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An investigation of Kurosh's theorem

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An Investigation of Kurosh's Theorem

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by

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Abstract

An algebra $\mathcal{A}$ is a vector space over a base field $\mathcal{F}$ which is not necessarily commutative nor unital. Though an algebra may be generated by a finite number of elements, this does not necessarily imply that the algebra is finite-dimensional over $\mathcal{F}$. This study will be investigation of Alekander Kurosh's problem, in the attempt to establish the necessary hypotheses to ensure that a finitely generated algebra is finite-dimensional.

An algebra $\mathcal{A}$ is algebraic if for each $a \in \mathcal{A}$, $\alpha_n a^n + \alpha_{n-1} a^{n-1} + \cdots + \alpha_0 = 0$ for some $\alpha_i \in \mathcal{F}$ and $n > 0$. Additionally, an algebra is said to satisfy a polynomial identity if there exists an $f \in \mathcal{F}\langle x_1, \ldots, x_d \rangle$ such that $f(a_1, \ldots, a_d) = 0$ for every $a_1, \ldots, a_d \in \mathcal{A}$. In this study we will arrive at the conclusion that if $\mathcal{A}$ is finitely generated, algebraic and satisfies a polynomial identity, then $\mathcal{A}$ is finite-dimensional, providing a sufficient condition to the Kurosh Problem.

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Chapter 1

Introduction

In this work, an algebra $\mathcal{A}$ is a vector space over a base field $\mathcal{F}$ which has a bilinear associative multiplication and is not necessarily commutative nor unital. If every finitely generated subalgebra of $\mathcal{A}$ is finite-dimensional then $\mathcal{A}$ is **locally finite.** In the case that $\mathcal{A}$ is finite-dimensional it will naturally occur that $\mathcal{A}$ is locally finite, since every finitely generated subalgebra is a subspace of $\mathcal{A}$. In the case that $\mathcal{A}$ is infinite-dimensional, would $\mathcal{A}$ be locally finite? The answer to this question is no since the algebra $\mathcal{F}[x]$ of the polynomials in x is infinite-dimensional, but is generated as an algebra by $\{1, x\}$.

In 1902, William Burnside posed a group theoretic conjecture on whether a finite collection of elements of finite order generates a finite group. Alekander Kurosh, in 1941 [Ami74, p.2], posed an analogous question in terms of algebras to that of Burnside's in an attempt to add a hypothesis to guarantee the local finiteness of algebras. An algebra $\mathcal{A}$ is said to be **algebraic** if for each element $a \in \mathcal{A}$, the subalgebra generated by a is finite-dimensional; that is for some $n \geq 1$ and for some $\alpha_0, \ldots, \alpha_n \in \mathcal{F}$, $\alpha_n a^n + \alpha_{n-1} a^{n-1} + \cdots + \alpha_0 = 0$. The conjecture proposed by Kurosh can then be stated as follows:

Suppose that an algebraic algebra $\mathcal{A}$ has a finite number of algebra generators, is $\mathcal{A}$ locally finite?

Although Kurosh validated this conjecture in the specific case for algebraic algebras each whose elements satisfy a minimal polynomial of degree no greater than three, in 1963 E.S. Golod and I.R. Shafarevitch disproved Kurosh's conjecture by constructing an infinite-dimensional finitely generated algebraic algebra proving Kurosh's general conjecture in the negative. In order for $\mathcal{A}$ to be finite-dimensional an additional hypothesis is needed to limit the length of the words $x_{i_1}\cdots x_{i_n} \in \mathcal{F}\langle x_1,\ldots,x_d\rangle$, the free associative, non-commutative algebra in the indeterminates $x_1,\ldots,x_d$.

For example, if the condition of commutativity is imposed on the algebra $\mathcal{A}$ then the subalgebra generated by $\{a_1,a_2\} \subseteq \mathcal{A}$ will then be locally finite, since the degree of each word $a_1^{n_1}a_2^{m_1}\cdots a_1^{n_j}a_2^{m_j} = a_1^{n_1+\cdots+n_j}a_2^{m_1+\cdots+m_j}$ in the subalgebra is bounded by the degrees of the polynomials satisfied by a_1 and a_2 respectively. To see this process in detail, let a_1 and a_2 be **nil** with $a_1^r = 0$ and $a_2^s = 0$, then a_1 and a_2 are algebraic and $a_1^{n_1+\cdots+n_j}a_2^{m_1+\cdots+m_j} = a_1^n a_2^m$, for some $0 \le n < r$, $0 \le m < s$. Thus the typical element of the subalgebra generated by a_1 and a_2 is spanned by these finitely many $a_1^n a_2^m$, and so is finite-dimensional. The same iterative process may be used to show in the algebraic, but not necessarily nil case, that $\mathcal{A}$ is locally finite. Though every commutative nil algebra is locally finite, in the absence of commutativity and nil-potency we will need the inclusion of a **polynomial identity**.

Let $\mathcal{A}$ be a algebra, then $\mathcal{A}$ satisfies a polynomial identity (P.I.) if there exists some $f(x_1,\ldots,x_d) \in \mathcal{F}\langle x_1,\ldots,x_d\rangle$ such that $f(a_1,\ldots,a_d) = 0$ for every $a_1,\ldots,a_d \in \mathcal{A}$. Kurosh's theorem can now be stated as,

Suppose that an algebraic algebra $\mathcal{A}$ satisfies a polynomial identity, then $\mathcal{A}$ is locally finite.

In order to comprehend Kurosh's throem, we will need to investigate the notion of a module over a ring. A **module** $\mathcal{M}$ over a ring $\mathcal{R}$ ($\mathcal{R}$-Module) is a ring homomorphism $\mathcal{R} \to End(\mathcal{M})$, the ring of all endomorphisms of the abelian group $\mathcal{M}$. For a given $\mathcal{R}$-module $\mathcal{M}$, a specific ring homomorphism that will be frequently used in this thesis will be the map $\mathcal{R} \to \mathrm{End}(\mathcal{M})$ which sends $a \mapsto S_a$ where $(b)S_a = ba$ $(b \in \mathcal{M})$. Furthermore a module is **faithful** if the ring homomorphism is 1-1, and is **irreducible** if there does not exist any proper submodules of $\mathcal{M}$ other than $\{0\}$.

For example $\mathbb{Z}_7$ is a right (or left) $\mathbb{Z}$-module given by the ring homomorphism $\Phi\colon \mathbb{Z} \to End(\mathbb{Z}_7)$ where for $r \in \mathbb{Z}$, S_r is well-defined by $(x)S_r = xr$ $(x \in \mathbb{Z}_7)$. Since the only proper subgroup of $\mathbb{Z}_7$ is $\langle 0\rangle$, $\mathbb{Z}_7$ does not contain any proper submodules, hence it is an irreducible $\mathbb{Z}$-module. In addition $\mathbb{Z}_7$ is not faithful as a $\mathbb{Z}$-module since any two multiples of 7 produce the same image in $\mathrm{End}(\mathbb{Z}_7)$. In particular for $7, 14 \in \mathbb{Z}$, $(x)S_7 = 7x = 0 = 14x = (x)S_{14}$, thus Φ is not 1-1 since $\mathrm{Ker}(\Phi)$=$7\mathbb{Z}$. To correct this we may consider the ring $\mathbb{Z}/\mathrm{Ker}(\Phi) = \mathbb{Z}/7\mathbb{Z}$, and by the first isomorphism theorem of rings

$\Psi : \mathbb{Z}/7\mathbb{Z} \to End(\mathbb{Z}_7)$ is a 1-1 mapping. We have constructed a $\mathbb{Z}/7\mathbb{Z}$-module $\mathbb{Z}_7$ that is faithful and irreducible. Alternatively, the abelian group $\mathbb{Z}_6$ is neither a faithful nor irreducible $\mathbb{Z}$ module, since multiples of 6 will produce the same image under Φ, and it contains the proper subgroups and hence submodules $\langle 2 \rangle$ and $\langle 3 \rangle$.

In this project we will also need to present the commuting ring Δ of an $\mathcal{R}$-module $\mathcal{M}$, which is the subring of $End(\mathcal{M})$ consisting of all endomorphisms of $\mathcal{M}$ that commute with the endomorphisms S_a $(a \in \mathcal{R})$. We will then examine **Jacobson Density**, which is a generalization of all Δ-linear endomorphisms on $\mathcal{M}$, and the concept of primitive rings (rings having a faithful irreducible module). These ideas will be necessary to prove **Kaplansky's theorem** an important breakthrough in P.I. theory and in the development of Kurosh's theorem; it states that a primitive algebra that satisfies a P.I. is finite dimensional over it's center.

This masters project will be an exposition of the Kurosh Theorem and the necessary and sufficient condition that $\mathcal{A}$ must be algebraic and satisfy a P.I. to be locally finite.

Chapter 2

$\mathcal{R}$ -Modules & Schur's Lemma

2.1 $\mathcal{R}$-modules

In this chapter we will introduce the concept of a module over a ring, which will be referred to as an $\mathcal{R}$-module. Generally an $\mathcal{R}$-module is a vector space over a ring. Formally an $\mathcal{R}$-module may be described using representation theory (Chapter 1) but it will be beneficial to the reader for an axiomatic description.

Definition 2.1. Let $\mathcal{R}$ be a ring and $\mathcal{M}$ an additive abelian group. Then $\mathcal{M}$ is a *right $\mathcal{R}$-module* if there is a map $\mathcal{M} \times \mathcal{R} \longrightarrow \mathcal{M}$, sending $(m, r) \mapsto mr$ and for which the following holds for all $m \in \mathcal{M}$ and $r \in \mathcal{R}$,

1. $(m_1 + m_2)r = m_1 r + m_2 r$
2. $m(r_1 + r_2) = mr_1 + mr_2$
3. $(mr_1)r_2 = m(r_1 r_2)$.

Though the rings that we consider do not necessarily have unity, an $\mathcal{R}$-module $\mathcal{M}$ is **unital** if there exists $1 \in \mathcal{R}$ such that $m(1) = m$ for every $m \in \mathcal{M}$. Additionally we could define a left $\mathcal{R}$-module by allowing the ring elements to act on the group elements on the left, but throughout this study an $\mathcal{R}$-module will simply be a right $\mathcal{R}$-module.

Example 2.2. $6\mathbb{Z}$ is a right $\mathbb{Z}$ module, $6\mathbb{Z} \times \mathbb{Z} \longrightarrow 6\mathbb{Z}$ with the action defined as $(x + 6\mathbb{Z})y = xy + 6\mathbb{Z} \quad (x, y \in \mathbb{Z})$. Since $6\mathbb{Z}$ is a two-sided ideal of $\mathbb{Z}$, the properties of $6\mathbb{Z}$ as a $\mathbb{Z}$-module are satisfied.

In the context of a vector space $\mathcal{V}$, a subspace of $\mathcal{V}$ is a subset which is a vector space itself under the operations of $\mathcal{V}$. It will be natural for us to define an analogous concept in terms of modules.

Definition 2.3. A *submodule* $\mathcal{N}$ of $\mathcal{M}$ is an abelian subgroup of $\mathcal{M}$ which is closed under scalar multiplication: if $x \in \mathcal{N}$, $r \in \mathcal{R}$ then $xr \in \mathcal{N}$.

Example 2.4. $\mathcal{R}$ itself is a right $\mathcal{R}$-module with the action defined as usual ring multiplication. In addition any right ideal ideal $\mathcal{U}$ of $\mathcal{R}$ is a submodule of $\mathcal{R}$.

We will often denote the action of $\mathcal{R}$ on $\mathcal{M}$ by multiplication. That is $\mathcal{MR} = \{mr \mid m \in \mathcal{M},\ r \in \mathcal{R}\}$. This will serve to remove any confusion that might arise by implementing function notation.

Proposition 2.5. *Let $\mathcal{M}$ be an $\mathcal{R}$-module then for a fixed $m \in \mathcal{M}$, $m\mathcal{R}$ is a submodule of $\mathcal{M}$.*

Proof. To verify this we will first show that $m\mathcal{R}$ is a subgroup of $\mathcal{M}$. $\mathcal{M}$ is an $\mathcal{R}$-module, hence $m\mathcal{R} \subseteq \mathcal{M}$. For $mr_1, mr_2 \in m\mathcal{R}$ we have $mr_1 - mr_2 = m(r_1 - r_2) \in m\mathcal{R}$. By the standard subgroup test $m\mathcal{R}$ is a subgroup of $\mathcal{M}$. To see that it preserves multiplication by ring elements, it follows from the fact that $\mathcal{M}$ is an $\mathcal{R}$-module that for $r_1 \in \mathcal{R}$, $(mr_1)r_2 = m(r_1r_2) \in m\mathcal{R}$. □

Example 2.6. Let $\mathcal{N}$ be a submodule of $\mathcal{M}$. The quotient module $\mathcal{M}/\mathcal{N}$ is an $\mathcal{R}$-module by defining the action $(\mathcal{N} + m)r = \mathcal{N} + mr$ for every $r \in \mathcal{R}$, $m \in \mathcal{M}$.

Proposition 2.7. *If $\mathcal{U}$ is an ideal of $\mathcal{R}$ then the submodules of $\mathcal{M}$ as an $\mathcal{R}$-module correspond to the submodules of $\mathcal{M}$ as a $\mathcal{R}/\mathcal{U}$-module.*

We will defer the reader to [Jac09, p.3] for the proof of this proposition.

Definition 2.8. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{R}$-modules then the mapping $\Psi : \mathcal{M} \to \mathcal{N}$ is a *module homomorphism* if and only if for $m_1, m_2 \in \mathcal{M}$, $r \in \mathcal{R}$

1. $(m_1 + m_2)\Psi = (m_1)\Psi + (m_2)\Psi$

2. $(m_1 r)\Psi = (m_1)\Psi r$.

Definition 2.9. A module $\mathcal{M}$ is *irreducible* if the action on $\mathcal{M}$ by $\mathcal{R}$ is non-trivial ($\mathcal{MR} \neq \{0\}$) and if the only submodules of $\mathcal{M}$ are $\{0\}$ and $\mathcal{M}$.

Definition 2.10. An ideal I of $\mathcal{R}$ is *maximal* if for every ideal S of $\mathcal{R}$ such that $I \subset S \subseteq \mathcal{R}$, then $S = \mathcal{R}$.

The next result provides a family of examples of irreducible modules.

Proposition 2.11. *Let I be a maximal ideal of $\mathcal{R}$ then $\mathcal{R}/I$ is an irreducible $\mathcal{R}$-module.*

Proof. Let I be a maximal right ideal of $\mathcal{R}$. Since $\mathcal{R}/I$ is an $\mathcal{R}$-module, it suffices to show that $\mathcal{R}/I$ contains no non-zero proper submodules. If $\mathcal{N}$ is a non-zero proper submodule of $\mathcal{R}/I$, then the preimage of $\mathcal{N}$ under the map $v\colon \mathcal{R} \to \mathcal{R}\ /I$ is a right ideal of $\mathcal{R}$. Furthermore this right ideal is not equal to $\mathcal{R}$ and properly contains I. A contradiction, since I is maximal. Hence $\mathcal{N} = \{0\}$. □

Example 2.12. For any prime p, $\mathbb{Z}/p\mathbb{Z}$ is an irreducible $\mathbb{Z}$-module.

Example 2.13. For a field $\mathcal{F}$, let $q(x)$ be an irreducible polynomial in $\mathcal{F}[x]$. Define the principle ideal of $\mathcal{F}[x]$ generated by $q(x)$ to be $\langle q(x)\rangle = \{f(x)q(x) \mid f(x) \in \mathcal{F}[x]\}$, then $\mathcal{F}[x]/\langle q(x)\rangle$ is an irreducible $\mathcal{F}[x]$-module.

