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The Universal coefficient theorem for cohomology

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THE UNIVERSAL COEFFICIENT THEOREM FOR COHOMOLOGY

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

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Abstract

This project is an expository survey of the Universal Coefficient Theorem for Cohomology, which allows one to express cohomology, with arbitrary coefficients, of a chain complex in terms of homology with integer (universal) coefficients. Algebraic preliminaries, homology, and cohomology are discussed prior to the proof of the theorem. The theorem's importance will be illustrated through a number of examples.

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Chapter 1

Introduction

The following work investigates the Universal Coefficient Theorem for Cohomology. This theorem says that the cohomology groups, with arbitrary coefficients, for a given chain complex $C = \{C_n, \partial\}$ of abelian groups can be expressed in terms of homology groups with integer coefficients. There are other Universal Coefficient Theorems to be studied, which allow one translate data from integer homology or cohomology to homology or cohomology with arbitrary coefficients; however, we focus on examining the Universal Coefficient Theorem for Cohomology.

To understand the significance of the Universal Coefficient Theorem, one must understand how homology is used in Algebraic Topology. Algebraic topologists use homology among other tools to analyze toplogical spaces in an algebraic setting as an alternative to applying a less practical and intuitive based geometric approach. Homology groups partition a topological space into homology classes (equivalence classes) that are designed to capture portraits of certain geometrical data. This in turn, allows one to differentiate between objects based on whether or not they share this homological data [Hil88]. In a sense, cohomology can be thought of as the dual to homology (we will be more precise in Chapter 3), although there are interesting differences in the construction of cohomology that illuminate different characteristics of the space. Cohomology is also used to examine topological spaces in a similar way, but it may possess information for a space that homology does not. For instance, there is an interesting ring structure in cohomology for general topological spaces that can distinguish spaces such as $S^1 \vee S^1 \vee S^2$ and $S^1 \times S^1$, but we will not discuss this structure in this thesis.

Given a topological space X, an algebraic topologist wishing to obtain algebraic topological data for the space could compute both homology and cohomology. We will see that the process for computing cohomology is essentially the same to that of homology; however, cohomology requires an intermediate step not required for homology. One could calculate both homology and cohomology for X, but it would be nice to have the machinery to calculate, say homology, and then be able to transfer that data in a way to obtain cohomology. The Universal Coefficient Theorem is the tool for this job.

In the ensuing chapters, we define and prove the preliminary notions required to not only prove the theorem, but to also understand its meaning. These topics form a prelude to the theorem, yet one should note that each may lead to a discussion on their own. For this study, we will focus on how the preliminaries build and are related to the Universal Coefficient Theorem.

After completing the preliminaries we turn our attention to the proof of the Universal Coefficient Theorem and its applications. The proof is quite lengthy, so Chapter 4 has been devoted to the proof.

The following is a brief outline of the thesis. In Chapter 2 we introduce the topic of R-modules, and investigate many of the algebraic ideas associated with it through proofs and examples. Chapter 3 will focus on defining homology and cohomology, supplemented by a series of examples and proofs, which will aid in understanding key notions pertinent to the proof of the Universal Coefficient Theorem. Chapter 4 features the proof of the Universal Coefficient Theorem, which has been divided into two parts. Finally, in Chapter 5 we illustrate the importance of the theorem through a number of examples.

Chapter 2

Algebra

As mentioned before, homology allows one to study topological spaces from an algebraic perspective. As a result, we begin our studies with some algebraic preliminaries. First, in Section 2.1, we define what it is to be a module over a ring R, and investigate special cases. In Section 2.2, we examine sequences of abelian groups and define exactness. In Section 2.3, we define free and projective modules over a ring, and discuss necessary and sufficient conditions to determine whether they are the same. Finally, in Section 2.4, we consider Hom groups, and learn how to map between them using duality.

2.1 R-Modules

Definition 2.1.1. [Hun74] *Let R be a ring. A (left) R-module is an additive abelian group A together with a function $R \times A \to A$ such that given any $r, s \in R$ and $a, b \in A$:*

1. $r(a + b) = ra + rb$;
2. $(r + s)a = ra + sa$;
3. $r(sa) = (rs)a$.

If R has an identity element 1_R and $1_R a = a$ for all $a \in A$ then A is said to be a unitary R-module.

In Definition 2.1.1, we see that the ring R acts on the abelian group A, which may or may not be closed under the operation of R. That said, one may wonder how ring elements act on the additive identity of A.

Lemma 2.1.2. *Suppose A is an R-module over a ring, and let 0_A be the additive identity of A. Then $r0_A = 0_A$ for all $r \in R$.*

Proof. Let 0_A be the additive identity of an R-module A, and let $r \in R$. Then we have $r0_A = r(0_A + 0_A) = r0_A + r0_A$. Now subtracting $r0_A$ from both sides yields $r0_A = 0_A$. $\square$

Modules over a ring R are abelian groups with additional algebraic structure. Every additive abelian group is a $\mathbb{Z}$-module, where $\mathbb{Z}$ is the integers (see Theorem 2.1.3); however, any arbitrary abelian group is not necessarily an R-module for certain R. In the following theorem, we show that any additive abelian group is a $\mathbb{Z}$-module with a very natural module structure. This fact will prove to be useful in Chapter 4.

Theorem 2.1.3. [Hun74] *Let G be an additive abelian group. Then G is a unitary $\mathbb{Z}$-module under the action $(n, g) \mapsto ng$ for all $n \in \mathbb{Z}$ and $g \in G$, where ng is repeated additions of g.*

Proof. We show that the action $\mathbb{Z} \times G \to G$ given by $(n, g) \mapsto ng$ satisfies Definition 2.1.1. Let $n, m \in \mathbb{Z}$ and $g, h \in G$. Then

$$
\begin{aligned}
n(g+h) &= \underbrace{(g+h)+(g+h)+\ldots+(g+h)}_{n-many} \\
&= \underbrace{(g+g+\ldots+g)}_{n-many} + \underbrace{(h+h+\ldots+h)}_{n-many} \\
&= ng + nh,
\end{aligned}
$$

$$
\begin{aligned}
(n+m)g &= \underbrace{g+g+\ldots+g}_{(n+m)-many} \\
&= \underbrace{(g+g+\ldots+g)}_{n-many} + \underbrace{(g+g+\ldots+g)}_{m-many} \\
&= ng + mg,
\end{aligned}
$$

and finally,

$$\begin{aligned} n(mg) &= n(\underbrace{g+g+\ldots+g}_{m-many}) \\ &= \underbrace{(\underbrace{g+g+\ldots+g}_{m-many}) + (\underbrace{g+g+\ldots+g}_{m-many}) + \ldots (\underbrace{g+g+\ldots+g}_{m-many}))}_{n-many} \\ &= (nm)g \end{aligned}$$

So G is a $\mathbb{Z}$-module. To show it is unitary, we see that the multiplicative identity of $\mathbb{Z}$, namely $1 \in \mathbb{Z}$, functions as the element in $\mathbb{Z}$ where $1g = g$. □

With groups there are the notions of group homomorphisms and subgroups. In the next two definitions we define similar notions for R-modules, namely R-module homomorphisms and submodules.

Definition 2.1.4. [Hun74] *Let A and B be modules over a ring R. A function $f : A \to B$ is an R-module homomorphism provided that for all $a, c \in A$ and $r \in R$:*

1. $f(a+c) = f(a) + f(c)$;
2. $f(ra) = rf(a)$.

Let $f : A \to B$ be an R-module homomorphism. We define the kernel of f to be its kernel as an additive group homomorphism, that is, Ker $f = \{a \in A \mid f(a) = 0_B\}$. Similarly, define the image of f as Im $f = \{f(a) \mid a \in A\}$.

Definition 2.1.5 (Submodule). [Hun74] *Let R be a ring, and A an R-module and B a nonempty subset of A. B is said to be a submodule of A provided B is an additive subgroup of A and $rb \in B$ for all $r \in R$.*

A submodule of an R-module A is itself an R-module. Let $f : A \to B$ be an R-module homomorphism. Then it is easy to verify that Ker f and Im f are submodules of A and B, respectively. We now turn our attention to two special submodules which we will refer to in Chapters 3 - 5.

Example 2.1.6 (Torsion Submodule). Suppose A is an R-module over an integral domain R Then the collection $T(A) = \{a \in A \mid ra = 0_A$ for some nonzero $r \in R\}$ is a submodule of A. $T(A)$ is said to be the collection of all torsion elements of A.

Proof. Let 0_A be the additive identity of A. Then $0_A \in T(A)$ since $0_A \in A$ and given any $r \in R$, $r0_A = 0_A$ by Lemma 2.1.2. Therefore, $T(A)$ is nonempty. Let $a, b \in T(A)$. We show $a - b \in T(A)$, that is, we will find a nonzero $r \in R$ such that $r(a-b) = 0_A$. Now $a, b \in T(A)$ implies there exists nonzero $r_1, r_2 \in R$ so that $r_1 a = 0_A$ and $r_2 b = 0_A$. Since R is an integral domain, choose the nonzero $r_1 r_2 \in R$ with

$$r_1 r_2(a-b) = r_1(r_2 a - r_2 b) = r_1(r_2 a - 0_A) = r_2 r_1 a = r_2 0_A = 0_A.$$

Thus, $a-b \in T(A)$. To complete the proof let $r \in R$ and $a \in T(A)$. Now $a \in T(A)$ means $r_1 a = 0$ for some nonzero $r_1 \in R$. Again, since R is an integral domain $r_1 r = r r_1 \neq 0$, and $r_1(ra) = (r_1 r)a = (r r_1)a = r(r_1 a) = r0_A = 0_A$. So $ra \in T(A)$. Therefore, $T(A)$ is a submodule of A. □

Theorem 2.1.7. [Hun74] *Let B be a submodule of a module A over a ring R. Then the quotient group A/B is an R-module over the quotient group A/B under the action given by $r(a+B) = ra + B$ for all $a \in A, r \in R$. The map $\pi : A \to A/B$ given by $\pi(a) = a + B$ is called the canonical map (projection), and it is easy to see Ker $\pi = B$.*

The theorem allows us to define another module structure. The cokernel of an R-module homomorphism $f : A \to B$ is defined to be $\text{Coker}f = B/\text{Im}f$ which is an R-module. The cokernel will be used in the proof of the Universal Coefficient Theorem. In the next section, we use the fact that we can map between R-modules via R-module homomorphisms to construct a sequence of module homomorphisms.

2.2 Exact Sequences

Definition 2.2.1. [Hun74] *A sequence of module homomorphisms*

$$\cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \xrightarrow{f_{n+2}} \cdots$$

is said to be exact provided $Imf_n = Kerf_{n+1}$ for all $n \in \mathbb{Z}$.

