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de Rham Cohomology, Homotopy Invariance and the Mayer-Vietoris Sequence

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A Thesis

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Abstract

This thesis will discuss the de Rham cohomology, homotopy invariance and the Mayer-Vietoris sequence. First the necessary information for this thesis is discussed such as differential p-forms, the exterior derivative as well as pull back of a map. The de Rham cohomology is defined explicitly, some properties of the de Rham cohomology will also be discussed. It will be shown that the de Rham cohomology is in fact a homotopy invariant as well as some examples using homotopy invariance are provided. Finally the Mayer-Vietoris sequence will be established, an example of using the Mayer-Vietoris sequence to compute the de Rham cohomology of groups of spheres is provided.

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Chapter 1

Introduction

The goal of this thesis will be to introduce and study the de Rham cohomology of smooth manifolds. Specifically, we show it is a homotopy invariant, establish the Mayer-Vietoris Sequence, and provide examples of how these results are useful.

In Chapter 2, we introduce all necessary background information for us to begin talking about the de Rham cohomology. We start with introducing what we will be working with, manifolds, the main building block for this thesis. We also discuss operators we use for manifolds, as well what a homotopy is, which is useful in Chapter 4.

Moving into Chapter 3, this is where we speak on what the de Rham cohomology is and formally define it (see Definition 3.1). Then after this some properties of the de Rham cohomology are presented, see Section 3.2.

Then in Chapter 4, we discuss homotopy invariance and why this is useful in the project. In this chapter we show how a property of the de Rham cohomology, induced cohomology maps (see Proposition 3.2), are equal when there exits a homotopy operator between the pull back maps. Moving on in the chapter we prove the existence of a homotopy operator, with the lemma coming from John Lee's book; Introduction to Smooth Manifolds [Lee13], but the proof being different than what he has shown in his book. Instead of using new techniques such as the Lie Derivative and Cartan's Magic Formula, we use slightly different techniques. With that we provide some computations of using homotopy invariance at the end of the chapter.

Finally, in Chapter 5, we introduce what the Mayer-Vietoris Sequence is. For us to talk about the Mayer-Vietoris Sequence we must understand what it means for a sequence to be exact, and then discuss for a sequence to be short exact and even long exact. Knowing this will help us in establishing the Mayer-Vietoris Sequence. For us to establish the Mayer-Vietoris Sequence we also need to know the δ map, which is brought to us by the Zig-Zag Lemma (see Lemma 5.2). After establishing the Mayer-Vietoris Sequence, at the end of the chapter there is an example of using the Mayer-Vietoris Sequence.

Chapter 2

Preliminaries

The de Rham cohomology requires some necessary knowledge before we are able to introduce and talk about what this truly is. We will be working with manifolds throughout this project, it is important for us to know what we are working with. For M to be a topological manifold of dimension n or a topological n-manifold, if M holds the following properties:

Property 2.1. *[Lee13] The topological space M is a manifold without boundary if:*

1. *M is Hausdorff space: for every pair of distinct points $p, q \in M$, there are disjoint open subset $U, V \subseteq M$ such that $p \in U$ and $q \in V$.*

2. *M is second-countable: there exists a countable basis for the topology M.*

3. *M is locally Euclidean of dimension n: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

One can define manifolds with boundary, see Definition 2.2. The above properties, Property 2.1, will be used to describe a manifold, as well as a manifold that is smooth and a manifold without boundary. Later on we will need to know what a manifold with boundary is, this will play an important role in proving the existence of a homotopy operator, Lemma 4.2.

Definition 2.2. *[Hat02] For a n-dimensional manifold with boundary, this implies we have a Hausdorff space M in which each point has an open neighborhood homeomorphic either to $\mathbb{R}^n$ or to the half space $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$.*

Let p be a point of a manifold M with or without boundary. A tangent vector to our manifold M at point p is a real valued function $v : C^\infty(M) \to \mathbb{R}$, where $C^\infty(M)$ is the set of all smooth real valued function on M, that has the following properties, for all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$,

1. $\mathbb{R}$-linear: $v(af + bg) = av(f) + bv(g)$.
2. The product rule holds: $v(fg) = v(f)g(p) + f(p)v(g)$.

The *tangent space at point* p, or more specifically at each point $p \in M$ let $T_p(M)$ be the set of all tangent vectors to M at p. The tangent space to M at point p is denoted by $T_p(M)$.

For each $p \in M$, we can define the *cotangent space at* p, which is written as $T_p^* M$, to be the dual space to the tangent space of M at point p, $T_p(M)$:

$$T_p^* M = (T_p M)^* = Hom_{\mathbb{R}}(T_p M, \mathbb{R}).$$

The elements of $T_p^* M$ are called the tangent covector at point p. We also have the cotangent bundle of M which is defined as the disjoint union:

$$T^* M = \coprod_{p \in M} T_p^* M,$$

topologized in the standard way, see pages $16 - 21$ of [Lee97].

Definition 2.3 (Coordinate Chart)**.** *[Lee13] Let M be a topological n-manifold. A coordinate chart on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.*

At this point we want to be able to perform partial differentiation on a manifold, which is defined precisely below.

Let $\xi = (x^1, \cdots, x^n)$ be a coordinate system on M at p. If $f \in C^\infty(M)$, let

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial(f \circ \xi^{-1})}{\partial u^i}(\xi p), \text{ for } (1 \leq i \leq n),$$

where $u^1, \cdots, u^n$ are the natural coordinates of $\mathbb{R}^n$. We define *coordinate vectors* by the following which, is useful in Theorem 2.4,

$$\partial_i|_p = \frac{\partial}{\partial x^i} \,|_p\colon C^\infty(M) \to \mathbb{R}.$$

The coordinate vectors will send each $f \in C^\infty(M)$ to $\frac{\partial f}{\partial x^i}$ which is a tangent vector to M at p.

Theorem 2.4. *[O'N83] If $\xi = (x^1, \cdots, x^n)$ is a coordinate system in M at p, then its coordinate vectors $\partial_1|_p, \cdots, \partial_n|_p$ form a basis for the tangent space $T_p(M)$; and*

$$v = \sum_{i=1}^{n} v(x^i)\partial_i|_p \qquad \textit{for all} \qquad v \in T_p(M).$$

Differential forms, the operator d and pullback play an important role for us to start using the de Rham Cohomology. We will also see these concepts throughout as we use these concepts in computations and proving new ideas.

Definition 2.5 (Differential k-forms)**.** *Let $\Lambda^k T^* M$ be a bundle of covariant k-tensors on M, where M is n-dimensional smooth manifold. A smooth section of $\Lambda^k T^* M$ is called a differential k-form, which is a smooth tensor field whose value at each point is an alternating tensor [Lee13]. We can denote the vector space of smooth differential k-forms by $\Omega^k(M)$. In any smooth chart, a differential k-form ω can be written locally as*

$$\omega = \sum_I \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I \omega_I dx^I,$$

where the coefficients ω_I are continuous functions on the coordinate domain.

Recall that a (smooth) section of a vector bundle $\pi : E \to M$ is a smooth map $s : M \to E$, where $\pi \circ s$ is the identity map on M. In other words, a section is a mechanism which, when given $x \in M$, it produces a vector in the fiber E_x. So in the case of $E = \Lambda^k T^* M$, a differential k-form is an alternating linear functional at each point of M.

The operator d is called the exterior derivative. The operator d can be applied to many ideas. First we will discuss the operator d on functions of $\mathbb{R}^n$. For a smooth real-valued function f,

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

Now let U be an open subset such that $U \subseteq \mathbb{R}^n$, there exist a ω which is a k-form on U where:

$$\omega = \sum_I \omega_I dx_I, \quad d\omega = \sum d\omega_I \wedge dx_I. \tag{2.1}$$

Now the following properties $(i - iv)$ hold,

Proposition 2.6 (Properties of the Exterior Derivative on $\mathbb{R}^n$)**.** *[Lee13]* *(i) The operator d is linear over $\mathbb{R}$.*

(ii) If ω is a smooth k-form and η is a smooth ℓ-form on an open subset $U \subseteq \mathbb{R}^n$ then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) It is true that, $d \circ d \equiv 0$.