Proposition 2.14. *Let $\mathcal{M}$ be an irreducible $\mathcal{R}$-module, then for every non-zero $m \in \mathcal{M}$, $m\mathcal{R} = \mathcal{M}$.*

Proof. Let $m \neq 0$ be an element of $\mathcal{M}$. Then $m\mathcal{R}$ and $\mathcal{N} = \{x \in \mathcal{M} \mid x\mathcal{R} = 0\}$ are both submodules of $\mathcal{M}$. From irreducibility, both submodules are either 0 or $\mathcal{M}$. We see that $\mathcal{N} = \{0\}$, otherwise $\mathcal{N}\mathcal{R} = \{0\} = \mathcal{M}\mathcal{R}$ which implies that the action is trivial. As a result $\mathcal{N} = \{0\}$ and since $m \neq 0$, we have that $m\mathcal{R} \neq \{0\}$. We see that $m\mathcal{R} = \mathcal{M}$. □

This result will be often referred to in Chapter 4.

For an $\mathcal{R}$-module $\mathcal{M}$ if $\mathcal{M}r = 0$, it does not necessarily imply that $r = 0$. In Example 2.2 for $6 \in \mathbb{Z}$ and for every $x \in \mathbb{Z}$ we have $(x + 6\mathbb{Z})6 = x6 + 6\mathbb{Z} = 0 + 6\mathbb{Z}$. In Example 2.13, any polynomials $g, h \in \mathcal{F}[x]$ with $g(x) = h(x)q(x)$, $g(x)$ will be mapped to 0 under the action on $\mathcal{F}[x]/\langle q(x)\rangle$ by $\mathcal{F}[x]$. We will look to classify these ring elements that annihilate the module by the given action.

Definition 2.15. $Ann(\mathcal{M}) = \{r \in \mathcal{R} \mid \mathcal{M}r = \{0\}\}$. Furthermore, a module $\mathcal{M}$ is *faithful* if $Ann(\mathcal{M}) = \{0\}$.

We will leave it to the reader to verify that $Ann(\mathcal{M})$ is a two-sided ideal of $\mathcal{R}$.

Proposition 2.16. *$\mathcal{M}$ is a faithful $\mathcal{R}/Ann(\mathcal{M})$-module.*

Proof. Consider $\Phi : \mathcal{M} \times \mathcal{R}/Ann(\mathcal{M}) \longrightarrow \mathcal{M}$, $(m, r + Ann(\mathcal{M})) \mapsto m(r + Ann(\mathcal{M})) = mr + Ann(\mathcal{M})$. With the presence of cosets we must first show that this mapping is well-defined. To see this suppose $(m, r + Ann(\mathcal{M})) = (m, r' + Ann(\mathcal{M}))$. Since $r + Ann(\mathcal{M}) = r' + Ann(\mathcal{M})$, we have $r - r' \in Ann(\mathcal{M})$. In particular, $m(r-r') + Ann(\mathcal{M}) = 0 + Ann(\mathcal{M})$. From this we obtain the desired result $mr + Ann(\mathcal{M}) = mr' + Ann(\mathcal{M})$. One can verify that the defined action preserves the necessary properties of an $\mathcal{R}/Ann(\mathcal{M})$-module.

We will proceed to show that Φ is faithful by proving that only the zero element in $\mathcal{R}/Ann(\mathcal{M})$ annihilates $\mathcal{M}$. Let $m(r + Ann(\mathcal{M})) = 0$ for every $m \in \mathcal{M}$. From the definition of $Ann(\mathcal{M})$, we see that $mr = 0$ for every $m \in \mathcal{M}$. This places $r \in Ann(\mathcal{M})$. We have the showed that $r + Ann(\mathcal{M}) = 0 + Ann(\mathcal{M})$. So $\mathcal{M}$ is a faithful $\mathcal{R}/Ann(\mathcal{M})$-module. □

2.2 Module Representation

Proposition 2.17. *1) Let $\mathcal{M}$ be an $\mathcal{R}$-module then there exists a ring homomorphism $\mathcal{R} \to End(\mathcal{M})$, the ring of all endomorphisms of $\mathcal{M}$. 2) Let $\mathcal{M}$ be an abelian group and let $\Phi : \mathcal{R} \to End(\mathcal{M})$ be a ring homomorphism. Then $\mathcal{M}$ is an $\mathcal{R}$-module.*

Proof. (1) For every $r \in \mathcal{R}$ let $S_r : \mathcal{M} \to \mathcal{M}$ with the evaluation $(m)S_r = mr$. Since $\mathcal{M}$ is an $\mathcal{R}$-module we see that for $m_1, m_2 \in \mathcal{M}$,

$$\begin{aligned}(m_1 + m_2)S_r &= (m_1 + m_2)r \\ &= m_1 r + m_2 r \\ &= (m_1)S_r + (m_2)S_r,\end{aligned}$$

so S_r is an endomorphism of the abelian group $\mathcal{M}$. $End(\mathcal{M})$ is a ring with respect to the binary operations of addition and multiplication given by: for $\phi, \psi \in End(\mathcal{M})$ we define addition as $(m)(\phi + \psi) = (m)\phi + (m)\psi$, and multiplication as $(m)(\psi\phi) = [(m)\psi]\phi$.

Let $\Phi : \mathcal{R} \to End(\mathcal{M})$ with $(r)\Phi = S_r$. Then for $r, s \in \mathcal{R}$ and $m \in \mathcal{M}$, $(m)S_{r+s} = (m)S_r + (m)S_s$ which results in $(r+s)\Phi = (r)\Phi + (s)\Phi$. In addition we see that $(m)S_{rs} = (m)(rs) = (mr)s = [(m)S_r]S_s$. Thus $(rs)\Phi = (r)\Phi(s)\Phi$. We see that Φ is a homomorphism.

(2) Define the map $\mathcal{M} \times \mathcal{R} \to \mathcal{M}$ with $(m, r) \mapsto (m)\Phi_r$. Where the evaluation at r of the endomorphism, $(r)\Phi$ is Φ_r. By applying the fact that Φ_r is an endomorphism of $\mathcal{M}$ we see that the axioms of $\mathcal{M}$ as a $\mathcal{R}$-module are satisfied.

1. $(m_1 + m_2)\Phi_r = (m_1)\Phi_r + (m_2)\Phi_r$
2. $(m)\Phi_{r+s} = (m)\Phi_r + (m)\Phi_s$
3. $(m)\Phi_{rs} = [(m)\Phi_r]\Phi_s$.

Thus the ring homomorphism Φ defines an $\mathcal{R}$-module. □

What arises from this proposition is an alternate way to define an $\mathcal{R}$-module. In addition to the axiomatic approach, we may consider an $\mathcal{R}$-module $\mathcal{M}$ to be a **representation** of $\mathcal{R}$. That is there exists a ring homomorphism from $\mathcal{R} \to End(\mathcal{M})$. This approach will be used primarily in Chapter 3, in an attempt to describe special characteristics of $\mathcal{R}$-modules. The definitions defined earlier in this chapter may be viewed in the context of Proposition 2.17. For instance, a **faithful** module may be viewed as a injective homomorpism $\mathcal{R} \to End(\mathcal{M})$.

In addition, a consequence that arises from Proposition 2.17 is that every $r \in \mathcal{R}$ may be identified with a specific endomorphism of $\mathcal{M}$, S_r. In the proceeding definition we will look to characterize the endomorphisms of $\mathcal{M}$ that commute with the particular endomorphisms S_r.

Definition 2.18. *$C(\mathcal{M})=\{\phi \in End(\mathcal{M}) \mid \phi S_r = S_r\phi$ for every $r \in \mathcal{R}\}$*

Proposition 2.19. *$C(\mathcal{M})$ is the ring of all module endomorphisms of $\mathcal{M}$.*

Proof. Since the identity endomorphism $1_m \colon m \mapsto m$ is in $C(\mathcal{M})$ we see it is a non-empty subset of $End(\mathcal{M})$. For $\psi, \phi \in C(\mathcal{M})$, we have $(m)S_r(\phi - \psi) = (mr)(\phi - \psi) = (mr)\phi - (mr)\psi = (mS_r)\phi - (mS_r)\psi = (m\phi)S_r - (m\psi)S_r = (m)(\phi - \psi)S_r$, which places $\phi - \psi \in C(\mathcal{M})$. To see that $\psi\phi \in C(\mathcal{M})$ observe that $(m)S_r(\phi\psi) = (m)(\phi S_r \psi) = (m)(\phi\psi)S_r$. It may be concluded that $C(\mathcal{M})$ is a subring of $End(\mathcal{M})$.

Let Υ be the collection of all module endomorphisms of $\mathcal{M}$. For an arbitrary $\Phi \in C(\mathcal{M})$, by definition $(m_1 + m_2)\Phi = (m_1)\Phi + (m_2)\Phi$ and $(mr)\Phi = (mS_r)\Phi = (m\Phi)S_r = (m\Phi)r$. As a result $C(\mathcal{M}) \subseteq \Upsilon$. Conversely, any module endomorphism of Υ must preserve the scalars of $\mathcal{R}$, hence must commute with every S_r so $\Upsilon \subseteq C(\mathcal{M})$. With both inclusions proved we see that $C(\mathcal{M}) = \Upsilon$. □

2.3 Schur's Lemma

Definition 2.20. A *division ring* is a ring in which every non-zero element has a multiplicative inverse.

Theorem 2.21. (Schur's Lemma) *Let $\mathcal{M}$ be an irreducible $\mathcal{R}$-module, then $C(\mathcal{M})$ is a division ring.*

Proof. For $C(\mathcal{M})$ to be a division ring we must show that any non-zero element of $C(\mathcal{M})$ is invertible. That is if $\phi \neq 0$ and $\phi \in C(\mathcal{M})$, there exists a ϕ^{-1} such that $\phi\phi^{-1} = \phi^{-1}\phi = 1_m$. Note that this can be reduced to proving that if $\phi \in C(\mathcal{M})$ that there is a $\phi^{-1} \in End(\mathcal{M})$. This is because if $S_r\phi = \phi S_r$ for every $r \in \mathcal{R}$, then $\phi^{-1}(S_r\phi)\phi^{-1} = \phi^{-1}(\phi S_r)\phi^{-1}$, which results in $\phi^{-1}S_r = S_r\phi^{-1}$ placing $\phi^{-1} \in C(\mathcal{M})$.

Let $\phi \neq 0 \in C(\mathcal{M})$ and denote $(\mathcal{M})\phi = N$. For every $r \in \mathcal{R}$ we see that $Nr = (N)S_r = (\mathcal{M}\phi)S_r = (\mathcal{M}S_r)\phi \subseteq (\mathcal{M})\phi \subseteq N$. Thus N is closed under multiplication of elements of the ring $\mathcal{R}$. From this we see that N is a submodule of $\mathcal{M}$. Since $\mathcal{M}$ is irreducible, N is either $\mathcal{M}$ or $\{0\}$. This implies that $(\mathcal{M})\phi = \mathcal{M}$ or $(\mathcal{M})\phi = \{0\}$. Since $\phi \neq 0$ the latter case cannot occur, thus $(\mathcal{M})\phi = \mathcal{M}$. We see that ϕ is surjective. From definition the kernel of this mapping, Ker(ϕ) is a submodule of $\mathcal{M}$. In addition it cannot be all of $\mathcal{M}$ thus, Ker(ϕ)=$\{0\}$. Note that ϕ is injective, since if $m_1, m_2 \in \mathcal{M}$ with $(m_1)\phi = (m_2)\phi$, then $(m_1 - m_2)\phi = 0$ which implies that $m_1 - m_2 \in \text{Ker}(\phi) = \{0\}$. Thus $m_1 = m_2$. We have proven that ϕ is a bijection. Thus there is an inverse ϕ^{-1} which is a endomorphism of $\mathcal{M}$. From the previous remarks, we have proven that $\phi^{-1} \in C(\mathcal{M})$. □

Chapter 3

The Density Theorem

3.1 The Density Theorem

Definition 3.1. A ring is *primitive* if and only if it has a faithful irreducible module.

From Schur's Lemma, for an irreducible module $\mathcal{M}$, it was proven that the commuting ring $C(\mathcal{M})$ is a division ring. With this result, we may view $\mathcal{M}$ as a right vector space over $C(\mathcal{M})$. In fact denoting $C(\mathcal{M}) = \Delta$, then for $v \in \mathcal{M}$ and $\alpha \in \Delta$, $v\alpha$ is the evaluation of v by the module homomorphism α. Since $\mathcal{M}$ is an $\mathcal{R}$-module, vector addition is identical to addition of group elements of $\mathcal{M}$. The scalars of $\mathcal{M}$ are elements of $End(\mathcal{M})$, so both distributive laws are satisfied and by the properties of composition of functions: $v(\alpha + \beta) = v\alpha + v\beta$, $v(\alpha\beta) = (v\alpha)\beta$. Since Δ is a division ring the scalars do not necessarily commute. Aside from this, most properties of a vector space over a field (i.e linear independence) are preserved.

Definition 3.2. $\mathcal{R}$ is a *dense ring of linear transformations* on $\mathcal{M}$ over Δ, if for any k linear independent elements (over Δ) $v_1, \ldots, v_k \in \mathcal{M}$ $(k \geq 1)$ and for any elements $m_1, \ldots, m_k \in \mathcal{M}$ there is an $r \in \mathcal{R}$ such that $v_i r = m_i$ for $i = 1, 2, \ldots, k$.

A dense ring of linear transformations is also said to act densely on $\mathcal{M}$.

Lemma 3.3. *Let $\mathcal{V}$ be a finite-dimensional subspace of $\mathcal{M}$ over Δ. Suppose $m \in \mathcal{M}$ and $m \notin \mathcal{V}$ then there exists an $r \in \mathcal{R}$ such that $\mathcal{V}r = \{0\}$ and $mr \neq 0$.*

Proof. Suppose that $\dim(\mathcal{V})=k$ $(k \geq 0)$. We will proceed by induction on the dimension of the subspace $\mathcal{V}$ over Δ.

If dim($\mathcal{V}$)=0, we have that $\mathcal{V} = \{0\}$. Since $\mathcal{M}$ is irreducible from Proposition 2.14, if $m \neq 0$ there exists a $r \in \mathcal{R}$ such that $mr \neq 0$. Naturally $\mathcal{V}r = \{0\}$ and the base case is proved.

Suppose that the hypothesis is valid for every subspace of $\mathcal{W}$ of $\mathcal{M}$ over Δ with $\dim(\mathcal{W}) \leq k-1$. If we let $v_1, \ldots, v_k$ be a basis of $\mathcal{V}$ then each element in $\mathcal{V}$ may be written as $\sum_{i=1}^{k} v_i\alpha_i = \sum_{i=2}^{k} v_i\alpha_i + v_1\alpha_1$ ($\alpha_i \in \Delta$). Thus $\mathcal{V}$ may be decomposed into $\mathcal{V} = \mathcal{W} + v\Delta$ ($v = v_1 \notin \mathcal{W}$). Define $A(\mathcal{W}) = \{x \in \mathcal{R} \mid \mathcal{W}x = 0\}$. Observe $\mathcal{W}$ is a finite-dimensional subspace with dim($\mathcal{W}$)= dim($\mathcal{V}$) $-1 = k - 1$. By induction, if $m \in \mathcal{M}$ and $m \notin \mathcal{W}$ then there exists $x \in \mathcal{R}$ with $\mathcal{W}x = 0$ and $mx \neq 0$. In short, for this particular m there exists $x \in A(\mathcal{W})$ such that $mx \neq 0$. The induction hypothesis may be stated that if $m \in \mathcal{M}$ and $mA(\mathcal{W}) = 0$ then $m \in \mathcal{W}$.

We see immediately that $A(\mathcal{W})$ is right ideal of $\mathcal{R}$. In fact from the converse of the above statement since $v \notin \mathcal{W}$, $vA(\mathcal{W}) \neq \{0\}$. We see that $vA(\mathcal{W})$ is a submodule of $\mathcal{M}$ that is a non-zero. By irreducibility we have that $vA(\mathcal{W}) = \mathcal{M}$.