If n is the degree of A_n in the sequence above, then we intend for an exact sequence to also be on which could decrease in degree. The following theorem supplies necessary and sufficient conditions for determining whether or not a given sequence of a special from, is exact without having to use Definition 2.2.1.

Theorem 2.2.2. [Hun74] *Let A and B be modules over a ring R. Then*

1. $0 \to A \xrightarrow{f} B$ *is exact if and only if f is a module monomorphism (injective).*

2. $B \xrightarrow{g} C \to 0$ *is exact if and only if g is a module epimorphism (surjective).*

Proof.

1. ($\Rightarrow$) Assume $0 \to A \xrightarrow{f} B$ is exact. Then $\text{Im } 0 = \{0\} = \text{Ker} f$. Hence, f is injective. ($\Leftarrow$) Assume f is injective. Then $\text{Ker} f = \{0\} = \text{Im } 0$. Hence, we have exactness at A.

2. ($\Rightarrow$) Assume $B \xrightarrow{g} C \to 0$ is exact. Then $\text{Im} g = \text{Ker } 0 = C$. Thus, g is surjective. ($\Leftarrow$) Assume g is surjective. Then $\text{Im} g = C = \text{Ker } 0$. Thus, we have exactness at C.

$\square$

Theorem 2.2.2 will aid us in determining whether or not a sequence of the form $0 \to A \to B \to C \to 0$ is exact at A and C. An exact sequence of such a form is of great importance to our studies, and is defined below.

Definition 2.2.3. [Hun74] *An exact sequence of the form* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is called a short exact sequence.*

Consider a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. By exactness, we know that $f : A \to B$ is injective and $g : B \to C$ is surjective. These observations lead to conditions (1) and (2) from the next theorem, yet we obtain a special short exact sequence, essential to proving and understanding the Universal Coefficient Theorem, when $B \cong A \oplus C$.

Theorem 2.2.4. [Hun74] *Let R be a ring and* $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ *a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent:*

1. *There is an R-module homomorphism $h : A_2 \to B$ such that $gh = 1_{A_2}$;*

2. *There is an R-module homomorphism $k : B \to A_1$ such that $kf = 1_{A_1}$;*

3. $B \cong A_1 \oplus A_2$.

Any short exact sequence of R-module homomorphisms satisfying any condition of Theorem 2.2.4 is said to be split exact. We will take Theorem 2.2.4 as fact without proof in this paper; refer to [Hun74] for a detailed proof. Next we establish the existence of a particular short exact sequence, which will be applied in Chapter 4.

Theorem 2.2.5. *Let A and B be $\mathbb{Z}$-modules, and $f : A \to B$ a surjective module homomorphism, then there exists the following short exact sequence*

$$0 \to \ Kerf \xrightarrow{i} A \xrightarrow{f} B \to 0,$$

where i is the inclusion map.

Proof. This is easily verified, using Theorem 2.2.2, and that $\operatorname{Im} i = \operatorname{Ker} f$. □

The proof of the Universal Coefficient Theorem and many of its preliminary notions will require examining special diagrams called commutative diagrams. The diagram found in Figure 2.1 is said to *commute* if for any $a \in A$, $g'f(a) = f'g(a)$, where f, f', g, and g' are homomorphisms. We use this defining property of commutative diagrams to prove the next theorem.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

Figure 2.1: Commutative Diagram

Theorem 2.2.6. *Consider the commutative diagram, where α_1, α_2, α_3 are isomorphisms:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha_1"', "\cong"] & B \arrow[r, "g"] \arrow[d, "\alpha_2"', "\cong"] & C \arrow[r] \arrow[d, "\alpha_3"', "\cong"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0. \end{tikzcd}$$

If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then so is $0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$.

Proof. By Theorem 2.2.2, to show exactness, it suffices to verify the following:

1. $f' : A' \to B'$ is injective.

2. $g' : B' \to C'$ is surjective.

3. $\mathrm{Im} f' = \mathrm{Ker} g'$.

1. Let $x \in \mathrm{Ker} f' \subset A'$. We will show $x = 0$. Now there exists a unique element $a \in A$ such that $a = \alpha_1^{-1}(x)$. Then $f(a) = f(\alpha_1^{-1}(x)) \in B$, and therefore, $\alpha_2 f(a) = \alpha_2 f \alpha_1^{-1}(x) \in B'$. So, by commutativity, $0 = f'(x) = \alpha_2 f \alpha_1^{-1}(x)$ which implies that $x = 0$ since $\alpha_2 f \alpha_1^{-1}$, being the composite of injective maps, is injective. Thus, f' is injective.
2. Let $c' \in C'$. Now there is a unique element $c \in C$ such that $c = \alpha_3^{-1}(c')$. By Theorem 2.2.2, $g : B \to C$ is surjective, so there exists $b \in B$ with $g(b) = c = \alpha_3^{-1}(c')$. Next $\alpha_2(b) = b'$ for a unique $b' \in B'$. So given $c' \in C'$ choose $b' \in B'$ so that $\alpha_3 g \alpha_2^{-1}(b') = c'$, then $g'(b') = \alpha_3 g \alpha_2^{-1}(b') = \alpha_3 g(b) = \alpha_3(c) = c'$. Therefore, g' is surjective.
3. ($\subseteq$) Let $f'(a') \in \mathrm{Im} f'$. Then $f\alpha_1^{-1}(a') \in \mathrm{Im} f = \mathrm{Ker} g$. By commutativity, we have $f'(a') = \alpha_2 f \alpha_1^{-1}(a')$. Now observe that for any $b' \in B'$, $g'(b') = \alpha_3 g \alpha_2^{-1}(b')$. Finally, $f'(a') \in B'$ and $g'(f'(a')) = \alpha_3 g \alpha_2^{-1}(\alpha_2 f \alpha_1^{-1}(a')) = \alpha_3(g f \alpha_1^{-1}(a')) = \alpha_3(0) = 0$. Thus, $f'(a') \in \mathrm{Ker} g'$. ($\supseteq$) Let $x \in \mathrm{Ker} g'$. Then $g'(x) = 0$, and by commutativity, $g\alpha_2^{-1}(x) = 0$. So $\alpha_2^{-1}(x) \in \mathrm{Ker} g = \mathrm{Im} f$. Thus, there exists $a \in A$ such that $f(a) = \alpha_2^{-1}(x)$. We claim that $f'(\alpha_1(a)) = x$. Now $f'(\alpha_1(a)) = \alpha_2 f(a) = \alpha_2(\alpha_2^{-1}(x)) = x$. So $x \in \mathrm{Im} f'$. □

We conclude this section by mentioning that any long exact sequence can be made into a sequence of short exact sequences and we describe that construction here. Consider the long exact sequence:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} C_{n-2} \xrightarrow{f_{n-2}} \cdots .$$

We show presently that the long exact sequence above can be made into the sequence of short exact sequences $0 \longrightarrow \mathrm{Coker} f_{n+2} \xrightarrow{\overline{f}_{n+1}} C_n \xrightarrow{f_n} \mathrm{Ker}\ f_{n-1} \longrightarrow 0$; we define the map $\overline{f}_{n+1}$ presently as well. By definition, $\mathrm{Coker} f_{n+2} = \frac{C_{n+1}}{\mathrm{Im} f_{n+2}}$. Now, by exactness, it follows that $\mathrm{Im} f_{n+2} = \mathrm{Ker} f_{n+1}$, and consequently, $f_{n+1}(\mathrm{Im} f_{n+2}) = f_{n+1}(\mathrm{Ker} f_{n+1}) = 0$. Thus, f_{n+1} induces a homomorphism $\overline{f}_{n+1}$ on $\frac{C_{n+1}}{\mathrm{Im} f_{n+2}} = \mathrm{Coker} f_{n+2}$. Observe that the sequence $C_n \xrightarrow{f_n} \mathrm{Ker} f_{n-1} \to 0$ is exact since $\mathrm{Im}\ f_n = \mathrm{Ker} f_{n-1}$. We verify exactness at $0 \to \mathrm{Coker} f_{n+2} \xrightarrow{\overline{f}_{n+1}} C_n$ by showing $\overline{f}_{n+1}$ is injective. Let $c + \mathrm{Im} f_{n+2} \in \mathrm{Coker} f_{n+2}$, where $c \in C_{n+1}$. Now $0 = \overline{f}_{n+1}(c + \mathrm{Im} f_{n+2}) = \overline{f}_{n+1}(c) = f_{n+1}(c)$, and $c \in \mathrm{Ker} f_{n+1}$.

Therefore, since $\text{Ker}f_{n+1} = \text{Im}f_{n+2}$, $\overline{f}_{n+1}$ is injective. Finally, we have exactness at $\text{Coker}f_{n+2} \xrightarrow{\overline{f}_{n+1}} C_n \xrightarrow{f_n} \text{Ker}f_{n-1}$ since $\text{Im}\,\overline{f}_{n+1} = \text{Im}f_{n+1} = \text{Ker}f_n$. Thus, the sequence in question is indeed short exact. The notion that a short exact sequence can be constructed from any long exact sequence plays a vital role in the proof of the Universal Coefficient Theorem, see Equation (4.3) of Chapter 4.

2.3 Free and Projective Modules

In this section we study two types of R-modules, namely, free and projective modules. We begin by examining each individually, then we show that over an arbitrary ring R they are not the same. Finally, we discover that free and projective modules coincide over a Principal Ideal Domain (PID).

Let A be an R-module, and let $\mathcal{B} = \{x \mid x \in A\}$ be a collection of elements of A. The set $\mathcal{B}$ is said to span A if for any $a \in A$, $a = r_1x_1 + r_2x_2 + \cdots r_kx_k$ where $r_i \in R$, and $x_i \in \mathcal{B}$. Now $\mathcal{B}$ is said to be linearly independent if for distinct $x_1, x_2, \ldots, x_n \in \mathcal{B}$ and $r_i \in R$, $r_1x_1 + r_2x_2 + \cdots r_nx_n = 0$ implies $r_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. A linearly independent subset of A which spans A is said to be a *basis* for A [Hun74].

Definition 2.3.1. [Hun74] *Let F be a module over a ring R with identity. F is said to be a free R-module if it has a nonempty basis.*

Since R-modules are abelian groups, one may be lead to believe that a submodule of a free module is free. However, in general, this is not the case when considering an arbitrary ring R, but it is the case when R is a principal ideal domain. Contrast the following result with Example 2.3.3.

