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A comparison of category and Lebesgue measure

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A Comparison of Category and Lebesgue Measure

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Abstract

The notions of category and Lebesgue measure are commonly used to describe the size of a set of real numbers (or of a subset of $\mathbb{R}^n$). Although cardinality is also a measure of the size of a set, category and measure are often the more important gauges of size when studying properties of classes of real functions, such as the space of continuous functions or the space of derivatives. Category can also be easily extended to complete metric spaces other than the real line, such as the space of continuous functions on a compact interval under uniform convergence. Thus through the study of Category one can study sets of functions as well as sets of real numbers. The following is a comparison of these two notions as well as a survey of useful results to which the study of them has led.

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Chapter 1

Introduction

The attempt to distinguish one set of numbers from another is a common goal among many mathematicians. Much of the foundational work in real analysis, topology, and abstract algebra is centered on this task. Many different properties have been explored to illuminate this topic, such as connectedness, compactness, and closure, to name only a few. One such property that can be studied to distinguish sets is size. In dealing with finite sets distinguishing sets by size is a relatively easy task. For instance, the set $\{1,2,3\}$ has exactly three elements in it so it clearly is distinguished from the set $\{1,2\}$ by size. In the case of infinite sets, however, this kind of distinction is not always easily accomplished. Take for example the set $\mathbb{Z}$ all integers in comparison to the interval $I = (0,1)$ of real numbers. It is obvious that they are not very similar except that they are both infinite. I is bounded, and $\mathbb{Z}$ is not, so in that way one might consider the integers larger. But at the same time we have that between any two integers there is a finite number of integers, whereas between any two real numbers there are infinitely more real numbers. So then I is in some respect bigger than $\mathbb{Z}$ because it seems to have more elements. In order to more precisely define the sizes of such sets, different classification systems have been invented. The most popular of which are cardinality, Lebesgue measure, and category. To classify sets by their cardinality is the simplest of these techniques. Though sometimes useful, this method cannot be used to draw as many conclusions about complicated sets as can Lebesgue measure and category. For this reason, we shall not make much mention of cardinality. Lebesgue Measure, defined by Henri Lebesgue at the beginning of the twentieth century, provides more information than does cardinality.

Take for instance the definition of a nullset, the Lebesgue measure version of a small set. A nullset is defined in such a way that it lends itself to analysis, requiring that the sum of the volumes of a sequence of intervals which covers the set to be measured be less than ϵ. This condition includes more basic properties such as countability, but clearly can also describe the size of even very complicated sets, such as the Cantor set, effectively. The notion of category, though defined very differently, has similar useful qualities. Published by Rene Louis Baire just a few years before Lebesgue's measure system, category was constructed by the observation that sets which Baire described as nowhere dense (dense in no interval) were closed under finite union, but not under countably infinite union. In fact, there was also a distinction made between infinite sets of numbers in that even dense subsets of $\mathbb{R}$ can be distinguished by whether or not they can be represented as a countable union of nowhere sense sets. Applications of the category method are quite varied as is shown in this paper. This is partially due to the fact that category arguments can be applied to topological spaces other than $\mathbb{R}^n$. The chapter which immediately follows this introduction is largely concerned with comparing category and Lebesgue measure by establishing mathematical properties possessed by either of them and then highlighting those properties which are analogous. It is also mentioned in a few places that there are sets of real numbers and subsets of $\mathbb{R}^n$ which have properties that involve both category and Lebesgue measure simultaneously. This is not surprising in itself but, as will be shown, it is surprising that the relationship between category and measure in such sets tends to in some cases be seemingly contradictory. The material for this chapter is completely expository and is primarily taken from *Measure and Category* by John Oxtoby. Chapter three is concerned with applications of the category method to characterize continuous functions. There are three results discussed. The first two results establish properties of typical continuous functions and the third is a theorem regarding the relationship betweenC^∞ functions and polynomial functions. Chapter four, in keeping with the pattern set by chapter two, then deals with an application of Lebesgue measure. In fact, the entire chapter is dedicated to the converse of the famous mean value theorem and the sets on which it holds (or doesn't hold). This is quite a complicated problem and only the case of continuously differentiable functions is considered, leaving the general case open for future research.

Chapter 2

Foundational Theorems in Measure and Category

2.1 "Small" Sets on The Real Line

2.1.1 First Category Sets

In the study of Category on the real line, infinite sets are separated into "small" sets and "large" sets, which we shall call sets of *first category* and *second category*, respectively. Before we explore the properties of these sets, let us define some preliminary terms.

We consider a set to be *countable* if it is either finite or is infinite and can be put into one-to-one correspondence with the natural numbers. A popular example is the rational numbers, since for each natural number k there are only finitely many ($\leq 2k - 1$ to be specific) rational numbers $\frac{p}{q}$ in reduced form for which $|p| + q = k$. A sequence for the rationals which gives them the required relationship with the natural numbers can therefore be found by numbering all of the rationals for which $|p| + q = 1$, then those for which $|p| + q = 2$, and then for which $|p| + q = 3$, and so on.

Now, a set A is considered to be *dense in the interval I* if A has a nonempty intersection with every subinterval of I. It is simply called *dense* if it is dense in all of $\mathbb{R}$. A set that is not dense in any interval of $\mathbb{R}$ is said to be *nowhere dense.* Two equivalent alternative definitions for nowhere dense that will prove to be useful are as follows: 1) A is nowhere dense if and only if its complement A' contains a dense open set, and 2) A is nowhere dense if and only if the closure of A (denoted either $\overline{A}, or A^{-}$ depending on the setting) has no interior points.

Theorem 2.1. *Any subset of a nowhere dense set is nowhere dense. The union of any two (or any finite number of) nowhere dense sets is nowhere dense. The closure of a nowhere dense set is nowhere dense.*

Proof. The first statement is obvious from the definition. As for the second, let there be two nowhere dense sets, A_1 and A_2. Then for each interval I there exists an interval $I_1 \subset I = A_1$ and an interval $I_2 \subset I_1 - A_2$. Therefore $I_2 \subset I - (A_1 \cup A_2)$. So we may conclude that $A_1 \cup A_2$ is nowhere dense. Now to prove the third statement notice that any open interval contained in A' is also contained in $A^{-'}$. □

While it has been shown that the union of finitely many nowhere dense sets is nowhere dense, it is important to note that unions of infinitely many nowhere dense sets are not necessarily nowhere dense. The set of rational numbers is again an illustrative example. Notice that this set is a countable union of singletons (sets containing only one element). This is interesting since singletons are of course nowhere dense, yet the set of rational numbers is dense. One might inquire, "Are there sets that are so large that they cannot be represented as even a countably infinite union of nowhere dense sets"? Indeed there are, and enough so that the contrast between such a set and sets which can be so represented is the basis for our study on Category. Now we may formally define the terms *first category* and *second category* as was alluded to earlier.

A set is said to be of *first category* if it can be represented as a countable union of nowhere dense sets. A subset of $\mathbb{R}$ that cannot be so represented is of *second category.*

Having established these definitions, we shall now look at a couple of important results that begin to characterize sets of first category. The first of which, found below, is a theorem by R. Baire known as the Baire Category Theorem.

Theorem 2.2. *The complement of any set of first category on the line is dense. No interval in $\mathbb{R}$ is of first category. The intersection of any sequence of dense open sets is dense.*

Proof. For the first statement, Let $A = \bigcup A_n$ be a representation of A as a countable union of nowhere dense sets. Now let I be an interval and define I_1 to be a closed subinterval of I such that $I_1 = I - A_1$. Similarly, I_2 will be closed and defined as $I_2 = I_1 - A_2$, and so on. Then $\bigcap I_n$ is a non-empty subset of $I - A$, so A' is dense. The second statement is an immediate corollary of the first. The third statement follows from the first by complementation.

□

Also worth mentioning is this theorem which will draw our first analogy between Category and Measure.

Theorem 2.3. *Any subset of a set of first category is of first category. The union of any countable family of first category sets is of first category.*

It is obvious that sets of first category have these closure properties, but the theorem is mentioned here to highlight a relationship. Any class of sets which, like those of first category, contains countable unions and arbitrary subsets of its members is called a σ-ideal. The class of countable sets is also a σ-ideal. Moreover, and perhaps more interestingly, the class of null sets is also a σ-ideal and this class will introduce us to the notion of "small" in the sense of Measure.

2.1.2 Null Sets

For any interval I, the length of I is denoted $|I|$. A set $A \subset \mathbb{R}$ is said to be a *null set*, or a set of *measure zero* if for each $\epsilon > 0$ there exists a sequence of intervals I_n such that $A \subset \cup I_n$ and $\sum |I_n| < \epsilon$.

To show that the class of null sets, like that of first category sets, is a σ-algebra, first notice that it is obvious that any subset of a null set is a null set. It may also be shown that a countable union of null sets is a null set. Begin by assuming A_i to be a null set for $i = 1, 2, \ldots$. Then for each i there is a sequence of intervals $I_{ij} (j = 1, 2, \ldots)$ such that $A_i \subset \cup_j I_{ij}$ and $\sum_j |I_{ij}| < \frac{\epsilon}{2^i}$. The set of all the interval I_{ij} covers A, and $\sum_{ij} |I_{ij}| < \epsilon$, so A is a null set.

Now that we have defined null sets and begun to draw an analogy to first category sets, let us further explore one of the properties of null sets which will provide greater illumination of the analogy.

Theorem 2.4. *If a finite or infinite sequence of intervals I_n covers an interval I, then* $\sum |I_n| \geq |I|$.

Assume first that I is a closed interval denoted $I = [a, b]$, and that all of the intervals I_n are open. Let (a_1, b_1) be the first interval that contains a. If $b_1 \leq b$, let (a_2, b_2) be the first interval of the sequence that contains b_1. If $b_{n-1} \leq b$, let (a_n, b_n) be the first interval that contains b_{n-1}. This procedure must terminate with some $b_N > b$. Otherwise the increasing sequence $\{b_n\}$ would converge to a limit $x \leq b$ and x would belong to I_k for some k. All but a finite number of intervals (a_n, b_n) would have to precede I_k in the given sequence. Specifically, every interval for which $b_{n-1} \in I_k$. This is impossible, since no two of these intervals are equal. We now have

$$b - a < b_n - a_1 = \sum_{i=2}^{N} (b_i - b_{i-1}) + b_1 - a_1 \leq \sum_{i=2}^{N} b_1 - a_1$$

Now for the general case. For any $\alpha > 1$ let J be a closed subinterval of I with $|J| = |I|/\alpha$, and let J_n be an open interval containing I_n with $|J_n| = \alpha |I_n|$. Then J is covered by e sequence $\{J_n\}$. We have already shown that $\sum |J_n| \geq |J|$. Hence $\alpha \sum |I_n| = \sum |J_n| \geq |J| = |I|/\alpha$. Letting $\alpha \to 1$ we obtain the desired conclusion. Notice that the implication of this theorem is that no interval is a null set.

2.1.3 The Odd Couple: An Unexpected Relationship

These two notions of smallness have shown some commonality and one might wonder if one includes the other or if somehow they are equivalent. One surprising theorem, as stated below, explicitly negates any possibility of such a relationship. At the same time this theorem establishes a surprising and even counterintuitive relationship between the two concepts, which strikes curiosity for further investigation.

Theorem 2.5. *The line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero.*

Proof. Let $a_1, a_2 ...$ be an enumeration of the set of rational numbers (or of any countable dense subset of $\mathbb{R}$). Let I_{ij} be the open interval with center a_i and length $1/2^{i+j}$. Let $G_j = \bigcup_{i=1}^{\infty} I_{ij} (j = 1, 2, ...)$ and $B = \bigcap_{i=1}^{\infty} G_j$. For any $\epsilon > 0$ we can choose j so that

$1/2^j < \epsilon$. Hence B is a null set. On the other hand, G_j is a dense open subset of $\mathbb{R}$, since it is the union of a sequence of open intervals and it includes all rational points. Therefore its complement G'_J is nowhere dense, and $A = B' = \bigcup_J G'_J$ is of first category. □

Corollary 2.6. *Every subset of the line can be represented as the union of a null set and a set of first category.*

Now that we have seen commonalities between the properties of the small sets in the studies of Measure and Category, let us explore further into each of those fields of study to continue the analogy.

2.2 Measure and Measurability

In this section we will introduce the concept of Lebesgue measure and highlight many of its most useful properties. We will conclude by establishing Lebesgue density and the Lebesgue Density Theorem, along with important results that follow from such.

2.2.1 Outer Measure

In order to understand Lebesgue measure, we must first define what is known as *outer measure.*The concept of outer measure will lay the foundation for Lebesgue measure (also referred to simply as "measure"). In fact, when a set A satisfies certain criteria we will define the measure of A, denoted $m(A)$, to be equal to the outer measure of A, denoted $m^*(A)$. Those criteria will be mentioned later in the section, after we have established some important properties about outer measure.
We must begin by clarifying some terminology and notation. An interval I in Euclidean r-space ($r = 1, 2, ...$) is to be understood as an r-dimensional rectangular parallelipiped with edges parallel to the axes. It is the Cartesian product of r 1-dimensional intervals. The r-dimensional volume of an interval I will be denoted $|I|$, just as it is for 1-dimensional volume (length).

Now, a sequence of intervals I_i is said to *cover* the set A if its union contains A. The *outer measure* of A, denoted $m^*(A)$,the greatest lower bound (infimum) of the sums

$\sum |I|$, for all sequences $\{I_i\}$ that cover A. Or, equivalently,

$$m^*(A) = inf\{\sum |I_i| : A \subset \cup I_i\}.$$

If, for every sequence that covers A, the series $\sum |I_i|$ diverges, then $m^*(A) = \infty$. Otherwise, $M^*(A)$ is a nonnegative real number.
We will now deduce several properties of outer Measure.