For $\mathcal{V} = \mathcal{W} + v\Delta$ we will choose a $m' \in \mathcal{M}$ with $m' \notin \mathcal{V}$. By contradiction, suppose that for every $r \in \mathcal{R}$, if $\mathcal{V}r = \{0\}$ then $m'r = 0$. We will show that this is not possible thus proving the theorem. Since $vA(\mathcal{W}) = \mathcal{M}$, for every $x \in \mathcal{M}$ there is an $a \in A(\mathcal{W})$ such that $va = x$. Consider the following map $\beta : \mathcal{M} \to \mathcal{M}$, $x \mapsto m'a$, where $x = va$. It follows immediately that β is well-defined and is an endomorphism of $\mathcal{M}$. In addition for $xr = (va)r = v(ar)$ we have,

$$(xr)\beta = m'(ar) = (m'a)r = (x)\beta r.$$

Hence β is a module homomorphism of $\mathcal{M}$ which places it in Δ. For $a \in A(\mathcal{W})$,

$$m'a = (x)\beta = (va)\beta = (v)\beta a.$$

From this we have that $m'a = (v)\beta a$ or equivalently,

$$(m' - (v)\beta)a = 0 \text{ for every } a \in A(\mathcal{W}).$$

Since $(m' - (v)\beta)A(\mathcal{W}) = \{0\}$, by the induction hypothesis $m' - (v)\beta \in \mathcal{W}$. But this leads us to conclude that $m' \in \mathcal{W} + (v)\beta \subseteq \mathcal{W} + v\Delta = \mathcal{V}$. This is a contradiction to the hypothesis that $m' \notin \mathcal{V}$. Thus for $m \in \mathcal{M}, m \notin \mathcal{V}$ there exists an $r \in \mathcal{R}$ such that $\mathcal{V}r = \{0\}$ and $mr \neq 0$. □

Theorem 3.4. *(Density Theorem). Let $\mathcal{R}$ be a primitive ring with a faithful irreducible module $\mathcal{M}$, then $\mathcal{R}$ is dense on $\mathcal{M}$.*

Proof. Let $v_1, \ldots, v_n \in \mathcal{M}$ be linearly independent over Δ, and let $w_1, \ldots, w_n \in \mathcal{M}$. Denote by $\mathcal{V}_i$ the linear span of v_j for $j \neq i$. Thus $\mathcal{V}_1 = \text{span}(v_2, v_3, \ldots, v_n)$. From Lemma 3.3, since $v_1 \in \mathcal{M}$ and $v_1 \notin \mathcal{V}_1$ there exists a $r_1 \in \mathcal{R}$ with $\mathcal{V}_1 r_1 = \{0\}$ and $v_1 r_1 \neq 0$. From Proposition 2.14 we have that $(v_1 r_1)\mathcal{R} = \mathcal{M}$.

From the above there exists a s_1 in $\mathcal{R}$ such that $w_1 = (v_1 r_1)s_1 = v_1 t_1$ $(t_1 \in \mathcal{R})$. In addition $\mathcal{V}_1 t_1 = \mathcal{V}_1(r_1 s_1) = (\mathcal{V}_1 r_1)s_1 = \{0\}$. This process may be conducted iteratively for each $\mathcal{V}_2, \ldots, \mathcal{V}_n$. As a result for every v_i there is a t_i such that $v_i t_i = w_i$ and $\mathcal{V}_i t_i = \{0\}$. Consider $t = t_1 + t_2 + \cdots + t_n$, then from the fact that $v_j \in \mathcal{V}_i$ for all $j \neq i$ we have,

$$v_i t = v_i(t_1 + t_2 + \cdots + t_n) = \sum_{j \neq i} v_i t_j + v_i t_i = w_i$$

exhibiting that $\mathcal{R}$ acts densely on $\mathcal{M}$. □

Theorem 3.5. *Let $\mathcal{R}$ be a primitive ring. Then for some division ring Δ either,*

1. *$\mathcal{R} \cong \Delta_n$, the ring of all $n \times n$ matrices with entries in Δ.*
2. *Given any integer k there exists a subring I_k of $\mathcal{R}$ which maps homomorphically onto Δ_k.*

Proof. $\mathcal{R}$ is primitive, thus by Theorem 3.4 it is dense on a vector space $\mathcal{V}$ over some division ring Δ. We will consider two cases:

Suppose $\mathcal{V}$ is finite-dimensional over Δ with $dim_\Delta(\mathcal{V}) = n$. It will be proven that $\mathcal{R}$ is isomorphic to the ring of all $n \times n$ matrices over Δ. Recall that we may define a ring homomorphism $\mathcal{R} \to End_\Delta(\mathcal{V})$, $r \mapsto S_r$ where for $v \in \mathcal{V}, r \in \mathcal{R}$ we have $(v)S_r = vr$. Note that $End_\Delta(\mathcal{V}) = Hom_\Delta(\mathcal{V}, \mathcal{V})$, the ring of all Δ-linear maps from $\mathcal{V} \to \mathcal{V}$. Since every Δ linear map (with respect to a given basis) is uniquely determined by a $n \times n$ matrix with entries in Δ, we see that $Hom_\Delta(\mathcal{V}, \mathcal{V}) \cong \Delta_n$. Our argument now reduces to proving $\mathcal{R} \cong End_\Delta(\mathcal{V})$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{V}$ over Δ. Then for $f \in End_\Delta(\mathcal{V})$, $(e_i)f = w_i$ $(w_i \in \mathcal{V})$. But from density there exists an r such that

$$(e_i)S_r = e_i r = w_i$$

Since S_r and f agree on the generators of $\mathcal{V}$, they are equal. Hence $r \mapsto S_r = f$, which proves surjectivity.

In addition $\mathcal{R}$ acts faithfully in $\mathcal{V}$. As a result for $\mathcal{R} \to End_{\Delta}(\mathcal{V})$ is an injective mapping. From above we have proved that $\mathcal{R} \cong End_{\Delta}(\mathcal{V})$, thus it is isomorphic to Δ_n.

Suppose $\mathcal{V}$ is not finite-dimensional over Δ, and let k be a positive integer. We will construct subring $I_k \cong \Delta_k$. Let $v_1, \ldots, v_k, \ldots \in \mathcal{V}$ be a infinite linear independent set. Consider the finite-dimensional subspace $\mathcal{Q} = v_1\Delta + v_2\Delta + \cdots + v_k\Delta$. In addition let $I_k = \{r \in \mathcal{R} \mid \mathcal{Q}r \subseteq \mathcal{Q}\}$. It then follows directly that I_k is a subring of $\mathcal{R}$. From density we may assert that $I_k \to End_{\Delta}(\mathcal{Q})$, $r \mapsto S_r$ is a surjective ring homomorphism. Thus I_k maps homomorphically onto Δ_k. □

It will be beneficial for the reader to note that the linear independence of $v_1, \ldots, v_n \in \mathcal{V}$ is a necessary condition for the Density Theorem. Suppose $v_1, \ldots, v_n$ are linearly dependent and take $w_1, \ldots, w_n$ to be a linearly independent collection in $\mathcal{V}$. From Theorem 3.4 there is a $t \in \mathcal{R}$ such that $v_i t = w_i$ $i = 1, 2, \ldots, n$. Suppose that from the assumption of dependence, v_1 can be written as

$$\begin{aligned} v_1 &= v_2\alpha_2 + \cdots + v_n\alpha_n \quad (\alpha_n \in \Delta) \\ v_1 t &= (v_2\alpha_2)t + \cdots + (v_n\alpha_n)t \\ w_1 &= (v_2 t)\alpha_2 + \cdots + (v_n t)\alpha_n \\ &= w_2\alpha_2 + \cdots + w_n\alpha_n. \end{aligned}$$

This would imply that $w_1, \ldots, w_n$ is a collection of linearly dependent elements over Δ which is a contradiction. Thus the independence of $v_1, \ldots, v_n$ is a necessary condition for the Density Theorem.

Chapter 4

The Jacobson Radical

4.1 The Jacobson Radical

Definition 4.1. Let $\mathcal{R}$ be a ring. The *radical of* $\mathcal{R}$, $\mathcal{J}(\mathcal{R})$ is the collection of all ring elements r such that $\mathcal{M}r = \{0\}$ for all irreducible $\mathcal{R}$-modules $\mathcal{M}$. If $\mathcal{M}$ has no irreducible modules then $\mathcal{J}(\mathcal{R}) = \mathcal{R}$.

It directly follows from Definition 4.1 that $\mathcal{J}(\mathcal{R})$ is a ideal of $\mathcal{R}$, and is equivalent to $\mathcal{J}(\mathcal{R}) = \bigcap Ann(\mathcal{M})$ where the intersection runs across all irreducible $\mathcal{R}$-modules $\mathcal{M}$. We will defer to [Her05, p.13], in which it is proven that $\mathcal{J}(\mathcal{R})$ when considering irreducible $\mathcal{R}$-modules is the same when considering irreducible left $\mathcal{R}$-modules. For a fixed $\mathcal{R}$, consider Ω to be the non-empty collection of all irreducible $\mathcal{R}$-modules. If $a \in \mathcal{J}(\mathcal{R})$ then for every $\mathcal{M} \subseteq \Omega$, $(\mathcal{M})S_a = \mathcal{M}a = \{0\}$. This means that for every $\mathcal{M} \subseteq \Omega$ the image of a under the representation determined by $\mathcal{M}$ is the zero endomorphism.

4.2 Characterization of the Jacobson Radical

Definition 4.2. A right ideal F of $\mathcal{R}$ is called *regular* if there exists a $b \in \mathcal{R}$ such that for every $x \in \mathcal{R}$, $x - bx \in F$.

Example 4.3. For a commutative ring $\mathcal{R}$ containing unity, every ideal is regular.

Example 4.4. Let $2\mathbb{Z}$ be the non-unital ring generated by the even integers. Then the ideal in $2\mathbb{Z}$, $\langle 6\rangle = \{6j \mid j \in 2\mathbb{Z}\}$ is regular. If $b = 4$ then for $x \in 2\mathbb{Z}$,

$$\begin{aligned} x - 4x &= 2q - 4(2q) \qquad q \in \mathbb{Z} \\ &= -6q \in \langle 6\rangle. \end{aligned}$$

Example 4.5. In the ring referenced in Example 4.4, the ideal $\langle 4\rangle = \{4j \mid j \in 2\mathbb{Z}\}$ is not regular. That is there does not exist a $b \in 2\mathbb{Z}$ such that $x - bx \in \langle 4\rangle$ for every $x \in 2\mathbb{Z}$. To see this, take $4 \in 2\mathbb{Z}$ then,

$$\begin{aligned} 4 - b(4) &= 4 - (2r)4 \qquad r \in \mathbb{Z} \\ &= 4 - 8r \\ &= 4(1 - 2r) \notin \langle 4\rangle. \end{aligned}$$

Proposition 4.6. *Let $\mathcal{M}$ be an irreducible $\mathcal{R}$-module, then $\mathcal{M} \cong \mathcal{R}/F$ for some maximal regular right ideal F of $\mathcal{R}$.*

Proof. Let $\mathcal{Q} = \{q \in \mathcal{M} \mid q\mathcal{R} = \{0\}\}$. We see that $\mathcal{Q}$ is a submodule of $\mathcal{M}$. Since $\mathcal{M}$ is irreducible, $\mathcal{Q} = \{0\}$ or $\mathcal{Q} = \mathcal{M}$. If $\mathcal{Q} = \mathcal{M}$, then $\mathcal{QR} = \{0\} = \mathcal{MR}$; a contradiction on the irreducibility of $\mathcal{M}$. This forces $\mathcal{Q} = \{0\}$. Since $\mathcal{MR} \neq \{0\}$ there exists a non-zero element m such that $m\mathcal{R} \neq \{0\}$. From Proposition 2.14 we may conclude that $m\mathcal{R} = \mathcal{M}$.

Consider the mapping $\psi \colon \mathcal{R} \to \mathcal{M}$ by $(r)\psi = mr$. We claim that ψ is a surjective module homomorphism from $\mathcal{R}$ onto $\mathcal{M}$. For $r_1, r_2 \in \mathcal{R}$ we have that ,

$$\begin{aligned} (r_1 + r_2)\psi &= m(r_1 + r_2) = mr_1 + mr_2 = (r_1)\psi + (r_2)\psi, \text{ and} \\ (r_1 a)\psi &= m(r_1 a) = (mr_1)a = (r_1)\psi a. \end{aligned}$$

This proves that ψ is a module homomorphism. To show that ψ is surjective, we recall that the direct image $(\mathcal{R})\psi = m\mathcal{R} = \mathcal{M}$. Denote the kernel of ψ as $\mathrm{Ker}(\psi) = F$, then by the standard isomorphism theorem,

$$\begin{aligned} \mathcal{R}/Ker(\psi) &\cong (\mathcal{R})\psi \text{ which is equivalent to,} \\ \mathcal{R}/F &\cong \mathcal{M}. \end{aligned}$$

To see that F is maximal, suppose there is a right ideal I of $\mathcal{R}$ such that $F \subset I \subseteq \mathcal{R}$. The image of I under ψ is a submodule of $\mathcal{M}$. By irreducibility of $\mathcal{M}$, $(I)\psi$ is equal to $\{0\}$ or $\mathcal{M}$. If $(I)\psi = \{0\}$, then $I \subseteq Ker(\psi) = F$ which results in $I = F$, which is a contradiction. If $(I)\psi = \mathcal{M}$ then $(I)\psi = (\mathcal{R})\psi$. Let $x \in \mathcal{R}$, then there exists a $i \in I$ such that $(x)\psi = (i)\psi$ or equivalently, $(x-i)\psi = 0$. This places $x - i \in Ker\psi = F$. Thus $(x-i)+i = x \in I$ and as a result $I = \mathcal{R}$. Since the only ideal that properly contains F is $\mathcal{R}$, we may conclude that F is maximal.

We will now prove the existence of a element $b \in \mathcal{R}$ such that $x - bx \in \mathcal{R}$ for every $x \in \mathcal{R}$. With $m\mathcal{R} = \mathcal{M}$ there exists a $b \in \mathcal{R}$ such that $mb = m$. For every $x \in \mathcal{R}$ we have $mx - (mb)x = 0$. Since $\mathcal{M}$ is an $\mathcal{R}$-module, $m(x - bx) = 0$ which places $x - bx \in F$. □

Proposition 4.7. *Let F be a proper regular right ideal of $\mathcal{R}$, then it can be embedded in a maximal right ideal that is regular.*

Proof. Since F is regular there exists a $b \in \mathcal{R}$ such that $x - bx \in F$ for every $x \in \mathcal{R}$. If $b \in F$ then as a right ideal $bx \in F$, which implies that $(x - bx) + bx = x \in F$ thus $F = \mathcal{R}$. This is a contradiction, and we have $b \notin F$.

Let $\mathcal{W}$ be a collection of proper ideals I_i that contain F. One can easily verify that $\mathcal{W}$ is a po-set with respect to the relation of inclusion. Denote $\mathcal{C} = \{I_i \mid I_i \subseteq \mathcal{W}\}$ to be a totally ordered subset of W. Note that $\bigcup_{\mathcal{C}} I_i$ serves as an upper bound for $\mathcal{C}$. This ideal is regular since $x - bx \in F \subseteq \bigcup_{\mathcal{C}} I_i$. It is important to note that $b \notin \bigcup_{\mathcal{C}} I_i$, otherwise this would lead to $\bigcup_{\mathcal{C}} I_i = \mathcal{R}$ which is a contradiction. Since every totally ordered set of regular ideals that contain F has an upper bound, Zorn's Lemma may be applied. There then exists a maximal regular ideal of $\mathcal{R}$ which clearly contains F. □

It is understood that a maximal regular right ideal of a ring $\mathcal{R}$, is a maximal right ideal of $\mathcal{R}$ which is regular.