Theorem 2.3.2. [Hun74] *Over a Principal Ideal Domain (PID), every submodule of a free module is free.*

We will take Theorem 2.3.2 as fact, but a proof may be found in [Hun74]. Notice that $\mathbb{Z}$ is a free $\mathbb{Z}$-module since it has a basis $\{1\}$. Now since $\mathbb{Z}$ is a PID, any submodule of free module is free, all submodules of $\mathbb{Z}$ are free. The next example shows that a submodule of a free module may not be free over an arbitrary ring R.

Example 2.3.3. Let $R = \mathbb{Z}_6$ and consider $2\mathbb{Z}_6 = \{0, 2, 4\}$. Then $2\mathbb{Z}_6$ is a submodule of $\mathbb{Z}_6$, but it is not free.

Proof. First observe that $2\mathbb{Z}_6 \subset \mathbb{Z}_6$ and $2\mathbb{Z}_6$ is an abelian group under addition. Let $[a]_6 \in \mathbb{Z}_6$ and $b \in 2\mathbb{Z}_6$. Now there is $[x]_6 \in \mathbb{Z}_6$ such that $b = 2[x]_6$. Notice that $[a]_6 b = 2[a]_6[x]_6 = 2[ax]_6 \in 2\mathbb{Z}_6$. Therefore $2\mathbb{Z}_6$ is a submodule of the $\mathbb{Z}_6$- module $\mathbb{Z}_6$. $\mathbb{Z}_6$ is not an integral domain, and therefore, not a PID. We claim $2\mathbb{Z}_6$ does not have a basis. Given any nonzero $a \in 2\mathbb{Z}_6$ one may choose $[b] = [3]_6 \in \mathbb{Z}_6$ such that $a[b]_6 = [0]_6$ even though $a \neq 0$. Thus $2\mathbb{Z}_6$ does not have a basis, and it is not free. □

Definition 2.3.4. [Hun74] *A module P over a ring R is said to be projective provided that given any diagram of R-module homomorphisms (with bottom row exact)*

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

there exists an R-module homomorphism $h : P \to A$ such that the diagram below commutes.

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "f"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

That is, $g \circ h = f$.

If an R-module F is free, then it is projective. However, in general, the converse is false. For example, $\mathbb{Z}_3$ is a projective $\mathbb{Z}_6$-module, yet $\mathbb{Z}_3$ is not a free $\mathbb{Z}_6$-module [Hun74].

In Theorem 2.3.6, we will establish a relationship between split exact sequences and projective modules. Before we introduce and prove Theorem 2.3.6 we refer to a useful fact about R-modules.

Lemma 2.3.5. [Hun74] *Every R-module A is the homomorphic image of a free R-module F.*

Theorem 2.3.6. [Hun74] *Let R be a ring and let P be a given R-module. Every short exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is split exact if and only if P is projective.*

Proof. ($\Rightarrow$) Assume $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is split exact. Given any diagram with bottom row exact

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

we must find an R-module homomorphism $h : P \to A$ so that $g \circ h = f$. First note that every projective module is the homomorphic image of a free module by Theorem 2.3.5. So there is a free module F and a surjective R-module homomorphism π such that $\pi(F) = P$. Let $K = \text{Ker}\ \pi$. Then $0 \to K \xrightarrow{i} F \xrightarrow{\pi} P \to 0$ is exact by Theorem 2.2.5. Now since F is free it is projective. So there exists an R-module homomorphism $p : P \to F$ such that $\pi \circ p = 1_P$ by Theorem 2.2.4. Now consider the diagram below.

$$\begin{tikzcd} & F \arrow[d, dotted, shift left, "\pi"] \arrow[ddl, dotted, "h_1"'] & \\ & P \arrow[u, shift left, "p"] \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

Since F is projective there is an R-module homomorphism $h_1 : F \to A$ with $g \circ h_1 = f\pi$. Set $h = h_1 \circ p : P \to A$. Then

$$g(h) = g(h_1 \circ p) = (g \circ h_1)(p) = (f\pi)(p) = f(\pi \circ p) = f \circ 1_P = f.$$

Therefore, P is projective.

($\Leftarrow$) Now suppose P is projective. Then by hypothesis for any diagram with bottom row exact

$$\begin{tikzcd} & P \arrow[d, "1_P"] \arrow[dl, dotted, "h"'] & \\ B \arrow[r, "g"] & P \arrow[r] & 0 \end{tikzcd}$$

there exists R-module homomorphism $h : P \to B$ such that $g \circ h = 1_P$. By Theorem 2.2.4, the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is split exact. This completes the proof. $\square$

We complete this section by referencing a fact which shows that free and projective modules coincide over a PID. This fact will be applied in Chapter 4.

Theorem 2.3.7. [Hun74] *Let R be a PID. Then a unitary R-module P is projective if and only if it is free.*

Refer to [Hun74] for a proof of Theorem 2.3.7. So far we have built a strong algebraic foundation, but the proof of the Universal Coefficient Theorem will require one more important piece of algebraic machinery.

2.4 Hom Groups and Duality

Definition 2.4.1. [Hun74] *Let A and B be modules over a ring R. Then $Hom_R(A, B)$ is the set of all R-module homomorphisms $f : A \to B$, which forms an abelian group under addition defined as follows: for $f, g \in Hom_R(A, B)$, $(f+g)(a) = f(a)+g(a)$ for all $a \in A$.*

Whenever $R = \mathbb{Z}$ the notation $\mathrm{Hom}(A, B)$ shall be used instead of $\mathrm{Hom}_{\mathbb{Z}}(A, B)$. In the next two examples and Theorem 2.4.4, we establish properties of the Hom group which will be used in ensuing chapters.

Example 2.4.2. *$Hom(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. Consider the map $\psi : \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}$ by $\psi(\varphi) = \varphi(1)$ for $\varphi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$. Now ψ is a homomorphism since given any $\varphi_1, \varphi_2 \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$,

$$\psi(\varphi_1 + \varphi_2) = (\varphi_1 + \varphi_2)(1) = \varphi_1(1) + \varphi_2(1) = \psi(\varphi_1) + \psi(\varphi_2).$$

To show ψ is injective let $\varphi_1, \varphi_2 \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ and suppose $\psi(\varphi_1) = \psi(\varphi_2)$. Then, $\varphi_1(1) = \varphi_2(1)$ implies $\varphi_1(n) = \varphi_2(n)$ for all $n \in \mathbb{Z}$. Therefore, ψ is injective. For $m \in \mathbb{Z}$ choose $\varphi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ such that $\varphi(1) = m$. Then, $\psi(\varphi) = \varphi(1) = m$. Thus, ψ is an isomorphism. □

Example 2.4.3. $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}) = 0$.

Proof. Let $\varphi \in \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z})$. Suppose $\varphi([1]) = m$ for some $m \in \mathbb{Z}$. We will show that $m = 0$. Now, $[1] = [1+n] \in \mathbb{Z}_n$. Observe, $\varphi([1]) = \varphi([1+n]) = \varphi([1])+n\varphi([1]) = m(1+n)$. So, $m = \varphi(1) = m(1+n)$. Thus, $m(1+n-1) = mn = 0$, which implies $m = 0$ since $n \neq 0$ and $\mathbb{Z}$ is an integral domain. Therefore, $\varphi = 0$. □

The previous two examples will aid in calculations found in Chapter 3. The next result shows that Hom can be split over a direct sum. This fact will be employed in Chapters 3 and 5.

Theorem 2.4.4. *Let A and B be $\mathbb{Z}$-modules and G an abelian group, then*

$$Hom(A \oplus B, G) \cong Hom(A, G) \oplus Hom(B, G).$$

Proof. Let $i_A : A \to A \oplus B$, and $i_B : B \to A \oplus B$ be inclusion homomorphisms defined by $i_A(a) = (a, 0)$ and $i_B(b) = (0, b)$ for all $a \in A$ and $b \in B$. Next consider the map $\psi : \mathrm{Hom}(A \oplus B, G) \to \mathrm{Hom}(A, G) \oplus \mathrm{Hom}(B, G)$ by $\psi(\varphi) = (\varphi \circ i_A, \varphi \circ i_B)$. Observe ψ, being the composite of group homomorphisms, is a group homomorphism. For notational purposes, let $P = \mathrm{Hom}(A \oplus B, G)$, and $Q = \mathrm{Hom}(A, G) \oplus \mathrm{Hom}(B, G)$. To prove ψ is an isomorphism we will find a homomorphism $h : Q \to P$ such that $h \circ \psi = 1_P$ and $\psi \circ h = 1_Q$. Let $\pi_A : A \oplus B \to A$ and $\pi_B : A \oplus B \to B$ be the projection homomorphisms to A and B, respectively. Consider the map $h : Q \to P$ defined by $h(\alpha, \beta) = \alpha \circ \pi_A + \beta \circ \pi_B$. The map h, being made of the point-wise sum of homomorphisms, is a homomorphism. Let $\varphi \in P$ and $(a, b) \in A \oplus B$. Then,

$$\begin{aligned}
h(\psi(\varphi))(a, b) &= h(\varphi \circ i_A, \varphi \circ i_B)(a, b) \\
&= \varphi \circ i_A[\pi_A(a)] + \varphi \circ i_B[\pi_B(b)] \\
&= \varphi(a, 0) + \varphi(0, b) \\
&= \varphi((a, 0) + (0, b)) \\
&= \varphi(a, b).
\end{aligned}$$

Thus, $h \circ \psi = 1_P$. Now, take any $(\alpha, \beta) \in Q$. Then,

$$\begin{aligned}
\psi(h(\alpha, \beta)) &= (h(\alpha, \beta) \circ i_A, h(\alpha, \beta) \circ i_B) \\
&= (\alpha, \beta)(a, 0) + (\alpha, \beta)(0, b) \\
&= (\alpha, \beta)(a, b), \text{ where } a \in A, b \in B.
\end{aligned}$$

So, $\psi \circ h = 1_Q$. Therefore, ψ is an isomorphism. □

Recall, $\mathrm{Hom}(A, B)$ is the collection of all $\mathbb{Z}$-module homomorphisms from A to B, which forms an abelian group under addition. As we will see, the Hom group will allow one to dualize a sequence of abelian groups. The reader should note that we may replace $\mathbb{Z}$ with any arbitrary ring R.