Theorem 2.7. *If $A \subset B$ then $m^*(A) \leq m^*(B)$.*

This can be concluded immediately, since any sequence $\{I_n\}$ that covers B also covers A.

Theorem 2.8. *If $A = \bigcup A_i$ then $m^*(A) \leq \sum m^*(A_i)$.*

Proof. We know that for any $\epsilon > 0$ there is a sequence of intervals $I_{ij}(j = 1, 2, ...)$ that covers A_i such that $\sum_j |I_{ij}| \leq m^*(A_i) + \epsilon/2^i$. Then $A \subset \cup_{i,j} I_i j$ and $\sum_{i,j} |I_{ij}| \leq \sum_i m^*(A_i) + \epsilon$. Therefore $m^*(A) \leq \sum_i m^*(A_i) + \epsilon$. Letting $\epsilon \to 0$, we reach the desired conclusion. This property is called *countable subadditivity.* □

Theorem 2.9. *For any interval I, $m^*(I) = |I|$.*

Proof. Since I covers itself, it is obvious that $m^*(I) \leq |I|$. Now we must show also that $m^*(I) \geq |I|$. Let $\epsilon > 0$, and let $\{I_i\}$ be an open covering of I such that $\sum |I_i| < m^*(I) + \epsilon$. Let also J be a closed subinterval of I such that $|J| > |I| - \epsilon$. By the Heine Borel Theorem, $J \subset \cup_1^k$ for some k. Let $K_1, ..., K_n$ be an enumeration of the closed intervals into which $\overline{I_1}, ..., \overline{I_k}$ are divided by all the $(r-1)$-dimensional hyperplanes that contain an $(r-1)$-dimensional face of one of the intervals $I_1, ..., I_k$, or J, and let $J_1, ..., J_m$ be the closed intervals into which J is divided by these same hyperplanes. Then each interval J_i in equal to at least one of the intervals K_j. Consequently,

$$|J| = \sum_{i=1}^{m} |J_i| \leq \sum_{j=1}^{n} |K_j| = \sum_{i=1}^{k} |I_i| < m^*(I) + \epsilon.$$

So, $|I| \leq m^*(I) + 2\epsilon$. Letting $\epsilon \to 0$ produces the desired inequality. □

Lemma 2.10. *If F_1 and F_2 are disjoint bounded closed sets, then $m^*(F_1 \cup F_2) = m^*(F_1) + m^*(F_2)$.*

Proof. Since F_1 and F_2 are disjoint bounded closed sets, there exists a $\delta > 0$ such that no interval of diameter less than δ meets both F_1 and F_2. Now for any $\epsilon > 0$ there is a sequence of intervals I_i of diameter less than δ such that $F_1 \cup F_2 \subset I_i$ and $\sum |I_i| \leq m^*(F_1 \cup F_2) + \epsilon$. Now let $\sum' |I_i|$ denote the sum over those intervals which meet F_1, and let $\sum'' |I_i|$ denote the sum over the remaining intervals, which cover F_2. Then

$$m^*(F_1) + m^*(F_2) \leq \sum' |I_i| + \sum'' |I_i| = \sum |I_i| \leq m^*(F_1 \cup F_2) + \epsilon.$$

Let then $\epsilon \to 0$ then we have

$$m^*(F_1) + m^*(F_2) \leq m^*(F_1 \cup F_2).$$

Then by theorem 2.9 we have

$$m^*(F_1 \cup F_2) = m^*(F_1) + m^*(F_2),$$

so equality is concluded. □

Lemma 2.11. *If $F_1, ..., F_n$ are disjoint bounded closed sets, then $m^*(\cup_1^n F_i) = \sum_1^n m^*(F_i)$.*

Proof. This follows from lemma 2.10 by induction on n. □

Lemma 2.12. *For any bounded open set G and $\epsilon > 0$ there exists a closed set F such that $F \subset G$ and $m^*(F) > m^*(G) - \epsilon$.*

Proof. Since G is open, it can be represented as a sequence of non-overlapping intervals I_i. Also, $m^*(G) \leq |I_i|$ by definition. Now determine n so that $\sum_1^n |I_i| > m^*(G) - \epsilon/2$, and let J_i be a closed interval contained in the interior of I_i such that $|J_i| > |I_i| - \epsilon/2n (i = 1, 2, ..., n)$. Then $F = \cup_1^n J_i$ is a closed subset of G. By theorem 2.9 and lemma 2.11 we have $m^*(F) = \sum_1^n |J_i| > \sum_1^n |I_i| - \epsilon/2 > m^*(G) - \epsilon$. □

Lemma 2.13. *If F is a closed subset of a bounded open set G, then $m^*(G - F) = m^*(G) - m^*(F)$.*

Proof. By lemma 2.12, for any $\epsilon > 0$ there is a closed subset F_1 of the open set $G - f$ such that $m^*(F_1) > m^*(G - F) - \epsilon$. By lemma 2.10 and theorem 2.7 we have

$$m^*(F) + m^*(G - F) < m^*(F) + m^*(F_1) + \epsilon = m^*(F \cup F_1) + \epsilon \leq m^*(G) + \epsilon.$$

Now letting $\epsilon \to 0$ we have

$$m^*(F) + m^*(G - F) \leq m^*(G) \Leftrightarrow m^*(G - F) \leq m^*(G) - m^*(F).$$

The reverse inequality follows from theorem 2.8 □

2.2.2 Measurability

We now have enough foundation to formally define what it means to be *measurable* in the sense of Lebesgue measure, and we will do so now.

Definition 2.14. *A set A is measurable if for each $\epsilon > 0$ there exists a closed set F and an open set G such that $F \subset A \subset G$ and $m^*(G - F) < \epsilon$.*

Lemma 2.15. *If A is measurable, then A' is measurable.*

Proof. This can be shown quickly since anytime we have $F \subset A \subset G$ then $F' \supset A' \supset G'$ and $F' - G' = G - F$. □

Lemma 2.16. *If A and B are measurable, then $A \cap B$ is measurable.*

Proof. Let F_1 and F_2 be closed sets, and let G_1 and G_2 be open sets, such that $F_1 \subset A \subset G_1$, $F_2 \subset B \subset G_2$, $m^*(G_1 - F_1) < \epsilon/2$, and $m^*(G_2 - F_2) < \epsilon/2$. Then $F = F_1 \cap F_2 \subset A \cap B \subset G_1 \cap G_2 = G$, say, and

$$G - F \subset (G_1 - F_1) \cup (G_2 - F_2).$$

Therefore $m^*(G - F) \leq m^*(G_1 - F_1)m^*(G_2 - F_2) < \epsilon$. □

Lemma 2.17. *A bounded set A is measurable if for each $\epsilon > 0$ there exists a closed set $F \subset A$ such that $m^*(F) > m^*(A) - \epsilon$.*

Proof. Let $\epsilon > 0$. Let F be a closed subset of A such that $m^*(F) > m^*(A) - \epsilon/2$. Since $m^*(A) < \infty$ (as A is bounded) there exists a covering sequence of open intervals I_i of diameter less than 1 such that $\sum |I_i| < m^*(A) + \epsilon/2$. Let G be the union of those intervals I_i that meet A. Then $F \subset A \subset G$, G is bounded, and by Lemma 2.13 $m^*(G - F) = m^*(G) - m^*(F) \leq \sum |I_i| - m^*(F) < m^*(A) + \epsilon/2 - m^*(F) < \epsilon$. So A is measurable. □

Lemma 2.18. *Any interval and any nullet is measurable.*

Proof. The first statement follows immediately from Lemma 2.17 and Theorem 2.9. Now if $m^*(A) = 0$, then for each $\epsilon > 0$ there is a covering sequence of open intervals I_i such that $\sum |I_i| < \epsilon$. Take $G = \cup I_i$ and $F = \emptyset$. Then F is closed, G is open, $F \subset A \subset G$, and $m^*(G - F) \leq \sum |I_i| < \epsilon$. □

Lemma 2.19. *Let $\{A_i\}$ be a disjoint sequence of measurable sets all contained in some interval I. If $A = \bigcup A_i$, then A is measurable and $m^*(A) = \sum m^*(A_i)$.*

Proof. For any $\epsilon > 0$ there exist closed sets $F_i \subset A_i$ such that $m^*(F_i) > m^*(A_i) - \epsilon/2^{i+1}$. By countable subadditivity we have

$$m^*(A) \leq \sum_1^\infty m^*(A_i).$$

Now determine k so that $\sum_1^k m^*(A_i) > m^*(A) - \epsilon/2$, and define $F = \bigcup_1^k m^*(F_i)$. Then by Lemma 2.11,

$$m^*(F) \leq \sum_1^k m^*(F_i) > \sum_1^k m^*(A_i) - \epsilon/2 > m^*(A) - \epsilon.$$

And so A is measurable by Lemma 2.17. For any n we have

$$\sum_1^n m^*(A_i) < \sum_1^n m^*(F_i) + \epsilon/2.$$

If we let $n \to 0$, and $\epsilon \to 0$ then we have $\sum_1^n m^*(A_i) \leq \sum_1^n m^*(A)$. The reverse inequality comes from the property of countable subadditivity established in Theorem 2.8. □

Lemma 2.20. *For any disjoint sequence of measurable sets A_i, the set $A = \cup A_i$ is measurable and $m^*(A) = \sum m^*(A_i)$.*

Proof. Let $I_j (j = 1, 2, ...)$ be a sequence of disjoint intervals whose union is the whole r-space such that any bounded set is covered by finitely many. By Lemmas 2.16, and 2.18, the sets $A_{ij} = A_i \cap I_j$ are measurable. They are also disjoint. Put $B_j = \bigcup_i A_{ij}$. By Lemma 2.19, B_j is a measurable subset of I_j. The sets B_j are disjoint, and $A = \bigcup B_j$. For any $\epsilon > 0$ there exist closed sets F_j and bounded open subsets G_j such that $F_j \subset B_j \subset G_j$ and $m^*(G_j - F_j) < \epsilon/2^j$. Let $F = \bigcup F_j$ and $G = \bigcup G_j$. The F is closed, since any convergent sequence contained in F is bounded and therefore contained in the union of a finite number of the sets F_j, which is a closed subset of F. Also, G is open. We have $F \subset A \subset G$ and $G - F = \bigcup (G_i - F) \subset \bigcup (G_i - F_i)$. Hence $m^*(G - F) \leq m^*(G_j - F_j) < \epsilon$. This shows that A is measurable. Since $A_i = \bigcup_j A_{ij}$, we have$m^*(A_i) \leq \sum m^*(A_{ij})$, and therefore

$$\sum m^*(A_i \leq \sum_{i,j} m^*(A_{ij} = \sum_j \sum_i m^*(A_{ij} = \sum_j m^*(B_j,$$

by Lemma 2.19. Also, for any n,

$$\sum_{i}^{n} m^*(B_j) \leq \sum_{i}^{n} m^*(F_j) + \sum_{i}^{n} m^*(G_j - F_j) \leq m^*(\bigcup_{1}^{n} F_j) + \epsilon \leq m^*(A) + \epsilon.$$

Letting $n \to \infty$ and $\epsilon \to 0$ we conclude that $\sum_j m^*(B_j) \leq m^*(A)$. Therefore $\sum m^*(A_i) \leq m^*(A)$. Again, the property of countable subadditivity provides the reverse inequality. □

2.2.3 Additional Properties of Lebesgue Measure

Now a foundation has been laid for the concepts of outer measure and measurability. It is therefore appropriate to at this time precisely define what is meant by measure. However, some additional Definitions are still needed to do so in an organized and efficient manner. A non-empty class S of subsets of a set X is called a ring of subsets of X if it contains the pairwise union and the difference of any two of its members. It is called a σ-ring if also contains the union of any arbitrarily large sequence of its members. Moreover, if X is a member of the ring (or σ-ring) , then the ring is called an algebra of subsets of X (or σ-Algebra, respectively). A set function μ on a ring S of subsets of X is said to be countably additive if the equation $\mu(A) = \sum \mu(A_i)$ holds whenever $\{A_i\}$ is a disjoint sequence of members of S whose union A also belongs to S. A measure is an extended real valued, nonnegative, countably additive set function μ, defined on a σ-ring of subsets of a set X, and such that $\mu(\emptyset) = 0$. A triple (X, S, μ), where S is a σ-ring of subsets of a set X and μ is a measure defined on S, is called a measure space. Sets belonging to S are called μ-measurable. A measure space is said to be complete if every subset of a set of μ-measure zero belongs to S (that is, when the sets of μ-measure zero constitute a σ-ideal. Now we may apply these concepts to r-dimensional Euclidean space. By Lemmas 2.15, 2.16, 2.18, and 2.20, the class S of measurable sets is a σ-algebra of subsets of r-space. Also m^* is countably additive on S. Therefore we have that m^* restricted to S is a measure. It is called Lebesgue Measure and is denoted by m. It is worth noting that since S includes all intervals, it includes all open sets, closed sets, countable unions of closed sets (denoted F_σ), and countable intersections of open sets (denoted G_δ). Having established these relationships, the following important results can be stated plainly.