Definition 4.8. Let I be an ideal of $\mathcal{R}$, then $(I : \mathcal{R}) = \{x \in \mathcal{R} \mid \mathcal{R}x \subseteq I\}$.

Proposition 4.9. $\mathcal{J}(\mathcal{R}) = \bigcap (F : \mathcal{R})$ *where the intersection runs across all maximal regular right ideals of $\mathcal{R}$.*

Proof. By definition, $\mathcal{J}(\mathcal{R}) = \bigcap Ann(\mathcal{M})$ where the intersection runs across all irreducible $\mathcal{R}$-modules. It is then required for us to show that $Ann(\mathcal{M}) = (F : \mathcal{R})$ for some maximal regular right ideal F. From Proposition 4.6 every irreducible $\mathcal{R}$-module is isomorphic to $\mathcal{R}/F$ for some maximal regular right ideal F. Let us denote $\mathcal{M} = \mathcal{R}/F$. If $x \in Ann(\mathcal{M})$ then $\mathcal{M}x = (r+F)x = F$ for every $r \in \mathcal{R}$. Thus $rx \in F$ for every $r \in \mathcal{R}$ which can be written as $\mathcal{R}x \subseteq F$. We see that $x \in (F\colon \mathcal{R})$, hence $Ann(\mathcal{M}) \subseteq (F\colon \mathcal{R})$.

To show the other inclusion let $x \in (F\colon \mathcal{R})$, then $\mathcal{R}x \subseteq F$. This implies that $rx \in F$ for every $r \in \mathcal{R}$ which is equivalent to $(r+F)x = F$ for every $r \in \mathcal{R}$. This places $x \in Ann(\mathcal{M})$. We see that for every irreducible $\mathcal{R}$-module, $\mathcal{M}$ that $Ann(\mathcal{M}) = (F\colon \mathcal{R})$ for a maximal regular right ideal. The Jacobson radical of a ring $\mathcal{R}$ is then $\mathcal{J}(\mathcal{R}) = \bigcap Ann(\mathcal{M}) = \bigcap (F\colon \mathcal{R})$. □

Proposition 4.10. *$\mathcal{J}(\mathcal{R})$ is the intersection of the maximal regular right ideals of $\mathcal{R}$.*

Proof. Let F be a maximal regular right ideal of $\mathcal{R}$ and let $b \in \mathcal{R}$ such that $x - bx \in F$ for every $x \in \mathcal{R}$. Then for any $r \in (F\colon \mathcal{R})$ we have $(r - br) + br = r \in F$ thus $(F\colon R) \subseteq F$, for every maximal regular right ideal F of $\mathcal{R}$. Intersecting over all maximal regular right ideals gives us $\bigcap (F\colon \mathcal{R}) \subseteq \bigcap F$. From Proposition 4.9 this results in $\mathcal{J}(\mathcal{R}) \subseteq \bigcap F$.

For the other inclusion let $x \in \bigcap F$. We will first construct a regular right ideal $\mathcal{R}' = \{xu + u | u \in \mathcal{R}\}$ $(x = -b)$ that is equal to $\mathcal{R}$. If $\mathcal{R}' \neq \mathcal{R}$ then by Proposition 4.7 $\mathcal{R}' \subseteq F'$ for some proper maximal regular right ideal F' of $\mathcal{R}$. Since $x \in \bigcap F$ we see that $x \in F'$. As a right ideal of $\mathcal{R}$, $xu \in F'$. Since $\mathcal{R}' \subseteq F'$, $xu + u \in F'$ thus $(xu + u) - xu = u \in F'$ for every $u \in \mathcal{R}$, implying that $F' = \mathcal{R}$. The proper ideal $\mathcal{R}'$ cannot be embedded in a maximal regular right ideal. A contraction, thus $R' = R$. From this equality, there exists a $b' \in \mathcal{R}$ such that $xb' + b' = -x$ or $x + xb' + b' = 0$.

If $\bigcap F \subseteq \mathcal{J}(\mathcal{R})$ then $\mathcal{J}(\mathcal{R}) = \bigcap F$ and we are done. Suppose that $\bigcap F \nsubseteq \mathcal{J}(\mathcal{R})$, then there exists an irreducible $\mathcal{R}$-module $\mathcal{M}$ such that $\mathcal{M}(\bigcap F) \neq \{0\}$. There then is a non-zero $m \in \mathcal{M}$ with $m(\bigcap F) \neq \{0\}$. It follows from Proposition 2.14 that the submodule, $m(\bigcap F) = \mathcal{M}$. From this there exists a $t \in \bigcap F$ such that $mt = -m$. It was established earlier that for $t \in \bigcap F$ that $\{tu + u | u \in \mathcal{R}\} = \mathcal{R}$, which implies $t + tv + v = 0$. With these two relations we have $0 = -m(0) = -m(t + tv + v) = -[mt + m(tv) + mv] = -(-m - mv + mv) = m$, which is a contradiction under the assumption that $m \neq 0$. Thus the hypothesis $\mathcal{M}(\bigcap F) \neq \{0\}$ is invalid. As a result $\bigcap F$ annihilates all irreducible $\mathcal{R}$-modules. From this we may conclude that $\bigcap F \subseteq \mathcal{J}(\mathcal{R})$. □

Definition 4.11. An element $x \in \mathcal{R}$ is *right-quasi-regular* if there exists a $b \in \mathcal{R}$ such that $x + xb + b = 0$. Furthermore, a right ideal, $\mathcal{I}$ is a right-quasi-regular ideal if every element in I is right-quasi-regular.

Corollary 4.12. *$\mathcal{J}(\mathcal{R})$ is a right-quasi-regular ideal of $\mathcal{R}$.*

Corollary 4.13. *$\mathcal{J}(\mathcal{R})$ contains all right-quasi-regular ideals of $\mathcal{R}$.*

Proposition 4.14. *Let $\mathcal{R}$ be commutative ring with unity. If the non units of $\mathcal{R}$ form an ideal $\mathcal{I}$, then $\mathcal{J}(\mathcal{R}) = \mathcal{I}$.*

Proof. First we will show that $\mathcal{I} \subseteq \mathcal{J}(\mathcal{R})$. Suppose $x \in \mathcal{I}$. If $1 + x$ is not a unit then we would have $(1 + x) - x = 1 \in \mathcal{I}$. This is a contradiction that every element of $\mathcal{I}$ is a non-unit. From this we see that $1 + x$ is a unit and there exists $b \in \mathcal{R}$ such that

$$\begin{aligned} (1+x)b &= -1 \\ b + xb &= -1 \\ -bx - x(bx) &= x \\ x + x(bx) + bx &= 0. \end{aligned}$$

Which proves that $\mathcal{I}$ is right-quasi-regular ideal. From Corollary 4.13, $\mathcal{I} \subseteq \mathcal{J}(\mathcal{R})$. Conversely if $x \in \mathcal{J}(\mathcal{R})$ and if x is a unit then $-1 \in \mathcal{J}(\mathcal{R})$. Since $\mathcal{J}(\mathcal{R})$ is a right-quasi-regular ideal for some b we have

$$\begin{aligned} -1 + (-1)b + b &= 0 \\ -1 - b + b &= 0 \\ 1 &= 0 \end{aligned}$$

which is a contradiction. We see that x is not a unit thus $\mathcal{J}(\mathcal{R}) = \mathcal{I}$. □

From this proposition we have the following example of the Jacobson radical of a ring.

Example 4.15. $\mathbb{R}[[x]]$ is the ring of the formal power series in one indeterminant with coefficients in $\mathbb{R}$. The non-units of $\mathbb{R}[[x]]$ form an ideal and its elements are the polynomials of $\mathbb{R}[[x]]$ with zero constant term (easy check). We see that $\mathcal{J}(\mathbb{R}[[x]]) = \langle x \rangle = \{xf(x) \mid f(x) \in \mathbb{R}[[x]]\}$.

4.3 Algebras

Definition 4.16. Let $\mathcal{A}$ be a ring and let $\mathcal{F}$ be a field. Then $\mathcal{A}$ is an *algebra* over $\mathcal{F}$ if,

1. *$\mathcal{A}$ is an $\mathcal{F}$-module with the action written as $(x, \alpha) \mapsto \alpha x$.*

2. *For every $\alpha \in \mathcal{F}$ and $x, y \in \mathcal{A}$,*

$$\alpha(xy) = (x\alpha)y = x(\alpha y).$$

Definition 4.17. A *right ideal* $\mathcal{I}$ of an algebra $\mathcal{A}$ over a field $\mathcal{F}$ is a linear subspace which for every $a \in \mathcal{A}$, $x \in \mathcal{I}$ then $xa \in \mathcal{I}$.

Definition 4.18. A set $\mathcal{S}$ is a *subalgebra* of $\mathcal{A}$ over $\mathcal{F}$ if $\mathcal{S}$ is both a subring and a submodule of $\mathcal{A}$.

Proposition 4.19. *Let $\mathcal{A}$ be a algebra over a field $\mathcal{F}$ then every maximal regular right ideal of $\mathcal{A}$ as a ring is a maximal regular right ideal of $\mathcal{A}$ as a algebra.*

Proof. Suppose F is a maximal regular right ideal in $\mathcal{A}$ as a ring. By definition there is a $b \in \mathcal{A}$ with $x - bx \in F$ for every $x \in \mathcal{A}$. Note that for $\alpha \in \mathcal{F}$, αF is a right ideal of $\mathcal{A}$.

If $\alpha F \not\subseteq F$, then $\alpha F + F$ is a right ideal of $\mathcal{A}$ which properly contains F. From the maximality of F we have $\alpha F + F = \mathcal{A}$. The element $b \in \mathcal{A}$ may be expressed as

$$\begin{aligned} b &= x_1 + \alpha x_2 \qquad (x_1, x_2 \in F) \\ b^2 &= (x_1 + \alpha x_2)b \\ &= x_1 b + x_2(\alpha b). \end{aligned}$$

From this we see that $b^2 \in F$. From the definition of F being a regular right ideal, $b - b^2 \in F$. This results in $(b - b^2) + b^2 = b \in F$. With this element in F it follows that $F = \mathcal{A}$, which is a contradiction of the maximality of F. Therefore $\alpha F \subseteq F$ for every $\alpha \in F$. Thus F is a subspace of $\mathcal{A}$ over $\mathcal{F}$ and is a regular right ideal of $\mathcal{A}$ as an algebra. Since any ideal that contains F is an ideal of $\mathcal{A}$ as a ring, we see that F must be a maximal regular right ideal of $\mathcal{A}$ as an algebra.

Suppose F is a maximal regular right ideal of $\mathcal{A}$ as an algebra. It immediately follows that F is a regular right ideal of $\mathcal{A}$ as a ring. By Proposition 4.7, F may be embedded in a maximal regular right ideal F'. From above, F' is a maximal regular right ideal of $\mathcal{A}$ as a algebra. Since F' is maximal $F = F'$. Thus F is a maximal regular right ideal of $\mathcal{A}$ as a ring. □

Corollary 4.20. *Let $\mathcal{A}$ be an algebra, then $\mathcal{J}(\mathcal{A}) = \bigcap F$ where the intersection runs across all maximal regular right ideals of $\mathcal{A}$ as an algebra.*

Definition 4.21. An algebra $\mathcal{A}$ is called *algebraic* if for every $a \in \mathcal{A}$ the subalgebra generated by a is finite-dimensional. That is there exists a n (dependent on a) and $\alpha_i \in \mathcal{F}$ such that $a^n + \alpha_{n-1}a^{n-1} + \cdots + \alpha_0 = 0$. The least such n is the *degree* of a. Furthermore an algebraic algebra is *bounded of degree n* if every $a \in \mathcal{A}$ has degree n.

Definition 4.22. An element a is *nilpotent* if there exists an integer $n > 0$ such that $a^n = 0$. An ideal $\mathcal{I}$ is *nil* if each of its elements is nilpotent.

Proposition 4.23. *If $\mathcal{A}$ be an algebraic algebra, then $\mathcal{J}(\mathcal{A})$ is nil.*

Proof. Since $\mathcal{J}(\mathcal{A})$ is a subset of $\mathcal{A}$, every element of $\mathcal{J}(\mathcal{A})$ is algebraic. Let $a \in \mathcal{J}(\mathcal{A})$, then let $\mathcal{U}$ be the subalgebra generated by a. This finite-dimensional subalgebra $\mathcal{U}$ consists of the elements $\sum_{i=1}^{n} \alpha_i a^i$ $(\alpha_i \in \mathcal{F})$. In addition $\mathcal{U} \supseteq a\mathcal{U}$, since for $au \in a\mathcal{U}$, $au = a\sum_{i=1}^{n} \alpha_i a^i = \sum_{i=2}^{n} \alpha_{i-1} a^i \in \mathcal{U}$. In general we have a descending chain condition where for $k = 0, 1, \ldots$ we have $a^k\mathcal{U} \supseteq a^{k+1}\mathcal{U}$. Since $\mathcal{U}$ is finite dimensional $a^k\mathcal{U} = a^{k+1}\mathcal{U}$ for some k. As a result $a^{k+1} \in a^k\mathcal{U} = a^{k+1}\mathcal{U}$ so there exists a $b \in \mathcal{U}$ such that $a^{k+1} = a^{k+1}b$ or equivalently, $a^{k+1} - a^{k+1}b = 0$. Since $b \in \mathcal{J}(\mathcal{A})$, there exists a right-quasi inverse b' such that $b + b' - bb' = 0$. We can now see that a is nilpotent since,

$$\begin{aligned} a^{k+1} &= a^{k+1} - a^{k+1}(b + b' - bb') \\ &= a^{k+1} - a^{k+1}b - a^{k+1}b' + a^{k+1}bb' \\ &= -a^{k+1}b' + a^{k+1}bb' \\ &= 0. \end{aligned}$$

Therefore $\mathcal{J}(\mathcal{A})$ is nil. □

4.4 Properties of Rings with no Nilpotent Elements

Definition 4.24. An element u of an algebra $\mathcal{A}$ is *idempotent* if $u^2 = u$.

Lemma 4.25. *Suppose $\mathcal{R}$ is a ring with no nilpotent elements, then all idempotent elements of $\mathcal{R}$ lie in its center, $\mathcal{Z}(\mathcal{R})$.*

Proof. Let u be a idempotent element of $\mathcal{R}$.Then for every $r \in \mathcal{R}$ we have,

$$(ur - uru)^2 = urur - ururu - uru^2r + uru^2ru = 0.$$

Similarly,

$$(ru - uru)^2 = 0.$$

Since $\mathcal{R}$ contains no nilpotent elements, $ur - uru = 0$, $ru - uru = 0$. This leaves us with $ur = uru = ru$, hence $ur = ru$. We have proven that $u \in \mathcal{Z}(\mathcal{R})$. □

Proposition 4.26. *Let $\mathcal{A}$ be an algebraic algebra that contains no nilpotent elements. Let $\mathcal{I}$ be an ideal and $F \subseteq \mathcal{I}$ a finite subset. Then there exists an idempotent of $\mathcal{A}$ that acts as unity on F.*

Proof. Suppose that $F = \{a_1, \ldots, a_k\}$. Let $a_1 \neq 0$ be a non-invertible element in F. As an element of $\mathcal{A}$ it satisfies a polynomial relation

$$a_1^m + \alpha_1 a_1^{m-1} + \cdots + \alpha_n a_1^{m-n} = 0, \qquad (\alpha_i \in \mathcal{F})$$

where $m - n > 0$. Note that,

$$a_1(a_1^n + \alpha_1 a_1^{n-1} + \cdots + \alpha_n) = (a_1^n + \alpha_1 a_1^{n-1} + \cdots + \alpha_n)a_1.$$

From this fact we see that

$$\begin{aligned}
& [a_1(a_1^n + \alpha_1 a_1^{n-1} + \cdots + \alpha_n)]^{m-n} \\
= \;& (a_1^m + \alpha_1 a_1^{m-1} + \cdots + \alpha_n a_1^{m-n})(a_1^n + \alpha_1 a_1^{n-1} + \cdots + \alpha_n)^{m-n-1} \\
= \;& 0.
\end{aligned}$$

It follows from the hypothesis that F contains no nilpotent elements thus, $a_1(a_1^n + \alpha_1 a_1^{n-1} + \cdots + \alpha_n) = 0$. Next, we will construct a specific idempotent u_1 from this polynomial with the property that $a_1 u_1 = a_1$. We have $0 = a_1^{n+1} + \alpha_1 a^n + \cdots + \alpha_n a_1$. Rewriting this relation we have $-\alpha_n a_1 = a_1^{n+1} + \alpha_1 a^n + \cdots + \alpha_{n-(j-1)} a^j$, $j \geq 2$. Factoring a^2 on the right leaves us with, $a_1 = a_1^2 p(a_1)$, which we will denote as $a_1 = a_1 u_1$. Where $u_1 = a_1 p(a_1)$.