(iv) The operator d commutes with pullbacks: if U and V are open subsets of $\mathbb{R}^n$, $F : U \to V$ is a smooth map, and $\omega \in \Omega^k(V)$, then

$$F^*(d\omega) = d(F^*\omega).$$

Now we can extend exterior derivative on function of $\mathbb{R}^n$ to manifolds.

Theorem 2.7 (Existence and Uniqueness of Exterior Differentiation)**.** *[Lee13] Suppose M is a smooth manifold with or without boundary. There are unique operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$, for all k, called exterior differentiation, satisfying the following properties:*

(i) The operator d is linear over $\mathbb{R}$.

(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) It is true that, $d \circ d \equiv 0$.

(iv) For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.

In any smooth coordinate chart, $\omega = \sum \omega_I dx_I$, so $d\omega$ is given as in Equation 2.1.

Definition 2.8 (Pullback)**.** *[Shi05] Let $U \subset M$ be open, where M is a n-dimensional smooth manifold, and let $g : U \to M$ be smooth. If $\omega \in \Omega^k(M)$, then we define $g^*\omega \in \Omega^k(U)$ (the pull back of ω by g) as follows. To pull back a function (0-form) f, we just compose functions:*

$$g^* f = f \circ g.$$

To pull back the basis 1-forms, if $g(u) = x$, then set

$$g^* dx_i = dg_i = \sum_{j=1}^{m} \frac{\partial g_i}{\partial u_j} du_j.$$

To pull on k-forms, assume ω is a k-form and U is a coordinate chart,

$$\omega = \sum \omega_I dx_I = \sum \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we then have

$$f^*\omega = \sum (f^*\omega_I)(f^*(dx_{i_1})) \wedge \cdots \wedge (f^*(dx_{i_k})).$$

Definition 2.9. *[Tho09] If X and Y are topological spaces and $f, g : X \to Y$ are continuous, then f is homotopic to g if there exists $F : X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, and F is continuous. For this we say that F is a homotopy and we write f is homotopic to g as $f \simeq g$. If F is smooth, then we say F is a smooth homotopy, and that f is smoothly homotopic to g.*

One uses the definition of homotopy of functions to define what it means for manifolds M and N to be homotopic. A manifold M is (smoothly) homotopic to N if there exists (smooth) functions $f : M \to N$ and $g : N \to M$ where $f \circ g$ is homotopic to the identity map on N, and $g \circ f$ is smoothly homotopic to the identity map on M.

Definition 2.10. *[Lee13] If M and N are smooth manifolds with or without boundary, a diffeomorphism from M to N is a smooth bijective map $F : M \to N$ that has a smooth inverse. We say that M and N are diffeomorphic if there exists a diffeomorphism between them. This is sometimes denoted as $M \approx N$.*

For example the function $F : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $F(x) = x^3$ is not a diffeomorphism. The function F is not a diffeomorphism because $F(x)$ does not have a smooth inverse, as the inverse of $F(x)$ is not differentiable at $x = 0$, even though F is a smooth bijection.

In Section 3.2, we will need to know a diffeomorphism is, see the below definition (Definition 2.10). We will show in Corollary 3.3, when we have smooth diffeomorphic manifolds they have isomorphic de Rham cohomology groups.

Chapter 3

de Rham Cohomology

3.1 What is de Rham Cohomology?

Definition 3.1. *[Lee13] Let M be a smooth manifold with or without boundary, and let p be a nonnegative integer. Since $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is linear, its kernel and image are linear subspaces. We define the following:*

$$\begin{aligned} \mathcal{Z}^p(M) &= \ker(d : \Omega^p(M) \to \Omega^{p+1}(M)) &= \{\textit{closed } p\textit{-forms on } M\} \\ \mathcal{B}^p(M) &= im(d : \Omega^{p-1}(M) \to \Omega^p(M)) &= \{\textit{exact } p\textit{-forms on } M\}. \end{aligned}$$

Since $d^2 = 0$, every exact closed form is closed so that $\mathcal{B}^p(M) \subseteq \mathcal{Z}^p(M)$. Thus it makes sense to define the de Rham cohomology group in degree p of M to be the quotient vector space,

$$H^p(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}.$$

3.2 Properties of the de Rham Cohomology

Proposition 3.2 (Induced Cohomology Maps)**.** *[Lee13] For any smooth map $F : M \to N$ between smooth manifolds with or without boundary, the pull back $F^* : \Omega^p(N) \to \Omega^p(M)$ carries $\mathcal{Z}^p(N)$ into $\mathcal{Z}^p(M)$ and $\mathcal{B}^p(N)$ into $\mathcal{B}^p(M)$. It thus descends to a linear map, still denoted by F^*, from $H^p(N)$ to $H^p(M)$, called the induced cohomology map. The induced cohomology map has the following properties:*

(a) If $G : N \to P$ is another smooth map, then

$$(G \circ F)^* = F^* \circ G^* : H^p(P) \to H^p(M).$$

(b) If Id denotes the identity map of M, then Id^* is the identity map of $H^p(M)$.

Proof. Let us begin by working with forms. We have $F : M \to N$ and the pull back $F^* : \Omega^p(N) \to \Omega^p(M)$, we want to know if the pullback F^* takes in a form that is in the kernel of d does it produce a form in the kernel of d. Let $\omega \in \ker d$, we know using the operator d, $d\omega = 0$. We want to check $F^*\omega \in \ker d$,

$$\begin{aligned} d(F^*\omega) &= 0, && (d \text{ and } F^* \text{ commute on the level of forms}) \\ F^*(d\omega) &= 0, && \text{and} \\ F^*(0) &= 0, && (d\omega = 0). \end{aligned}$$

We now know that the pullback, F^* takes in a form that is in the kernel of d and produced a form in the kernel of d, meaning $F^* : \ker d \to \ker d$. With that we want to define $F^* : \dfrac{\ker d}{imd} \to \dfrac{\ker d}{imd}$, and we need to check that $F^*[\omega] = [F^*\omega]$ is well defined. It suffices to check that $F^*(d\tau) = 0$.

Let $d\tau \in imd$, so then $F^*(d\tau) = d(F^*\tau) \in imd$, since d and F^* commute on the level of forms. So now F^* is well defined map, thus we have $F^* : H^p(N) \to H^p(M)$.

Now we want to show the two properties, we know these properties to be true when working with p-forms,

$$\begin{aligned} (F \circ G)^*\omega &= (G^* \circ F^*)(\omega), \\ Id^*(\omega) &= \omega. \end{aligned}$$

Since the pull back map Id^* does not change a p-form and Id does not change a p-form then $Id^* = Id$, which will not change when working with cohomology classes either since,

$$\begin{aligned} Id^*[\omega] &= [Id^*(\omega)] \\ &= [\omega], \text{ since on forms } Id^*(\omega) = \omega. \end{aligned}$$

The pull back map Id^* took in the equivalence class of omega and gave us the equivalence class of omega so,

$$Id^* = Id : H^p(M) \to H^p(M).$$

So now,

$$(F \circ G)^*[\omega] = [(F \circ G)^*(\omega)],$$

we know $(F \circ G)^* = G^* \circ F^*$ on forms so we have,

$$\begin{aligned} [(F \circ G)^*(\omega)] &= [(G^* \circ F^*)(\omega)] \\ &= G^*[F^*(\omega)] \\ &= (G^* \circ F^*)[\omega]. \end{aligned}$$

So we started with $(F \circ G)^*[\omega]$ and ended up showing that $(F \circ G) * [\omega] = (G^* \circ F^*)[\omega]$. Therefore $(F \circ G)^* = G^* \circ F^*$ is true for cohomology. □

We will use Proposition 3.2 in the proof of the following corollary, (see Corollary 3.3), and also throughout Chapter 4. For the following result, Corollary 3.3, it may be useful to refer back to what a diffeomorphism is (see Definition 2.10).

