss:=[7]^T7[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..2640] do if cst[i] ne []
```

```
then m:=m+1; end if; end for;  m;

Orbits(N7);

N71:=Stabiliser(N7,1);
S:={[7,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1] eq n*(ts[(Rep(SSS[i]))[1]])*
(ts[(Rep(SSS[i]))[2]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,1]^n eq [7,8]
then N71:=sub<N|N71,n>; end if; end for;
for n in N do if [7,1]^n eq [14,8]
then N71:=sub<N|N71,n>; end if; end for;
for n in N do if [7,1]^n eq [14,1]
then N71:=sub<N|N71,n>; end if; end for;

Orbits(N71);

T71:=Transversal(N,N71);
for i in [1..#T71] do
ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

N712:=Stabiliser(N71,2);
S:={[7,1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1]*ts[2] eq n*(ts[(Rep(SSS[i]))[1]])*
(ts[(Rep(SSS[i]))[2]])*(ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
```

```
end for;
end for;

for n in N do if [7,1,2]^n eq [5,1,9]
then N712:=sub<N|N712,n>; end if; end for;

Orbits(N712);

T712:=Transversal(N,N712);
for i in [1..#T712] do
ss:=[7,1,2]^T712[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

N713:=Stabiliser(N71,3);
S:={[7,1,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1]*ts[3] eq n*(ts[(Rep(SSS[i]))[1]])*
(ts[(Rep(SSS[i]))[2]])*(ts[(Rep(SSS[i]))[3]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,1,3]^n eq [12,1,10]
then N713:=sub<N|N713,n>; end if; end for;

T713:=Transversal(N,N713);
for i in [1..#T713] do
ss:=[7,1,3]^T713[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

N7123:=Stabiliser(N712,3);
S:={[7,1,2,3]};
SS:=S^N;
SSS:=Setseq(SS);
```

```
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1]*ts[2]*ts[3] eq n*(ts[(Rep(SSS[i]))[1]])*
(ts[(Rep(SSS[i]))[2]])*(ts[(Rep(SSS[i]))[3]])*(ts[(Rep(SSS[i]))[4]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,1,2,3]^n eq [ 14, 6, 2, 10 ]
then N7123:=sub<N|N7123,n>; end if; end for;

T7123:=Transversal(N,N7123);
for i in [1..#T7123] do
ss:=[7,1,2,3]^T7123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

N7124:=Stabiliser(N712,4);
S:={[7,1,2,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1]*ts[2]*ts[4] eq n*(ts[(Rep(SSS[i]))[1]])*
(ts[(Rep(SSS[i]))[2]])*(ts[(Rep(SSS[i]))[3]])*(ts[(Rep(SSS[i]))[4]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,1,2,4]^n eq [ 10, 8, 2, 5 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 13, 12, 2, 7 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 4, 10, 2, 13 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 5, 7, 2, 6 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 1, 5, 2, 3 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 14, 6, 2, 12 ]
```

```
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 12, 3, 2, 8 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 11, 14, 2, 1 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 8, 4, 2, 14 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 3, 13, 2, 11 ]
then N7124:=sub<N|N7124,n>; end if; end for;
for n in N do if [7,1,2,4]^n eq [ 6, 11, 2, 10 ]
then N7124:=sub<N|N7124,n>; end if; end for;

T7124:=Transversal(N,N7124);
for i in [1..#T7124] do
ss:=[7,1,2,4]^T7124[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

N7139:=Stabiliser(N713,9);
S:={[7,1,3,9]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in N1 do
if ts[7]*ts[1]*ts[3]*ts[9] eq n*(ts[(Rep(SSS[i]))[1]])*
(ts[(Rep(SSS[i]))[2]])*(ts[(Rep(SSS[i]))[3]])*(ts[(Rep(SSS[i]))[4]])
then  print Rep(SSS[i]);
end if;
end for;
end for;

for n in N do if [7,1,3,9]^n eq [ 14, 11, 3, 2 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 12, 14, 3, 6 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 11, 12, 3, 1 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 9, 4, 3, 7 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 4, 6, 3, 8 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 2, 8, 3, 14 ]
```

```
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 1, 13, 3, 11 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 5, 2, 3, 13 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 6, 9, 3, 12 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 13, 7, 3, 5 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 7, 1, 3, 9 ]
then N7139:=sub<N|N7139,n>; end if; end for;
for n in N do if [7,1,3,9]^n eq [ 8, 5, 3, 4 ]
then N7139:=sub<N|N7139,n>; end if; end for;


T7139:=Transversal(N,N7139);
for i in [1..#T7139] do
ss:=[7,1,3,9]^T7139[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..2640] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
```

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