2.2.4 Noteworthy Properties of Measurable Sets

Theorem 2.21. *A set A is measurable if and only if it can be represented as an F_σ set plus a nullset (or as a G_δ set minus a nullset).*

Proof. If A is measurable, then for each n there exists a closed set F_n and an open set G_n such that $F_n \subset A \subset G_n$ and $m^*(G_n - F_n) < \frac{1}{n}$. Put $E = \cup F_n$ and $N = A - E$. Then E is an F_σ set. N is a nullset, since $N \subset G_n - F_n$ and $m^*(N) < \frac{1}{n}$ for every n. A is the disjoint union of E and N. It follows by complementation that A can also be represented as a G_δ set minus a nullset. Conversely, any set that can be so represented is measurable, by lemma 2.18 and the fact that S is a σ-algebra. □

Now, for any class of subsets of a set X there is a smallest σ-algebra of subsets of X that contains it. This σ-algebra is found by intersecting all such σ-algebras, and is called the σ-algebra generated by the class. When studying r-space, the sets of the σ-algebra generated by the open sets are called Borel sets. By theorem 21 the Borel sets combined with the nullsets generate the class of measurable sets. The following theorem is nothing but a summary of these facts. The following theorem is functionally equivalent to the property of countable additivity, but is in many situations more useful.

Theorem 2.22. *The class S of measurable sets is the σ-algebra of subsets of r-space X generated by the open sets together with the nullsets. Lebesgue measure m is a measure on S such that $m(I) - |I|$ for every interval I. (X, S, m) is a complete measure space.*

Proof. In the first case, let $B_1 = A_1$ and $B_i = A_i - A_i - 1$ for $i > 1$. Then $\{B_i\}$ is a disjoint sequence of measurable sets, with $A = \bigcup B_i$. Hence

$$m(A) = \sum m(B_i) = \lim \sum_1^n m(B_i) = \lim m(A_n),$$

where the limit may be equal to ∞. In the second case, we may assume $m(A_1) < \infty$. Let $B_1 = A_1 - A_i$ and $B = A_1 - A$. Then $B_i \subset B_{i+1}$ and $\bigcup B_i = B$. So $m(A_1) - m(A) = m(B) = \lim m(B_i) = \lim m(A_1) - m(A_i) = m(A_1) - \lim m(A_i)$, and so $m(A) = \lim m(A_i)$, both members being finite. □

The following theorem indicates the manner in which the outer measure function (m^*) is determined by its values on open and closed sets.

Theorem 2.23. *If A_i is measurable , and $A_i \subset S_{i+1}$ for each i, then the set $A = \cup A_i$ is measurable and $m(A) = \lim m(A_i)$. If If A_i is measurable , and $A_i \supset S_{i+1}$ for each i, then the set $A = \cap A_i$ is measurable and $m(A) = \lim m(A_i)$ provided $m(A_i) < \infty$ for some i.*

Theorem 2.24. *The outer measure of any set A is expressed by the formula*

$$m^*(A) = \inf\{m(G) : A \subset G, G \text{ open}\}.$$

Further, if A is measurable, then

$$m^*(A) = \sup\{m(F) : A \supset F, F \text{ bounded and closed}\}.$$

Conversely, if this equation holds and $m(A) < \infty$ then A is measurable.

Proof. The first statement is clear, since the union of any covering sequence of open intervals is an open superset of A. To prove the second, let α be any real number less than $m(A)$, and let $A_i = A \cap (-i, i)^r$. by Theorem 2.23, $m(A) = \lim m(A_i)$, meaning that we can choose i so that $m(A_i) > \alpha$. By measurability, A_i (which is bounded) contains a closed set F with $m(F) > \alpha$, and F is also a subset of A. Conversely, if $m^*(A) < \infty$ and F isa closed subset of A with $m(F) > m^*(A) - \epsilon/2$, let G be an open superset of A such that $m(G) < m^*(A) + \epsilon/2$. Then $F \subset A \subset G$ and $m(G - F) < \epsilon$. Therefore A is measurable. □

Theorem 2.25. *If A is congruent by translation to a measurable set B, then A is measurable and $m(A) = m(B)$.*

Proof. This is clear from the definition and from the fact that congruent intervals have equal volume. □

A natural inquiry that arises from any discussion of measurability and density is regarding the distribution of a measurable set within a given space. For instance, the set of rational numbers within the real numbers is dense over all of $\mathbb{R}$, implying that the rational numbers are spread evenly throughout the real line. But is is possible that there are subsets of the rational numbers which are "more dense" than others? The Lebesgue Density Theorem illuminates this topic. In fact, according to this theorem a measurable set is either highly concentrated or highly rarified at almost all points. Before this theorem is stated formally, we must first define a couple of terms.

First, speaking only in terms of subsets of $\mathbb{R}$, a measurable set $E \subset \mathbb{R}$ is said to have density d at x if the

$$\lim_{h\to 0} \frac{m(E \cap [x-h, x+h])}{2h}$$

exists and is equal to d. Now, denote the set of all points of $\mathbb{R}$ at which E has density 1 by $\phi(E)$. Then of course $\phi(R - E)$ would be the set of all points at which E has density 0. ϕ is called the Lebesgue lower density. The Lebesgue Density Theorem asserts that $\phi(E)$ is measurable and differs from E by only a nullset, meaning that E has density 1 at almost every point of E and has density 0 at almost all points of $\mathbb{R} - E$. This makes it impossible for a set and its complement to each include, say exactly one half of the outer measure of a given interval, since a set as such would have density $\frac{1}{2}$ everywhere.

We now must define one more term. Given two sets, A and B, the set of all points that belong to either A or B but not both is called the symmetric difference. It is denoted $A\triangle B$. An equivalent definition is

$$A\triangle B = (A - B) \cup (B - A).$$

And now we will finally state the Lebesgue Density Theorem

Theorem 2.26. *For any measurable set $E \subset \mathbb{R}$, $m(E\triangle\phi(E)) = 0$.*

Proof. It suffices to show that $E - \phi(E)$ is a nullset, since $\phi(E) - E \subset E' - \phi(E')$ and E is measurable. We may also assume that E is bounded. Furthermore, $E - \phi(E) = \bigcup_{\epsilon>0} A_\epsilon$, where

$$A_\epsilon = \left\{x \in E : \liminf_{h\to 0} \frac{m(E \cap [x-h, x+h])}{2h} < 1 - \epsilon\right\}.$$

So it is sufficient to show that A_ϵ is a nullset for every $\epsilon > 0$. Let now $A = A_\epsilon$, and assume that $m^*(A) > 0$. If $m^*(A) > 0$, there exists a bounded open set G containing A such that $m(G) < m^*(A)/(1-\epsilon)$. Let ξ denote the class of all closed intervals I such that $I \subset G$ and $m(E \cap I) \leq (1-\epsilon)|I|$. Observe that (i) ξ includes arbitrarily short intervals about each point of A, and (ii) for any disjoint sequence $\{I_n\}$ of members of ξ we have $m^*(A - \bigcup I_n) > 0$. Property (ii) follows from the fact that

$$m * (A \cap \bigcup I_n) \leq \sum m(E \cap I_n) \leq \sum |I_n| \leq (1-\epsilon)m(G) < m^*(A).$$

We construct inductively a disjoint sequence I_n of members of ξ as follows. Choose I_1 arbitrarily from ξ. Having chosen $I_1, ..., I_n$, let ξ_n be the set of members of ξ that are disjoint to $I_1, ..., I_n$. Properties (i) and (ii) imply that ξ_n is non-empty. Let d_n be the least upper bound of the lengths of members of ξ_n, and choose $I_n + 1 \in \xi$ such that $|I_n + 1| > d_n/2$. Put $B = A - \bigcup_1^\infty I_n$. By (ii) we have $m^*(B) > 0$. Hence there exists a positive integer such that

$$\sum_{N+1}^{\infty} |I_n| < m^*(B)/3.$$

Call this inequality (1). For each $n > N$ let J_n denote the interval concentric with I_n with $|J_n| = 3|I_n|$. Then (1) implies that $J_n (n > N)$ do not cover B, hence there exists a point $x \in B - \bigcup_{N+1}^{\infty} J_n$. Since $x \in A - \bigcup_1^N I_n$, it follows from (i) that there exists an interval $I \in \xi_N$ with caenter x. I must meet some interval I_n with $n > N$. (Otherwise $|I| \leq d_n < 2|I_{n+1}|$ for all n, contrary to $\sum_1^\infty |I_n| \leq m(G) < \infty$.) Let k be the least integer such that I meets I_k. Then $k > N$ and $|I| \leq d_{k-1} < 2|I_k|$. It follows that the center x of I belongs to J_k, contrary to $x \notin \bigcup_{N+1}^{\infty} J_n$. □

Theorem 2.27. *For any measurable set A, let $\phi(A)$ denote the set of points of $\mathbb{R}$ where A has density 1.Then ϕ has the following properties:*

(i) $m(\phi(A) \triangle B) = 0$ implies $\phi(A) = \phi(B)$

(ii) $\phi(\emptyset) = \emptyset$ and $\phi(\mathbb{R}) = \mathbb{R}$

(iii) $m(\phi(A) \cup B) = \phi(A) \cup \phi(B)$

(iv) $A \subset B$ implies $\phi(A) \subset \phi(B)$

Proof. The first and second assertions are immediate consequences of the definition of ϕ. To prove (iii), notice that for any interval I we have $I - (A \cap B) = (I - A) \cup (I - B)$. Hence $m(I) - m(I \cap A \cap B) \leq m(I) - m(I \cap A) + m(I) - m(I \cap B)$. Therefore

$$\frac{m(I \cap A)}{|I|} + \frac{m(I \cap B)}{|I|} - 1 \leq \frac{m(I \cap A \cap B)}{|I|}.$$

Taking $I = [x - h, x + h]$ and letting $h \to 0$ it follows that $\phi(A) \cap \phi(B) \subset \phi(A \cap B)$. The opposite inclusion is obvious. Property (iv) is a consequence of (iii). □

2.3 The Property of Baire: A Match Made in Heaven

We found earlier that the category analog of a nullset is a set of first category. Now having discussed measurable sets one might wonder if there is an appropriate analog for them also. Indeed it seems that the *property of Baire* provides such an analog. This section is dedicated to studying the property of Baire and highlighting analogous concepts to those studies in the previous section.

2.3.1 Properties of Symmetric Difference

Before we define what it means for a set to have the property of Baire it is important to make mention of some of the properties of the symmetric difference relation, as these will be utilized in our studies.
First, remember that the definition of symmetric difference is as follows:

$$A\triangle B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

It is worth noting (although we will not be proving this here) that symmetric difference is commutative, associative, and satisfies the distributive law $A\cap(B\triangle C) = (A\cap B)\triangle(A\cap C)$. Also worth mentioning, though somewhat obvious, is the fact that $A\triangle B \subset A \cup B$ as well as that $A\triangle A = \emptyset$.

2.3.2 The Property of Baire

We say that a subset A of r-space (or any topological space) has the *property of Baire* if it can be represented in the form $A = G\triangle P$ where G is open and P is of first category. We need no further information to be able to discuss fully several relevant Theorems regarding the property of Baire.

Theorem 2.28. *A set A has the property of Baire if and only if it can be represented in the form $A = F\triangle Q$, where F is closed and Q is of first category.*

Proof. If $A = G\triangle P$, where G is open and P is of first category, then $N = \overline{G} - G$ is a nowhere dense closed set, and $Q = N\triangle P$ is of first category. Let $F = \overline{G}$. Then $A = G\triangle P = (\overline{G}\triangle N)\triangle P = \overline{G}\triangle(N\triangle P) = F\triangle Q$. Conversely, if $A = F\triangle Q$, where F is closed and Q is of first category, let G be the interior of F. Then $N = F - G$ is nowhere dense, $P = N\triangle Q$ is of first category, and $A = F\triangle Q = (G\triangle N)\triangle Q = G\triangle(N\triangle Q) = G\triangle P$. □

Theorem 2.29. *If A has the property of Baire, then so does its complement.*

Proof. For any two sets A and B we have $(A\triangle B)' = A'\triangle B'$. Hence if $A = G\triangle P$, then $A' = G'\triangle P$, and since G' is closed and P is of first category the conclusion follows from Theorem 2.28. □

Theorem 2.30. *The class of sets having the property of Baire is a σ-algebra. It is the σ-algebra generated by the open sets together with the sets of first category.*

Proof. Let $A_i = G_i\triangle P_i (i = 1, 2, ...)$ be any sequence of sets having the property of Baire. Let $G = \bigcup G_i, P = \bigcup P_i, and A = \bigcup A_i$. Then G is open, P is of first category, and $G - P \subset A \subset (G \cup P)$. Therefore $G\triangle A \subset P$ is of first category, and $A = G\triangle(G\triangle A)$ has the property of Baire. This result, in combination with Theorem 2.29, shows that such a class of sets is a σ-algebra. More than that, it is the smallest σ-algebra that includes all open sets and all sets of first category. □

Theorem 2.31. *A set has the property of Baire if and only if it can be represented as a G_δ set plus a set of first category (or an f_σ set minus a set of first category).*

Proof. Since the closure of any nowhere dense set is nowhere dense, any set of first category is contained in an F_σ set of first category. If G is open and P is of first category, let Q be an F_σ set of first category that contains P. Then the set $E = G - Q$ is a G_δ set, and we have

$$G\triangle P[(G - Q)\triangle(G \cap Q)]\triangle(P \cap Q) = E\triangle[(G\triangle P) \cap Q].$$

The set $(G\triangle P) \cap Q$ is of first category and disjoint to E. Therefore any set having the property of Baire can be represented as the disjoint union of a G_δ set and a set of first category. Conversely, and set that can be so represented belongs to the σ-algebra generated by the open sets and the sets of first category; it therefore has the property of Baire. The parenthetical statement follows by complementation, along with Theorem 2.29. □

Before we proceed it is necessary to introduce a new definition.