Corollary 3.3 (Diffeomorphism Invariance of de Rham Cohomology)**.** *Diffeomorphic smooth manifolds (with or without boundary) have isomorphic de Rham cohomology groups.*

Proof. Let M and N be diffeomorphic smooth manifolds with or without boundary. Let the maps F and G be inverse diffeomorphisms, the maps are defined as the following, $F : M \to N$ and $G : N \to M$. We also know the pull back maps of F and G are defined as, $F^* : H^k(N) \to H^k(M)$ and $G^* : H^k(M) \to H^k(N)$. Since M and N are diffeomorphisms then we know $G \circ F = Id_M$ and $F \circ G = Id_N$. Using Property (a) of Proposition 3.2,

$$\begin{aligned} (G \circ F)^* &= Id_M^* = F^* \circ G^* : H^k(M) \to H^k(M) \\ (F \circ G)^* &= Id_N^* = G^* \circ F^* : H^k(N) \to H^k(N). \end{aligned}$$

Since Id_M is surjective map we know that F^* is surjective, also since Id_N is injective map we then know that F^* is injective. With that we know that F^* is a bijection, so there must exist a isomorphism thus,

$$H^k(M) \cong H^k(N).$$

□

Proposition 3.4 (Cohomology in Degree Zero)**.** *[Lee13] If M is a connected smooth manifold with or without boundary, then $H^0(M)$ is equal to the space of constant functions and is therefore* 1*-dimensional.*

Proof. Since there are no nonzero forms for $p < 0$, then the $im d = \mathcal{B}^0(M) = 0$. A closed 0-form is a smooth real-valued function, f, such that $df = 0$, since our manifold M is connected then it must be true if and only if f is a constant function. Thus $H^0(M) = \mathcal{Z}^0(M) = \{\text{constants}\}$. $\square$

Chapter 4

Homotopy Invariance

Homotopy invariants are useful in this project because, as we discovered cohomology is hard to compute. With the use of homotopy invariants, this takes away some of the hard computations. In this chapter, we will prove the existence of a homotopy operator (see Lemma 4.2). While this lemma comes from John Lee's book; Introduction to Smooth Manifolds [Lee13], the proof he has included uses new ideas such as the Lie Derivative and Cartan's Magic Formula rather we use slightly different techniques to accomplish the proof. With that, we prove that the induced cohomology maps are equal when we have homotopic smooth maps (see Proposition 4.3). Then finally we prove the homotopy invariance of de Rham cohomology. The proof is short since a lot of the work was prepped beforehand. Now, in Section 4.1, there are some computations of using homotopy invariance.

Suppose $F, G : M \to N$ are smooth maps, also let $F^* = G^*$. For $F^* = G^*$ to be true, it must be the case that when given a closed p-form ω on N, we will need to come up with a $(p-1)$-form on M such that,

$$G^*[\omega] - F^*[\omega] = [d\eta] = 0.$$

We can then create a map $h : \Omega^p(N) \to \Omega^{p-1}(M)$, which takes closed p-forms on N to $(p-1)$-forms on M, such that the following is true:

$$d(h\omega) = G^*\omega - F^*\omega. \tag{4.1}$$

Unfortunately what we have defined above is only true when ω is closed. We want to be able to define a map h for all smooth p-forms on N, rather than just closed p-forms, to

$(p-1)$-forms on M. To accomplish this, we use the following equation:

$$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega. \tag{4.2}$$

In fact, Equation 4.2 gives us Equation 4.1 when ω is closed, since $d\omega = 0$. If we have smooth maps $F, G : M \to N$, a collection of linear maps is defined by $h : \Omega^p(N) \to \Omega^{p-1}(M)$ so that the above equation, Equation 4.2, holds true for all ω. This is called a homotopy operator between the the maps F^* and G^*.

Proposition 4.1. *[Lee13] Suppose M and N are smooth manifolds with or without boundary. If $F, G : M \to N$ are smooth maps and there exists a homotopy operator between the pull back maps F^* and G^*, then the induced cohomology maps $F^*, G^* : H^p(N) \to H^p(M)$ are equal.*

Proof. Assume $F, G : M \to N$ are smooth maps and there exists a homotopy operator, h, between the maps F^* and G^*. Let us define $h : \Omega^p(N) \to \Omega^{p-1}(M)$, which will satisfy Equation 4.2. Our end goal is to show that $F^*[\omega] = G^*[\omega]$. Since we have Equation 4.2 we want to know what $G^*\omega - F^*\omega$ equals so we can arrive at our conclusion. We want $[G^*\omega - F^*\omega] = 0$ for $\omega \in \ker d$ where $d\omega = 0$.

$$\begin{aligned}
[G^*\omega] - [F^*\omega] &= [G^*\omega - F^*\omega] \\
&= [d(h\omega) + h(d\omega)] \\
&= [d(h\omega)] + [h(d\omega)] \\
&= 0.
\end{aligned}$$

We can say that $[d(h\omega)] + [h(d\omega)]$ is zero since $d(h\omega)$ is in the image of d. Therefore, is zero on cohomology, and $d\omega = 0$ so $h(d\omega) = h(0) = 0$ so we have $[0] + [0] = 0$. Now putting all of it together, we have,

$$\begin{aligned}
0 &= [G^*\omega] - [F^*\omega] \\
[F^*\omega] &= [G^*\omega] \\
F^*[\omega] &= G^*[\omega].
\end{aligned}$$

$\square$

We want to construct a homotopy operator. To accomplish this, let M be a smooth manifold with or without boundary, and for each $t \in I$, let $i_t : M \to M \times I$ be the map $i_t(x) = (x, t)$. So if M has empty boundary, then $M \times I$ is smooth manifold with boundary and thus the results above apply. Although if the boundary of M is nonempty, then $M \times I$ has to be considered as a smooth manifold with corners.

The following results will help us prepare for the main result of the chapter, Theorem 4.4. We will first prove the existence of a homotopy operator, then show that when we have two homotopic smooth maps the induced cohomology of the maps are in fact equal. The following results, Lemma 4.2 and Proposition 4.3, provides as the major leg work for the proof of the homotopy invariance of the de Rham cohomology (see Theorem 4.4).

Lemma 4.2 (Existence of a Homotopy Operator)**.** *[Lee13] For any smooth manifold M with or without boundary, there exists a homotopy operator between the two maps $i_0^*, i_1^* : \Omega^*(M \times I) \to \Omega^*(M)$.*

Before we jump into this proof we will be using a new operator, which is commonly referred to as interior multiplication. Let V be a finite dimensional vector space, for each $v \in V$, we define a linear map $i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$,

$$i_v(\omega)(w_1, \cdots, w_{k-1}) = \omega(v, w_1, \cdots, w_{k-1}).$$

It commonly denoted as $v \lrcorner \omega$ which is read as v into ω, what this operator is doing is taking ω and inserting v into the first slot.

Proof. For each p we want to define a map,

$$h : \Omega^p(M \times I) \to \Omega^{p-1}(M)$$

such that Equation 4.2 is true. That is, we want the following to be true:

$$d(h\omega) + h(d\omega) = i_1^*\omega - i_0^*\omega. \tag{4.3}$$

If we can show that Equation 4.3 holds then we will have shown there is a homotopy operator between our two maps, i_0^* and i_1^*.