Observe that $u_1^2 = a_1^2 p^2(a_1) = a_1 p(a_1) = u_1$. Since a_1 is non-invertible nor 0, $u_1 \neq 0, 1$. In general for a given non-invertible, non-zero element $a_i \in F$ a specific idempotent element u_i exists with $a_i = a_i u_i$. Note that the idempotent $u_i \in F$ is constructed from the polynomial that a_i satisfies. Induction will be used to show that there is an idempotent u such that $a_i u = a_i$ for all $a_i \in F$.

From the previous paragraph there exists a u_1 such that $a_1 u_1 = a_1$. Next, suppose that $a_2 u_1 = a_2, a_3 u_1 = a_3, \ldots, a_{k-1} u_1 = a_{k-1}$. If $a_k u_1 = a_k$, we may take $u = u_1$ and we are done. If $a_k u_1 \neq a_k$, from Lemma 4.25 all idempotent elements of $\mathcal{I}$ lie in $\mathcal{Z}(\mathcal{I})$. Then by using the idempotent u_k we have $(a_k - a_k u_1) u_k = a_k - a_k u_1$. From this we see that, $a_k u_k - a_k u_1 u_k = a_k - a_k u_1$, and we may rearrange this relation to get $a_k = a_k u_k - a_k u_1 u_k + a_k u_1$. Factoring an a_k, we have,

$$a_k = a_k(u_k - u_1 u_k + u_1).$$

Let $u = u_k - u_1 u_k + u_1$. It can be easily verified that $u^2 = u$. Furthermore for $i = 1, 2, \ldots, k-1$, we see that $a_i u = a_i(u_k - u_1 u_k + u_1) = a_i u_k - a_i u_1 u_k + a_i u_1 = a_i$. Thus an idempotent $u \in F$ has been constructed such that for every $a \in F$, $au = a$. $\square$

4.5 Free Algebra

Definition 4.27. A set M is a *monoid* if there exists a binary operation $(a, b) \mapsto a \cdot b$ called multiplication which satisfies the following for every $a, b, c \in M$.

1. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. *There exists* $1 \in M$ *with* $1 \cdot a = a \cdot 1 = a$.

Consider the set $\mathcal{X} = \{x_1, \ldots, x_n\}$, then the **free monoid** generated by $\mathcal{X}$ is a monoid whose elements consists of all finite sequences of $\mathcal{X}$. That is the set consisting of 1 and elements which are written as,

$$x_{i_1} x_{i_2} \cdots x_{i_m}.$$

These elements are called **monomials**. Multiplication is defined as

$$(x_{t_1} x_{t_2} \cdots x_{t_j})(x_{q_1} x_{q_2} \cdots x_{q_k}) = x_{t_1} x_{t_2} \cdots x_{t_j} x_{q_1} x_{q_2} \cdots x_{q_k}.$$

In addition, $x_{t_1}x_{t_2}\cdots x_{t_j} = x_{q_1}x_{q_2}\cdots x_{q_k}$ if and only if $t_1 = q_1, t_2 = q_2, \ldots, t_j = q_k$.

Let $\mathcal{F}$ be a field, then $\mathcal{F}\langle x_1,\ldots,x_n\rangle$ is the **free algebra** generated by the non-commuting variables $x_1,\ldots,x_n$. This algebra is spanned by all sums of products of the indeterminates $x_1,\ldots,x_n$. From the previous explanation of the free monoid on M, elements of $\mathcal{F}\langle x_1,\ldots,x_n\rangle$ may be expressed as a finite sum

$$f = \sum \alpha_{(i_1,i_2,\ldots,i_m)} x_{i_1}^{\sigma(i_1)} x_{i_2}^{\sigma(i_2)} \cdots x_{i_m}^{\sigma(i_m)} \qquad \sigma(i_k) \in \mathbb{Z}^+.$$

The **degree** of each monomial occurring in f, $x_{i_1}^{\sigma(i_1)} x_{i_2}^{\sigma(i_2)} \cdots x_{i_m}^{\sigma(i_m)}$ is the sum $\sigma(i_1) + \sigma(i_2) + \cdots \sigma(i_m)$. The degree of f is the greatest degree of all of the monomials occurring in f. In most cases $f \in \mathcal{F}\langle x_1,\ldots,x_n\rangle$ will be expressed as $f(x_1,\ldots,x_n)$.

Definition 4.28. $f \in \mathcal{F}\langle x_1,\ldots,x_n\rangle$ is *multilinear* if for $k = 1,2,\ldots,n$ and for every $\alpha \in \mathcal{F}$,

1. $f(x_1, x_2, \ldots, \overbrace{x + x'}^{kth}, \ldots, x_n) = f(x_1, x_2, \ldots, \overbrace{x}^{kth}, \ldots, x_n) + f(x_1, x_2, \ldots, \overbrace{x'}^{kth}, \ldots, x_n)$
2. $f(x_1, x_2, \ldots, \overbrace{\alpha x}^{kth}, \ldots, x_n) = \alpha f(x_1, x_2, \ldots, \overbrace{x}^{kth}, \ldots, x_n)$

As a consequence f is of the form, $f = \sum \alpha_{(i_1,i_2,\ldots,i_n)} x_{i_1}x_{i_2}x_{i_3}\cdots x_{i_n}$ $(\alpha \in \mathcal{F})$, where the monomials $x_{i_1}x_{i_2}\ldots x_{i_n}$ in the summation range over some permutations of $x_1, x_2, \ldots, x_n$.

Example 4.29. In $\mathcal{F}\langle x_1,\ldots,x_n\rangle$, $f(x_1,x_2,x_3) = \alpha x_1x_2x_3 - \beta x_2x_1x_3$ $(\alpha,\beta \in \mathcal{F})$ is a multi-linear polynomial. While $g(x_1,x_2,x_3) = \gamma x_1^2x_2x_3 - \delta x_2^3x_1x_3^2$ $(\gamma,\delta \in \mathcal{F})$ is not.

Chapter 5

Kaplansky's Theorem

5.1 Polynomial Identities

Definition 5.1. Let $\mathcal{A}$ be a algebra. Then $\mathcal{A}$ satisfies a *polynomial identity* (P.I.) if there exists some $f(x_1, \ldots, x_d) \in \mathcal{F}\langle x_1, \ldots, x_d\rangle$, such that $f(a_1, \ldots, a_d) = 0$ for every $a_1, \ldots, a_d \in \mathcal{A}$.

Example 5.2. If $\mathcal{A}$ is a commutative algebra then it satisfies the polynomial identity $f(x_1, x_2) = x_1x_2 - x_2x_1$.

Example 5.3. Let $\mathcal{F}_2$ be the algebra of 2×2 matrices with entries in field $\mathcal{F}$. Then $\mathcal{F}_2$ satisfies the polynomial identity $f(x, y, z) = z(xy - yx)^2 - (xy - yx)^2z$.

Example 5.4. Let $\mathcal{F}$ be field such that $3x = 0$ for every $x \in \mathcal{F}$. Then $\mathcal{F}\langle x_1, x_2 \mid x_1x_2 - x_2x_1 = 1\rangle$ satisfies the polynomial identity $f(x, y) = (xy + yx)^2 + 2xyxy + xy + 1$. This is a variation of the **Weyl algebra** algebra where the characteristic of $\mathcal{F}$ is 3.

Lemma 5.5. *Let $f \neq 0$ be in $\mathcal{F}\langle x_1, \ldots, x_n\rangle$ then there is a positive integer m such that $\mathcal{F}_m$ does not satisfy f.*

Proof. Let f be of degree t. Consider $\mathcal{Q}$ to be the ideal of $\mathcal{F}\langle x_1, \ldots, x_n\rangle$ generated by the monomials in $x_1, \ldots, x_n$ of degree greater than t. As a result the algebra, $\mathcal{A} = \mathcal{F}\langle x_1, \ldots, x_n\rangle/\mathcal{Q}$ is spanned by the representatives that are contained in subspace consisting of all monomials of degree no greater than t. Since $\mathcal{A}$ is finite-dimensional over $\mathcal{F}$ it may be represented as a subalgebra of $\mathcal{F}_m$ ($m = Dim_{\mathcal{F}}\mathcal{A}$), where $\mathcal{F} \cong End_{\mathcal{F}}(\mathcal{A})$. Let

$\overline{f}$ be the image of f under the map $\mathcal{F}\langle x_1, \ldots, x_n\rangle \to \mathcal{F}\langle x_1, \ldots, x_n\rangle / \mathcal{Q}$. Since f is of degree t it is not contained in $\mathcal{Q}$, thus $\overline{f}$ is not zero. Since $1 \in \mathcal{A}$ the representation of of $\overline{f}$ in $\mathcal{F}_m$ not zero as well. There then exists matrices $a_1, \ldots, a_n \in \mathcal{F}_m$ with $f(a_1, \ldots, a_n) \neq 0$. This establishes the lemma. □

Proposition 5.6. *Let $\mathcal{A}$ be an algebra that satisfies a polynomial identity f of degree d. Then $\mathcal{A}$ also satisfies a multilinear identity of degree $\leq d$.*

Proof. We will defer to [Her05, p.157] for the proof of this proposition. □

We will demonstrate the following process as described in Proposition 5.6. Let $f(x_1, x_2, x_3) = x_1^2 x_3 x_2 - x_2 x_3^2$ be a polynomial identity of degree 4. We will now construct a multilinear polynomial identity from f. By letting $h(x_1, x_2, x_3, x_4) = f(x_1+x_4, x_2, x_3) - f(x_1, x_2, x_3) - f(x_4, x_2, x_3)$ it directly follows that $\mathcal{A}$ satisfies h. The calculation of h gives us,

$$\begin{aligned} h(x_1, x_2, x_3, x_4) &= (x_1 + x_4)^2 x_3 x_2 - x_2 x_3^2 - (x_1^2 x_3 x_2 + x_2 x_3^2) - (x_4^2 x_3 x_2 + x_2 x_3^2) \\ &= (x_1 x_4 + x_4 x_1) x_3 x_2 - 3 x_2 x_3^2 \\ &= x_1 x_4 x_3 x_2 + x_4 x_1 x_3 x_2 - 3 x_2 x_3^2 \end{aligned}$$

which produces an identity that is linear in x_1. By applying the same iterative process to x_3 we have, $g(x_1, x_2, x_3, x_4, x_5) = h(x_1, x_2, x_3+x_5, x_4) - h(x_1, x_2, x_3, x_4) - h(x_1, x_2, x_5, x_4)$. Simplifying this gives us,

$$\begin{aligned} g(x_1, x_2, x_3, x_4, x_5) &= x_1 x_4 (x_3 + x_5) x_2 + x_4 x_1 (x_3 + x_5) x_2 - 3 x_2 (x_3 + x_5)^2 \\ &\quad -(x_1 x_4 x_3 x_2 + x_4 x_1 x_3 x_2 - 3 x_2 x_3^2) \\ &\quad -(x_1 x_4 x_5 x_2 + x_4 x_1 x_5 x_2 - 3 x_2 x_5^2) \\ &= -3 x_2 x_3 x_5 - 3 x_2 x_5 x_3. \end{aligned}$$

We see that $g(x_1, x_2, x_3, x_4, x_5)$ is a polynomial identity of degree 3 that is multilinear. Since any algebra homomorphism preserves both products and sums we have the following result.

Proposition 5.7. *Let $\mathcal{A}$ be a algebra that satisfies a polynomial identity f. If Φ is an algebra homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, then $\Phi(\mathcal{A})$ satisfies the same identity.*

5.2 Tensor Product

Definition 5.8. Let E be a field, then E is a *field extension* of $\mathcal{F}$ if $\mathcal{F} \subseteq E$ and if $\mathcal{F}$ is a field with respect to the operations of E restricted to $\mathcal{F}$.

Definition 5.9. Let $\mathcal{A}$ be an algebra over field $\mathcal{F}$ with a field extension E. Then the *tensor product* of $\mathcal{A}$ and E over $\mathcal{F}$, written as $\mathcal{A} \otimes_{\mathcal{F}} E$ is an algebra constructed by "extending the base field to E." If $\{a_i\}$ is an $\mathcal{F}$ basis of $\mathcal{A}$ then $\{a_i \otimes 1\}$ is an E basis of $\mathcal{A} \otimes_{\mathcal{F}} E$. The elements of $\mathcal{A} \otimes_{\mathcal{F}} E$ are expressed as a finite sum of $\alpha(a \otimes e)$ for $a \in \mathcal{A}$, $e \in E, \alpha \in \mathcal{F}$ and which satisfies the following properties

1. $(a_1 \otimes e_1)(a_2 \otimes e_2) = a_1a_2 \otimes e_1e_2$
2. $(a_1 + a_2) \otimes e = a_1 \otimes e + a_2 \otimes e$
3. $a \otimes (e_1 + e_2) = a \otimes e_1 + a \otimes e_2$
4. $\alpha(a \otimes e) = \alpha a \otimes e = a \otimes \alpha e$
5. $0 \otimes e = a \otimes 0 = 0.$

It follows from the above properties that the tensor product is bi-linear and that $Dim_E(\mathcal{A} \otimes_{\mathcal{F}} E) = Dim_{\mathcal{F}}\mathcal{A}$. We will refer the reader to [Jac09, $p.215-220$], [Hun74, $p.207-216$], for further details on the construction of this algebra.

Lemma 5.10. *If $\mathcal{A}$ satisfies a multilinear polynomial identity f then for any extension field E of field $\mathcal{F}$, $\mathcal{A} \otimes_{\mathcal{F}} E$ satisfies f.*

We will demonstrate a particular example of Lemma 5.10 where $\mathcal{A}$ is commutative. Note that $\mathcal{A}$ satisfies the identity $f(x_1, x_2) = x_1x_2 - x_2x_1$. Then for the tensor product $\mathcal{A} \otimes_{\mathcal{F}} E$ we have

$$\begin{aligned} f(a_1 \otimes e_1, a_2 \otimes e_2) &= (a_1 \otimes e_1)(a_2 \otimes e_2) - (a_2 \otimes e_2)(a_1 \otimes e_1) \\ &= a_1a_2 \otimes e_1e_2 - a_2a_1 \otimes e_2e_1 \\ &= a_1a_2 \otimes e_1e_2 - a_2a_1 \otimes e_1e_2 \\ &= (a_1a_2 - a_2a_1) \otimes e_1e_2 \\ &= 0 \otimes e_1e_2 \\ &= 0. \end{aligned}$$

This shows that $\mathcal{A} \otimes_{\mathcal{F}} E$ satisfies f.