Definition 2.32. *A* regular open *set is a set that is equal to the interior of its closure. Equivalently, any set of the form $A^{-'-'}$ is regular open.*

Theorem 2.33. *Any open set H is of the form $H = G - \overline{N}$, where G is regular open and N is nowhere dense.*

Proof. Let $G = H^{-'-'}$ and $N = G - H$. Then G is regular open, N is nowhere dense, and $H = G - N$. We have $\overline{N} \subset \overline{G} - H$. Therefore $G - \overline{N} \supset G - (\overline{G} - H) = G \cap H = H$. Also, $H = G - N \supset G - \overline{N}. And so H = G - \overline{N}$. □

Theorem 2.34. *Any set having the property of Baire can be represented in the form $A = G \triangle P$, where G is a regular open set and P is of first category. This representation is unique in any space in which every non-empty open set is of second category (that is, not of first category).*

Proof. The existence of such a representation follows from Theorem 2.33; in any representation we can always replace the open set by the interior of its closure. To prove uniqueness, suppose $G \triangle P = H \triangle Q$, where G is a regular open set, H is open, and P and Q are of first category. Then $H - \overline{G} \subset H \triangle G = P \triangle Q$. So then $H - \overline{G}$ is an open set of first category, and therefore empty. We have $H \subset \overline{G}$, and therefore $H \subset ofG^{-'-'} = G$. Thus in the regular open representation the open set G is maximal. If both G and H are regular open, then each contains the other. Then finally $G = H$ and $P = Q$. □

Theorem 2.35. *The intersection of any two regular open sets is a regular open set.*

Proof. Let $G = G^{-'-'} and H = H^{-'-'}$. Since $G \cap H$ is open, it follows that $G \cap H \subset (G \cap H)^{-'-'}$. Further, $(G \cap H)^{-'-'} \subset G^{-'-'} = G$ and $(G \cap H)^{-'-'} \subset H^{-'-'} = H$. Therefore $(G \cap H)^{-'-'} \subset (G \cap H)$. Now we have by double inclusion that $(G \cap H)^{-'-'} = G \cap H$. □

2.3.3 Together Again: The Odd Couple Revisited

Theorem 2.5 was an exciting result in that it clearly defined a relationship between nullsets and first category sets. The following Theorem is an equally exciting result for the same reason. However, unlike Theorem 2.5, it does not have any counterintuitive surprises. Rather, this theorem shows an interesting property common to measurable sets and second category sets which have the property of Baire.

Theorem 2.36. *For any linear set A of second category having the property of Baire, and for any measurable set A with $m(A) > 0$, there exists a positive number δ such that $(x + A) \cap A \neq \emptyset$ whenever $|x| < \delta$*

Proof. In the first case, let $A = G \triangle P$. Since G is non-empty, it contains an interval I. For any x, we have

$$(x + A) \cap A \supset [(x + I) \cap I] - [P \cup (x + P)].$$

If $|x| < |I|$, the right member represents an interval minus a set of first category; it is therefore non-empty. Hence we may take $\delta = |I|$.
In the second case, let F be a bounded closed subset of A with $m(F) > 0$ (Theorem 24). Enclose F in a bounded open set G with $m(G) < \frac{4}{3}m(F)$. G is the union of a sequence of mutually disjoint intervals. For at least one of these, say I, we must have $m(F \cap I) > \frac{3}{4}m(I)$. Take $\delta = \frac{m(I)}{2}$. If $|x| < \delta$, then $(x + I) \cup I$ is an interval of length less than $\frac{3}{2}m(I)$ that contains both $F \cap I$ and $x + (F \cap I)$. These sets cannot be disjoint, because $m(x + (F \cap I)) = m(F \cap I) > \frac{3}{4}m(I)$. Since $(x + A) \cap A \supset [x + (F \cap I)] \cap [F \cap I]$, it follows that the left member is non-empty. □

2.3.4 Drawing More Analogies

At this point it is pertinent to draw attention to some of the analogous results of our study of Measurable sets and set having the property of Baire. First, notice the analogy between Theorem 2.30 and Theorem 2.22. These theorems state that a measurable set is similar to set which has the property of Baire in that either forms a σ-algebra generated by "small" sets together with open sets. Also, there is a strange analogy between Theorem 2.31 and Theorem 2.21. This is interesting in that, just as was seen in Theorem 2.5, these two analogous classes of sets seem to act oppositely in some ways.

2.4 Missing Property

We have discussed many different facets of classes of sets which are measurable and/or have the property of Baire. But we have not touched on the subject of sets which do not possess these qualities. It may seem ridiculous to try to imagine a set which is not measurable. But as we are soon to discuss, it is not only possible to have non-measurable sets and sets which don't have the property of Baire. It is unavoidable.

2.4.1 Before We Build: Gathering Materials to Construct Non-Measurable Sets

Lemma 2.37. *Any uncountable G_δ subset of $\mathbb{R}$ contains a nowhere dense closed set C of measure zero that can be mapped continuously onto* $[0,1]$.

Proof. Let $E - \bigcap G_n$, where G_n is open, be an uncountable G_δ set. Let F denote the set of all condensation points of E that belong to E, that is,all points in E such that every neighborhood of x contains countably many points of E. F is non=empty; otherwise, the family of intervals that have rational endpoints and contain only countably many points of E would cover E, and E would be countable. Similar reasoning shows that F has no isolated points. Let $I(0)$ and $I(1)$ be two disjoint closed intervals of lentgh at most $1/3$ whose interiors meet F and whose union is contained in G_1. Proceeding inductively, if 2^n disjoint closed intervals $I(i_1, ..., i_n)$ have been defined, let $I(i_1, ..., i_{n+1})$ with ($I_{n+1} = 0$ or 1) be disjoint closed intervals of length at most $1/3^{n+1}$ contained in $G_{n+1} \cap I(i_1, ..., i_n)$ whose interiors meet F. From the fact that G has no isolated points and that $E \subset G_{n+1}$ it is clear that such intervals exist. Thus a family of intervals $I(i_1, ..., i_n)$ having the stated properties can be defined. Let

$$C = \bigcap_n \bigcup_{i_1,...,i_n} I(i_1, ..., i_n).$$

Then C is a closed nowhere dense subset of E. C has measure zero for the same reason as the Cantor set (More than that, it is actually homeomorphic to the Cantor set). For each x in C there is a unique sequence $\{i_n\}$, $i_n = 0$ or 1, such that $x \in I(i_1, ..., i_n)$ for every n, and every such sequence corresponds to some point of C. Let $f(x)$ be the real number having the binary development $i_1 i_2 i_3 ....$ Then f maps C onto $[0,1]$. Hence C has power c. Also, f is continuous because $|f(x) - f(x)'| \leq 1/2^n$ when x and x' both belong to $C \cap I(i_1, ..., i_n)$. □

Lemma 2.38. *The class of uncountable closed subsets of $\mathbb{R}$ has power c.*

Proof. The class of pen intervals with rational endpoints is countable, and every open set is the union of some subclass. Therefore there are at most c open sets and therefore, by complementation, at most c closed sets. At the same time, there are at least c uncountable closed sets, since there are that many closed intervals. So then there must be exactly c uncountable closed subsets of the line. □

Theorem 2.39. *There exists a set B of real numbers such that both B and B' meet every uncountable closed subset of the line.*

Proof. By the well ordering principle and Lemma 2.38, the class $\widehat{F}$ of uncountable closed subsets of the line can be indexed by the ordinal numbers less than ω_c, where ω_c is the first ordinal having c predecessors, say $\widehat{F} = \{F_\alpha : \alpha < \omega_c\}$. We may assume that $\mathbb{R}$, and therefore each member of $\widehat{F}$, has been well ordered. Note that each member of $\widehat{F}$ has power c, by Lemma 2.37, since any closed set is a G_δ. Let p_1 and q_1 be the first two members of F_1. Let p_2 and q_2 be the first two members of F_2 different from both p_1 and q_1. If $1 < \alpha < \omega_c$, and if p_β and q_β have been defined for all $\beta > \alpha$, let p_α and q_α be the first two elements of $F_\alpha - \bigcup_{\beta<\alpha}\{p_\beta, q_\beta\}$. This set is non-empty (it has power c) for each α and so p_α and q_α are defined for all $\alpha < \omega_c$. Put $B = \{p_\alpha : \alpha < \omega_c\}$. Since $p_{alpha} \in B \cap F_\alpha$ and $q_\alpha \in B' \cap F_\alpha$ for each $\alpha < \omega_c$, the set B has the property that both it and its complement meet every uncountable closed set. Let us call any set with this property a *Bernstein* set. □

2.4.2 A Comparison of Non-Measurable Sets and Those Lacking the Property of Baire

Having discussed Bernstein sets, we will now proceed to a few important important results regarding non-measurable sets, sets lacking the property of Baire, and analogies between the two.

Theorem 2.40. *Any Bernstein set B is non-measurable and lacks the Property of Baire. Indeed, every measurable subset of either B of B' is a nullset, and any subset of B or B' that has the property of Baire is of first category.*

Proof. Let A be any measurable subset of B. Any closed set F contained in A must be countable (since every uncountable closed set meets B'), so $m(F) = 0$. Therefore $m(A) = 0$, by Theorem 2.24. Similarly, if A is a subset of B having the property of Baire, then $A = E \sup P$, where E is G_δ and P is of first category. The set E must be uncountable, since every uncountable G_δ set contains an uncountable closed set, by

Lemma 2.37, and therefore meets B'. Therefore A is of first category. The same reasoning applies to B'. □

Theorem 2.41. *Any set with positive outer measure as a non-measurable subset. Any set of second category has a subset that lacks the property of Baire.*

Proof. If A has a positive outer measure and B is a Bernstein set, Theorem 2.37 shows that the subsets $A \cap B$ and $A \cap B'$ cannot both be measurable. If A is of second category, these two subsets cannot both have the property of Baire. □

2.4.3 A Generalization on Non-Measurable Sets

To this point we have concerned ourselves with Measurability with respect to Lebesgue measure, and therefore we have made conclusions about non-measurable sets also with respect to Lebesgue measure. But is it the case that we could define a different measure in such a way that we would not have any non-measurable sets? In fact, that is not the case for any set of power $\aleph_1$. The following theorems will prove that for any such set it is not possible to define a measure that assigns a measure to every element (given that all singletons have measure zero).

Theorem 2.42. *A finite measure μ defined for all subsets of a set X of power $\aleph_1$ vanishes identically if it is equal to zero for every one-element subset.*

Proof. By hypothesis, there exists a well ordering of X such that for each y in X the set $\{x : x < y\}$ is countable. Let $f(x, y)$ be a one-to-one mapping of this set onto a subset of the positive integers. Then f is an integer-valued function defined for all pairs (x, y) of elements of X for which $x < y$. It has the following property:

$$x < x' < y \Rightarrow f(x, y) \neq f(x', y). \tag{2.1}$$

For each x in X and each positive integer n, define

$$F_x^n = \{y : x < y, f(x, y) = n\}.$$

We may picture these sets as arranged in an array

$$\begin{array}{l} F^1_{x_1} F^1_{x_2} \ldots F^1_x \ldots \\ F^2_{x_1} F^2_{x_2} \ldots F^2_x \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ F^n_{x_1} F^n_{x_2} \ldots F^n_x \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

with $\aleph_0$ rows and $\aleph_1$ columns. This array has the following properties:

(2) The sets in any row are mutually disjoint.

(3) The union of the sets in any column is equal to X minus a countable set.

To verify (2), suppose $y \in F^n_x \cap F^n_{x'}$, for some n and some y, x, and x', with $x \leq x'$. Then $x < y$, $x' < y$, and $f(x, y) = f(x', y) = n$. Hence $x = x'$, by (1). Therefore, for any fixed n, the sets $F^n_x (x \in X)$ are disjoint. To verify (3), observe that if $x < y$, then y belongs to one of the sets F^n_x, namely, that one for which $n = f(x, y)$. Therefore the union of the sets $F^n_x (n = 1, 2, ...)$ differs from X by the countable set $\{y : y \leq x\}$. By (2), in any row there can be at most countably many sets for which $\mu(F^n_x) > 0$ (since $\mu(X)$ is finite). Therefore there can be at most countably many such sets in the whole array. Since there are uncountably many columns it follows that there exists an element x in X such that $\mu(F^n_x) = 0$ for every n. The union of the sets in this column has measure zero, and the complementary countable set also has measure zero. Therefore $\mu(X) = 0$ and so μ is identically zero. □

Notice that this theorem, as stated above, does not apply to Lebesgue measure directly, unless we accept the *continuum hypothesis*, which states that there is no set whose cardinality is strictly between that of the integers and that of the real numbers, or in other words $\aleph_1 = c$. If however, we assume this to be true then the following theorem is an equivalent assertion which applies to r-space.

Theorem 2.43. *A finite measure defined for all subsets of a set of power c vanishes identically if it is zero for points.*

And therefore, since Lebesgue measure assigns measure zero to all singletons, and yet does not vanish identically, it must not be defined for all subsets.

2.5 Results Concerning Continuity

2.5.1 Functions of First Class

Let us begin with a useful definition. Let f be a real-valued function on $\mathbb{R}$. For any interval I, the quantity

$$\omega(x) = \lim_{\delta \to 0} \omega((x-\delta, x+\delta)),$$

called the *oscillation of f at x*. $\omega(x)$ is an extended real-valued function on $\mathbb{R}$. Evidently, $\omega(x_0) = 0$ if and only if f is continuous at x_0. When it is not zero, $\omega(x_0)$ is a measure of the size of the discontinuity of f at x_0. If $\omega(x_0) < \epsilon$, then $\omega(x) < \epsilon$ for all x in a neighborhood of x_0. Hence the set $\{x : \omega(x) < \epsilon\}$ is open. The set D of all points at which f is discontinuous can be represented in the form

$$D = \bigcup_{n=1}^{\infty} \{x : \omega(x) \geq 1/n\},$$

so D is always an F_σ set. This fact produces the following theorem

Theorem 2.44. *If f is a real-valued function on $\mathbb{R}$, then the set of points of discontinuity of f is an F_σ.*

Having this fact, we may now consider the converse.