Let $i_t : M \to M \times I$, where $i_t(x) = (x, t)$ and $t \in [0, 1]$ as above. We define $h\omega \in \Omega^{p-1}(M)$ by

$$h\omega = \int_{s=0}^{1} i_s^*(\partial t \lrcorner \omega) ds.$$

We know i_t to be defined as, $i_t : M \to M \times I$ then it must be the case that i_t^* is defined as the following, $i_t^* : \Omega^k(M \times I) \to \Omega^k(M)$. Let $\omega \in \Omega^p(M \times I)$. Choose the global coordinates system (t) on I, and for any point $p \in M$, choose any coordinates $(x_1, \cdots x_n)$ on M near p. Then, $(x_1, \cdots, x_n, t)$ are coordinates on $M \times I$, and t may be any element in $[0,1]$. Then in coordinates:

$$\omega = \sum_I f_I(x,t)dt \wedge dx_I + \sum_J g_J(x,t)dx_J.$$

With now knowing ω we are able to find $\partial_t \lrcorner \omega$, this will move dt into the 'first slot', if the component does not have dt then it will become zero. For example in ω above the sum over J does not contain dt so when calculating $\partial_t \lrcorner \omega$ we omit it.

$$\begin{aligned} \partial_t \lrcorner \omega &= \sum_I f_I(x,t)dx_I \\ i_t^*(\partial_t \lrcorner \omega) &= i_t^*\left(\sum_I f_I(x,t)dx_I\right) \\ &= \sum_I f_I(x,t)dx_I. \end{aligned}$$

Now with $i_t^*(\partial_t \lrcorner \omega)$, we are able to easily find $d(h\omega)$,

$$\begin{aligned} d(h\omega) &= d\int_{s=0}^1 i_s^*(\partial_t \lrcorner \omega)ds \\ &= d\int_{s=0}^1 \left(\sum_I f_I(x,s)dx_I\right) ds \\ &= \sum \int_{s=0}^1 \left(\frac{\partial f_I}{\partial x_i}(x,s)dx_i \wedge dx_I\right) ds. \end{aligned}$$

Now we need to find $h(d\omega)$, so what we want to do is find first is $d\omega$.

Since $\omega = \sum_I f_I(x,t)dt \wedge dx_I + \sum_J g_J(x,t)dx_J$,

$$\begin{aligned} d\omega &= d(\sum_I f_I(x,t)dt \wedge dx_I + \sum_J g_J(x,t)dx_J) \\ &= \sum \frac{\partial f_I}{\partial x_i}dx_i \wedge dt \wedge dx_I + \frac{\partial g_J}{\partial x_i}dx_i \wedge dx_J + \frac{\partial f_I}{\partial t}dt \wedge dt \wedge dx_I + \frac{\partial g_J}{\partial t}dt \wedge dx_J. \end{aligned}$$

Note that $dt \wedge dt = 0$, so $\sum \frac{\partial f_I}{\partial t}dt \wedge dt \wedge dx_I = 0$. So now continuing on with $d\omega$,

$$d\omega = \sum \frac{\partial f_I}{\partial x_i}dx_i \wedge dt \wedge dx_I + \frac{\partial g_J}{\partial x_i}dx_i \wedge dx_J + \frac{\partial g_J}{\partial t}dt \wedge dx_J.$$

Now we can find $h(d\omega)$,

$$h(d\omega) = \int_{s=0}^1 i_s^*(\partial_t \lrcorner d\omega)ds.$$

For ease let us find $\partial_t \lrcorner d\omega$,

$$\partial_t \lrcorner d\omega \quad = \quad \sum \frac{-\partial f_I}{\partial x_i} dt \wedge dx_i \wedge dx_I + \frac{\partial g_J}{\partial t} dt \wedge dx_J.$$

Note that since $\sum \frac{\partial g_J}{\partial x_i} dx_i \wedge dx_J$ does not have a dt when we find $\partial_t \lrcorner d\omega$ it becomes 0. Continuing now with $h(d\omega)$,

$$\begin{aligned} h(d\omega) &= \int_{s=0}^{1} i_s^* \left(\sum \frac{-\partial f_I}{\partial x_i} dt \wedge dx_i \wedge dx_I + \frac{\partial g_J}{\partial t} dt \wedge dx_J \right) ds \\ &= \sum \int_{s=0}^{1} \left(-\frac{\partial f_I}{\partial x_i}(x,s) dx_i \wedge dx_I + \frac{\partial g_J}{\partial t}(x,s) dx_J \right) ds. \end{aligned}$$

Now we have found the necessary components to see if Equation 4.3 holds.

$$\begin{aligned} d(h\omega) + h(d\omega) &= \int_{s=0}^{1} \left(\frac{\partial f_I}{\partial x_i}(x,s) dx_i \wedge dx_I \right) ds + \int_{s=0}^{1} \left(-\frac{\partial f_I}{\partial x_i}(x,s) dx_i \wedge dx_I \right) ds \\ &\quad + \int_{s=0}^{1} \left(\frac{\partial g_J}{\partial t}(x,s) dx_J \right) ds \\ &= \int_{s=0}^{1} \left(\frac{\partial g_J}{\partial t}(x,s) dx_J \right) ds. \end{aligned}$$

Now for us to show that Equation 4.3 is true to get our result we are looking for, let us look at the following:

$$\begin{aligned} i_t^*(\omega) &= i_t^* \left(\sum f_I(x,s) dt \wedge dx_I \right) + i_t^* \left(\sum g_J(x,s) dx_J \right) \\ &= \sum f_I(x,t) i_t^*(dt) \wedge i_t^*(dx_I) + \sum g_J(x,t)(dx_J) \\ &= \sum f_I(x,t) 0 \wedge dx_I + \sum g_J(x,t) dx_J \\ &= \sum g_J(x,t) dx_J. \end{aligned}$$

Now differentiating each term:

$$\begin{aligned} \frac{d}{dt}(i_t^*(\omega)) &= \frac{d}{dt} \left(\sum g_J(x,t) dx_J \right) \\ &= \sum \frac{\partial g_J}{\partial t}(x,t) dx_J. \end{aligned}$$

Now continuing from above we have,

$$\begin{aligned} d(h\omega) + h(d\omega) &= \int_{t=0}^{1} \left(\frac{\partial g_J}{\partial S}(x,t) dx_J \right) dt \\ &= \int_{t=0}^{1} \frac{d}{dt}(i_t^*(\omega)) dt \\ &= i_1^*\omega - i_0^*\omega, \end{aligned}$$

by the Fundamental Theorem of Calculus. Therefore there exists a homotopy operator between the two maps, $i_0^*, i_1^* : \Omega^*(M \times I) \to \Omega^*(M)$. $\square$

Proposition 4.3. *[Lee13] Suppose M and N are smooth manifolds with or with boundary, and $F, G : M \to N$ are homotopic smooth maps. For every p, the induced cohomology maps $F^*, G^* : H^p(N) \to H^p(M)$ are equal.*

Proof. By the Existence of the Homotopy Operator, Lemma 4.2, we know the maps i_0^* and i_1^* are equal from $H^p(M \times I)$ to $H^p(M)$. Since we have $F, G : M \to N$, which are homotopic smooth maps then there must exist a homotopy, $H : M \times I \to M$ from F to G. We know, $F = H \circ i_0$ and $G = H \circ i_1$, so we have

$$F^* = (H \circ i_0)^* = i_0^* \circ H^*. \tag{4.4}$$

By the Existence of the Homotopy Operator, Lemma 4.2 and the earlier discussion, we know $i_0^* = i_1^*$. So then using, Equation 4.4 we have

$$F^* = (H \circ i_0)^* = i_0^* \circ H^* = i_1^* \circ H^* = (H \circ i_1)^* = G^*, \tag{4.5}$$

which gives us the result we wanted, $F^* = G^*$. □

Theorem 4.4 (Homotopy Invariance of de Rham Cohomology)**.** *[Lee13] If M and N are homotopic smooth manifolds with or with out boundary, then $H^p(M) \cong H^p(N)$ for each p. The isomorphisms are induced by any smooth homotopy equivalence $F : M \to N$.*

Proof. There exists $F : M \to N$ and $G : N \to M$. It must be the case that $G \circ F \simeq Id_M$ and $F \circ G \simeq Id_N$. We know that $F^* : H^p(N) \to H^p(M)$ and $G^* : H^p(M) \to H^p(N)$. By Proposition 3.2 and Proposition 4.3 we have,

$$\begin{aligned} (G \circ F)^* &= F^* \circ G^* = Id_M^* = Id_{H^p(M)} \\ (F \circ G)^* &= G^* \circ F^* = Id_N^* = Id_{H^p(N)}. \end{aligned}$$

Since the map Id_M^* is a surjective map then F^* is surjective. Also since Id_N^* is an injective map then F^* is an injective map. So we have F^* being a bijection, so it must be the case it is an isomorphism. Thus we have, $H^p(M) \cong H^p(N)$. □

4.1 Computations Using Homotopy Invariance

Before we jump into the following computations, some knowledge of zero dimensional manifolds will be help in our first computation, Theorem 4.6. If M is a manifold of dimension n, then, as a vector space, the dimension of the p-forms at any given point of M is $\binom{n}{p}$. So for example if we have a manifold of dimension 3 the dimension of the p-forms for $p = 2$ is $\binom{3}{2} = 3$, meaning the dimension of the 2-forms is 3. Then the dimension of 3-forms on our manifold of dimension 3 would be 1, which make sense since there is only one way to get a basis vector: $dx_1 \wedge dx_2 \wedge dx_3$, so that would mean that there are not any nonzero 4-forms, 5-forms and so on. So if we are working with a zero dimensional manifold there are no nonzero forms for $p \geq 1$. Which then we know that $H^p(M) = 0$ when $p > \dim(M)$, since there are non nonzero p-forms on M when $p > \dim(M)$.