Definition 2.4.5. *Let $f : A \to B$ and $\varphi : B \to G$ be R-module homomorphisms. We define the dual of f, denoted f^*, to be $f^*(\varphi) = \varphi \circ f$.*

Definition 2.4.5 allows one to map between Hom groups. For example, if we take $f : A \to B$ and any $\varphi \in \mathrm{Hom}(B, G)$. By definition, to evaluate $f^*(\varphi)$, one needs to find the composite of φ and f. Using the commutative diagram below, one can see that the definition of f^* is quite natural.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f^*(\varphi)"'] & B \arrow[d, "\varphi"] \\ & G \end{tikzcd}$$

So $f^*(\varphi) : A \xrightarrow{f} B \xrightarrow{\varphi} G$. This composite gives us a way to transform homomorphisms $B \to G$ to homomorphisms $A \to G$. Notice, for $A \xrightarrow{f} B$ by duality $\mathrm{Hom}(A, G) \xleftarrow{f^*} \mathrm{Hom}(B, G)$. Thus, duality is order reversing. Finally, duality will be required when calculating cohomology, so we conclude this section by considering a concrete example which illustrates how it works.

Example 2.4.6. Consider the map $f : \mathbb{Z} \to \mathbb{Z}_6$ defined by $f(n) = [n]_6$ for all $n \in \mathbb{Z}$, and take $\varphi \in \mathrm{Hom}(\mathbb{Z}_6, \mathbb{Z}_3)$ as $\varphi([n]_6) = [n]_3$. We describe how one may construct a map from $\mathrm{Hom}(\mathbb{Z}_6, \mathbb{Z}_3)$ to $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_3)$ using duality.

Examine the following commutative diagram with $\varphi([n]_6) = [n]_3$ for $[n]_6 \in \mathbb{Z}_6$.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f"] \arrow[dr, "f^*(\varphi)"'] & \mathbb{Z}_6 \arrow[d, "\varphi"] \\ & \mathbb{Z}_3 \end{tikzcd}$$

By definition, for all $n \in \mathbb{Z}$, $f^*(\varphi)(n) = \varphi(f(n)) = \varphi([n]_6) = [n]_3$. So $\mathbb{Z} \xrightarrow{f} \mathbb{Z}_6$, and by duality, $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_3) \xleftarrow{f^*(\varphi)} \mathrm{Hom}(\mathbb{Z}_6, \mathbb{Z}_3)$. This concludes the algebraic preliminaries. In the next chapter we introduce homology and cohomology.

Chapter 3

Homology and Cohomology

Now that the algebraic preliminaries are complete we are ready to discuss homology and cohomology. We begin by introducing a special sequence of abelian groups called a chain complex in Section 3.1. In Section 3.2, we define the homology of a chain complex. The next phase is to examine cohomology, and expose the differences between homology and cohomology through examples in Section 3.3. Finally, in Section 3.4, we introduce Ext, and prove a number of properties that will be of use subsequently in Chapter 5.

3.1 Chain Complexes

Before studying homology and cohomology one must be introduced to a special sequence of abelian groups. We note that such a sequence in Definition 3.1.1 need not be exact, and the most interesting cases are when this happens.

Definition 3.1.1. [Hat02] *Let C_n be a sequence of abelian groups with maps (homomorphisms) $\partial_n : C_n \to C_{n-1}$ satisfying $\partial_{n-1}\partial_n = 0$. A chain complex is the collection $\mathcal{C} = \{C_n, \partial_n\}$.*

An important consequence of $\partial_{n-1}\partial_n = 0$ is that $\mathrm{Im}\partial_n$ is a normal subgroup of $\mathrm{Ker}\partial_{n-1}$. From this, one may define the *nth homology group* of a chain complex $\mathcal{C}$. We will refer to elements of $\mathrm{Ker}\partial$ as cycles and elements of $\mathrm{Im}\partial$ as boundaries.

3.2 Homology

Definition 3.2.1. [Hat02] *Consider the following chain complex $\mathcal{C}$, where $C_n = 0$ for $n < 0$.*

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots$$

The nth homology group of the chain complex $\mathcal{C}$ (with integer coefficients), denoted $H_n(\mathcal{C})$, is defined as the quotient group

$$H_n(\mathcal{C}) := \frac{Ker\partial_n}{Im\partial_{n+1}}.$$

The map ∂_n is called the *differential* on homology, and unless there is the possibility of confusion, we drop the subscript for convenience and write $\partial_n = \partial$ because, in most cases, the context will be clear. One easily sees that if the chain complex $\mathcal{C}$ is exact, then $H_n(\mathcal{C})$ is trivial for each n because exactness implies that $\mathrm{Im}\partial_{n+1} = \mathrm{Ker}\partial_n$, and therefore, $H_n(\mathcal{C}) = \{0\}$. The reader should bear in mind that one may choose any arbitrary coefficient group G to define homology. Then the nth homology group with coefficients in G is denoted $H_n(\mathcal{C}; G)$; however, we will not concern ourselves with this since its construction is not pertinent nor necessary to proving the Universal Coefficient Theorem. For the remainder of this paper, the term homology will be referring to the integer homology in Definition 3.2.1. The following example presents a sample chain complex to illustrate the process for computing homology groups.

Example 3.2.2. [Hat02] Compute the homology groups of the chain complex $\mathcal{C}$

$$\mathcal{C} : 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0,$$

where $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 2n$ for all $n \in \mathbb{Z}$ and 0 denotes the zero homomorphism.

By definition, $H_0(\mathcal{C}) = \frac{\mathrm{Ker}\ 0}{\mathrm{Im}\ 0} = \frac{\mathbb{Z}}{0} \cong \mathbb{Z}$. Now, Im $f = 2\mathbb{Z}$, and Ker $0 = \mathbb{Z}$. So, $H_1(\mathcal{C}) = \frac{\mathrm{Ker}\ 0}{\mathrm{Im}\ f} = \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2$. Next, since f is injective, Ker $f = 0$. In addition, Im $0 = 0$. Thus, $H_2(\mathcal{C}) = \frac{\mathrm{Ker}\ f}{\mathrm{Im}\ 0} = \frac{0}{0} \cong 0$. Finally, at dimension 3, Im $0 = 0$ and Ker $0 = \mathbb{Z}$. So, $H_3(\mathcal{C}) = \frac{\mathrm{Ker}\ 0}{\mathrm{Im}\ 0} = \frac{\mathbb{Z}}{0} \cong \mathbb{Z}$. The homology groups of $\mathcal{C}$ have been placed in Equation (3.1). Next, we focus our attention on cohomology.

$$H_n(C) \cong \begin{cases} \mathbb{Z}, & \text{for } n = 0, 3 \\ \mathbb{Z}_2, & \text{for } n = 1 \\ 0, & \text{for } n = 2. \end{cases} \tag{3.1}$$

3.3 Cohomology

In this section we define cohomology of a chain complex $\mathcal{C} = \{C_n, \partial\}$ with coefficients in a fixed abelian group G. Cohomology will require an intermediate step where we dualize $\mathcal{C}$. Take any abelian group G, and for each n, replace C_n with the group $C_n^* = \mathrm{Hom}(C_n, G)$ in the chain complex. With G fixed, this process produces the sequence of abelian groups $\{C_n^*, d\}$, where $d : C_{n-1}^* \to C_n^*$ is the dual map to ∂. In the following lemma, we will see that the sequence $\{C_n^*, d\}$ is a chain complex, and we will refer to this dualized chain complex as the co-chain complex corresponding to $\mathcal{C}$ with coefficients in G.

Lemma 3.3.1. *Let $\mathcal{C} = \{C_n, \partial\}$ be a chain complex, then $\mathcal{C}^* = \{C_n^*, d\}$ is a chain complex as well, where $C_n^* = Hom(C_n, G)$.*

Proof. Let 0 be the zero map. We must show that $dd = 0$. Let $\varphi \in C_{n-1}^*$. By definition, $dd(\varphi) = d(\varphi\partial) = \varphi(\partial\partial) = \varphi(0) = 0$. Therefore, $dd = 0$. □

Since $d^2 = 0$ it follows that Im d is a normal subgroup of Ker d. This relationship allows one to define the *nth cohomology group*, with coefficients in G, of a chain complex $\mathcal{C}$.

Definition 3.3.2. [Hat02] *Let $\mathcal{C} = \{C_n, \partial\}$ be a chain complex and let G be an abelian group. The nth cohomology group, with coefficients in G, of the chain complex $\mathcal{C}$, denoted $H^n(\mathcal{C}; G)$, is defined as the quotient group*

$$H^n(\mathcal{C}; G) := \frac{Ker\ d}{Im\ d}.$$

Definition 3.3.2 says that in order to compute the cohomology groups for a given chain complex $\mathcal{C} = \{C_n, \partial_n\}$ one must first choose an abelian group G, then replace the C_n with $C_n^* = \mathrm{Hom}(C_n, G)$, and then compute homology on the co-chain complex $\mathcal{C}^*$.

The map $d : C^*_{n-1} \to C^*_n$ is called the *differential* on cohomology with coefficients in G. The next example demonstrates this process.

Example 3.3.3. Compute the cohomology groups, with coefficients in $G = \mathbb{Z}$, of the chain complex C from Example 3.2.2.

To compute cohomology of C, we will replace $\mathbb{Z}$ with $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ in C at each dimension. Next, we must compute the dual maps, namely f^* and 0^*. It is clear that $0^* = 0$ by the composite $0^* : \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}$. Next f^* acts as the "times two" map from $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ to $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ since for any $\varphi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$, $f^*\varphi(n) = \varphi f(n) = \varphi(2n) = 2\varphi(n)$ where $n \in \mathbb{Z}$. So we have the resulting co-chain complex

$$0 \longleftarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{0} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{f^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{0} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longleftarrow 0.$$

By definition, $H^0(C; \mathbb{Z}) = \frac{\mathrm{Ker}\ 0}{\mathrm{Im}\ 0} = \frac{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})}{0} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. Now f^* is injective since for any $\varphi \in \mathrm{Ker}\ f^*$, $0 = f^*(\varphi(n)) = 2\varphi(n)$ for all $n \in \mathbb{Z}$, which implies $\varphi = 0$. Thus, $\mathrm{Ker}\ f^* = 0$. So $H^1(C; \mathbb{Z}) = \frac{\mathrm{Ker}\ f^*}{\mathrm{Im}\ 0} = \frac{0}{0} \cong 0$. Next,

$$H^2(C; \mathbb{Z}) = \frac{\mathrm{Ker}\ 0}{\mathrm{Im}\ f^*} = \frac{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})}{2\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2.$$

Finally, $H^3(C; \mathbb{Z}) = \frac{\mathrm{Ker}\ 0}{\mathrm{Im}\ 0} = \frac{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})}{0} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. The cohomology groups of C have been placed in Equation (3.2).

$$H^n(C; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{for } n = 0, 3 \\ 0, & \text{for } n = 1 \\ \mathbb{Z}_2, & \text{for } n = 2 \end{cases} \tag{3.2}$$

Contrasting the results from Equations (3.1) and (3.2), we see that $H^n(C; \mathbb{Z})$ is not isomorphic to $\mathrm{Hom}(H_n(C), \mathbb{Z})$ for $n = 1, 2$ and so the intermediate step of dualizing a chain complex to form cohomology has a further reaching effect as its end result. We complete this section with a discussion regarding exact sequences. Example 3.3.4 shows that the dual to a short exact sequence need not be exact. This result may seem insignificant now, but its importance shall be made more transparent in the next section and in ensuing chapters.