Theorem 2.45. *For any F_σ set E there exists a bounded function f having E for its set of points of discontinuity.*

Proof. Let $E = \bigcup F_n$, where F_n is closed. We may assume that $F_n \subset F_{n+1}$ for all n. Let A_n denote the set of rational points interior to F_n. For any set A, the function χ_A defined by

$$\chi_a(x) = \begin{cases} 1 & \text{when } x \in A \\ 0 & \text{when } x \notin A \end{cases}$$

is called the *characteristic function* of A. The function $f_n = \chi_{F_n} - \chi_{A_n} = \chi_{F_n} - A_n$ has oscillation equal to 1 at each point of F_n, and equal to 0 elsewhere. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n > \sum_{i>n} a_i$ for every n. Then the series $\sum_{n=1}^{\infty} a_n f_n(x)$ converges uniformly on $\mathbb{R}$ to a bounded function f. f is continuous at any point where all of the terms are continuous, so at each point of $\mathbb{R} - E$. On the other hand, at each point of $F_n - F_{n-1}$ the oscillation points of discontinuity of f is exactly E. □

If a function can be represented as the limit of an everywhere convergent sequence of continuous functions, it is said to be a function of *first class.* Such a function does not need to be continuous itself. However, as it would seem, a first class function also cannot be everywhere discontinuous either. In fact, speaking in terms of category, it may only be discontinuous on a "small" set of points as is shown in the following theorem.

Theorem 2.46. *If f can be represented as the limit of an everywhere convergent sequence of continuous functions, then f is continuous except at a set of points of first category.*

Proof. It suffices to show that, for each $\epsilon > 0$, the set $F = \{x : \omega(x) \geq 5\epsilon\}$ is nowhere dense. Let $f(x) = \lim f_n(x), f_n$ continuous, and define

$$E_n = \bigcap_{i,j\geq n} \{x : |f_i(x) - f_j(x)| \leq \epsilon\}(n = 1, 2, ...).$$

Then E_n is closed, $E_n \subset E_{n+1}$, and $\bigcup E_n$ is the whole line. Consider any closed interval I. Since $I = \bigcup(E_n \cap I)$, the sets $E_n \cap I$ cannot all be nowhere dense. Hence, for some positive integer n, $E_n \cap I$ contains an open interval J. We have $|f_i(x) - f_j(x)| \leq \epsilon$ for all x in $J; i, j]geq$. Putting $j = n$ and letting $i \to \infty$, it follows that $|f(x) - f_n(x)| \leq \epsilon$ for all x in J. For any x_0 in J there is a neighborhood $I(x_0) \subset J$ such that $|f_n(x) - f_n(x_0)| \leq \epsilon$ for all x i $I(x_0)$. Hence $|f(x) - f_n(x_0)| \leq 2\epsilon$ for all x i $I(x_0)$. Therefore $\omega(x_0) \leq 4\epsilon$, and so no point of J belongs to F. Thus for every closed interval I there is an open interval $J \subset I - F$. This shows that F is nowhere dense. □

A similar, and perhaps even more useful result further describes the conditions under which the set D of points of discontinuity of a function is of first category. This result is seen below.

Theorem 2.47. *Let f be areal-valued function on $\mathbb{R}$. The set of points of discontinuity of f is of first category if and only if f is continuous at a dense set of points.*

This is easily shown, as it is an immediate consequence of Theorem 44 and the fact that an F_σ set is of first category if and only if its complement is dense. Now that we have found the conditions under which the set D of points of discontinuity is of first category, it is time to explore under what conditions this set is a nullset, hoping once again to draw an analogy. Without further ado, the following result gives us exactly that for which we are looking.

Theorem 2.48. *In order that a function f be Reimann-integrable on every finite interval it is necessary and sufficient that f be bounded on every finite interval and that its set of points of discontinuity be a nullset.*

Before we prove this result, we will establish a supporting lemma as well as a corollary to that lemma. Let f be a function bounded on the interval I and let $\{I_1, ..., I_n\}$ be a subdivision of I. Define $F(I)$ to be the greatest lower bound of all sums of the form

$$\sum_{i=1}^{n} \omega(I)|I|.$$

Notice that $F(I)$ is the difference between the upper and lower integrals of f on I,and that whenever $F(I) = 0$ is exactly the case when f is integrableon I. Notice also that given any particular subdivision $\{I_1, ..., I_n\}$ of I we have that $F(I) = \sum_1^n F(I_i)$. Now we may proceed to prove the following helpful lemma.

Lemma 2.49. *If $\omega(x) < \epsilon$ for each x in I, then $F(I) < \epsilon|I|$.*

Proof. Assume that this is not true, meaning that with the given hypothesis we have that $F(I) \geq \epsilon|I|$. If we bisect I then we have for at least one of the resulting subintervals, call it I_1, that $F(I_1) < \epsilon|I|/2$. Then bisecting I_1 similarly yields a subinterval I_2 such that $F(I_2) < \epsilon|I_1|/2$. Continuing in this way produces a sequence of nested closed intervals I_n such that $F(I_n) < \epsilon|I|/2^n$. These intervals intersect at a point x in I. But by hypothesis $\omega(x) < \epsilon$ for each x in I and therefore there exists a subinterval J containing x we have $\omega(J) < \epsilon$. Now choose n such that $I_n \subset J$. We then have

$$F(I_n) \leq \omega(I_n)|I_n| \leq \omega(J)|I|/2^n \leq F(I_n),$$

which is a contradiction. □

Corollary 2.50. *Any continuous function on a closed interval is integrable.*

Now we finally prove Theorem 2.48.

Proof. First, assume that f is integrable on I. Then for any positive integer k, I can be divided into intervals $I_1, ...I_n$ such that

$$\sum_{i=1} \omega(I_i)|I_i| < 1/k^2.$$

Let $\sum'$ denote the sum over those intervals I_i for which $\omega(x) \geq 1/k$ at some interior point. Then

$$1/k^2 > \sum{}' \omega(I_i)|I_i| \geq (1/k)\sum{}' |I_i|.$$

Therefore $\sum' |I_i| < 1/k$. The set

$$F_k = \{x \in I : \omega(x) \geq 1/k\}$$

is entirely covered by these intervals, except perhaps for a finite number of points (endpoints of intervals, except perhaps for a finite number of points (endpoints of intervals of the subdivision). Therefore $m(F_k) < 1/k$. If D is the set of points of discontinuity of f, then $D \cap I$ is the union of the increasing sequence F_k, and we have

$$m(D \cap I) = \lim_{k \to \infty} m(F_k) = 0.$$

If f is integrable on every finite interval, it follows that D is a nullset. Conversely, suppose D is a nullset and that f is bounded on I, with upper and lower bounds M and m, respectively. For any $\epsilon > 0$, choose k so that $(M - m) + |I| < k\epsilon$. Since F_k is a bounded closed nullset, it is possible to cover F_k with a finite number of disjoint open intervals, the sum of whose lengths is less than $1/k$. The endpoints of these intervals that belong to I determine a subdivision of I into nonoverlapping intervals I_i and J_j such that $\sum |I_i| < 1/k$ and $\omega(x) < 1/k$ on each of the intervals J_j. Hence, by lemma 2.49,

$$\begin{aligned} F(I) &= \sum F(I_i) + \sum F(J_i) \\ &\leq (M-m)\sum |I| + \sum (1/k)|J_i| \\ &\leq (M-m)/k + |I|/k \\ &< \epsilon. \end{aligned}$$

consequently, f is Riemann-integrable on I. □

2.6 On Continuity and Convergence

2.6.1 Continuous Functions

When classifying functions, typically the main consideration is the properties of the inverse images of open sets in the range of a given function. The classic case is of course the definition of continuous functions. We consider a function f on $\mathbb{R}$ to be continuous if and only if for every open set U in $\mathbb{R}$ we have $f^{-1}(U)$ is also open. The same is true for classifying functions as being *measurable* or having the *property of Baire.*

Definition 2.51. *We consider a function to be* measurable *if and only if for every open set U in $\mathbb{R}$ we have $f^{-1}(U)$ is measurable.*

Definition 2.52. *We consider a function to have the* property of Baire *if and only if for every open set U in $\mathbb{R}$ we have $f^{-1}(U)$ has the property of Baire.*
As one might conjecture, there are indeed strong relationships between continuous functions and functions which have the above mentioned properties. The following theorems describe those relationships.

Theorem 2.53. *A real-valued function f on $\mathbb{R}$ has the property of Baire if and only if there exists a set P of first category such that the restriction of f to $\mathbb{R} - P$ is continuous.*

Proof. Let $U_1, U_2, ...$ be a countable base for the topology of $\mathbb{R}$, say for example, the open intervals with rational endpoints. If f has the property of Baire, then $f^{-1}(U_i) = G_i \triangle P_i$, where G_i is open and P_i is of first category. Put $P = \bigcup_1^\infty P_i$. Then P is of first category. Define $g = f|_{\mathbb{R}-P}$. Then g is continuous, since $g^{-1}(U_i) = f^{-1}(U_i) - P = (G_i \triangle P_i) - P = G_i - P$ is open relative to $\mathbb{R} - P$ for each i, and therefore so is $g^{-1}(U)$, for every open set U.
Conversely, if the restriction g of f to the complement of some set P of first category is continuous, then for any open set U, $g^{-1}(U) = G - P$ for some open set G. Since

$$g^{-1}(U) \subset f^{-1}(U) \subset (g^{-1}(U) \cup P),$$

we have

$$G - P \subset f^{-1}(U) \subset G \cup P.$$

Therefore $f^{1-}(U) = G \triangle Q$ for some set $Q \subset P$. Thus f has the property of Baire. □

Similar to the above relationship is this theorem, the analog for measurable functions, called *Lusin's Theorem.*

Theorem 2.54. *A real-valued function f on $\mathbb{R}$ is measurable if and only if for each $\epsilon > 0$ there exists a set E with $m(E) < \epsilon$ such that the restriction of f to $\mathbb{R} - E$ is continuous.*

Proof. Let $U_1, U_2, ...$ be a countable base for the topology of $\mathbb{R}$. If f is measurable, then for each i there exists a closed set F_i and an open set G_i such that

$$F_i \subset f^{-1}(U_i) \subset G_i \text{ and } m(G_i - F_i) < \epsilon/2^i.$$

Put $E = \bigcup_1^\infty (G_i - F_i)$. Then$m(E) < \epsilon$. If g denotes the restriction of f to $\mathbb{R} - E$, then

$$g^{-1}(U_i) = f^{-1}(U_i) - E = F_i - E = G_i - E.$$

Hence $g^{-1}(U_i)$ is both closed and open relative to $R-E$, and it follows that g is continuous. Conversely, if f has the stated property there is a sequence of sets E_i with $m(E_i) < 1/i$ such that the restriction f_i to $R - E_i$ is continuous. For any open set U there are open sets G_i such that $F_i^{-1}(U) = G_i - E_i (i = 1, 2, ...)$. Putting $E = \bigcap_1^\infty E_i$, we have

$$f_1(U) - E = \bigcup_{i=1}^{\infty}(f^{-1}(U) - E_i) = \bigcup_{i=1}^{\infty} f_i^{-1}(U).$$

Consequently,

$$f_1(U) = [f_1(U) \cap E] \cup \bigcup_{i=1}^{\infty}(G_i - E_i).$$

All of these sets are measurable, since $m(E) = 0$, and therefore f is a measurable function. □

It is important to mention that this theorem does not state or imply that every measurable function is continuous on the complement of a nullset.

2.6.2 Convergent Functions

A fascinating result in the study of measurable functions is the following, known as *Erogoff's Theorem*, which establishes a relationship between convergence and absolute convergence.

Theorem 2.55. *If a sequence of measurable functions f_n converges to f at each point of a set E of finite measure, then for each $\epsilon > 0$ there is a set $F \subset E$ with $m(F) < \epsilon$ such that f_n converges to$f$ uniformly on $E - F$.*

Proof. For any two positive integers n and k let

$$E_{n,k} = \bigcup_{i=1}^{\infty}\{x \in E : ||f_i(x) - f(x) \geq 1/k\}.$$

Then $E_{n,k} \subset E_{n+1,k}$ and $\bigcup_{n=1}^{\infty} E_{n,k} = \emptyset$, for each k. Given $\epsilon > 0$, for each k there is an integer $n(k)$ such that $m(E_{n(k),k}) < \epsilon/2^k$. Put $F = \bigcup_{k=1}^{\infty} E_{n(k),k}$. Then $m(F) < \epsilon$. For each k we have $E - F \subset E - E_{n(k),k}$. Therefore $|f_i(x) - f(x)| < 1/k$ for all $i \geq n(k)$ and all $x \in E - F$. Thus f_n converges to f uniformly on $E - F$. □

A perplexing fact is that the analog of *Egoroff's Theorem* with respect to category is not true. The following example demonstrates this fact.