The first computation below requires knowledge of a contractible manifold, this is defined below, see Definition 4.5.

Definition 4.5. *[Lee13] For a topological space X to be contractible then the identity map of X is homotopic to a constant map.*

Now with this we are able to do the following computation.

Theorem 4.6 (Cohomology of Contractible Manifolds)**.** *[Lee13] If M is a contractible smooth manifold with or without boundary, then $H^p(M) = 0$ for $p \geq 1$.*

Proof. Assume M is a contractible smooth manifold with or without boundary. By Definition 4.5, there exists is some $s \in M$ such that the IdM is homotopic to the constant map $c : M \to M$, defined by $c(s) = s$. So we know that $H^p(\{s\}) = 0$ for $p \geq 1$, by the earlier discussion there are no nonzero p-forms for $p \geq 1$ since s is a zero dimensional manifold. □

Theorem 4.7 (The Poincaré Lemma)**.** *[Lee13] If U is star-shaped open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, then $H^p(U) = 0$, for $p \geq 1$.*

Proof. It is known that star-shaped subsets are contractible to a point. Let U be contractible to the point u, where $u \in U$. Since U is contractible to a point by the above theorem, Theorem 4.6, we know $H^p(\{u\}) = 0$. Thus we have,

$$H^p(U) \cong H^p(\{u\}) = 0, \text{ for } p \geq 1.$$

□

Chapter 5

Mayer-Vietoris Sequence

The main goal of using the Mayer-Vietoris Sequence is to compute $H^*(M)$ in terms of $H^*(U), H^*(V)$, and $H^*(U \cap V)$ where $\{U, V\}$ is an open cover of M. We need to introduce the concept of an exact sequence of Abelian groups in order to properly state this result.

Suppose $C^0, C^1, \ldots$ are abelian groups, and $F_p : C^p \to C^{p+1}$ are homomorphisms. Now consider the following sequence

$$\cdots \to C^{p-1} \xrightarrow{F_{p-1}} C^p \xrightarrow{F_p} C^{p+1} \xrightarrow{F_{p+1}} C^{p+2} \to \cdots .$$

For this sequence to be *exact*, this would mean that the image of each map is equal to the kernel of the next: that is for each p,

$$imF_{p-1} = \ker F_p.$$

More specifically if we have the following sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

this sequence is short exact if the following hold:

1. f is injective.

2. g is surjective.

3. $imf = \ker g$.

One can adapt the concept of a short exact sequence of groups to a short exact sequence of chain complexes, and arrive at the following well-known result.

If A^*, B^* and C^* are complexes, a co-chain map from A^* to B^* will be denoted by $f : A^* \to B^*$ and a co-chain map from B^* to C^* will be denoted by $g : B^* \to C^*$. These co-chain maps are collection of linear maps, $f : A^p \to B^p$ and $g : B^p \to C^p$, such that the following commutes for each p:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \mathcal{A}^{p-1} \arrow[r, "d"] \arrow[d, "f"] \arrow[dr, phantom, "\circ"] & \mathcal{A}^{p} \arrow[r, "d"] \arrow[d, "f"] \arrow[dr, phantom, "\circ"] & \mathcal{A}^{p+1} \arrow[r, "d"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r, "d"] & \mathcal{B}^{p-1} \arrow[r, "d"] \arrow[d, "g"] \arrow[dr, phantom, "\circ"] & \mathcal{B}^{p} \arrow[r, "d"] \arrow[d, "g"] \arrow[dr, phantom, "\circ"] & \mathcal{B}^{p+1} \arrow[r, "d"] \arrow[d, "g"] & \cdots \\
\cdots \arrow[r, "d"] & \mathcal{C}^{p-1} \arrow[r, "d"] & \mathcal{C}^{p} \arrow[r, "d"] & \mathcal{C}^{p+1} \arrow[r, "d"] & \cdots .
\end{tikzcd}$$

A short exact of co-chain complexes have the same properties of a short exact sequence as describe earlier in the chapter, the following is an example of short exact sequence of complexes, $0 \to \mathcal{A}^* \xrightarrow{f} \mathcal{B}^* \xrightarrow{g} C^* \to 0$. Since these maps f and g are just a collection of linear maps the following commutes for each p:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "d"] & \mathcal{A}^{p-1} \arrow[r, "d"] \arrow[d, "f"] \arrow[dr, phantom, "\circ"] & \mathcal{A}^{p} \arrow[r, "d"] \arrow[d, "f"] \arrow[dr, phantom, "\circ"] & \mathcal{A}^{p+1} \arrow[r, "d"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r, "d"] & \mathcal{B}^{p-1} \arrow[r, "d"] \arrow[d, "g"] \arrow[dr, phantom, "\circ"] & \mathcal{B}^{p} \arrow[r, "d"] \arrow[d, "g"] \arrow[dr, phantom, "\circ"] & \mathcal{B}^{p+1} \arrow[r, "d"] \arrow[d, "g"] & \cdots \\
\cdots \arrow[r, "d"] & \mathcal{C}^{p-1} \arrow[r, "d"] \arrow[d] & \mathcal{C}^{p} \arrow[r, "d"] \arrow[d] & \mathcal{C}^{p+1} \arrow[r, "d"] \arrow[d] & \cdots . \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Theorem 5.1. *A short exact sequence of co-chain complexes gives rise to a long exact sequence in cohomology. More precisely if*

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

is exact then the following is exact:

$$\begin{aligned}
\cdots \xrightarrow{\delta} H^p(\mathcal{A}) \to H^p(\mathcal{B}) &\to H^p(\mathcal{C}) \\
&\xrightarrow{\delta} H^{p+1}(\mathcal{A}) \to H^{p+1}(\mathcal{B}) \to H^{p+1}(\mathcal{C}) \\
&\xrightarrow{\delta} \cdots .
\end{aligned}$$

Theorem 5.1, helps us prove that Mayer-Vietoris Sequence is exact, which is useful for the proof (see Theorem 5.4).

Suppose M is a smooth manifold, let U and V be open subsets of M such that $U \cup V = M$, we have the following inclusions,

$$\begin{tikzcd}
 & U \arrow[dr, "k"] & \\
U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\
 & V \arrow[ur, "\ell"'] &
\end{tikzcd}$$

which then we induce pullback maps on differential forms,

$$\begin{tikzcd}
 & \Omega^p(U) \arrow[dl, "i^*"'] & \\
\Omega^p(U \cap V) & & \Omega^p(M). \arrow[ul, "k^*"'] \arrow[dl, "\ell^*"] \\
 & \Omega^p(V) \arrow[ul, "j^*"] &
\end{tikzcd}$$

Note that these pullback are in fact just restrictions for example, if we take a $\omega \in \Omega^p(M)$ and apply the ℓ^* map we have $\ell^*(\omega) = \omega \mid_V$, meaning ω restricted to V. Same for if have the same ω and apply the map $k^* \oplus \ell^*$ we have $(k^* \oplus \ell^*)\omega = (k^*\omega, \ell^*\omega) = (\omega \mid_U, \omega \mid_V)$ as well if we take a $(\omega, \eta) \in \Omega^p(U) \oplus \Omega^p(V)$ and apply the $i^* - j^*$ we have $(i^* - j^*)(\omega, \eta) =$