Example 3.3.4. Let $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 2n$ and $g : \mathbb{Z} \to \mathbb{Z}_2$ by $g(n) = [n]_2$. Then $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \to 0$ is exact, but its dual with $G = \mathbb{Z}$ fails to be exact.

Proof. By Theorem 2.2.2, $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \to 0$ are exact since f is injective and g is surjective. Finally, the sequence $\mathfrak{M} : 0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \to 0$ is exact at $\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2$ since $\mathrm{Im} f = 2\mathbb{Z} = \mathrm{Ker} g$. Let $G = \mathbb{Z}$ and dualize $\mathfrak{M}$. The resulting co-chain complex is

$$\mathfrak{M}^* : 0 \leftarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{f^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{g^*} \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}) \leftarrow 0.$$

We claim that exactness fails at $0 \leftarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{f^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$, in other words, f^* is not surjective. Take any $\varphi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ and $n \in \mathbb{Z}$. Then, for $n \in \mathbb{Z}$,

$$f^*(\varphi(n)) = \varphi(f(n)) = \varphi(2n) = 2\varphi(n).$$

Hence, as before, f^* acts as the "times two" map from $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ to $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$. But, by Example 2.4.2, $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, and f^* clearly does not map onto $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$. Thus, for example, $id \notin \mathrm{Im}\, f^*$, where id is the identity of $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$. $\square$

Consider the exact sequence $A \to B \to C \to 0$. Then, the corresponding dual sequence $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ is also exact [Hun74]. Now, from Example 3.3.4, if $0 \to A_1 \to A_0 \to A \to 0$ is exact, then $0 \leftarrow A_1^* \leftarrow A_0^* \leftarrow A^* \leftarrow 0$ need not be exact. From this, if we place a 0 to the left of A in $A \to B \to C \to 0$, then exactness is no longer preserved on the dual sequence. As a result, duality is said to be *right exact* since it preserves exactness on the right on the corresponding dual sequence. If we think of the subscript on A_i as a degree, then in the previous example we see exactness fail at degree one. In general, when working with an exact chain complex, the first place one may look for nontrivial homology will be at degree one on the corresponding co-chain complex. This location to look for nontrivial homology will be defined in the next section, but first we consider one more important fact about short exact sequences, which will used in the proof of the Universal Coefficient Theorem.

Theorem 3.3.5. *Let R be a PID and G an R-module. If $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ where P is projective, then $0 \leftarrow \mathrm{Hom}(A, G) \xleftarrow{f^*} \mathrm{Hom}(B, G) \xleftarrow{g^*} \mathrm{Hom}(P, G) \leftarrow 0$ is exact.*

Proof. By Theorem 2.3.6, since P is projective, $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ splits. That is, $B \cong A \oplus P$. Consider the commutative diagram below.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "id"'] & B \arrow[r, "g"] \arrow[d, "h"', "\cong"] & P \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i_1"] & A \oplus P \arrow[r, "\pi_2"] & P \arrow[r] & 0
\end{tikzcd}$$

Let G be an abelian group. Now dualize both short exact sequences(rows of the diagram). Denote $\mathrm{Hom}(A, G)$ as A^*, and similarly for B^* and P^*. In the following diagram, we do not assume the top row is exact although by Theorem 2.3.6 we know the bottom row is exact.

$$\begin{tikzcd} 0 & A^* \arrow[l] & B^* \arrow[l, "f^*"'] & P^* \arrow[l, "g^*"'] & 0 \arrow[l] \\ 0 & A^* \arrow[l] \arrow[u, "id"] & A^* \oplus P^* \arrow[l, "i_1^*"'] \arrow[u, "h^*", "\cong"'] & P^* \arrow[l, "\pi_2^*"'] \arrow[u, "id"] & 0 \arrow[l] \end{tikzcd}$$

From Example 3.3.4 and the discussion thereafter, we know exactness may only fail at A^*. Let $\alpha \in A^*$. We seek $(\varphi, \psi) \in A^* \oplus P^*$ such that $i_1^*(\varphi, \psi)(a) = \alpha(a)$. Notice that i_1^* injects into the left component of (φ, ψ). So, $i_1^*(\varphi, \psi)(a) = \varphi(a)$. Thus, $\alpha = \varphi$, with $\psi = 0$. So given $\alpha \in A^*$ there exists $(\alpha, 0) \in A^* \oplus P^*$ with $i_1^*(\alpha, 0) = \alpha$. Set $\beta = h^*(\alpha, 0)$. Then commutativity guarantees that $f^*(\beta) = i_1^*(h^*)^{-1}(\beta) = i_1^*(\alpha, 0) = \alpha$. Therefore, f^* is onto, and by Theorem 2.2.2, $0 \leftarrow A^* \overset{f^*}{\leftarrow} B^* \overset{g^*}{\leftarrow} C^* \leftarrow 0$ is exact. $\square$

3.4 Ext Groups

Recall, in Example 3.3.4, we saw that the dual of an exact sequence need not be exact. As mentioned before, the first place one may look for nontrivial homology is at degree one on the corresponding co-chain complex. Now computing homology on the co-chain yields $H^1(\mathfrak{M}; \mathbb{Z}) \cong \mathbb{Z}_2$.

Definition 3.4.1. [Hat02] *Let H be a $\mathbb{Z}$-module. Consider the following long exact sequence, $\{F_i, f_i\}$, of free abelian groups*

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

Then, we define $Ext^n(H, G) = H^n(F; G)$ for $n \geq 0$.

Using the notation from Definition 3.4.1, we say $F = \{F_i, f_i\}$ is a *free resolution* for H if F_i is free for all $i \geq 0$. Every $\mathbb{Z}$-module possesses a free resolution of $\mathbb{Z}$-modules of the form $0 \to F_1 \to F_0 \to H \to 0$ obtainable in the following manner: take the collection $H = \{h_\alpha : \alpha \in I\}$, set $F_0 = \{\sum_{i=1}^k n_i h_{\alpha_i}\}$ a free abelian group generated by h_α for all $\alpha \in I$, define the map $\varepsilon : F_0 \to H$ by $\varepsilon\left(\sum_{i=1}^k n_i h_{\alpha_i}\right) = \sum_{i=1}^k n_i h_{\alpha_i}$ where ε takes finite summands of elements of H with no relation to each other, then imposes a relation under the addition in H. Finally, set $F_1 = \text{Ker } \varepsilon$. After applying the aforementioned process, one may say $0 \to F_1 \to F_0 \to H \to 0$ is a free resolution for H since, by construction, F_0 is free and F_1, a submodule of F_0 , is free by Theorem 2.3.2 since $\mathbb{Z}$ is a PID.

This statement is very strong since, as we shall see in Theorem 3.4.7 and Corollary 3.4.9, any two free resolutions for H produce isomorphic Ext groups. The ability to construct such a free resolution, and obtain the same cohomology, is motivated by the following argument: given any free resolution of an abelian group H, one may use a free resolution of the form $0 \to F_1 \to F_0 \to H \to 0$ to compute Ext.

Definition 3.4.2. [Hat02] *Consider the following diagram containing rows that are chain complexes. Here, $\alpha_i : C_i \to D_i$, although we omit the subscript for convenience.*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{f_{n+2}} & C_{n+1} & \xrightarrow{f_{n+1}} & C_n & \xrightarrow{f_n} & C_{n-1} & \xrightarrow{f_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \\
\cdots & \xrightarrow{g_{n+2}} & D_{n+1} & \xrightarrow{g_{n+1}} & D_n & \xrightarrow{g_n} & D_{n-1} & \xrightarrow{g_{n-1}} & \cdots
\end{array}$$

The map $\alpha : C_i \to D_i$ is said to be a chain map provided $g_n\alpha = \alpha f_n$ for each n.

Lemma 3.4.3. *Let $\mathcal{C} = \{C_n, f_n\}$ and $\mathcal{D} = \{D_n, g_n\}$ be chain complexes with chain maps $\alpha_i, \alpha_i' : C_i \to D_i$. Then $\mu_i = \alpha_i - \alpha_i'$ is a chain map.*

Proof. Begin by considering the following diagram $\mathfrak{C}$.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{f_{n+1}} & C_n & \xrightarrow{f_n} & C_{n-1} & \xrightarrow{f_{n-1}} & C_{n-2} & \longrightarrow & \cdots \\
 & & {\scriptstyle\mu_n}\downarrow & & {\scriptstyle\mu_{n-1}}\downarrow & & {\scriptstyle\mu_{n-2}}\downarrow & & \\
\cdots & \xrightarrow{g_{n+1}} & D_n & \xrightarrow{g_n} & D_{n-1} & \xrightarrow{g_{n-1}} & D_{n-2} & \longrightarrow & \cdots
\end{array}$$

The collection $\{\mu_n : C_n \to D_n\}$ will be a chain map provided $g_n\mu_n = \mu_{n-1}f_n$ for all n. Let $x_n \in C_n$. Then

$$\begin{aligned}
g_n\mu_n(x_n) &= g_n(\alpha_n - \alpha_n')(x_n) \\
&= g_n\alpha_n(x_n) - g_n\alpha_n'(x_n) \\
&= \alpha_{n-1}f_n(x_n) - \alpha_{n-1}'f_n(x_n) \\
&= (\alpha_{n-1} - \alpha_{n-1}')f_n(x_n) \\
&= \mu_{n-1}f_n(x_n).
\end{aligned}$$

Therefore, the collection $\{\mu_n : C_n \to D_n\}$ is a chain map. □

If $\alpha : C_i \to D_i$ is a chain map, then it sends boundaries in C_i to boundaries in D_i and cycles in C_i to cycles in D_i. In that case, there exists a well-defined homomorphism $\alpha_* : H_n(\mathcal{C}) \to H_n(\mathcal{D})$.