Example 2.56. Let $\phi(x)$ be the piece-wise linear continuous function defined by $\phi(x) = 2x$ on$[0, 1/2]$, $\phi(x) = 2-2x$ on $[1/2, 1]$, and $\phi(x) = 0$ on $\mathbb{R}-[0, 1]$. Then $\lim_{n\to\infty} \phi(2^n x) = 0$ for every x in $\mathbb{R}$. Let $\{r_i\}$ be a dense sequence in $\mathbb{R}$, and define $f_n(x) = \sum_{i=1}^{\infty} 2^{-i}\phi(2^n(x-r_i))$. As the sum of a uniformly convergent series of continuous functions, f_n is continuous on $\mathbb{R}$, and $\lim_{n\to\infty} f_n(x) = 0$ for each x in $\mathbb{R}$. If (a, b) is any open interval, then $r_i \in (a, b)$ for some i, and we have $\sup_{a<x<b} f_n(x) \geq 1/2^i$ for all sufficiently large n. This shows that f_n does not converge uniformly on (a, b). Let E be any set on which f_n does not converge uniformly. This means that if we let $\alpha_n = sup_{x\in E} f_n(x)$, then $\alpha - n \to 0$. Because f_n is continuous, α_n is also the supremum of f_n on $\overline{E}$. Hence f_n converges to 0 uniformly on $\overline{E}$. From what we have shown, $\overline{E}$ cannot contain an interval. Therefore any set on which the sequence $\{f_n\}$ converges uniformly is nowhere dense.

2.7 Topological and Metric Spaces

As some of the topics to be discussed require a basic knowledge of Topology, we will at this point review some relevant definitions regarding general topological spaces and metric spaces. Also, this section will include two important theorems, one of which is a general version of the Baire Category Theorem, as well as a useful example of a metric space.

2.7.1 Topological Spaces

We define a *topology* on a set X to be a collection τ of subsets of X having the following properties:

(i) $\emptyset$ and x are in τ

(ii) The union of the elements of any subcollection of τ is in τ

(iii) The intersection of the elements of any finite subcollection of τ is in τ.

A *topological space* is a set X for which a topology τ has been specified, and is denoted (X, τ). We say that a subset U of X is *open* if it belongs to the collection τ. Appropriately, any set which is the complement of an open set is called *closed.* A *base* (or *basis*) for a topology is a subcollection β of τ such that every element of τ is the union of elements of β. As we mentioned earlier, a function is defined to be continuous if the inverse image of every open set in the range of the function is an open set in the domain. We define now a *homeomorphism* to be a function $f : X \to Y$ which is a bijective mapping such that both f and f^{-1} are continuous. If there exists a homeomorphism between two sets, the sets are considered to be *topologically equivalent* or *homeomorphic.* Homeomorphisms are useful in that they allow for the study of unfamiliar spaces and sets by comparison to familiar ones.

2.7.2 Metric Spaces

One of the most frequently used methods of imposing a topology on a set is to define a topology in terms of a metric on the set. This is especially useful in modern analysis. We define a *metric space* to be a set X paired with a *distance function* or *metric* $\varrho(x, y)$ defined for all pairs of points x, y of X satisfying the following conditions:

(i) $\varrho(x, y) \geq 0, \varrho(x, x) = 0$

(ii) $\varrho(x, y) = \varrho(y, x)$

(iii) $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$ triangle inequality

(iv) $\varrho(x, y) = 0$ implies $x = y$.

Such a metric will form a base for a topology (called a *metric topology*) on the set X. We denote a metric space X with metric ϱ by (X, ϱ). Consider the set

$$B_{\varrho}(x, \epsilon) = \{y : \varrho(x, y) < \epsilon\}.$$

This is the set of all points y whose distance from x is less than ϵ, as defined by ϱ. It is called the *ϵ-ball* (or the *ϵ-neighborhood* centered at x). ϵ-balls are the open sets in a topology generated by a metric. Often the metric on a topological space is simply used to provide a homeomorphism to a more familiar space. Such a homeomorphism is typically defined via *convergent sequences.* A sequence $x_1, x_2, \ldots$ of a metric space (X, ϱ) is said to *converge to the point* x if $\varrho(x_n, x) \to 0$ as $n \to \infty$. We then write $x_n \to x$. If a sequence converges to a point in X, it is called *convergent.* Now, we say that in a set X a metric ϱ is topologically equivalent to a metric σ if and only if the identity mapping of X onto itself is a homeomorphism of (X, ϱ) onto (X, σ). This is equivalent to saying that $\varrho(x_n, x) \to 0$ if and only if $\sigma(x_n, x) \to 0$. Also, a sequence x_n of points in a metric space is considered to be a *Cauchy sequence* if for each $\epsilon > 0$ there is a positive integer n such that $\varrho(x_i, x_j) < \epsilon$ for all $i, j \geq n$. Every convergent sequence is indeed Cauchy, but not every Cauchy sequence is convergent. At least that is not the case in all spaces. In fact, a space in which every Cauchy sequence is convergent is called *complete.* The real line with the standard Euclidean metric $|x - y|$ is an example of a complete space. A space which is not necessarily complete can sometimes be considered *topologicallly complete* if it is homeomorphic to a complete space. If f is a homeomorphism from (X, ϱ) onto a complete space (Y, σ) then $\sigma(f(x), f(y))$ is a metric in X which is topologically equivalent to ϱ. So then a metric space is topologically complete if and only if it can be remetrized with a topologically equivalent metric to be complete. One important property of topologically complete spaces is that the Baire category theorem holds.

Theorem 2.57. *If X is a topologically complete metric space, and if A is of first category in X, the $X - A$ is dense in X.*

Proof. Let $A = \bigcup A_n$, where A_n is nowhere dense, let q be a metric with respect to which X is complete, and let S_0 be a non-empty open set. Choose a nested sequence of balls S_n of radius $r_n < 1/n$ such that $\overline{S}_n \subset S_{n-1} - A_n (n \geq 1)$. This can be done step by step, taking for S_n a ball with center x_n in $S_{n-1} - \overline{A}_n$ (which is non-empty because $\overline{A}_n$ is nowhere dense) and with sufficiently small radius. Then $\{x_n\}$ is a Cauchy sequence, since

$$\varrho(x_i, x_j) \leq \varrho(x_i, x_n) + \varrho(x_n, x_j) < 2r_n \text{ for } i, j \geq n.$$

Hence $x_n \to x$ for some x in X. Since $x_i \in \overline{S}_n$ for $i \geq n$, it follows that $x \in \overline{S}_n \subset S_0 - A$. This shows that $X - A$ is dense in X. □

A topological space if called a *Baire space* if every non-empty open set in X is of second category, or equivalently, if the complement of every set of first category is dense.

We call the complement of a first category set a *residual set.*

Theorem 2.58. *In a Baire space X, a set E is residual if and only if E contains a dense G_δ subset of X.*

Proof. Suppose $B = \bigcup G_n, G_n$ open, is a G_δ subset of E that is dense in X. Then each G_n is dense, and $X - E \subset X - B = \bigcup(X - G_n)$ is of first category. Conversely, if $X - E = \bigcup A_n$, where A_n is nowhere dense, let $B = \bigcap(X - \overline{A}_n)$. Then B is a G_δ set contained in E. Its complement $X - B = \bigcup \overline{A}_n$ is of first category. Since X is a Baire space, it follows that B is dense in X. □

Example 2.59. Let C denote the set of real-valued continuous functions f on the interval $[a, b]$. Define the metric

$$\varrho(f,g) = \sup_{a \leq x \leq b} |f(x) - g(x)|.$$

That ϱ is a metric is easy to verify. In fact, properties (1), (2), and (4) are obvious from the definition. As for the triangle inequality, let $f, g, h \in C$. Then

$$\begin{aligned} \varrho(f,h) = |f(x) - h(x)| &= |f(x) - g(x) + g(x) - h(x)| \\ &\leq |f(x) - g(x)| + |g(x) - h(x)| \\ &\leq \varrho(f,g) + \varrho(g,h) \end{aligned}$$

for all x in $[a, b]$. This metric is called the *uniform metric.* The space (C, ϱ) is a *complete* metric space, as we will show now. Let $\{f_n\}$ be any Cauchy sequence in C, say $\varrho(f_i, f_j) \leq$ $\epsilon$for all $i, j \geq n(\epsilon)$. Then

$$|f(x_i) - f(x_j)| \leq \epsilon \text{ for all } i,j \geq n(\epsilon) \text{ and } a \leq x \leq b.$$

So for each x in $[a, b]$, $\{f_n(x)\}$ is a Cauchy sequence of real numbers. It therefore converges to a limit $f(x)$. Letting $j \to \infty$ we see that $|f(x_i) - f(x)| \leq \epsilon$ for all $i \geq n(\epsilon)$ and all z in $[a, b]$. Therefore f_i converges to f uniformly on $[a, b]$. By the *uniform convergence theorem,* it follows that f is continuous on $[a, b]$. So now we have that $f_n \to f$ in C, which shows that the space (C, ϱ) is complete.

Chapter 3

Applications of Category

3.1 Nowhere Differentiable Functions

We now begin our study of applications of measure and category by introducing the concept of a continuous, yet nowhere differentiable function. Below is a proof that not only does such a function exist, but rather the set of nondifferentiable functions is of second category in the space of continuous functions. Let C be the space of continuous functions on $[0,1]$ with the uniform metric. Now define E_n to be the set of all functions f in C such that for some x in $[0,1-1/n]$ the inequality $|f(x+h)-f(x)| \leq nh$ for all $h \in (0,1-x)$. That is to say that E_n is the set of all functions with bounded difference quotients in the interval $[0,1]$. Notice that $\bigcup E_n$ contains the set of all functions in C that are differentiable somewhere. Our objective is now to show that this set is a countable union of nowhere dense sets, or of first category, and therefore the set of functions with unbounded difference quotients is that of all continuous functions except for a set of first category. It is useful to first prove that E_n is closed. To do so let f be any function in the closure of E_n. We will now show that f is in E_n. Let now $\{f_k\}$ be a sequence in E_n. We then have a sequence of numbers x_k such that the following three properties hold for each k:

(i) $x \in [0,1-1/n]$

(ii) $|f(x_k+h)-f(x_k)| \leq nh$ for all $h \in (0,1-x_k)$

(iii) $x_k \to x$ for some $x \in [0,1-1/n]$

Notice that we my assume that the third property holds because if it were to not we could simply choose a subset of $\{f_k\}$ so that it were to. Since we have that $h \in (0, 1-x)$ we also have that $h \in (0, 1-x_k)$ for sufficiently large k. Therefore we have the following string of inequalities:

$$\begin{aligned}
&|f(x+h) - f(x)| \\
&\leq |f(x+h) - f(x_k+h)| - f(x_k+h) + f(x_k+h) - f_k(x_k+h)| \\
&+|f_k(x_k+h) - f_k(x_k)| + |f_k(x_k) - f(x_k)| + |(x_k) - f(x)| \\
&\leq |f(x+h) - f(x_k+h)| + \varrho(f, f_k) + nh + \varrho(f_k, f) + |f(x_k) - f(x)|
\end{aligned}$$

Now if we let $k \to \infty$, recalling that f is continuous at x and $x+h$ we have that

$$|f(x+h) - f(x)| \leq nh \text{ for all } h \in (0, 1-x).$$

Therefore f is in E_n as desired and E_n is closed. Next, we will show that E_n is nowhere dense by showing that every neighborhood about every point of E_n contains a point that is not in E_n. Recall that any continuous function can be approximated uniformly and arbitrarily closely by a piecewise linear function g. Now we will show that for any such function g there exists a function h in $C - E_n$ such that $\varrho(g, h) \leq \epsilon$ for all $\epsilon > 0$. Let now M be the maximum value absolute value of the slopes of the segments which compose the graph of g and choose an integer m such that $\epsilon m > M + n$, or equivalently $\epsilon m - M > n$. Further, let ϕ be defined by $\phi(x) = (x - [x], x+1-x)$, which is the function which measures the distance from x to the nearest integer, and define $h(x) = g(x) + \epsilon m\phi(x) = \epsilon\phi(mx)$. Then since $\epsilon\phi(mx)$ has a right sided derivative of $\pm\epsilon m$ and since $g(x)$ has a right sided derivative of no more than M we have that $h(x)$ has a right sided derivative greater than n at each point of $[0, 1)$ Therefore we have that $h \in C - E_n$. Notice that by construction $\varrho(g, h) = \epsilon/2$, and therefore E_n is nowhere dense in C. Moreover, $\bigcup E_n$ is of first category. The argument at this point only covers the case of continuous functions with bounded right side difference quotients in $[0, 1)$. However, a simple substitution of $x - 1$ for x will show you that the same result can be attained for continuous functions with bounded left difference quotients on $(0, 1]$. Therefore the union of the two arguments includes all functions in C which have a finite one-sided derivative somewhere in $[0, 1]$.

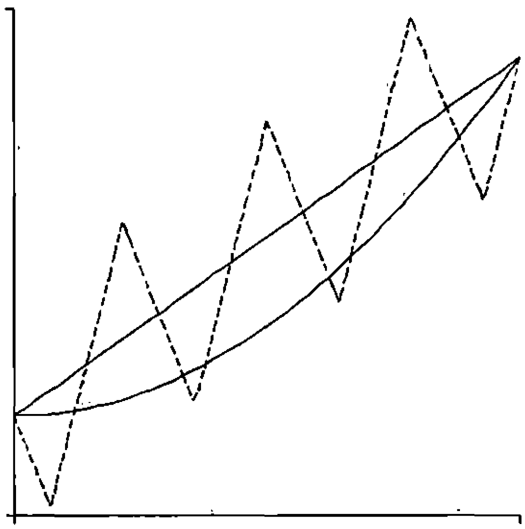

Figure 3.1: Graphs of f, g (linear) and h (dashed)

3.2 Nowhere Monotonic Functions

The notion of category can further describe the behavior of a typical continuous function with respect to how likely it is for such a function to be monotonic. It is obvious that most functions are not monotonic over their entire domain, but more interestingly we shall now prove that the typical continuous function is indeed nowhere monotonic. That is, it is not monotonic on any interval.