$\omega\mid_{U\cap V} - \eta\mid_{U\cap V}$. This can also be applied to the induced cohomology maps,

$$\begin{tikzcd}[column sep=small] & H^p(U) \arrow[dl, "i^*"'] & \\ H^p(U\cap V) & & H^p(M). \arrow[ul, "k^*"'] \arrow[dl, "\ell^*"] \\ & H^p(V) \arrow[ul, "j^*"] & \end{tikzcd}$$

Lemma 5.2 (Zig-Zag Lemma). *[Lee13] Given a short exact sequence of complexes,*

$$0 \to \mathcal{A}^* \to \mathcal{B}^* \to \mathcal{C}^* \to 0$$

as in Theorem 5.1, for each p there is a linear map δ which is defined as:

$$\delta : H^p(\mathcal{C}^*) \to H^{p+1}(\mathcal{A}^*),$$

this is called the connecting homomorphism, such that the following sequence is exact:

$$\cdots \xrightarrow{\delta} H^p(\mathcal{A}^*) \xrightarrow{F^*} H^p(\mathcal{B}^*) \xrightarrow{G^*} H^p(\mathcal{C}^*) \xrightarrow{\delta} H^{p+1}(\mathcal{A}^*) \xrightarrow{F^*} \cdots.$$

The Zig-Zag Lemma (Lemma 5.2) will be useful for the proof of the Mayer-Vietoris Sequence, Theorem 5.4, below. Before we dive into the proof of the Mayer Vietoris Sequence it will be helpful to know what partition of unity is.

Theorem 5.3. *Let us suppose $\{U_\alpha\}$ is an open cover of the manifold M. There exists a partition of unity subordinate to $\{U_\alpha\}$: this is a collection of smooth functions $\varphi_\alpha : M \to \mathbb{R}$ and the following properties hold;*

1. *For all $x \in M$, only a finite number of φ_α are nonzero at x.*
2. *The support of φ_α is contained in U_α, meaning supp$\varphi_\alpha \subseteq U_\alpha$.*
3. *For any $x \in M, \sum_\alpha \varphi_\alpha(x) = 1$.*

Theorem 5.4 (Mayer-Vietoris Sequence)**.** *[Lee13] Let M be smooth manifold with or without boundary, and let U, V be open subsets of M whose union is M. For each p, there is a linear map $\delta : H^p(U \cap V) \to H^{p+1}(M)$ such that the following sequence, called the Mayer-Vietoris sequence for the open cover $\{U, V\}$, is exact:*

$$\cdots \xrightarrow{\delta} H^p(M) \xrightarrow{k^* \oplus \ell^*} H^p(U) \oplus H^p(V) \xrightarrow{i^* - j^*} H^p(U \cap V)$$
$$\xrightarrow{\delta} H^{p+1}(M) \xrightarrow{k^* \oplus \ell^*} \cdots .$$

Proof. In this proof we will want to use the Zig-Zag Lemma (Lemma 5.2) to help us establish the Mayer-Vietoris Sequence. In the Zig-Zag Lemma we have a short exact sequence of chain complexes,

$$\cdots \xrightarrow{\delta} H^p(\mathcal{A}^*) \xrightarrow{F^*} H^p(\mathcal{B}^*) \xrightarrow{G^*} H^p(\mathcal{C}^*) \xrightarrow{\delta} H^{p+1}(\mathcal{A}^*) \xrightarrow{F^*} \cdots .$$

Let us define the following:

$$\mathcal{A}^* = \cdots \to \mathcal{A}^p \to \mathcal{A}^{p+1} \to \cdots$$
$$\mathcal{B}^* = \cdots \to \mathcal{B}^p \to \mathcal{B}^{p+1} \to \cdots$$
$$\mathcal{C}^* = \cdots \to \mathcal{C}^p \to \mathcal{C}^{p+1} \to \cdots$$

where for any p,

$$0 \to \mathcal{A}^p \to \mathcal{B}^p \to \mathcal{C}^p \to 0$$

is a short exact sequence. We want to show the following is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}^p \arrow[r, "k^* \oplus \ell^*"] \arrow[d, "d"] \arrow[dr, phantom, "\circ"] & \mathcal{B}^p \arrow[r, "i^* - j^*"] \arrow[d, "d \oplus d"] \arrow[dr, phantom, "\circ"] & \mathcal{C}^p \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \mathcal{A}^{p+1} \arrow[r, "k^* \oplus \ell^*"] & \mathcal{B}^{p+1} \arrow[r, "i^* - j^*"] & \mathcal{C}^{p+1} \arrow[r] & 0.
\end{tikzcd} \quad (5.1)$$

Let $\mathcal{A}^p = \Omega^p(M), \mathcal{B}^p = \Omega^p(U) \oplus \Omega^p(V), \mathcal{C}^p = \Omega^p(U \cap V)$. Consider the following sequence,

$$0 \to \Omega^p(M) \xrightarrow{k^* \oplus \ell^*} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{i^* - j^*} \Omega^p(U \cap V) \to 0. \quad (5.2)$$

Suppose this above sequence, in Equation 5.2, is a short exact sequence. Take a $\omega \in \Omega^p(M)$, if we first apply the map $k^* \oplus \ell^*$ to ω then apply $d \oplus d$ to the output we should be able to get the same result if we first do $d\omega$ then apply the map $k^* \oplus \ell^*$,

$$\omega \xrightarrow{k^* \oplus \ell^*} (\omega|_U, \omega|_V) \xrightarrow{d \oplus d} (d\omega|_U, d\omega|_V).$$

Now let us apply d then the map $k^* \oplus \ell^*$,

$$\omega \xrightarrow{d} d\omega \xrightarrow{k^* \oplus \ell^*} (k^* \oplus \ell^*)(d\omega).$$

The two outputs we found above are equal,

$$\begin{aligned}(d(\omega\mid_U), d(\omega\mid_V)) &= (d(k^*\omega), d(\ell^*\omega)) \\ &= (k^*(d\omega), \ell^*(d\omega)) \\ &= (k^* \oplus \ell^*)(d\omega).\end{aligned}$$

This shows that the first diagram in Equation 5.1 is a commutative one. Now we want to show the rest of the diagram in Equation 5.1 is commutative. Meaning if we take $(\omega, \tau) \in \Omega^p(U) \oplus \Omega^p(V)$ and fist apply the $i^* - j^*$ map then apply the operator d to the output we should get the same result if we fist apply the operator $d \oplus d$ to (ω, τ) then apply the $i^* - j^*$ map,

$$(\omega, \tau) \xrightarrow{i^* - j^*} i^*(\omega) - j^*(\tau) \xrightarrow{d} d(i^*(\omega)) - d(j^*(\tau)),$$

now let us apply $d \oplus d$ then the map $i^* - j^*$,

$$(\omega, \tau) \xrightarrow{d \oplus d} (d\omega, d\tau) \xrightarrow{i^* - j^*} i^*(d\omega) - j^*(d\tau).$$

The two outputs we found above are equal because the operator d commutes with pull-back, by Proposition 2.6 property (iv) we have,

$$i^*(d\omega) - j^*(d\tau) = d(i^*(\omega)) - d(j^*(\tau)).$$

It has now been shown that the whole diagram in Equation 5.1 is a commutative one. Now all that is left is to show that the Sequence 5.2 is a short exact sequence. We will be able to show that Sequence 5.2 is a short exact sequence by showing the following:

1. Show the map $k^* \oplus \ell^*$ is an injective map.
2. Show the $im(k^* \oplus \ell^*) = \ker(i^* - j^*)$.
3. Show the map $i^* - j^*$ is a surjective map.

For us to show that the map $k^* \oplus \ell^*$ is an injective map we want to show that the $\ker(k^* \oplus \ell^*) = 0$. For this let us start with $\omega \in \Omega^p(M)$ and apply the map $k^* \oplus \ell^*$,

$$\begin{aligned}(k^* \oplus \ell^*)(\omega) = 0 \\ \Rightarrow (k^*\omega, \ell^*\omega) = 0 \\ \Rightarrow (\omega \mid_U, \omega \mid_V) = 0.\end{aligned}$$

For $(\omega \mid_U, \omega \mid_V) = 0$, this would mean that both $\omega \mid_U$ and $\omega \mid_V$ must be equal to zero, which clearly ω must be zero as well since $\omega \in M$ and $M = U \cup V$, it must then be the case that the map $k^* \oplus \ell^*$ is an injective map.