5.3 Kaplansky's Theorem

Definition 5.11. Let $\mathcal{R}_n$ be the ring of $n \times n$ matrices with entries in a commutative unital ring $\mathcal{R}$. Then $\mathcal{R}(1_n)$ is the subring of $\mathcal{R}_n$ consisting of all diagonal matrices of the form $r(1_n)$ $(r \in \mathcal{R})$, where 1_n is the $n \times n$ identity matrix of $\mathcal{R}_n$.

Theorem 5.12. *Let $\mathcal{R}$ be a ring with unity. Then $\mathcal{Z}(\mathcal{R}_n) = \mathcal{Z}(\mathcal{R})(1_n)$.*

Proof. Since an arbitrary element in $\mathcal{Z}(\mathcal{R})(1_n)$ may be written as $z(1_n)$ for some $z \in \mathcal{Z}(\mathcal{R})$, it follows directly from the definition of the center that it lies in $\mathcal{Z}(\mathcal{R}_n)$. For $X \in \mathcal{R}_n$ we see that

$$X(z1_n) = (Xz)1_n = (zX)1_n = z(X1_n) = zX = (z1_n)X$$

thus $\mathcal{Z}(\mathcal{R})(1_n) \subseteq \mathcal{Z}(\mathcal{R}_n)$. For the other inclusion, let $Y \in \mathcal{Z}(\mathcal{R}_n)$. We will denote the entries of Y as z_{kl} for $k, l = 1, 2, \ldots, n$. Let 1_{ij} denote the $n \times n$ matrix with 1 in the i^{th} row and j^{th} column and all remaining entries 0. Since Y commutes with every element of $\mathcal{R}_n$, $Y(1_{ii}) = (1_{ii})Y$. From this we have the following equivalent i^{th} rows and columns that my be compared.

$$\overset{i^{th} colum}{\begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ z_{i1} & z_{i2} & \cdots & z_{ii} & \cdots & z_{in} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}} = \overset{i^{th} colum}{\begin{pmatrix} 0 & \cdots & 0 & z_{1i} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & z_{2i} & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & z_{ii} & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & z_{ni} & 0 & \cdots & 0 \end{pmatrix}}$$

By equating the entries of each matrix we have $z_{ik} = z_{ki} = 0$ for $i \neq k$. From repeating this process and and equating $Y(1_{kk}) = (1_{kk})Y$ for $k = 1, 2, \ldots, n$ we may conclude that $z_{kl} = 0$ for every $k \neq l$. We have shown that all non-diagonal entries of Y

are zero. We will now prove that the remaining entries are equal. For $i < j$ the relation $Y(1_{ij}) = (1_{ij})Y$ will result in,

$$
\begin{array}{c} \\ \\ \\ \\ i^{th}row \\ \\ \\ \\ \end{array}
\begin{pmatrix}
0 & \dots & \overset{j^{th}colum}{\dots} & \dots & 0 \\
\vdots & \dots & \dots & \dots & \vdots \\
0 & \dots & \dots & \dots & 0 \\
0 & \dots & \dots & \dots & 0 \\
z_{j1} & \dots & z_{jj} & \dots & z_{jn} \\
0 & \dots & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & \dots & \dots & 0
\end{pmatrix}
=
\begin{array}{c} \\ \\ \\ \\ i^{th}row \\ \\ \\ \\ \end{array}
\begin{pmatrix}
0 & \dots & 0 & \overset{j^{th}colum}{z_{1i}} & 0 & \dots & 0 \\
\vdots & \dots & \vdots & z_{2i} & \vdots & \dots & \vdots \\
\vdots & \dots & \vdots & z_{3i} & \vdots & \dots & \vdots \\
\vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
\vdots & \dots & \vdots & z_{ii} & \vdots & \dots & \vdots \\
\vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & z_{ni} & 0 & \dots & 0
\end{pmatrix}
$$

Since the entries of each matrix must be equal we have that $z_{ii} = z_{jj}$. By continuing this process for all $j \neq i$ we see that $z_{11} = z_{22} = \cdots = z_{nn} = z$. Since Y must commute with any element in $\mathcal{R}_n$ we see that $z \in \mathcal{Z}(\mathcal{R})$. this shows that $Y = z(1_n)$, thus $\mathcal{Z}(\mathcal{R}_n) \subseteq \mathcal{Z}(\mathcal{R})(1_n)$. □

Theorem 5.13. *Let $\mathcal{R}$ be a ring with unity. Then $\mathcal{Z}(\mathcal{R})(1_n) \cong \mathcal{Z}(\mathcal{R})$, as sub $\mathcal{R}$-modules.*

Proof. Define $f\colon \mathcal{Z}(\mathcal{R}) \to \mathcal{Z}(\mathcal{R})(1_n)$, $z \mapsto z(1_n)$. We see that f is well defined since if $z = z'$, then $f(z - z') = (z - z')1_n = 0$. Thus $z(1_n) = z'(1_n)$. It follows directly from the properties of matrix multiplication that f preserves addition and scalar multiplication. Hence f is a module homomorphism. Since any element in $\mathcal{Z}(\mathcal{R})(1_n)$ may be written as $z(1_n)$ for some $z \in \mathcal{Z}(\mathcal{R})$, f is surjective. Lastly, to show that f is injective suppose that $f(z) = f(z')$. Then $z(1_n) = z'(1_n)$ which gives us the $(z - z')1_n = 0$. By equating the entries of the two matrices, we see that $z = z'$. We have proven that $\mathcal{Z}(\mathcal{R})(1_n) \cong \mathcal{Z}(\mathcal{R})$. □

Corollary 5.14. *Let $\mathcal{R}$ be a ring with unity, then $\mathcal{Z}(\mathcal{R}_n) \cong \mathcal{Z}(\mathcal{R})$.*

Definition 5.15. Let S be a subfield of a division ring Δ. Then S is *maximal* if for every proper subfield Q of Δ with $S \subset Q \subseteq \Delta$ then $Q = \Delta$.

Lemma 5.16. *Let Δ be a division ring with center $\mathcal{Z}(\Delta)$. If K is a maximal subfield of Δ then $\Delta \otimes_{\mathcal{Z}(\Delta)} K$ is a dense ring of linear transformations on Δ as a vector space over K.*

Proof. We will defer the proof of this lemma to [Jac64, p.95]. □

Theorem 5.17. *(Kaplansky's Theorem). Let $\mathcal{A}$ be a primitive algebra that satisfies a polynomial identity. Then $\mathcal{A}$ is finite-dimensional over its center $\mathcal{Z}(\mathcal{A})$.*

Proof. Since $\mathcal{A}$ is primitive, from Theorem 3.5 it is either isomorphic to Δ_n for some integer n, or for every integer k there exists a subalgebra of $\mathcal{A}$ that maps homomorphically onto Δ_k.

Suppose that the latter of the two occured. For each k, let S_k be the subalgebra of $\mathcal{A}$ that maps homomorphically onto Δ_k. From Proposition 5.6 we may assume f be the multilinear identity that $\mathcal{A}$ satisfies. Since any subalgebra or homomorphic image of $\mathcal{A}$ satifies the polynomial identity on $\mathcal{A}$, Δ_k satisfies f as well. From this the center of Δ_k, $\mathcal{Z}(\Delta_k)$ (a field) satisfies f for every k. From Lemma 5.5 we see that this is an impossibility. As a result the first case must occur, thus $\mathcal{A} \cong \Delta_n$.

Let K be a maximal subfield of Δ. From the above isomorphism, Δ satisfies the polynomial identity f. With this result and the multi-linearity of f, $\Delta \otimes_{\mathcal{Z}(\Delta)} K$ satisfies this identity as well. In addition from Lemma 5.16, $\Delta \otimes_{\mathcal{Z}(\Delta)} K$ is a dense ring of linear transformations on Δ over K thus the above argument may be applied. It follows that $\Delta \otimes_{\mathcal{Z}(\Delta)} K \cong K_q$ for some positive integer q. From the definition of the tensor product we have $q^2 = Dim_K(\Delta \otimes_{\mathcal{Z}(\Delta)} K) = Dim_{\mathcal{Z}(\Delta)}(\Delta)$. In addition Δ_n is finite-dimensional over Δ, thus $Dim_\Delta(\Delta_n) = n^2$.

From the fact that Δ_n over Δ and Δ over $\mathcal{Z}(\Delta)$ are finite-dimensional, the spanning set of Δ_n over $\mathcal{Z}(\Delta)$ is finite. Thus Δ_n is finite-dimensional over $\mathcal{Z}(\Delta)$. In other words $Dim_{\mathcal{Z}(\Delta)}(\Delta_n) = p$, for some positive integer p. From Corollary 5.14, $\mathcal{Z}(\Delta) \cong \mathcal{Z}(\Delta_n)$, thus $Dim_{\mathcal{Z}(\Delta_n)}(\Delta_n) = p$. Since it was established earlier that $\Delta_n \cong \mathcal{A}$, we may conclude that $p = Dim_{\mathcal{Z}(\Delta_n)}(\Delta_n) = Dim_{\mathcal{Z}(\mathcal{A})}(\mathcal{A})$. We have proven that $\mathcal{A}$ is finite-dimensional over its center. □

Chapter 6

Locally Finite Algebras

6.1 Locally Finite Algebras

Definition 6.1. Let $\mathcal{X} = \{x_1, x_2, \ldots\}$ be a subset of $\mathcal{A}$, then the *subalgebra generated by* $\mathcal{X}$, denoted $\langle \mathcal{X} \rangle$ is the intersection of all subalgebras of $\mathcal{A}$ containing $\mathcal{X}$.

Definition 6.2. $\mathcal{A}$ is *locally finite* if and only if every finite subset $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ of $\mathcal{A}$ generates a finite-dimensional subalgebra.

Proposition 6.3. *Let $\mathcal{A}$ be an algebraic algebra that is commutative, then $\mathcal{A}$ is locally finite.*

Proof. Let $\{a_1, \ldots, a_k\}$ be a finite subset of $\mathcal{A}$ and take $\mathcal{U}$ be the subalgebra of $\mathcal{A}$ generated by this set. Since $\mathcal{A}$ is algebraic there exists a polynomial f_i of degree $n_i \geq 0$ that a_i satisfies for $i = 1, 2, \ldots k$. Note that each generator a_i has its corresponding n_i. $\mathcal{A}$ is commutative thus the multiplication of two monomials in $\mathcal{U}$ may be rewritten as follows

$$a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k} a_1^{s_1} a_2^{s_2} \cdots a_k^{s_k} = a_1^{r_1+s_1} a_2^{r_2+s_2} \cdots a_k^{r_k+s_k}, \quad r_i, s_i \geq 0.$$

From this, every element in $\mathcal{U}$ is a finite sum of monomials of the above form. We will prove that $\mathcal{U}$ is finite dimensional by showing that for an arbitrary element each monomial in the summand has generators a_i with an exponent no greater than n_i, the degree of the polynomial satisfied by a_i. It will suffice to show that the generator a_1 can be iteratively reduced to that of degree less than n_1. Given $u = \sum a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k} \in \mathcal{U}$ we will use induction on the exponent of a_1. Assume that $n_1 \leq m_1$. If $m_1 = n_1$ then

$$u = \sum a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k} = \sum a_1^{n_1} a_2^{m_2} \cdots a_k^{m_k}.$$

$\mathcal{U}$ is algebraic thus $a_1^{n_1} = \alpha_1 a_1^{n_1-1} + \cdots + \alpha_{n_1}$, $(\alpha_i \in \mathcal{F})$. We will denote this as $a_1^{n_1} = \alpha_1 a_1^{n_1-1} + p(a_1)$, where the highest exponent of a_1 in $p(a_1)$ is less than $n_1 - 1$. Substituting this into the sum we now have

$$u = \sum [\alpha_1 a_1^{n_1-1} + p(a_1)] a_2^{m_2} \cdots a_k^{m_k} = \sum \alpha_1 a_1^{n_1-1} a_2^{m_2} \cdots a_k^{m_k} + \sum p(a_1) a_2^{m_2} \cdots a_k^{m_k}.$$

The exponent of a_1 has been iteratively reduced to that of less than n_1. Suppose that for $m_1 = n_1 + t_1$ and for some $t_1 \geq 1$ the exponent of a_1 may be reduced to that which is less than $n_1 + t_1$. Let $m_1 = n_1 + (t_1 + 1)$ then applying the previous technique we have

$$\begin{aligned} u &= \sum a_1^{n_1} a_1^{t_1+1} a_2^{m_2} \cdots a_k^{m_k} \\ &= \sum [\alpha_1 a_1^{n_1-1} + p(a_1)] a_1^{t_1+1} a_2^{m_2} \cdots a_k^{m_k} \\ &= \sum [\alpha_1 a_1^{n_1+t_1} + a_1^{t_1+1} p(a_1)] a_2^{m_2} \cdots a_k^{m_k} \\ &= \sum \alpha_1 a_1^{n_1+t_1} a_2^{m_2} \cdots a_k^{m_k} + \sum a_1^{t_1+1} p(a_1) a_2^{m_2} \cdots a_k^{m_k} \end{aligned}$$

Where the highest exponent of a_1 in $a_1^{t_1+1} p(a_1)$ is less than $n_1 + t_1$. By induction, u may be expressed as a sum where a_1 has an exponent less than n_1. This process may be applied to $a_2, \ldots, a_k$ successively and as a result the finite set of monomials $a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}$ $(m_i < n_i)$ spans $\mathcal{U}$. Therefore, $\mathcal{U}$ is finite-dimensional which proves that $\mathcal{A}$ is locally finite. □

Proposition 6.4. *Let $\mathcal{U}$ be a finitely generated algebraic algebra containing unity. If $\mathcal{U}$ is finite-dimensional over its center $\mathcal{Z}(\mathcal{U})$ (a field), then $\mathcal{U}$ is finite-dimensional over $\mathcal{F}$.*

Proof. By hypothesis $\mathcal{U}$ is finite-dimensional over $\mathcal{Z}(\mathcal{U})$. There then exists elements $e_1, \ldots, e_m$ in $\mathcal{U}$ such that every element in $\mathcal{U}$ is a linear combination of the e_i's with scalars in $\mathcal{Z}(\mathcal{U})$. From this fact if $a_i \in \mathcal{U}$, then it may be expressed as $a_i = \sum_{t=1}^{m} z_{it} e_i$ $(z_{it} \in \mathcal{Z}(\mathcal{U}))$. In addition if $a_i, a_j \in \mathcal{U}$, then $a_i a_j = \sum_{k=1}^{m} z_{ijk} e_k$ $(z_{ijk} \in \mathcal{Z}(\mathcal{U}))$.

Note that for a finitely generated algebra $\mathcal{U}$ over $\mathcal{F}$, if it is also locally finite over $\mathcal{F}$ then $\mathcal{U}$ is finite-dimensional over $\mathcal{F}$. Let $\mathcal{X} = \{a_1, \ldots, a_n\}$ be a subset of $\mathcal{U}$. We will show that $\langle \mathcal{X} \rangle$ the algebra over $\mathcal{F}$ generated by this set, is finite-dimensional over $\mathcal{F}$. Let $\mathcal{I}$ be the subalgebra of $\mathcal{Z}(\mathcal{U})$ generated by the set $\{z_{it}, z_{ijk}\}$ $i, j, k = 1, 2, \ldots, m$. Since $\mathcal{I} \subseteq \mathcal{Z}(\mathcal{U}) \subseteq \mathcal{U}$, we see that $\mathcal{I}$ is commutative and algebraic. From Proposition 6.3, $\mathcal{I}$ is locally finite. With the added property that $\mathcal{I}$ is finitely generated we may concluded that $\mathcal{I}$ is finite-dimensional over $\mathcal{F}$.