Let C again be the space of continuous functions on $I = [0, 1]$, with the uniform metric. Let A be the set of all functions in C that are monotonic on some interval. Let J_n be the set of all intervals with rational endpoints, a countable collection. Define A_n to be the set of all continuous functions which are monotonic in J_n for each given n. Then $A = \bigcup A_n$. We will proceed as in the previous theorem by showing that A_n is closed. Therefore let f be in the closure of A_n and, without loss of generality, define a sequence $\{f_k\} \in A_n$ of monotone increasing functions that converges to f. Then for each k we have

$$f_k(y) - f_k(x) \geq 0 \text{ for all } x \leq y.$$

Letting $k \to \infty$ we have then

$$f(y) - f(x) \geq 0 \text{ for all } x \leq y$$

as desired.

Now we must show that A_n is nowhere dense. To do this it is helpful to recall that one statement of the definition states that a set is nowhere dense if the complement contains a dense open set. We already have that the complement is open since A_n is closed, so now to show that the complement is dense we will find a continuous function which is not monotonic but is within ϵ distance of an arbitrarily chosen function in A_n. Let now $f \in A_n$. We know that f is continuous, so for any $x < y \in I$ we have that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon/3$. Now partition I into subintervals of length less than δ. Define a function g such that $g = f$ everywhere except for one subinterval $[a, b]$. On $[a, b]$ we have $f(a) = g(a)$ and $f(b) = g(b)$, but on (a, b) g is composed of two line segments, one connecting $(a, f(a))$ to $((a + b)/2, f(a) - \epsilon/4)$ and the other connecting $((a + b)/2, f(a) - \epsilon/4)$ to $(b; f(b))$.

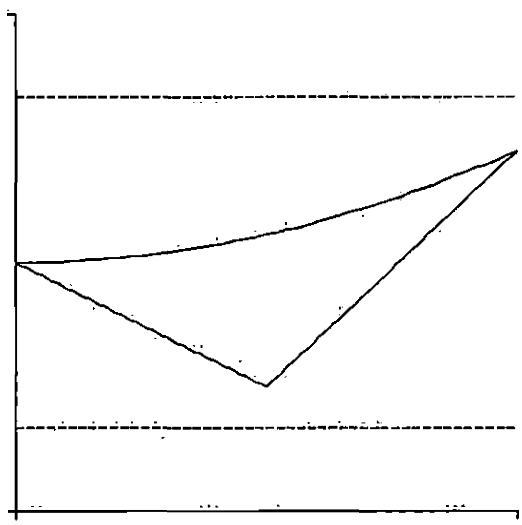

Figure 3.2: Graphs of f and g on (a, b)

3.3 A Polynomial Problem

Theorem 3.1. *Let $f : [0,1] \to \mathbb{R}$ be a C^∞ function with derivative eventually zero at each point. Then f is a polynomial.*

Proof. Let J be an arbitrary subinterval of $I = [0,1]$ and define

$$Z_n = \{x \in J | f^k(x) = 0 \text{ for all } k \geq n\}.$$

Since f is C^∞ all of its derivatives are continuous and $Z_n = \cap_{k=n}^{\infty}\{x \in J | f^k(x) = 0\}$ is the intersection of closed sets and is therefore itself a closed set. Since f has derivative eventually 0 at each point we know $\cup_{n=1}^{\infty} Z_n = J$. This implies that there exists an i so that Z_i is second category. Therefore we have that Z_i contains an interval, say (c,d) on which f is a polynomial of degree at most $i-1$. Define $G \neq \emptyset$ to be the union of all open intervals in I on which f is a polynomial. Since J is arbitrary we have that G is dense in I. We claim that f is a polynomial on each component (a,b) of G. To see this, by the definition of G there is a an interval $(a',b') \subset (a,b)$ containing $\frac{a+b}{2}$ on which f is a polynomial. Let $\beta = \sup\{x | f \text{ is a polynomial on } (a',x)\}$. If $\beta < b$ then, again by the definition of G there is an interval $(c,d) \subset (a',b)$ containing β on which f is a polynomial. We now have $f = p_1$ on $(a', \frac{c+\beta}{2})$ and $f = p_2$ on (c,d). These two intervals overlap so in fact the two polynomials p_1 and p_2 are the same polynomial. Thus $\beta = b$. By a similar argument, $a = \inf\{x | f \text{ is a polynomial on } (x,b')\}$. Thus f is a polynomial on (a,b). To show that G is not only dense, but indeed all of I, we will now consider the complement. Denote $G^c = F$. Suppose now that G does not contain all of I. Then F might have isolated points. Let b be an isolated point in F. Then f is composed of a polynomial P_1 on and interval immediately to the left of b, say $(a,b]$, and another polynomial P_2 on an interval immediately to the right of b, say $[b,c)$. But for $x \in (a,b]$ we have

$$\begin{aligned} P_1(x) &= \sum_{k=0}^{\infty} \frac{P_1{}^{(k)}(b)}{k!}(x-b)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(b)}{k!}(x-b)^k. \end{aligned}$$

But similarly for $x \in [b,c)$ we have

$$\begin{aligned} P_2(x) &= \sum_{k=0}^{\infty} \frac{{P_2}^{(k)}(b)}{k!}(x-b)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(b)}{k!}(x-b)^k, \end{aligned}$$

so we now have that $P_1 = P_2$, so f is in fact a polynomial on (a, c) and no such isolated point exists in F.

Since G is dense and there are no isolated points in F then either F is a perfect set or F is empty. Let us assume that F is perfect. Now define

$$A_n = \{x \in F | f^{(k)}(x) = 0 \text{ for all } k \geq n\}.$$

As above we have that $\cup_{n=1}^{\infty} A_n = F$. Notice that F, as a perfect set, is itself then a Baire space, and so there exists an i so that A_i is of second category in F. But this implies that there exists an interval U such that $U \cap F \subset A_i$. So then $f^{(k)}(x) \equiv 0$ on $U \cap F$ for all $k \geq i$, while $f^{(i)}$ is a polynomial on each complementary interval. Let (a', b') represent such a complementary interval. We know immediately that $f^{(i)}$ is not a nonzero constant because if it were then, due to continuity, neither $f^{(i)}(a')$ nor $f^{(i)}(b')$ would equal zero. So then it is a polynomial of degree at least 1 and therefore takes the form

$$f^{(i)}(x) = a_0 + a_1 x^1 + \ldots + a_n x^n.$$

Then differentiating n times yields $f^{(i+n)}(x) = n! a_n$.This is a contradiction, since $f^{(i+n)}(a') = 0$. So then $f^{(i)}(x) \equiv 0$ on (a', b'). Since (a', b') was arbitrary we have $f^{(i)}(x) \equiv 0$ on U making f a polynomial of degree at most $i - 1$ which implies that $U \subset G$ and then in turn that $U \cap F = \emptyset$, contradicting the choice of U. Thus F is empty and we are done. □

Chapter 4

Mean Value Theorem Converse: An Application of Measure

4.1 General Results

In 1999 a short article by Fejzic and Rinne [FR99] appeared in the MAA Monthly in which the authors discussed the set of points on which a differentiable function on an interval could satisfy or fail a Mean Value Theorem converse. In that paper the authors did not attempt to characterize the sets on which a converse could fail, concentrating on demonstrating that such sets could be large in measure. A couple of papers dealing with a converse theorem have appeared since then but none that we know of addresses the characterization question. The most general question of characterizing the set on which a differentiable function could fail the Strong form may be very deep and difficult. The question appears to involve the level sets and associated sets of derivatives. A 1982 paper by D. Preiss [Pre82] characterizes level sets of derivatives and that work is lengthy and quite involved.

The two forms of a converse of the Mean Value Theorem are called the Weak Form and the Strong Form. We say that for $c \in (a, b)$, a continuous function F on $[a, b]$ that is differentiable on (a, b) satisfies the:

1. Weak Form at c if $F'(c) = \frac{F(\beta)-F(\alpha)}{\beta-\alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$, and the
2. Strong Form at c if $F'(c) = \frac{F(\beta)-F(\alpha)}{\beta-\alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$ with $c \in (\alpha, \beta)$.

Let W_F (resp. S_F) be the set of x for which F fails the Weak Form (resp. Strong Form). Obviously, $W_F \subset S_F$.

Theorem 4.1. *Let F be differentiable on an interval I. If $c \in W_F$ then $F'(c)$ is either a maximum or minimum value of F'.*

Proof. Suppose $c \in I$ and $F'(c)$ is not a max or min of F'. Then there exist c_0, c_1 such that $F'(c_0) < F'(c) < F'(c_1)$. This implies that there exist b_0, b_1 such that $\frac{F(c_0)-F(b_0)}{c_0-b_0} < F'(c) < \frac{F(c_1)-F(b_1)}{c_1-b_1}$. We may assume that $c_0 > b_0$ and $c_1 > b_1$. The function $g(x,y) = \frac{F(y)-F(x)}{y-x}$ is continuous on the half-plane $y > x$ and therefore has the intermediate value property along the line segment between (b_0, c_0) and (b_1, c_1). Since $g(c_1, b_1) < F'(c) < g(c_0, b_0)$, $F'(c) = g(\alpha, \beta)$ for some $\alpha, \beta \in (a, b)$. Thus $c \notin W_F$ and we are done. □

□

It is important to mention that a max or min of F' need not correspond to a point at which the Weak Form fails. This is easy to demonstrate with linear functions. The function $F(x) = 3x$ has derivative $F'(x) = 3$. Then 3 is the maximum of the derivative, but obviously f has many (uncountably infinitely many) difference quotients that equal 3. Still, it is possible for there to be infinitely many points at which the Weak Form fails. For instance, take $F(x) = \sin x$ on the entire real line which of course has derivative $F'(x) = \cos x$. The max and min of F' are 1 and -1 yet no difference quotient for F is equal to 1 or -1.

Lemma 4.2. *The set W_F is of type G_δ for any differentiable F and if F' is continuous then S_F is of type G_δ.*

Proof. By Theorem 4.1, W_F is a level set of F' or a union of two levels sets. Since a derivative is Baire 1 those level sets are G_δ sets. To show that S_F is a G_δ if F' is continuous we show that its complement is type F_σ. Let A_n be the set of x such that there exist $z \leq x - \frac{1}{n} < x + \frac{1}{n} \leq w$ with $\frac{F(w)-F(z)}{w-z} = F'(x)$. Then $S_F' = \cup_{n=1}^{\infty} A_n$. We show that each A_n is closed. Let $\{x_k\}$ be a sequence in A_n converging to x. By passing to subsequences if necessary we may assume that the corresponding sequences $\{z_k\}$ and $\{w_k\}$ converge to z and w respectively. Since F, F' are continuous we see that $\frac{F(w)-F(z)}{w-z} = F'(x)$. Thus A_n is closed.□ □

We should say more about level sets of derivatives at this time. D. Preiss [Pre82] characterized the levels sets of derivatives as being the complements of sets he called of type M_3^*. The definition of type M_3^* is rather complicated and involves a class of F_σ sets M_3 defined by Zahorski [Zah50] with additional conditions imposed.

Zahorski defined a hierarchy of sets he called type M_0 through M_5. Three of those are relevant to our work.

Definition 4.3. *A set E is type M_3 if E is an F_σ set and for all $x \in E$ if $\{I_n\}$ is a sequence of closed intervals not containing x but converging to x so that $\lambda(E \cap I_n) = 0$ for all n then $\lim_{n\to\infty} \frac{\lambda(I_n)}{d(x,I_n)} = 0$. A set E is type M_4 if there is a sequence of closed sets $\{K_n\}$ and a sequence of positive numbers $\{\eta_n\}$ such that $E = \cup K_n$ and for each $x \in K_n$ and each $c > 0$ there is an $\epsilon > 0$ so that if h, k satisfy $0 < \frac{h}{k} < c$ and $|h+k| < \epsilon$ then $\frac{\lambda(E\cap(x+h,x+h+k))}{|k|} > \eta_n$. A set E is type M_5 if E is an F_σ set and every point of E is a density point of E.*

In essence, E is required to be "thicker" around its points as the subscript on M increases. C. Weil [Wei65] defined property (Z) as follows.

Definition 4.4. *A set P has property (Z) with respect to the set E if for every open set $H \subset E'$ which intersects each component of $R - (\overline{P \cap E})$ in a connected set, the set $(P \cap E) \cup (R - (\overline{H} \cup (\overline{P \cap E})))$ is type M_4.*

Preiss then defined M_3^* to be $M^* \cap M_3$ where M^* is as follows.

Definition 4.5. *A set E is M^* if E is an F_σ set and for each closed set P, some portion Q of P is contained in E or there is a portion Q of P such that Q has property (Z) with respect to E and if $x \in Q \cap E$ and $c > 0$ then there is an $\epsilon > 0$ such that $0 < \frac{h}{k} < c$ and $|h+k| < \epsilon$ imply $E \cap (x+h, x+h+k) \neq \emptyset$.*

Preiss' characterization of level sets of derivatives is based on the following theorem.

Theorem 4.6. *If F possesses a finite approximate derivative f on an open interval then $f^{-1}(a,\infty)$ and $f^{-1}(-\infty,a)$ are of type M_3^* for all a.*

If E is of type M_3^* then there is a nondecreasing absolutely continuous function F possessing a finite derivative f with $E = f^{-1}(0,\infty)$.

We define a set S to have property T if S' is of type M_3^* with positive measure in every interval. Note that the empty set has property T. Combining Theorems 4.1 and 4.6 we obtain the following.