For us to show that the $im(k^* \oplus \ell^*) = \ker(i^* - j^*)$ this will be a simple double inclusion argument. We will show that,

1. $im(k^* \oplus \ell^*) \subseteq \ker(i^* - j^*)$.
2. $\ker(i^* - j^*) \subseteq im(k^* \oplus \ell^*)$.

We will first start with showing $im(k^* \oplus \ell^*) \subseteq \ker(i^* - j^*)$. We know for something to be in the kernel of a map that means the map will send it to zero. Take a $\omega \in \Omega^p(M)$, now apply the $k^* \oplus \ell^*$ map, $(k^* \oplus \ell^*)(\omega) = (\omega \mid_U, \omega \mid_V)$, so $(\omega \mid_U, \omega \mid_V) \in im(k^* \oplus \ell^*)$. Now we want to see if $(\omega \mid_U, \omega \mid_V)$ is in the $\ker(i^* - j^*)$:

$$\begin{aligned}(i^* - j^*)(\omega \mid_U, \omega \mid_V) &= i^*(\omega \mid_U) - j^*(\omega \mid_V) \\ &= \omega \mid_{U\cap V} - \omega \mid_{U\cap V} \\ &= 0.\end{aligned}$$

We have just shown the first inclusion argument $im(k^* \oplus \ell^*) \subseteq \ker(i^* - j^*)$. Now we want to show $\ker(i^* - j^*) \subseteq im(k^* \oplus \ell^*)$. Suppose $(\eta, \eta') \in \ker(i^* - j^*)$, which means we have $(i^* - j^*)(\eta, \eta') = \eta \mid_{U\cap V} - \eta' \mid_{U\cap V} = 0$. Since $\eta \mid_{U\cap V} - \eta' \mid_{U\cap V} = 0$, we know η and η' agree on $U \cap V$, so let us define the following, which is smooth since η and η' are smooth:

$$\omega = \begin{cases} \eta & \text{in } U \\ \eta' & \text{in } V, \end{cases}$$

$(k^* \oplus \ell^*)(\omega) = (\omega \mid_U, \omega \mid_V) = (\eta, \eta') \in im(k^* \oplus \ell^*)$, which now we have shown $\ker(i^* - j^*) \subseteq im(k^* \oplus \ell^*)$. Now we have completed our double inclusion argument thus $\ker(i^* - j^*) = im(k^* \oplus \ell^*)$.

Now the last thing we must show is that the map $i^* - j^*$ is a surjective map. So let us consider $\omega \in \Omega^p(U \cap V)$, and we want to find (η, η') such that $(i^* - j^*)(\eta, \eta') = \omega$. Let $\{\varphi, \psi\}$ be a partition of unity subordinate to $\{U, V\}$. Now we want to define,

$$\eta = \begin{cases} \psi\omega & \text{on } U \cap V \\ 0 & \text{on } U - supp(\psi) \end{cases} \qquad \eta' = \begin{cases} -\varphi\omega & \text{on } U \cap V \\ 0 & \text{on } V - supp(\varphi). \end{cases}$$

We have $\eta \in \Omega^p(U)$ and $\eta' \in \Omega^p(V)$,

$$\begin{aligned} (i^* - j^*)(\eta, \eta') &= \eta\mid_{U\cap V} - \eta'\mid_{U\cap V} \\ &= \psi\omega - (-\varphi\omega) \\ &= \psi\omega + \varphi\omega \\ &= (\psi + \varphi)\omega. \end{aligned}$$

So since ψ and φ make up a partition of unity, $\psi + \varphi = 1$, by the third property of partition of unity (see Theorem 5.3), so in the end of the calculation of $(i^* - j^*)(\eta, \eta')$ we end up being left with just ω, which is exactly what we wanted to show so the map $i^* - j^*$ is surjective. Now we have a short exact sequence of complexes so we can apply the Zig-Zag Lemma (see Lemma 5.2), and with that we arrive at the conclusion that the following sequence is exact,

$$\begin{aligned} \cdots \xrightarrow{\delta} H^p(M) \xrightarrow{k^* \oplus \ell^*} H^p(U) \oplus H^p(V) &\xrightarrow{i^* - j^*} H^p(U \cap V) \\ &\xrightarrow{\delta} H^{p+1}(M) \xrightarrow{k^* \oplus \ell^*} \cdots . \end{aligned}$$

□

5.1 Example of Using the Mayer-Vietoris Sequence

Here is an example that uses the Mayer-Vietoris sequence to compute the de Rham cohomology of groups of spheres.

In the following example we will be working with the unit sphere, which is given by, for $n \geq 0$, the unit n-sphere is the subset $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ defined by

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

Let N and S denote the north and south poles respectively in $\mathbb{S}^n$, let $U = \mathbb{S}^n \setminus \{S\}$, $V = \mathbb{S}^n \setminus \{N\}$, also note that $\mathbb{S}^n = U \cup V$.

The following proposition is helpful for the base case in the following theorem, cohomology of spheres (see Theorem 5.6), the computation of $H^1(\mathbb{S}^1)$ is interesting on its own and shows how cohomology is hard to compute.

Proposition 5.5. *$H^0(\mathbb{S}^1) \cong \mathbb{R} \cong H^1(\mathbb{S}^1)$ and $H^p(\mathbb{S}^1) \cong 0$ for $p \geq 2$.*

Proof. We know that $H^0(\mathbb{S}^1) \cong \mathbb{R}$ by Proposition 3.4, since $\mathbb{S}^1$ is path connected. We also know that $H^p(\mathbb{S}^1) \cong 0$ for $p \geq 2$, by the earlier discussion in Chapter 4.1. All that is now left for us to show is that $H^1(\mathbb{S}^1) \cong \mathbb{R}$. Identify $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{Z}$. We have the following:

$$\Omega^0(\mathbb{S}^1) = \{2\pi\text{-periodic, } f : \mathbb{R} \to \mathbb{R}\}$$
$$\Omega^1(\mathbb{S}^1) = \{f(x)dx, f \text{ is } 2\pi\text{-periodic}\},$$

above the functions we have are in fact smooth functions. Now define a map T to be the following,

$$T = \Omega^1(\mathbb{S}^1) \to \mathbb{R}$$

as $T(\omega) = \displaystyle\int_0^{2\pi} \omega$ more specifically, $T(f(x)dx) = \displaystyle\int_0^{2\pi} f(x)dx$.

We will show that T is onto, as well as showing that $\ker T = imd$. Notice that $\ker d : \Omega^1(\mathbb{S}^1) \to \Omega^2(\mathbb{S}^1) = 0$, so then the $\ker d = \Omega^1(\mathbb{S}^1)$. Since $\ker d = \Omega^1(\mathbb{S}^1)$, it would then follow by the First Isomorphism Theorem that

$$H^1(\mathbb{S}^1) = \frac{\ker d}{imd} = \frac{\Omega^1(\mathbb{S}^1)}{\ker T} \cong \mathbb{R}.$$

The map T is onto since $T(kdx) = \displaystyle\int_0^{2\pi} kdx = 2\pi k = \alpha$, where $k = \dfrac{\alpha}{2\pi}$ is a constant.

Now all that is left for us to show is that the $imd = \ker T$, we will do this by a double inclusion argument, first it will be shown that $imd \subseteq \ker T$, then it will be shown that $\ker T \subseteq imd$.

Let $\omega \in imd : \Omega^0(\mathbb{S}^1) \to \Omega^1(\mathbb{S}^1)$. Which then we have, $\omega = dg = g'(x)dx$, so

$$T(\omega) = \int_0^{2\pi} g'(x)dx = g(2\pi) - g(0) = 0$$

since g is 2π-periodic. Thus we have $imd \subseteq \ker T$.