Let $W = \{\sum \beta_i e_i \mid \beta_i \in \mathcal{I}\}$. Since $\mathcal{I}$ contains 1 it may be considered as a field, thus W is a finitely generated algebraic algebra over $\mathcal{I}$. From this we see that W is a finite-dimensional algebra over $\mathcal{I}$. Since $\mathcal{I}$ is a finite-dimensional algebra over $\mathcal{F}$ we may conclude that W is a finite-dimensional algebra over $\mathcal{F}$. With $z_{it}, z_{ijk} \in \mathcal{I}$ the subalgebra generated $\{a_1, \dots, a_n\}$, $\langle \mathcal{X} \rangle \subseteq W$. Thus $\langle \mathcal{X} \rangle$ is finite-dimensional, hence $\mathcal{U}$ is locally finite. Since $\mathcal{U}$ itself is finitely generated over $\mathcal{F}$, it may be concluded that it is finite-dimensional over $\mathcal{F}$. □

Lemma 6.5. *If $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are finite-dimensional vector spaces then $\mathcal{A}$ is finite dimensional.*

Proof. Let $v \colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$. Then v is a surjective linear map with $\mathcal{B}$ as its kernel. A well know linear algebra result [Hun74, p.5] is that $\dim(\text{kernel}(v)) + \dim(\text{image}(v)) = \dim(\mathcal{A})$. Which is equivalent to $\dim(\mathcal{B}) + \dim(\mathcal{A}/\mathcal{B}) = \dim(\mathcal{A})$. Since from the hypothesis $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are finite-dimensional, it follows that $\mathcal{A}$ is finite-dimensional. □

Definition 6.6. An ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is a *locally finite ideal* if when regarded as an algebra it is locally finite.

Proposition 6.7. *Let $\mathcal{C}$ be an ideal of a algebra $\mathcal{A}$. If $\mathcal{A}/\mathcal{C}$ and $\mathcal{C}$ are locally finite, then $\mathcal{A}$ is locally finite.*

Proof. Let $\{a_1, \dots, a_k\} \subseteq \mathcal{A}$. We will show that the subalgebra of $\mathcal{A}$ generated by this set is finite-dimensional. We will denote $\{\overline{a}_1, \dots, \overline{a}_k\}$ to be the image of this set under the map $\mathcal{A} \to \mathcal{A}/\mathcal{C}$. Since $\mathcal{A}/\mathcal{C}$ is locally finite, the subalgebra generated by $\{\overline{a}_1, \dots, \overline{a}_k\}$ is finite-dimensional. Let $\{\overline{a_1}, \dots, \overline{a}_k, \overline{a}_{k+1}, \dots, \overline{a}_n\}$ be a spanning set of this subalgebra. From this every element in $\mathcal{A}/\mathcal{C}$ may be expressed as $\sum \alpha_i \overline{a}_i$, $(\alpha_i \in \mathcal{F})$. The multiplication of two elements $\overline{a}_i, \overline{a}_j \in \mathcal{A}/\mathcal{C}$ may be expressed as $\overline{a}_i \overline{a}_j = \sum \alpha_{ijl} \overline{a_l}$. Using the properties of cosets, a inverse image of this product will be $a_i a_j = \sum \alpha_{ijl} a_l + c_{ij}$, $(c_{ij} \in \mathcal{C})$.

Denote $\mathcal{S}$ to be the subalgebra generated by $\{a_1, \dots, a_n\}$. It will be beneficial for the reader to observe the product of elements in $\mathcal{A}$. We see that,

$$\begin{aligned}(a_ia_j)a_q &= (\sum \alpha_{ijk}a_k + c_{ij})a_q \\ &= \sum \alpha_{ijk}a_ka_q + c_{ij}a_q \\ &= \sum \alpha_{ijk}(\beta_{kqp}a_p + c_{kq}) + c_{ij}a_q \\ &= \sum (\alpha_{ijk}\beta_{kqp})a_p + \alpha_{ijk}c_{kq} + c_{ij}a_q.\end{aligned}$$

In addition,

$$a_i(a_ja_q) = \sum (\alpha_{ipk}\beta_{jqp})a_k + \beta_{jqp}c_{ip} + a_ic_{jq}.$$

Lastly, we need to consider the product above by an additional a_r,

$$(a_ia_ja_q)a_r = \sum (\alpha_{ipk}\beta_{jqp}\gamma_{krs})a_s + (\alpha_{ipk}\beta_{jqp})c_{pq} + \beta_{jqp}c_{ip}a_r + a_ic_{jq}a_r.$$

We will let $\mathcal{Q}$ be the subalgebra generated by $\{c_{ij}, a_ic_{jq}, c_{ij}a_q, a_ic_{jq}a_r\}$. Since each of these elements are in $\mathcal{C}$, a locally finite ideal, $\mathcal{Q}$ is finite dimensional. In addition $\mathcal{Q}$ is a subspace of $\mathcal{S}$ in which clearly $qa, aq \in \mathcal{Q}$ ($a \in \mathcal{A}, q \in \mathcal{Q}$), hence is an ideal of $\mathcal{S}$.

Consider the map $\mathcal{S} \to \mathcal{S}/\mathcal{Q}$, sending $a \mapsto \overline{\overline{a}} = a + \mathcal{S}$. Since $\mathcal{C} \subseteq \mathcal{Q}$, the image of the product of a_ia_j is $\overline{\overline{a}}_i\overline{\overline{a}}_j = \sum \alpha_{ijk}\overline{\overline{a}}_k$, in which we see that $\{\overline{\overline{a}}_i, \dots, \overline{\overline{a}}_n\}$ is a spanning set of $\mathcal{S}/\mathcal{Q}$. This set of vectors can be reduced to a finite basis by removing any vectors that are linearly dependent, thus $\mathcal{S}/\mathcal{Q}$ is finite-dimensional over $\mathcal{F}$. Since $\mathcal{S}/\mathcal{Q}$ and $\mathcal{Q}$ are finite-dimensional over $\mathcal{F}$ from Lemma 6.5 we may conclude that $\mathcal{S}$ is finite-dimensional over $\mathcal{F}$. This proves that $\mathcal{A}$ is locally finite. □

Lemma 6.8. *Let $\phi\colon \mathcal{U} \to \mathcal{V}$ be a ring homomorphism. If $\mathcal{U}$ is locally finite, then the image of $\mathcal{U}$ under ϕ, $\phi(\mathcal{U})$ is locally finite.*

We will leave the proof of this lemma to the reader.

Proposition 6.9. *Let $\mathcal{U}$ and $\mathcal{V}$ be a locally finite ideals of $\mathcal{A}$, then $\mathcal{U} + \mathcal{V}$ is locally finite.*

Proof. Define $\phi\colon \mathcal{U} + \mathcal{V} \to \mathcal{U}/\mathcal{U} \cap \mathcal{V}$, sending $u + v \mapsto u + \mathcal{U} \cap \mathcal{V}$. Note that this map is surjective, since for any coset $u' + \mathcal{U} \cap \mathcal{V}$, we have that, $\phi(u' + v) = u' + \mathcal{U} \cap \mathcal{V}$. Next, we will show that ϕ is a homomorphism. Let $u + v, u' + v' \in \mathcal{U} + \mathcal{V}$ then,

$$\begin{aligned}\phi[(u+v)+(u'+v')] &= \phi[(u+u')+(v+v')] = (u+u')+\mathcal{U}\cap\mathcal{V}\\ &= u+\mathcal{U}\cap\mathcal{V}+u'+\mathcal{U}\cap\mathcal{V}\\ &= \phi(u+v)+\phi(u'+v').\end{aligned}$$

In addition, we may note that the kernel of this mapping satisfies, Ker(ϕ) $=\mathcal{V}$. From the standard isomorphism theorem,

$$\mathcal{U}+\mathcal{V}/Ker\phi \cong \phi(\mathcal{U}+\mathcal{V})$$

Since ϕ is surjective we have,

$$\mathcal{U}+\mathcal{V}/\mathcal{V} \cong \mathcal{U}/\mathcal{U}\cap\mathcal{V}.$$

From the use of Lemma 6.8 and the fact that $\mathcal{U}/\mathcal{U}\cap\mathcal{V}$ is a homomorphic image of $\mathcal{U}$, we see that $\mathcal{U}/\mathcal{U}\cap\mathcal{V}$ is locally finite. Applying this lemma to the isomorphism above we may conclude that $\mathcal{U}+\mathcal{V}/\mathcal{V}$ is locally finite. By hypothesis $\mathcal{V}$ is locally finite. Then by applying Theorem 6.7 implies that $\mathcal{U}+\mathcal{V}$ is locally finite. □

Proposition 6.10. *For every algebra $\mathcal{A}$ there exists a maximal locally finite ideal which contains all locally finite ideals of $\mathcal{A}$.*

Proof. We will first show the existence of a maximal locally finite ideal of $\mathcal{A}$ then proceed to show it contains all locally finite ideals of $\mathcal{A}$. Let $\mathcal{W}$ be a collection of locally finite ideals of $\mathcal{A}$. One may verify that $\mathcal{W}$ is a po-set related by containment. Let us denote $\mathcal{C} = \{\mathcal{W}_i \mid \mathcal{W}_i \subseteq \mathcal{W}\}$ to be a totally ordered subset of $\mathcal{W}$. We will show that $\bigcup_{\mathcal{C}} \mathcal{W}_i$ is a upper bound of $\mathcal{C}$. If X is a finite subset of $\bigcup_{\mathcal{C}} \mathcal{W}_i$, then since $\mathcal{C}$ is totally ordered there exists a $\mathcal{W}_\beta$ of $\mathcal{A}$ such that $X \subseteq \mathcal{W}_\beta$, thus $\langle X\rangle$ is finite-dimensional. This implies that $\bigcup_{\mathcal{C}} \mathcal{W}_i$ is an element of $\mathcal{C}$. In addition, $\bigcup_{\mathcal{C}} \mathcal{W}_i$ clearly contains $W_j \in \mathcal{C}$, thus is an upper bound of $\mathcal{C}$. Since every totally ordered subset of $\mathcal{W}$ contains an upper bound, Zorn's Lemma may be applied. As a result, there exists a maximal locally finite ideal denoted as $L(\mathcal{A})$.

Next we will show that $L(\mathcal{A})$ contains all locally finite ideals of $\mathcal{A}$. Let $\mathcal{U}$ be a locally finite ideal of $\mathcal{A}$. By Proposition 6.9, $L(\mathcal{A})+\mathcal{U}$ is also a locally finite ideal of $\mathcal{A}$. As a result, $L(\mathcal{A}) \subseteq L(\mathcal{A})+\mathcal{U}$. By maximality we see that $L(\mathcal{A}) = L(\mathcal{A})+\mathcal{U}$, which implies that $\mathcal{U} \subseteq L(\mathcal{A})$. This completes the proof. □

From here on we will denote $L(\mathcal{A})$ to be the maximum locally finite ideal of $\mathcal{A}$.

Corollary 6.11. *$\mathcal{A}$ is locally finite if and only if $L(\mathcal{A}) = \mathcal{A}$.*

The proof of this theorem follows directly from Proposition 6.10. We will leave the proof to the reader.

Theorem 6.12. $L(\mathcal{A}/L(\mathcal{A})) = \{0\}$.

Proof. Consider the homomorphism $v\colon \mathcal{A} \to \mathcal{A}/L(\mathcal{A})$. Let $\overline{C}$ be a locally finite ideal of $\mathcal{A}/L(\mathcal{A})$. From the correspondence theorem $\overline{C} = C/L(\mathcal{A})$ for some ideal C of $\mathcal{A}$ containing $L(\mathcal{A})$. With $\overline{C}$ and $L(\mathcal{A})$ as locally finite ideals of $\mathcal{A}$, from Proposition 6.7 it may be concluded that C is locally finite as well. Since C is locally finite, $C \subseteq L(\mathcal{A})$ and as a result $L(\mathcal{A}) = C$. This implies that the locally finite ideal, $\overline{C} = C/L(\mathcal{A}) = \{0\}$. We see that every locally finite ideal of $\mathcal{A}/L(\mathcal{A})$ is $\{0\}$. Thus it follows from Proposition 6.10 that $L(\mathcal{A}/L(\mathcal{A})) = \{0\}$. □

In Proposition 6.10 it was proven that $L(\mathcal{A})$ contains all locally finite ideals of $\mathcal{A}$. We will look to extend this result by proving $L(\mathcal{A})$ contains all one-sided locally finite ideals of $\mathcal{A}$ as well.

Proposition 6.13. *Let $\mathcal{U}$ be a locally finite right (or left) ideal of $\mathcal{A}$ then $\mathcal{U} \subseteq L(\mathcal{A})$.*

Proof. Let $\overline{\mathcal{U}}$ be the image of the right ideal $\mathcal{U}$ under the homomorphism $v : \mathcal{A} \to \mathcal{A}/L(\mathcal{A}) = \overline{\mathcal{A}}$. Since $L(\overline{\mathcal{A}}) = L(\mathcal{A}/L(\mathcal{A})) = \{0\}$, if $\overline{\mathcal{U}}$ is locally finite as a two-sided ideal $\overline{\mathcal{U}} = \{0\}$. From the properties of cosets this would imply that $\mathcal{U} \subseteq L(\mathcal{A})$, hence being locally finite. With this we see that the proof of this theorem reduces to showing that the locally finite right ideal $\overline{\mathcal{U}} = \{0\}$.

Since v is a surjective map $\overline{\mathcal{U}}$ is a right ideal of $\overline{\mathcal{A}}$. To prove that $\overline{\mathcal{U}} = \{0\}$ we will first show that the ideal $\overline{\mathcal{A}}\,\overline{\mathcal{U}}$ is locally finite ideal of $\overline{\mathcal{A}}$. Let $\{x_1, \ldots, x_n\}$ be a non-empty subset of $\overline{\mathcal{A}\mathcal{U}}$. Then

$$\begin{aligned} x_i &= \sum a_{ij}u_{ij}, \quad a_{ij} \in \overline{\mathcal{A}}, u_{ij} \in \overline{\mathcal{U}} \quad \text{and,} \\ x_i x_k &= \sum a_{ij}u_{ij}a_{kl}u_{kl}. \end{aligned}$$

We will denote $u_{ij}a_{kl} = q_{ijkl}$. Let W be the subalgebra generated by $\{q_{ijkl}, u_{kl}\}$. From Proposition 5.7, $\overline{\mathcal{U}}$ is locally finite. In addition, $q_{ijkl} \in \overline{\mathcal{U}}$ so the subalgebra W is finite-dimensional over $\mathcal{F}$. The product $x_i x_k = \sum a_{ij} q_{ijkl} u_{kl} \subseteq \sum a_{ij} W$. Let $Q = \sum a_{ij} W$. Since W is finite-dimensional and Q is a finite sum, it follows that Q is finite dimensional and that it contains any product any two x_i 's. To show that Q is closed under multiplication it will suffice just to compute the following

$$\begin{aligned} x_t x_i x_k &\subseteq x_t \sum a_{ij} W \\ &\subseteq (\sum a_{tp} u_{tp}) \sum a_{ij} W = \sum a_{tp} q_{tpij} W \\ &\subseteq \sum a_{tp} W \\ &\subseteq Q. \end{aligned}$$

Since the product of any collection of x's is contained in Q, the subalgebra generated by $\{x_1, \ldots, x_n\}$ will be contained in this finite dimensional vector space as well. Thus the subalgebra generated by $\{x_1, \ldots, x_n\}$ is finite-dimensional over $\mathcal{F}$. It has been shown that $\overline{\mathcal{A}}\,\overline{\mathcal{U}}$ is a locally finite ideal of $\overline{\mathcal{A}}$ hence $\overline{\mathcal{A}}\,\overline{\mathcal{U}} = \{0\}$. With the additional fact that $\overline{\mathcal{U}}$ is a right ideal of $\overline{\mathcal{A}}$ we see that $\overline{\mathcal{U}}$ is a two sided ideal of $\overline{\mathcal{A}}$. Since $\overline{\mathcal{U}}$ is locally finite, we see that $\overline{\mathcal{U}} = \{0\}$. In reference to the earlier remarks, we have that $U \subseteq L(\mathcal{A})$, the desired result. □

Theorem 6.14. *Let $\mathcal{A} \neq \{0\}$ be finitely generated algebraic algebra that satisfies a polynomial identity. If $\mathcal{A}$ contains no nilpotent elements then $L(\mathcal{A}) \neq \{0\}$.*

Proof. Since $\mathcal{A}$ is algebraic, from Proposition 4.23 $\mathcal{J}(\mathcal{A})$ is nil. With the additional hypothesis that $\mathcal{A}$ contains no nilpotent elements, we may conclude that $\mathcal{J}(\mathcal{A}) = \{0\}$. Since $\mathcal{J}(\mathcal{A}) \neq \mathcal{A}$, there exists an irreducible $\mathcal{A}$-module $\mathcal{N}$. From Proposition 2.16 we see that $\mathcal{N}$ is a faithful $\mathcal{A}/Ann(\mathcal{N})$-module. A problem that may occur is $\mathcal{N}$ over $\mathcal{A}/Ann(\mathcal{N})$ may not be irreducible. Since from Proposition 2.7 the submodules of $\mathcal{N}$ over $\mathcal{A}$ correspond to the submodules of $\mathcal{N}$ over $\mathcal{A}/Ann(\mathcal{N})$, $\mathcal{N}$ is a faithful irreducible $\mathcal{A}/Ann(\mathcal{N})$-module. Generally, we may conclude that there exists an ideal $\mathcal{I}$ such that $\mathcal{A}/\mathcal{I}$ is primitive.