Lemma 3.4.4. [Hat02] *A chain map between two chain complexes induces a homomorphism on the homology groups of the complexes.*

Proof. Let $\mathcal{C} = \{C_n, \partial\}$ and $\mathcal{D} = \{D_n, \partial\}$ be chain complexes with $\alpha_n : C_n \to D_n$ a chain map. Consider the following commutative diagram where $\alpha = \alpha_n$.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] \arrow[d, "\alpha"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "\alpha"] & C_{n-2} \arrow[r, "\partial"] \arrow[d, "\alpha"] & \cdots \\
\cdots \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r, "\partial"] & D_{n-2} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Since α is a chain map $\partial\alpha = \alpha\partial$. We claim that α takes cycles to cycles and boundaries to boundaries. Let $\beta \in \text{ Ker } \partial \subset C_n$. Then, $\partial\alpha\beta = \alpha\partial\beta = \alpha(0) = 0$. So $\alpha\beta \in \text{ Ker } \partial \subset D_n$. Now take $\partial(c) \in \text{ Im } \partial \subset C_n$ where $c \in C_{n+1}$. Then, $\alpha\partial(c) = \partial\alpha(c) \in \text{ Im } \partial \subset D_n$. So, α induces a homomorphism $\alpha_* : H_n(\mathcal{C}) \to H_n(\mathcal{D})$. $\square$

Definition 3.4.5. [Hat02] *Let $\mathcal{C} = \{C_n, \partial\}$ and $\mathcal{D} = \{D_n, \partial\}$ be chain complexes with chain maps $\alpha_n : C_n \to D_n$ and $\alpha'_n : C_n \to D_n$. Consider the commutative diagram.*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, dotted, shift right, "\alpha'_{n+1}"'] \arrow[d, shift left, "\alpha_{n+1}"] & C_n \arrow[r, "\partial"] \arrow[d, dotted, shift right, "\alpha'_n"'] \arrow[d, shift left, "\alpha_n"] \arrow[dl, dotted, "P_n"'] & C_{n-1} \arrow[r, "\partial"] \arrow[d, dotted, shift right, "\alpha'_{n-1}"'] \arrow[d, shift left, "\alpha_{n-1}"] \arrow[dl, dotted, "P_{n-1}"'] & \cdots \\
\cdots \arrow[r, "\partial"] & D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Then α_n and α'_n are said to be chain homotopic if there is a map $P_n : C_n \to D_{n+1}$ such that $\alpha_n - \alpha'_n = \partial P_n + P_{n-1}\partial$ for each n.

Lemma 3.4.6. [Hat02] *Let α_n and α'_n be chain homotopic, then α_n and α'_n induce the same homomorphism on homology.*

Proof. In this proof we adopt the notation and refer to the commutative diagram from Definition 3.4.5. Since α_n and α'_n are chain homotopic, there exists a chain homotopy

$P_n : C_n \to D_{n+1}$ such that $\alpha_n - \alpha'_n = \partial P_n + P_{n-1}\partial$. If $\beta \in \operatorname{Ker}\partial \subset C_n$, then since $\partial\beta = 0$, $\alpha_n(\beta) - \alpha'_n(\beta) = \partial P_n(\beta) + P_{n-1}\partial(\beta) = \partial P_n(\beta) \in \operatorname{Im}\partial \subset D_n$. So we see that $\alpha_n(\beta) - \alpha'_n(\beta) + \operatorname{Im}\partial = \operatorname{Im}\partial \in H_n(\mathcal{D})$. Thus, $\alpha_n(\beta) + \operatorname{Im}\partial = \alpha'_n(\beta) + \operatorname{Im}\partial$ for $\beta \in \operatorname{Ker}\partial$. Consequently, $\alpha_n(\beta)$ and $\alpha'_n(\beta)$ determine the same homology class in $H_n(\mathcal{D})$. Therefore, α_n and α'_n induce the same homomorphism on homology. $\square$

From Lemma 3.4.4, we know that given any chain map α between two chain complexes, say $\mathcal{C}$ and $\mathcal{D}$, α induces a homomorphism between the homology groups of the complexes. Now, by Lemma 3.4.6, for any chain map α' between $\mathcal{C}$ and $\mathcal{D}$ which is chain homotopic to α, α and α' will induce the same homomorphism on homology. We will use these results to help prove the next theorem which shows that one can extend a homomorphism between abelian groups to a chain map between respective free resolutions. Most importantly, all such extensions will be chain homotopic, and therefore, induce the same homomorphism on homology.

Theorem 3.4.7. [Hat02] *Given any free resolutions F and F' of abelian groups H and H', then every homomorphism $\alpha : H \to H'$ can be extended to a chain map from F to F'. Furthermore, any two such chain maps extending α are chain homotopic.*

Proof. Consider the following diagram, $\mathfrak{F}$, with F_i and F'_i free for all i.

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{f_4} & F_3 & \xrightarrow{f_3} & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\cdots & \xrightarrow{f'_4} & F'_3 & \xrightarrow{f'_3} & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0
\end{array}$$

We will begin the proof by constructing the chain map $\{\alpha_i\}$ extending the map $\alpha : H \to H'$ from $F = \{F_i, f_i\}$ to $F' = \{F'_i, f'_i\}$. In constructing the chain maps α_i, one must ensure the diagram $\mathfrak{F}$ commutes, that is, $\alpha_{n-1}f_n = f'_n\alpha_n$ for $n \geq 0$, where $\alpha_{-1} = \alpha$. We construct α_i on the basis elements of the free groups F_i. Let $\{\beta_0^\gamma\}$ be a basis for F_0 for some indexing set $\{\gamma : \gamma \in I\}$. Observe, by Theorem 2.2.2, f'_0 is surjective. Therefore, for any $\alpha f_0(\beta_0^\gamma) \in H'$ there exists $(\beta_0^\gamma)' \in F'_0$ such that $f'_0((\beta_0^\gamma)') = \alpha f_0(\beta_0^\gamma)$. So, define $\alpha_0 : F_0 \to F'_0$ by $\beta_0^\gamma \mapsto (\beta_0^\gamma)'$. Since F_0 is free, this map is well-defined. The map $\alpha_1 : F_1 \to F'_1$ will be constructed similarly to α_0. Let $\{\beta_1^\gamma\}$ be a basis for F_1, where we consider $\{\gamma : \gamma \in I\}$ to be a new indexing set. We seek to find $(\beta_1^\gamma)' \in F'_1$ with the property

$f_1'((\beta_1^\gamma)') = \alpha_0 f_1(\beta_1^\gamma)$. The existence of such an element is guaranteed provided $\alpha_0 f_1(\beta_1^\gamma)$ is a member of $\mathrm{Im} f_1'$. By exactness, it suffices to show $\alpha_0 f_1(\beta_1^\gamma) \in \mathrm{Ker} f_0' = \mathrm{Im} f_1'$. Let $\beta \in F_1$, and let 0 be the zero map. Observe, $f_0'(\alpha_0 f_1(\beta)) = \alpha f_0 f_1(\beta) = \alpha 0(\beta) = \alpha(0) = 0$. Set $\alpha_1 : F_1 \to F_1'$ by $\beta_1^\gamma \mapsto (\beta_1^\gamma)'$. The remaining α_i's may be constructed in the same manner as α_1.

Next, take another chain map extending α, say with maps $\alpha_i' : F_i \to F_i'$. Set $\mu_i = \alpha_i - \alpha_i'$. Applying Lemma 3.4.3 we conclude that the μ_i's form a chain map extending $\mu = \alpha - \alpha = 0 : H \to H'$. We claim that the two chain maps are chain homotopic, that is, there exist maps $P_i : F_i \to F_{i+1}'$ such that $\mu_i = f_{i+1}' P_i + P_{i-1} f_i$. The P_i's will be constructed similar to the α_i's. When $i = 0$ set $P_{-1} = 0 : H \to F_0'$. So, for $i = 0$, we wish to establish $\mu_0 = f_1' P_0$. To achieve the relation, let P_0 send a basis element $\beta_0^\gamma \in F_0$ to an element $(\beta_1^\gamma)' \in F_1'$ so that $\mu_0(\beta_0^\gamma) = f_1'((\beta_1^\gamma)')$. Now, such a $(\beta_1^\gamma)' \in F_1'$ exists since $\mathrm{Im} f_1' = \mathrm{Ker} f_0'$ and $f_0' \mu_0(\beta_0^\gamma) = \mu f_0(\beta_0^\gamma) = \mu(0) = 0$. In other words, $\mu_0(\beta_0^\gamma) \in \mathrm{Im} f_1'$ implies there exists $(\beta_1^\gamma)' \in F_1'$ such that $\mu_0(\beta_0^\gamma) = f_1'((\beta_1^\gamma)')$, as desired. To define the remaining P_i's we wish to send a basis element $\beta_i^\gamma \in F_i$ to some $P_i(\beta_i^\gamma) = (\beta_{i+1}^\gamma)' \in F_{i+1}'$ with $f_{i+1}' P_i(\beta_i^\gamma) = \mu_i(\beta_i^\gamma) - P_{i-1} f_i(\beta_i^\gamma)$. That is, we need $\mu_i(\beta_i^\gamma) - P_{i-1} f_i(\beta_i^\gamma) \in \mathrm{Im} f_{i+1}' = \mathrm{Ker} f_i'$. Now,

$$\begin{aligned}
f_i'(\mu_i(\beta_i^\gamma) - P_{i-1} f_i(\beta_i^\gamma)) &= f_i' \mu_i(\beta_i^\gamma) - f_i' P_{i-1} f_i(\beta_i^\gamma) \\
&= f_i'(f_{i+1}' P_i(\beta_i^\gamma) + P_{i-1} f_i(\beta_i^\gamma)) - f_i' P_{i-1} f_i(\beta_i^\gamma) \\
&= f_i' f_{i+1}' P_i(\beta_i^\gamma) + f_i' P_{i-1} f_i(\beta_i^\gamma) - f_i' P_{i-1} f_i(\beta_i^\gamma) \\
&= f_i' f_{i+1}' P_i(\beta_i^\gamma) \\
&= 0.
\end{aligned}$$

So, there exists $P_i(\beta_i^\gamma) = (\beta_{i+1}^\gamma)' \in F_{i+1}'$ such that $f_{i+1}'((\beta_{i+1}^\gamma)') = \mu_i(\beta_i^\gamma) - P_{i-1} f_i(\beta_i^\gamma)$. For any basis element $\beta_i^\gamma \in F_i$ set $P_i(\beta_i^\gamma) = (\beta_{i+1}^\gamma)'$. This completes the construction of the chain homotopy, and therefore the two extensions of α are chain homotopic. □

Remark 3.4.8. If α_n and α_n' are chain homotopic, then so are α_n^* and $\alpha_n'^*$.