Now suppose that $f dx \in \ker T$, we will want to show $f = g'$ for some g, meaning $f dx = g' dx = dg$. We want to define $g(x) = \int_0^x f(t)dt$, we know that with this $g' = f$. So we want to show that $g(x) = g(x + 2\pi)$ if $\int_0^{2\pi} f(t)dt = 0$.

$$\begin{aligned} g(x+2\pi) &= \int_0^{x+2\pi} f(t)dt \\ &= \int_0^x f(t)dt + \int_x^{x+2\pi} f(t)dt \\ &= g(x) + 0 \\ &= g(x). \end{aligned}$$

We finish the proof by proving that $\int_x^{x+2\pi} f(t)dt = 0$ for any x by the following argument, if f is 2π-periodic, then

$$\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt.$$

First we show that when f is $2\pi-$periodic, for any a and b we have

$$\int_a^b f(t)dt = \int_{a+2\pi}^{b+2\pi} f(t)dt. \tag{5.3}$$

In the above Equation 5.3, make the substitution $u = t + 2\pi$. We have $du = dt$, and when $t = a$ or b, we have $u = a + 2\pi$ or $b + 2\pi$, respectively. Then

$$\int_a^b f(t)dt = \int_{a+2\pi}^{b+2\pi} f(u-2\pi)dt = \int_{a+2\pi}^{b+2\pi} f(u)dt,$$

where the last equality follows since f is $2\pi-$periodic.

Now, for any x, there exists an integer multiple of 2π between x and $x + 2\pi$, say, for the integer k we have

$$x \le 2\pi k < x + 2\pi.$$

Then using the above for $a = x$ and $b = 2\pi k$ and subsequently combining the integrals, we have

$$\begin{aligned} \int_x^{x+2\pi} f(t)dt &= \int_x^{2\pi k} f(t)dt + \int_{2\pi k}^{x+2\pi} f(t)dt \\ &= \int_{x+2\pi}^{2\pi k+2\pi} f(t)dt + \int_{2\pi k}^{x+2\pi} f(t)dt \\ &= \int_{2\pi k}^{2\pi k+2\pi} f(t)dt. \end{aligned}$$

Now apply the above fact repeatedly for $a = 2\pi k$ and $b = 2\pi k + 2\pi$ to find that

$$\int_{2\pi k}^{2\pi k+2\pi} f(t)dt = \cdots = \int_0^{2\pi} f(t)dt.$$

So since we already know that $\int_0^{2\pi} f(t)dt = 0$, we then now know that $\int_x^{x+2\pi} f(t)dt = 0$. Therefore we know that $\ker T \subseteq imd$, and now we have $imd = \ker T$, and we know that $H^1(\mathbb{S}^1) \cong \mathbb{R}$. $\square$

Now we have all necessary information to compute the cohomology of spheres. The work done in the previous proposition, Proposition 5.5, is in fact our base case in Theorem 5.6.

Theorem 5.6 (Cohomology of Spheres)**.** *[Lee13] For $n \geq 1$, the de Rham cohomology groups of $\mathbb{S}^n$ are*

$$H^p(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } p = n, \\ 0 & \text{if } 0 < p < n \text{ or } p > n. \end{cases}$$

Proof. We will prove this by induction.

Base case: When $n = 1$ the following is true, $H^0(\mathbb{S}^1) = \mathbb{R}$, $H^1(\mathbb{S}^1) = \mathbb{R}$ and $H^p(\mathbb{S}^1) = 0$ for $p \neq 0, 1$, see Proposition 5.5.

Induction Step: Assume the following is true for $n \geq 1$,

$$H^p(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } p = n, \\ 0 & \text{if } 0 < p < n \text{ or } p > n. \end{cases}$$

We want to show this is true for $n + 1$, meaning,

$$H^p(\mathbb{S}^{n+1}) \cong \begin{cases} \mathbb{R} & \text{if } p = 0, n+1 \\ 0 & \text{if } 0 < p < n+1, p > n+1. \end{cases}$$

Recall that $\mathbb{S}^n$ is the n-unit sphere, for $n \geq 0$ which is defined by $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, also recall that N and S denote the north and south poles where U and V are defined as, $U = \mathbb{S}^n \setminus \{S\}$ and $V = \mathbb{S}^n \setminus \{N\}$. Note that $U \cap V$ deformation retracts to $\mathbb{S}^n$, so $H^p(U \cap V) \cong H^p(\mathbb{S}^p)$. For $p \geq 1$, let us write out the relevant portion of the Mayer Vietoris Sequence,

$$H^p(U) \oplus H^p(V) \to H^p(\mathbb{S}^n) \to H^{p+1}(\mathbb{S}^{n+1}) \to H^{p+1}(U) \oplus H^{p+1}(V).$$

We know that $H^p(U) \oplus H^p(V) = 0$ as well as $H^{p+1}(U) \oplus H^{p+1}(V) = 0$ since U and V are contractible to a point. So our sequence is now

$$0 \to H^p(\mathbb{S}^n) \to H^{p+1}(\mathbb{S}^{n+1}) \to 0.$$

Next, we know that $H^p(\mathbb{S}^n) \cong H^{p+1}(\mathbb{S}^{n+1})$ via the connecting homomorphism δ in the Zig-Zag Lemma (see Lemma 5.2). For $1 \leq p < n$, we have $H^{p+1}(\mathbb{S}^{n+1}) = 0$ since $H^p(\mathbb{S}^n) = 0$. For $p = n$ we have $H^{p+1}(\mathbb{S}^{n+1}) = \mathbb{R}$ since, by assumption, $H^p(\mathbb{S}^n) = \mathbb{R}$.

We have proven that,

$$H^k(\mathbb{S}^{n+1}) \cong \begin{cases} 0 & \text{for } k = 2, 3, \cdots, n \text{ and } k > n+1 \\ \mathbb{R} & \text{for } k = n+1. \end{cases}$$

We know that $H^0(\mathbb{S}^{n+1}) = \mathbb{R}$, so the only part that we need to show is $H^1(\mathbb{S}^{n+1}) = 0$. Let us write out the relevant portion of the Mayer-Vietoris Sequence,

$$0 \to H^0(\mathbb{S}^{n+1}) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(\mathbb{S}^{n+1}) \to H^1(U) \oplus H^1(V).$$

We know that $H^1(U) \oplus H^1(V) = 0$, again, since U and V are contractible:

$$0 \to H^0(\mathbb{S}^{n+1}) \to H^0(U) \oplus H^0(V) \xrightarrow{i^* - j^*} H^0(U \cap V) \xrightarrow{\delta} H^1(\mathbb{S}^{n+1}) \xrightarrow{f} 0.$$

We will show that the map $i^* - j^* : H^0(U) \oplus H^0(V) \to H^0(U \cap V)$, is onto. Let $[\omega] \in H^0(U \cap V)$, where $d\omega = 0$. Since $\omega \in \Omega^0(U \cap V)$ is a function and $d\omega = 0$ then ω must the be constant function. So let $\omega = c$, where c is a constant function, have c be the map, $c : U \cap V \to \mathbb{R}$. Now define $c : U \to \mathbb{R}$. Then $i^*(c) = c$, so $(i^* - j^*)(c, 0) = c$. Now, $im(i^* - j^*) = H^0(U \cap V) = \ker \delta$, so δ must be the 0 map. Since δ is the 0 map the $im\delta = \ker f$ so f is injective.

We have $f : H^1(\mathbb{S}^{n+1}) \to 0$, which tells us that f is surjective. Now we have f being injective and surjective, so it must be the case that f is an isomorphism, implying $H^1(\mathbb{S}^{n+1}) = 0$. □

Chapter 6

Conclusion

In this thesis we have introduced the basic information needed to speak on the de Rham cohomology, homotopy invariance, and the Mayer-Vietoris Sequence. With that we, discussed precisely what the de Rham cohomology is and also provided some properties the de Rham cohomology. Then we discussed homotopy invariance, where we established the existence of a homotopy operator. We also established that the de Rham cohomology is in fact a homotopy invariant, then provided some examples using the homotopy invariance. Finally we established the Mayer-Vietoris sequence and provided an example that used the Mayer-Vietoris sequence to compute the de Rham cohomology of groups of spheres.

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