Let $v : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I}$, sending $a \mapsto \overline{a}$. By hypothesis, $\mathcal{A}$ satisfies a polynomial identity. Since v is a homomorphism, from Lemma 5.7 $\mathcal{A}/\mathcal{I}$ satisfies the same polynomial identity. With the previous paragraph in mind, we have shown that $\mathcal{A}/\mathcal{I}$ is a primitive algebra that satisfies a polynomial identity. From Kaplansky's Theorem, $\mathcal{A}/\mathcal{I}$ is finite dimensional over its center $\mathcal{Z}(\mathcal{A}/\mathcal{I})$.

From the hypothesis $\mathcal{A}$ is generated by some finite set $\{a_1, \ldots, a_k\}$. It is evident that the image of the set under the mapping v,$\{\overline{a_1}, \ldots, \overline{a_k}\}$ generates $\mathcal{A}/\mathcal{I}$. Since $\mathcal{A}/\mathcal{I}$ is finitely generated, we may now apply Proposition 6.4 and conclude that $\mathcal{A}/\mathcal{I}$ is finite-dimensional over $\mathcal{F}$.

Let $\{\overline{e}_1, \ldots, \overline{e}_m\}$ be a basis of $\mathcal{A}/\mathcal{I}$. Although v is not an injective map, we may still determine a inverse image $\{e_1, \ldots, e_m\}$ of $\{\overline{e}_1, \ldots, \overline{e}_m\}$. For an $\overline{a}_i \in \mathcal{A}/\mathcal{I}$, we have $\overline{a}_i = \sum \alpha_{ij}\, \overline{e}_j, (\alpha_{ij} \in \mathcal{F})$. In $\mathcal{A}$, this element is of the form $a_i - \sum \alpha_{ij} e_j = b_i, (b_i \in \mathcal{I})$. We have showed that

$$a_i = \sum \alpha_{ij} e_j + b_i.$$

Similarly for $e_i\, e_j \in \mathcal{A}$,

$$e_i e_j = \sum \beta_{ijk} e_j + b_{ij} \quad b_{ij} \in \mathcal{I}.$$

Let $\mathcal{I}'$ be an ideal of $\mathcal{A}$ generated by finite set $\{b_i, b_{ij}\}$, $i, j = 1, \ldots, n$. We will now prove that $\mathcal{I} = \mathcal{I}'$, showing that $\mathcal{I}$ is finitely generated. Clearly the generators of $\mathcal{I}'$ imply $\mathcal{I}' \subseteq \mathcal{I}$. Let $a \in \mathcal{I}$ then from above we may conclude that $a = \sum \gamma_i e_i + b'$ $(b' \in \mathcal{I}'$, $\gamma_i \in \mathcal{F})$. Since $a, b' \in \mathcal{I}$, $a - b' = \sum \gamma_i e_i \in \mathcal{I}$. The image $\overline{a - b'} = \sum \gamma_i\, \overline{e_i} = \overline{0}$. The $\overline{e_i}$'s form a basis for $\mathcal{A}/\mathcal{I}$. By linear independence $\gamma_i = 0$ for every i, which results in $a = b'$, hence $\mathcal{I} \subseteq \mathcal{I}$. This proves that $\mathcal{I} = \mathcal{I}'$.

From Proposition 4.26 there is an idempotent element $u \in \mathcal{Z}(\mathcal{I})$ where $ub_i = b_i$ and $ub_{ij} = b_{ij}$ for all i, j. With this idempotent element we can apply the *left Pierce decomposition* [Jac64, p48] of $\mathcal{A}$ which results in

$$\mathcal{A} = \mathcal{A}u \oplus \mathcal{A}(1-u)$$

where $\mathcal{A}u$ and $\mathcal{A}(1-u) = \{a - au \mid a \in \mathcal{A}\}$ are two sided ideals of $\mathcal{A}$. Since u is in the center of $\mathcal{I}$, $\mathcal{I} = \mathcal{A}u$ and the left Pierce decomposition can be reduced to,

$$\mathcal{A}/\mathcal{I} = \mathcal{A}/\mathcal{A}u \cong \mathcal{A}(1-u).$$

$\mathcal{A}/\mathcal{I}$ corresponds to a two sided ideal $\mathcal{A}(1-u)$ of $\mathcal{A}$. It was previously proven that $\mathcal{A}/\mathcal{I}$ is finite dimensional over $\mathcal{F}$. Its isomorphic image $\mathcal{A}(1-u)$ must also be finite dimensional, hence locally finite. We may conclude that $\mathcal{A}(1-u)$ is a non-zero locally finite ideal of $\mathcal{A}$ which is contained in $L(\mathcal{A})$. We have proven that $L(\mathcal{A}) \neq \{0\}$. □

Chapter 7

Kurosh's Theorem

7.1 Overview

Recall that a finitely generated algebra is not necessarily finite-dimensional. There are numerous examples that can confirm this. In particular the algebra $\mathcal{F}[x]$ is not finite-dimensional but is generated as an algebra by 1 and x. Conversely there are natural examples in which a finitely generated algebra is finite-dimensional. In the case in which $\mathcal{A}$ is algebraic and commutative, any finite subset of $\mathcal{A}$ generates a finite-dimensional algebra. In general is there a condition that is both necessary and sufficient to ensure that an algebra is locally finite?

Alekander Kurosh in 1962 discovered that an algebra that is algebraic and satisfies a polynomial identity is locally finite. Note that $\mathcal{F}[x]$ is not locally finite, nor is it algebraic, since there does not exists a non-zero polynomial that x satisfies. We will conclude our exposition by proving Kurosh's Theorem.

7.2 Kurosh's Theorem

Theorem 7.1. (Kurosh's Theorem) *Let $\mathcal{A}$ be an algebraic algebra over field $\mathcal{F}$ that satisfies a polynomial identity, then $\mathcal{A}$ is locally finite.*

Proof. From Proposition 5.6 we may assume that the polynomial identity is multilinear. Since any finitely generated subalgebra of $\mathcal{A}$ will satisfy the same identity we may assume that $\mathcal{A}$ is finitely generated. Our argument will be reduced to proving that a finitely generated algebraic algebra $\mathcal{A}$ that satisfies a multilinear polynomial identity of degree d is locally finite.

Recall that the maximum locally finite ideal of $\mathcal{A}$, $L(\mathcal{A})$ contains all locally finite ideals of $\mathcal{A}$. From Corollary 6.11, $\mathcal{A}$ is locally finite if and only if $L(\mathcal{A}) = \mathcal{A}$. In this proof we will consider the quotient $\overline{\mathcal{A}} = \mathcal{A}/L(\mathcal{A})$, and arrive at the conclusion that $\overline{\mathcal{A}} = \{0\}$ which will result in $L(\mathcal{A}) = \mathcal{A}$.

From Theorem 6.12 we have that $L(\overline{\mathcal{A}}) = L(\mathcal{A}/L(\mathcal{A})) = \{0\}$. We will assume that $\overline{\mathcal{A}} \neq \{0\}$ and distinguish two cases both resulting in $L(\overline{\mathcal{A}}) \neq \{0\}$, which is a contradiction. This will show that $\overline{\mathcal{A}} = \{0\}$ and prove that $\mathcal{A}$ is locally finite. We will proceed with the first case.

Case 1. ($\overline{\mathcal{A}}$ contains no non-zero nilpotent elements)

Since $\overline{\mathcal{A}}$ is the homomorphic image of the natural map, $v : \mathcal{A} \to \mathcal{A}/L(\mathcal{A})$, $\overline{\mathcal{A}}$ satisfies the polynomial identity of $\mathcal{A}$. Let $f(x_1, \ldots, x_d)$ be the identity satisfied by $\overline{\mathcal{A}}$. By assumption $\mathcal{A}$ is generated by some non-empty set $\{a_1, \ldots, a_k\}$ and the image of these elements under v will also generate $\overline{\mathcal{A}}$. Thus $\overline{\mathcal{A}}$ is finitely generated algebraic algebra that satisfies an identity. From Theorem 6.14 we have that $L(\overline{\mathcal{A}}) \neq \{0\}$.

Case 2. ($\overline{\mathcal{A}}$ contains a non-zero nilpotent element)

We may assume there exists a non-zero nilpotent element $u \in \overline{\mathcal{A}}$ is such that $u^2 = 0$. We will look at the left ideal $\overline{\mathcal{A}}u = \{\overline{a}u \mid \overline{a} \in \overline{\mathcal{A}}\}$ of $\overline{\mathcal{A}}$.

If $\overline{\mathcal{A}}u = \{0\}$, then $u \in Ann(\overline{\mathcal{A}})$. Note that $Ann(\overline{\mathcal{A}})$ is locally finite since for any finite subset $\{u_1, u_2, \ldots, u_k\} \subseteq Ann(\overline{\mathcal{A}})$ the subalgebra generated by this set will be spanned by $\{u_1, u_2, \ldots, u_k\}$ $(u_i u_j = 0,\ i, j = 1, 2, \ldots n)$. Hence $Ann(\overline{\mathcal{A}})$ is a non-empty locally finite ideal of $\overline{\mathcal{A}}$. This implies $L(\overline{\mathcal{A}}) \neq \{0\}$.

In the case that $\overline{\mathcal{A}}u \neq \{0\}$, we will use induction on the degree of the polynomial identity to show that $\overline{\mathcal{A}}u$ is locally finite and is thus contained in $L(\overline{\mathcal{A}})$. As previously stated we may assume $\overline{\mathcal{A}}$ to satisfy a multilinear polynomial identity $f(x_1, \cdots, x_d)$ of degree d. The proof will proceed by induction on d.

If $d = 2$, then the polynomial identity that is satisfied by $\overline{\mathcal{A}}$ is of the form $f(x_1, x_2) = \alpha x_1 x_2 + \beta x_2 x_1$ $(\alpha, \beta \in \mathcal{F})$. As a result $\overline{\mathcal{A}}$ is either commutative or $\overline{\mathcal{A}}^2 = \{0\}$. In the former case, Proposition 6.3 may be used to prove that $\overline{\mathcal{A}}$ is locally finite. If $\overline{\mathcal{A}}^2 = \{0\}$ then clearly any subalgebra generated by a finite subset of $\overline{\mathcal{A}}$ will be finite-dimensional.

We will now assume that every algebraic algebra that satisfies a polynomial identity of degree less than d is locally finite. By assumption $\overline{\mathcal{A}}$ satisfies a multilinear identity of degree d. Since $\overline{\mathcal{A}}u$ is a subspace of $\overline{\mathcal{A}}$ it satisfies f as well. Next, we will decompose f into

$$f(x_1, \ldots, x_d) = x_1 g(x_2, \ldots, x_d) + h(x_1, \ldots, x_d)$$

where x_1 never appears first in any of the monomials in h. Setting $x_1 = u$, $x_2 = \overline{a_2}u, x_3 = \overline{a_3}u, \ldots,\ x_d = \overline{a_d}u$, we have

$$f(u, \overline{a_2}u, \ldots, \overline{a_d}u) = ug(\overline{a_2}u, \ldots, \overline{a_d}u) + h(u, \ldots, \overline{a_d}u).$$

From the fact that h is multilinear the evaluation $h(u, \overline{a_2}u, \ldots, \overline{a_d}u) = 0$. This is because every monomial in the summand has a factor of the form $x_j x_1$ which substitutes to $(\overline{a_j}u)u = \overline{a_j}u^2 = 0$ for some $j = 2, \ldots, d$. We are left with $0 = f(u, \ldots, \overline{a_d}u) = ug(\overline{a_2}u, \ldots, \overline{a_d}u)$.

In $\overline{\mathcal{A}}u$, let $\mathcal{D} = \{x \in \overline{\mathcal{A}}u \mid ux = 0\}$. Since $(\overline{\mathcal{A}}u)\mathcal{D} = \{0\}$, $\mathcal{D}$ is a two sided ideal of $\overline{\mathcal{A}}u$. In addition $\mathcal{D}^2 \subseteq (\overline{\mathcal{A}}u)\mathcal{D}$, hence $\mathcal{D}^2 = \{0\}$ and from the above remarks we may conclude that $\mathcal{D}$ is locally finite. It follows directly from $\mathcal{D}$ that $\overline{\mathcal{A}}u/\mathcal{D}$ satisfies $g(x_2, \ldots, x_d)$ under the map sending $\overline{a}u \mapsto \overline{a}u + \mathcal{D}$.

Since $\overline{\mathcal{A}}u/\mathcal{D}$ satisfies a polynomial of degree $d-1$, by induction we can conclude that $\overline{\mathcal{A}}u/\mathcal{D}$ it is locally finite. Since $\mathcal{D}$ is locally finite, by Proposition 6.7 it follows that $\overline{\mathcal{A}}u$ is locally finite. We have proven that $\overline{\mathcal{A}}u$ is a non-empty locally finite left ideal of $\overline{\mathcal{A}}$. From Proposition 6.13, $\overline{\mathcal{A}}u \subseteq L(\overline{\mathcal{A}})$ which implies that $L(\overline{\mathcal{A}}) \neq \{0\}$.

From both cases it has been shown that if $L(\overline{\mathcal{A}}) \neq \{0\}$, but from Theorem 6.12, $L(\overline{\mathcal{A}}) = L(\mathcal{A}/L(\mathcal{A})) = \{0\}$. For this to be valid it must be that $\overline{\mathcal{A}} = \{0\}$, which implies that $L(\mathcal{A}) = \mathcal{A}$ revealing to us that $\mathcal{A}$ is locally finite. $\square$

Chapter 8

Conclusion

The origin of Kurosh's initial question can be traced to the Burnside Problem. Similar to the Kurosh Theorem, the Burnside Problem's hypothesis is whether a group in which any finite collection of group elements all of which have finite order generates a finite group. In 1963 Golod and Shafarevitch introduced a technique in which a nilpotent algebra was constructed that is not locally finite. Thus showing that in absence of a polynomial identity an algebraic algebra may not be locally finite. With this, Kurosh's Theorem provides the necessary and sufficient conditions for an algebra to be locally finite. From the Golod/Shafarevitch result an analogous group may be constructed that provides a negative answer to the Burnside Problem. We will recommend [Her05, p.187-193] to the reader that is curious of Golod and Shafarevitch's construction of an algebraic algebra that is not locally finite.

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