Theorem 4.7. *If F has a derivative f on I then W_F is the disjoint union of two sets having property T. If S is a set having property T then there is a differentiable function F with $S = W_F$.*

Proof. Let F be differentiable with derivative f. From Theorem 4.1 we have that W_F is either empty, $f^{-1}(M)$, $f^{-1}(m)$ or $f^{-1}(M) \cup f^{-1}(m)$ where M and m are the max and min of f. If $W_F = \emptyset$ we are done. If $W_F = f^{-1}(M) \neq \emptyset$ then W_F' is of type M_3^* by the first half of Theorem 4.6. If there exists an interval $J = (a, b)$ so that W_F is of full measure in J then in fact f is identically M on J and $\frac{F(b)-F(a)}{b-a} = M$ contradicting the fact that $W_F = f^{-1}(M)$. Thus W_F has property T. A similar argument applies if $W_F = f^{-1}(m) \neq \emptyset$. Lastly, if $W_F = f^{-1}(M) \cup f^{-1}(m)$, both nonempty, then W_F is the union of those two nonempty sets having property T.

Now suppose S has property T. By the second half of Theorem 4.6 there is a nondecreasing absolutely continuous function P with derivative p and $p^{-1}(0) = S$. Since S has property T no difference quotient of P can be 0 so $S \subset W_P$. We modify p to create a derivative f with $f^{-1}(0) = S$ that is unbounded above so that $S = W_F$. For $n \geq 2$ define g_n as follows. Pick an M_5 set E_n in $S' \cap [1-\frac{1}{n}, 1-\frac{1}{n+1}]$ so that $\lambda(E_n) \leq \frac{1}{2^n}$. Using a method of Zahorski [Zah50] there is a bounded approximately continuous function g_n so that g_n is identically 0 on E_n' and has a maximum value of n. Since g_n is bounded approximately continuous it is the derivative of its integral on $[0, 1]$. Define $f(x) = p(x) + \sum_{n=2}^{\infty} g_n(x)$ and let $F(x) = \int_0^x f(t)dt$. Then $F' = f$ on $(0, 1)$, $f'(0) = S$ and f' is unbound so $S = W_F$ as desired. □

4.2 Continuous Derivatives

We can improve on the previous theorem is we assume that F' is continuous. We have this version of Theorem 4.7 for continuously differentiable functions.

Theorem 4.8. *A set $K \subset (0, 1)$ is the set W_F for some continuously differentiable function F if and only if K is a closed nowhere dense set. In fact, if $K = K_1 \cup K_2$ where*

K_1, K_2 are closed nowhere dense sets we can find F so that if $K_1 \neq \emptyset$ then F' is minimal on K_1 and if $K_2 \neq \emptyset$ then F' is maximal on K_2.

Proof. First, given such a set K with $K = K_1 \cup K_2$ we will show that there exists a function F on $I = [0, 1]$ such that $W_F = K$. We shall proceed with four cases:

1. If $K_1, K_2 = \emptyset$ then for $F(x) = x$ we have that $W_F = \emptyset$.

2. If $K_1, K_2 \neq \emptyset$ then let $f(x) = \frac{d(x,K_1)}{d(x,K_1)+d(x,K_2)}$ and define F by $F(x) = \int_0^x f(t)dt$. Then $F' = f$, $f^{-1}(0) = K_1$, and $f^{-1}(1) = K_2$. Since K_1, K_2 are nowhere dense, $0 < \int_a^b f(t)dt < b - a$ for all $a < b \in I$. In other words, $0 < \frac{F(b)-F(a)}{b-a} < 1$ and $W_F = K_1 \cup K_2$.

3. If $K_1 \neq \emptyset, K_2 = \emptyset$ let $g(x) = d(x, K_1)$ and $f(x) = \min\{g(x), \frac{B}{2}\}$ where B is the maximum of g. Let $F(x) = \int_0^x f(t)dt$. Then there is an interval (c, d) such that $f(x) = \frac{B}{2}$ on (c, d) and $\frac{\int_c^d f(t)dt}{d-c} = \frac{F(d)-F(c)}{d-c} = \frac{B}{2}$. Thus $W_F \cap f^{-1}(\frac{B}{2}) = \emptyset$ and we have $W_F = f^{-1}(0) = K_1$ as desired.

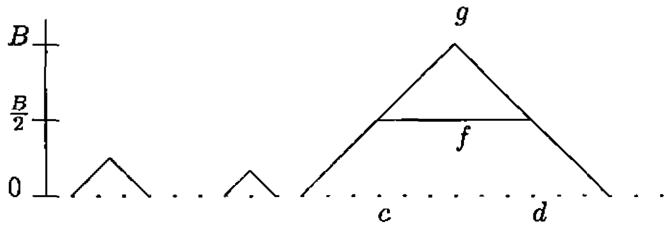

Figure 4.1: Graphs of g and $F' = f$

4. If If $K_1 = \emptyset, K_2 \neq \emptyset$ we can construct F in the same way as described in case 3 but make the maximum of f 0 and the minimum of f equal to $-B$.

By all of the above cases we now have that for K satisfying the hypotheses in the above theorem, there exists a continuously differentiable function F with $W_F = K$.

In the other direction, if we have a continuously differentiable function F with $W_F = K$ and derivative f we must show that K is nowhere dense. We know from Theorem 4.1 that $W_F \subset f^{-1}(m) \cup f^{-1}(M)$ where m and M are minimum and maximum values (if they exist) of F'. Suppose $W_F \supset f^{-1}(m)$ is not nowhere dense. Then $f^{-1}(m)$ contains an interval $[a,b]$ on which $f^{-1} \equiv m$ and so $\frac{F(b)-F(a)}{b-a} = m$. This is a contradiction. The same reasoning can be applied to $f^{-1}(M)$. We also know that the union of two nowhere dense sets is also nowhere dense, so then $K = W_F$ is nowhere dense. □

We now turn to the set S_F.

Lemma 4.9. *If F is differentiable and F' is monotonic, and thus continuous, on an interval (a,b) then $S_F \cap (a,b) = \emptyset$.*

Proof. For $z < x < w$ contained in (a,b) we have $\frac{F(w)-F(z)}{w-z} = \frac{1}{w-z}\int_z^w F'(t)dt$. It is easy to see that if F' is monotonic we can pick z, w so that $\frac{1}{w-z}\int_z^w F'(t)dt = F'(x)$. □

Theorem 4.10. *Let E be a nonempty closed nowhere dense set. Then there is a continuously differentiable function F with $S_F = E$ and $W_F = \emptyset$.*

Proof. If E is a singleton $\{c\} \subset (0,1)$ we define $f(x) = \begin{cases} 1-\frac{x}{c} \text{ if } 0 \le x \le c \\ 0 \text{ if } c < x \le 1 \end{cases}$ and let $F(x) = \int_0^x f(t)dt$. Since f is monotonic on $(0,c)$ and $(c,1)$, $S_F \cap \big((0,c) \cup (c,1)\big) = \emptyset$. If $z < c < w$ then $\frac{F(w)-F(z)}{w-z} = \frac{1}{w-z}\int_z^w f(t)dt = \frac{1}{w-z}\int_z^c f(t)dt = \frac{(c-z)^2}{w-z} > 0$. Thus F fails the Strong Form at c and $S_F = \{c\}$. It is easy to see that $W_F = \emptyset$ since $W_F \subset S_F$ and F passes the Weak Form at c since $\frac{F(1)-F(c)}{1-c} = 0$.

Now assume E contains at least two points and pick $0 < c < d < 1$ in E. We define f on $[c,d]$ to be the line segment joining the point $(c,0)$ to the point $(d,1)$. Let f be 0 on $E \cap [0,c]$ and 1 on $E \cap [d,1]$. On one complementary interval of $E \cap [0,c]$ let f be constant 0 and on one complementary interval of $E \cap [d,1]$ let f be constant 1. On each remaining complementary interval (a,b) of $E \cap [0,c]$ define f to be piecewise linear as follows. Join the points $(a,0)$, $(\frac{2}{3}a + \frac{1}{3}b, \gamma)$, $(\frac{1}{3}a + \frac{2}{3}b, \gamma)$, and $(b,0)$ to form a trapezoidal shape. We pick γ to be the minimum of $\frac{b-a}{2}$ and $\frac{1}{d}\int_c^d f(t)dt = \frac{d-c}{2d}$. This choice of γ ensures that f is continuous on $[0,c]$ On each remaining complementary interval (a',b') of $E \cap [d,1]$ define f to be piecewise linear by joining the points $(a',1)$, $(\frac{2}{3}a' + \frac{1}{3}b', \gamma')$, $(\frac{1}{3}a' + \frac{2}{3}b', \gamma')$, and $(b',1)$ to form a trapezoidal shape. We pick γ' to be the maximum of $1 - \frac{b'-a'}{2}$ and $\frac{1}{1-c}(\int_c^d f(t)dt + 1 - d) = \frac{2-c-d}{2(1-c)}$. The choice of γ' here ensures that f is continuous on $[d,1]$.

Define $F(x) = \int_0^x f(t)dt$. Since $F' = f$, we know that F passes the Strong Form on every interval on which f is linear. At every point x of $E \cap (0,c]$ we have $f(x) = 0$ while

$\int_z^w f(t)dt > 0$ for all $z < x < w$ so $x \in S_F$. Similarly, for every point x of $E \cap [d, 1)$ we have $f(x) = 1$ while $\int_z^w f(t)dt < w - z$ for all $z < x < w$ so $x \in S_F$. Thus $E \subset S_F$. To establish equality we must show that the values of x forming the corners of the trapezoidal shapes are not in S_F. Let $(a, b) \subset (0, c)$ be a subinterval on which the graph of f is trapezoidal and let $x = \frac{2}{3}a + \frac{1}{3}b$. We then have $f(x) = \gamma$ as chosen above. It is easy to see that $\frac{F(b)-F(a)}{b-a} = \frac{1}{b-a}\int_a^b f(t)dt < \gamma$ while $\frac{F(d)-F(0)}{d} = \frac{1}{d}\int_0^d f(t)dt \geq \frac{1}{d}\int_c^d f(t)dt \geq \gamma$. Since F is continuous there must exist $z < x < w$ so that $\frac{F(w)-F(z)}{w-z} = \frac{1}{w-z}\int_z^w f(t)dt = \gamma$. The same calculation shows that F satisfies the Strong Form at $x = \frac{1}{3}a + \frac{2}{3}b$. A similar argument shows that F passes the Strong Form at those values locating the corners of the trapezoidal shapes in $(d, 1)$. Thus $E = S_F$.

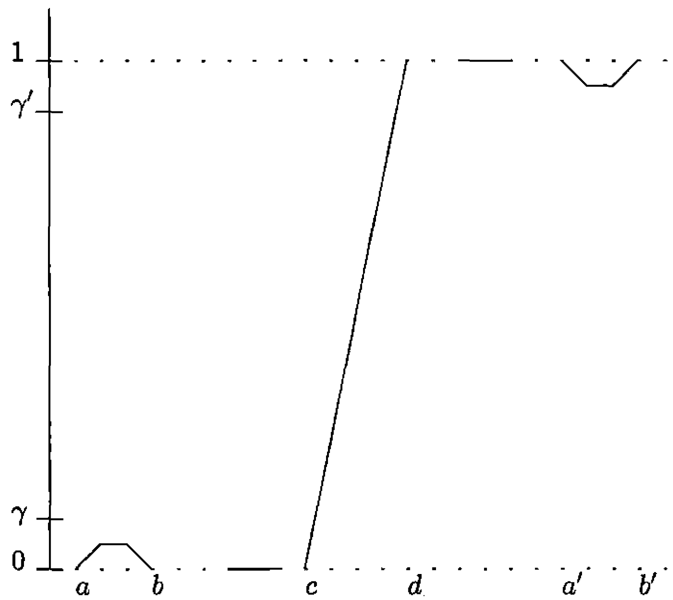

Figure 4.2: Graph of $F' = f$

Lastly, we show that $W_F = \emptyset$. We have $W_F \subset f^{-1}(1) \cup f^{-1}(0)$. But f is constant 0 on one subinterval (a, b) and constant 1 on another (a', b'). Since $\int_a^b f(t)dt = 0$ and $\int_{a'}^{b'} f(t)dt = b' - a'$, $W_F = \emptyset$. □

Chapter 5

Conclusion

Over the course of this study, both Lebesgue measure and category have proven to be useful tools in describing the sizes of sets. We have established analogous relationships between the two through the investigation of different properties held by either of them. Chapter 2 provided a direct comparison as particular theorems regarding these two notions were compared. It also provided a foundation for the applications found in the following chapters. Chaptesr 3 and 4 then provided a demonstration of the similar functions of category and Lebesgue measure in proving theorems.

Within Chapter 2 we have in section 2.1 that a study of nullsets and first category sets on the real line proved that there was indeed a strong relationship between category and Lebesgue measure with respect to these "small" sets. The relationship was complicated only by theorems 2.5 and 2.6 which highlighted a surprising and almost paradoxical result. Thereafter though, when sets of positive measure and second category sets having the property of Baire were considered the relationship was again strengthened by theorem 2.36. Later in the chapter it was established that the class of measurable sets and those which have the property of Baire are both σ-algebras. This gave rise to more observations of similarities. In particular, theorem 2.21 and theorem 2.31 provided another strange dynamic when F_σ sets and G_δ sets were explored. The direct comparison of Lebesgue measure and category then ended in section 2.4.2 with theorems 2.40 and 2.41 displaying a relationship between nonmeasurable sets and sets which do not have the property of Baire.

Chapter 3 demonstrated the usefulness of category arguments for establishing properties of continuous functions and derivatives. In fact, sections 3.1 and 3.2 provided counterintuitive results about continuous functions that would be difficult to prove without the use of this method. Then a result concerning differentiable functions was proven in section 3.3.These all serve to exhibit the practicality of studying category in Real Analysis.

Finally, Chapter 4 focused on the converse of the famous mean value theorem. With Lebesgue measure arguments scattered throughout, there were many important results proven concerning the sets on which both the Weak form and the Strong form hold. A full characterization of these sets was given for both forms with respect to functions with continuous derivatives. Still, the general case remains unsolved.

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