Proof. Consider the diagram.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[rr, "\partial"] \arrow[d, dotted, shift right, "\alpha'_{n+1}"'] \arrow[d, shift left, "\alpha_{n+1}"] & & C_n \arrow[rr, "\partial"] \arrow[lld, dotted, "P_n"'] \arrow[d, dotted, shift right, "\alpha'_n"'] \arrow[d, shift left, "\alpha_n"] & & C_{n-1} \arrow[r, "\partial"] \arrow[lld, dotted, "P_{n-1}"'] \arrow[d, dotted, shift right, "\alpha'_{n-1}"'] \arrow[d, shift left, "\alpha_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial"] & D_{n+1} \arrow[rr, "\partial"] & & D_n \arrow[rr, "\partial"] & & D_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Since α_n and α'_n are chain homotopic, there is a chain homotopy $P_n : C_n \to D_{n+1}$ such that $\alpha_n - \alpha'_n = \partial P_n + P_{n-1}\partial$. Let G be any abelian group and dualize the diagram above.

$$\begin{tikzcd}
\cdots & C^*_{n+1} \arrow[l, "d"'] & & C^*_n \arrow[ll, "d"'] & & C^*_{n-1} \arrow[ll, "d"'] & \cdots \arrow[l, "d"'] \\
\cdots & D^*_{n+1} \arrow[l, "d"'] \arrow[u, shift left, "\alpha^*_{n+1}"] \arrow[u, dotted, shift right, "(\alpha'_{n+1})^*"'] \arrow[rru, dotted, "P^*_n"] & & D^*_n \arrow[ll, "d"'] \arrow[u, shift left, "\alpha^*_n"] \arrow[u, dotted, shift right, "(\alpha'_n)^*"'] \arrow[rru, dotted, "P^*_{n-1}"] & & D^*_{n-1} \arrow[ll, "d"'] \arrow[u, shift left, "\alpha^*_{n-1}"] \arrow[u, dotted, shift right, "(\alpha'_{n-1})^*"'] & \cdots \arrow[l, "d"']
\end{tikzcd}$$

Claim $P^*_n : D^*_{n+1} \to C^*_n$ defines a chain homotopy on the co-chain complex, that is, $\alpha^*_n - (\alpha'_n)^* = P^*_n d + dP^*_{n-1}$ for each n. Let $\varphi \in D^*_n$. For $x \in D_n$

$$\begin{aligned}
\left(\alpha^*_n - (\alpha'_n)^*\right)(\varphi)(x) &= \alpha^*_n(\varphi)(x) - (\alpha'_n)^*(\varphi)(x) \\
&= \varphi(\alpha_n)(x) - \varphi(\alpha'_n)(x) \\
&= \varphi(\alpha_n(x) - \alpha'_n(x)) \\
&= \varphi\left(\partial P_n(x) + P_{n-1}\partial(x)\right) \\
&= (\varphi\partial)P_n(x) + (\varphi P_{n-1})\partial(x) \\
&= P^*_n(\varphi\partial)(x) + d(\varphi P_{n-1})(x) \\
&= P^*_n d(x) + dP^*_{n-1}(x).
\end{aligned}$$

Thus, P^*_n defines a homotopy, and α^*_n and $(\alpha'_n)^*$ are chain homotopic. □

The following corollary says that if we have two free resolutions F and F' for abelian groups H and H' with chain maps between them, then the cohomology groups are the same up to ismorphism. The significance is that if we take two free resolutions for a single group, say H, then the cohomology groups are the same using either free resolution. Thus, the cohomology groups are independent of the choice of the free resolution, and only depend on the coefficient group G and H. Consequently, $\mathrm{Ext}(H, G)$ does not depend on the free resolution F. Recall that every $\mathbb{Z}$-module H has a free resolution of the form $0 \to F_1 \to F_0 \to H \to 0$. So, to compute $\mathrm{Ext}(H, G)$ with respect to any other free resolution $\cdots \to F_2 \to F_1 \to F_0 \to H \to 0$, it suffices to calculate cohomology of the shorter sequence. Adopt the notation from Theorem 3.4.7 in the proof of Corollary 3.4.9.

Corollary 3.4.9. [Hat02] *Given any two free resolutions F and F' of abelian groups H and H', with isomorphism $\alpha : H \to H'$, there are canonical isomorphisms such that $H^n(F; G) \cong H^n(F'; G)$ for all n. Therefore, considering $\alpha = id : H \to H$, Ext(H, G) depends only on H and G, and not on the free resolution chosen to build Ext(H, G).*

Proof. Let G be an abelian group. By definition, if $F_n \xrightarrow{\alpha_n} F'_n$ then $\alpha_n^* : F'^*_n \to F^*_n$ because of the composition $F_n \xrightarrow{\alpha_n} F'_n \xrightarrow{\varphi} G$. So, α_n dualize to $\alpha_n^* : F'^*_n \to F^*_n$, where $F'^*_n = \mathrm{Hom}(F'_n, G)$ and $F^*_n = \mathrm{Hom}(F_n, G)$, forming a chain map extending α^* between the co-chain complexes F^* and F'^*. Thus, by Lemma 3.4.4, α^* induces a homomorphism between the cohomology groups of F and F'. Furthermore, if one takes another chain map extending α^*, say α'^*_n, then by Remark 3.4.8, α^*_n and α'^*_n are chain homotopic by the chain homotopy P^*. Moreover, α^*_n and α'^*_n will induce the same homomorphism on the homology of the co-chain complexes, that is, induce the same homomorphism on the cohomology groups of F and F'.

Consider the case when $\alpha : H \to H'$ is an isomorphism. Then, there is a map $\sigma : H' \to H$ so that $\alpha\sigma = 1_{H'}$ and $\sigma\alpha = 1_H$. Examine the following commutative diagram.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\cdots & \longrightarrow & F'_2 & \longrightarrow & F'_1 & \longrightarrow & F'_0 & \longrightarrow & H' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \sigma} & & \\
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

By commutativity, we may extend $\sigma\alpha = 1_H$ by $\sigma_n\alpha_n$ forming a chain map. Similarly, extend $\alpha\sigma$ by $\alpha_n\sigma_n$. Let G be an abelian group and dualize, then $(\sigma\alpha)^* = (1_H)^*$, and $(\sigma_n\alpha_n)^*$ extends $(1_H)^*$. Similarly, since $(\alpha\sigma)^* = (1_{H'})^*$, we have that $(\alpha_n\sigma_n)^*$ extends $(1_{H'})^*$. Next, observe $(\alpha_n\sigma_n)^* = \alpha_n^*\sigma_n^*$ and $(\sigma_n\alpha_n)^* = \sigma_n^*\alpha_n^*$. So, $\alpha_n^*\sigma_n^* = 1_{F'^*_n}$ and $\sigma_n^*\alpha_n^* = 1_{F^*_n}$. Thus, α_n^* and σ_n^* are isomorphisms. From this, if α^* is an isomorphism then so are the induced homomorphisms $\alpha_n^* : H^n(F;G) \to H^n(F';G)$ for all n. So, if F and F' are free resolutions for H, then we may take $\alpha = 1_H$, and the co-chain complexes are chain homotopic. Therefore, since $\mathrm{Ext}(H,G) = H^1(F;G)$, $\mathrm{Ext}(H,G)_F \cong \mathrm{Ext}(H,G)_{F'}$. □

The corollary below shows that for any free resolution F for a $\mathbb{Z}$-module H, it suffices to compute $\mathrm{Ext}^1(H,G) = \mathrm{Ext}(H,G)$ since $\mathrm{Ext}^n(H,G) = H^n(F;G)$ is trivial for all $n > 1$.

Corollary 3.4.10. *Let H and G be $\mathbb{Z}$-modules. Then $\mathrm{Ext}^n(H,G) = 0$ for $n > 1$.*

Proof. Recall that since H is a $\mathbb{Z}$-module it has a free resolution F of the form

$$0 \to F_1 \to F_0 \to H \to 0.$$

Also, by Corollary 3.4.9, we know that $\mathrm{Ext}^n(H,G)$ is independent of the free resolution used. Therefore, for any other free resolution F' for H, $\mathrm{Ext}^n(H,G)_F \cong \mathrm{Ext}^n(H,G)_{F'}$ for each n. We can view F as

$$\cdots \to 0 \to 0 \to 0 \to F_1 \to F_0 \to H \to 0.$$

Take any abelian group G and dualize to obtain the co-chain complex

$$\cdots \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0.$$

Then it is clear for $n > 1$, that $\mathrm{Ext}^n(H,G) = \dfrac{\mathrm{Ker}\ 0}{\mathrm{Im}\ 0} = \dfrac{\{0\}}{\{0\}} \cong \{0\}$. □

For the remainder of this chapter we will explore some properties of Ext, which will prove to be useful in Chapter 5. In order to do so, we first study how one may obtain a free resolution for a direct sum of two abelian groups.

Lemma 3.4.11. *Suppose G_1 and G_2 are abelian groups, and $H_1 \le G_1, H_2 \le G_2$. Then*

$$\frac{G_1}{H_1} \oplus \frac{G_2}{H_2} \cong \frac{G_1 \oplus G_2}{H_1 \oplus H_2}.$$

Proof. Let $\varphi : \frac{G_1}{H_1} \oplus \frac{G_2}{H_2} \to \frac{G_1 \oplus G_2}{H_1 \oplus H_2}$ be defined as $\varphi(g_1 + H_1, g_2 + H_2) = (g_1, g_2) + H_1 \oplus H_2$ We show φ is (1) well-defined, (2) a homomorphism, (3) injective, and (4) surjective.

1. Suppose that $(g_1 + H_1, g_2 + H_2) = (\widetilde{g_1} + H_1, \widetilde{g_2} + H_2)$. Then $\widetilde{g_1} = g_1 + h_1$ and $\widetilde{g_2} = g_2 + h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$. To prove φ is well-defined we show $\varphi(g_1 + H_1, g_2 + H_2) = \varphi(\widetilde{g_1} + H_1, \widetilde{g_2} + H_2)$. Now

$$\begin{aligned}\varphi(\widetilde{g_1} + H_1, \widetilde{g_2} + H_2) &= (\widetilde{g_1}, \widetilde{g_2}) + H_1 \oplus H_2 \\ &= (g_1 + h_1, g_2 + h_2) + H_1 \oplus H_2 \\ &= (g_1, g_2) + (h_1, h_2) + H_1 \oplus H_2 \\ &= (g_1, g_2) + H_1 \oplus H_2 \\ &= \varphi(g_1 + H_1, g_2 + H_2).\end{aligned}$$

   Thus, φ is well-defined.

2. For any $(g_1 + H_1, g_2 + H_2), (\widetilde{g_1} + H_1, \widetilde{g_2} + H_2) \in \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}$,
   $\varphi((g_1 + H_1, g_2 + H_2) + (\widetilde{g_1} + H_1, \widetilde{g_2} + H_2))$

$$\begin{aligned} &= \varphi(g_1 + \widetilde{g_1} + H_1, g_2 + \widetilde{g_2} + H_2) \\ &= (g_1 + \widetilde{g_1}, g_2 + \widetilde{g_2}) + H_1 \oplus H_2 \\ &= (g_1, g_2) + H_1 \oplus H_2 + (\widetilde{g_1}, \widetilde{g_2}) + H_1 \oplus H_2 \\ &= \varphi((g_1 + H_1, g_2 + H_2)) + \varphi((\widetilde{g_1} + H_1, \widetilde{g_2} + H_2))\end{aligned}$$

   So φ is a homomorphism.

3. Let $(g_1 + H_1, g_2 + H_2), (\widetilde{g_1} + H_1, \widetilde{g_2} + H_2) \in \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}$, and suppose that we have $\varphi(g_1 + H_1, g_2 + H_2) = \varphi(\widetilde{g_1} + H_1, \widetilde{g_2} + H_2)$. Then, by the definition of φ,

$$(g_1, g_2) + H_1 \oplus H_2 = (\widetilde{g_1}, \widetilde{g_2}) + H_1 \oplus H_2.$$

   Thus, $(g_1, g_2) = (\widetilde{g_1}, \widetilde{g_2}) + (h_1, h_2)$ for some $(h_1, h_2) \in H_1 \oplus H_2$. Component-wise, $g_1 = \widetilde{g_1} + h_1$ and $g_2 = \widetilde{g_2} + h_2$, where $h_1 \in H_1$ and $h_2 \in H_2$. Then

$$(g_1 + H_1, g_2 + H_2) = (\widetilde{g_1} + h_1 + H_1, \widetilde{g_2} + h_2 + H_2) = (\widetilde{g_1} + H_1, \widetilde{g_2} + H_2).$$

   Therefore, φ is injective.

4. This map is clearly surjective since the arbitrary element $(g_1, g_2) + H_1 \oplus H_2$ is the image of the element $(g_1 + H_1, g_2 + H_2)$.

So φ is an isomorphism. This completes the proof. $\square$

Theorem 3.4.12. *Let F and F' be free resolutions for abelian groups H and H', respectively. Then $F \oplus F'$ is a free resolution for $H \oplus H'$.*

Proof. Let $F = \{F_i, f_i\}$ and $F' = \{F_i', f_i'\}$ be free resolutions for H and H', respectively. Set $\overline{f}_i = f_i \oplus f_i' : F_i \oplus F_i' \to F_{i-1} \oplus F_{i-1}'$ by $\overline{f}_i(a, b) = (f_i(a), f_i'(b))$. Notice $F_i \oplus F_i'$ is free for all i, as the direct sum of free groups. So it suffices to show that the sequence $F \oplus F' = \{F_i \oplus F_i', \overline{f}_i\}$ is exact. Now Im $\overline{f}_{i+1} \subset$ Ker $\overline{f}_i$ since, for $(a, b) \in F_{i+1} \oplus F_{i+1}'$,

$$\overline{f}_i\overline{f}_{i+1}(a, b) = \overline{f}_i(f_{i+1}(a), f_{i+1}'(b)) = (f_if_{i+1}(a), f_i'f_{i+1}'(b)) = (0, 0).$$

To show the reverse inclusion, let $(\alpha, \beta) \in$ Ker $\overline{f}_i$. Then $(0, 0) = \overline{f}_i(\alpha, \beta) = (f_i(\alpha), f_i'(\beta))$, and we see $\alpha \in$ Ker $f_i =$ Im f_{i+1} and $\beta \in$ Ker $f_i' =$ Im f_{i+1}'. So there are $c \in F_{i+1}$ and $c' \in F_{i+1}'$ such that $\alpha = f_{i+1}(c)$ and $\beta = f_{i+1}'(c')$. Finally, Im $\overline{f}_{i+1} \subset$ Ker $\overline{f}_i$ since

$$(\alpha, \beta) = (f_{i+1}(c), f_{i+1}'(c')) = \overline{f}_{i+1}(c, c') \in \text{Im } \overline{f}_{i+1}.$$

Therefore, the sequence is exact, and $F \oplus F'$ is a free resolution for $H \oplus H'$. $\square$

Theorem 3.4.13. *Let F and F' be free resolutions for abelian groups H and H', respectively. Let G be an abelian group. Then the following are true :*

1. *$Ext(H, G) = 0$ if H is free.*

2. *$Ext(H \oplus H', G) \cong Ext(H, G) \oplus Ext(H', G)$.*

3. *$Ext(\mathbb{Z}_n, G) \cong G/nG$*

Proof.

1. We begin the proof by first noting, since H is free, it has a free resolution of the form $0 \to H \xrightarrow{1} H \to 0$. Let G be an abelian group. Recall, $\text{Ext}(H, G)$ does not depend on the free resolution by Corollary 3.4.9. If we dualize $0 \to H \xrightarrow{1} H \to 0$ with coefficient group G, we obtain the sequence $0 \leftarrow H^* \xleftarrow{1^*} H^* \leftarrow 0$. Then

$$\text{Ext}(H, G) = \frac{\{0\}}{\{0\}} = \{0\}.$$

This completes the proof of (1).

2. Since $F = \{F_i, f_i\}$ and $F' = \{F_i', f_i'\}$ are free resolutions for H and H', $F \oplus F'$ is a free resolution for $H \oplus H'$ by Theorem 3.4.12. So we have the following sequence

$$\cdots \xrightarrow{\overline{f_3}} F_2 \oplus F_2' \xrightarrow{\overline{f_2}} F_1 \oplus F_1' \xrightarrow{\overline{f_1}} F_0 \oplus F_0' \xrightarrow{\overline{f_0}} H \oplus H' \longrightarrow 0.$$

By Theorem 2.4.4, we know $(A \oplus B)^* = A^* \oplus B^*$. Let G be an abelian group, and dualize the sequence while applying Theorem 2.4.4 to obtain

$$\cdots \xleftarrow{\overline{f_3^*}} F_2^* \oplus (F_2')^* \xleftarrow{\overline{f_2^*}} F_1^* \oplus (F_1')^* \xleftarrow{\overline{f_1^*}} F_0^* \oplus (F_0')^* \xleftarrow{\overline{f_0^*}} H^* \oplus (H')^* \longleftarrow 0.$$

In the above sequence we claim $\overline{f_n}^* = f_n^* \oplus (f_n')^*$. Let $(a_n, b_n) \in F_n \oplus F_n'$. Then for $\psi \in (F_{n-1} \oplus F_{n-1}')^*$.

$$\overline{f_n}^*(\psi)(a_n, b_n) = \psi(\overline{f_n}(a_n, b_n)) = \psi(f_n(a_n), f_n'(b_n)) = (f_n \oplus f_n')^*(\psi)(a_n, b_n).$$

So, $\overline{f_n}^* = (f_n \oplus f_n')^* = f_n^* \oplus (f_n')^*$.

Claim 1: $\text{Im}\overline{f_n}^* = \text{ Im } f_n^* \oplus \text{Im } (f_n')^*$.

Let $(\alpha', \beta') \in \text{ Im } \overline{f_n}^*$. Then, for $(a, b) \in F_n \oplus F_n'$, there exists $(\alpha, \beta) \in F_n^* \oplus (F_n')^*$ such that $\overline{f_n}^*(\alpha, \beta)(a, b) = (\alpha', \beta')(a, b)$. Now

$$\begin{aligned}
\overline{f_n}^*(\alpha, \beta)(a, b) &= (\alpha, \beta)\overline{f}_n(a, b) \\
&= (\alpha, \beta)(f_n(a), f_n'(b)) \\
&= (\alpha f_n(a), \beta f_n'(b)) \\
&= (f_n^*\alpha(a), (f_n')^*\beta(b)) \in \text{ Im } f_n^* \oplus \text{Im } (f_n')^*.
\end{aligned}$$

So $(\alpha', \beta') \in \text{ Im } f_n^* \oplus \text{Im } (f_n')^*$, and $\text{Im}\overline{f_n}^* \subset \text{ Im } f_n^* \oplus \text{Im } (f_n')^*$. For the reverse inclusion, let $(\alpha, \beta) \in \text{Im } f_n^* \oplus \text{ Im } (f_n')^*$. Then, for $(a, b) \in F_n \oplus F_n'$, there are $\alpha \in F_n^*$ and $\beta \in (F_n')^*$ such that $(\alpha', \beta')(a, b) = (f_n^*\alpha(a), (f_n')^*\beta(b))$. So

$$\begin{aligned}
(\alpha', \beta')(a, b) &= (\alpha'(a), \beta'(b)) \\
&= (f_n^*\alpha(a), (f_n')^*\beta(b)) \\
&= (\alpha f_n(a), \beta f_n'(b)) \\
&= (\alpha, \beta)(f_n(a), f_n'(b)) \\
&= (\alpha, \beta)\overline{f}_n(a, b) \\
&= \overline{f_n}^*(\alpha, \beta)(a, b) \in \text{ Im } \overline{f_n}^*.
\end{aligned}$$

Thus, $\mathrm{Im}\overline{f}_n^* = \mathrm{Im}\ f_n^* \oplus \mathrm{Im}\ (f_n')^*$.

Claim 2: $\mathrm{Ker}\overline{f}_n^* = \mathrm{Ker}\ f_n^* \oplus \mathrm{Ker}\ (f_n')^*$.

Let $(a,b) \in F_n \oplus F_n'$, and let $(\alpha,\beta) \in F_n^* \oplus (F_n')^*$. Then

$$\begin{aligned}(\alpha,\beta) \in \mathrm{Ker}\ \overline{f}_n^* &\iff \overline{f}_n^*(\alpha,\beta)(a,b) = (0,0)\\ &\iff (f_n^*\alpha(a), (f_n')^*\beta(b)) = (0,0)\\ &\iff (\alpha,\beta) \in \mathrm{Ker}\ f_n^* \oplus \mathrm{Ker}\ (f_n')^*.\end{aligned}$$

Therefore, $\mathrm{Ker}\overline{f}_n^* = \mathrm{Ker}\ f_n^* \oplus \mathrm{Ker}\ (f_n')^*$.

Now, by definition and claims (1) and (2),

$$\mathrm{Ext}^n(H \oplus H', G) = \frac{\mathrm{Ker}\ \overline{f}_n^*}{\mathrm{Im}\ \overline{f}_{n-1}^*} = \frac{\mathrm{Ker}\ f_n^* \oplus\ \mathrm{Ker}\ (f_n')^*}{\mathrm{Im}\ f_{n-1}^* \oplus\ \mathrm{Im}\ (f_{n-1}')^*} \quad \text{and}$$

$$\mathrm{Ext}^n(H,G) \oplus\ \mathrm{Ext}^n(H',G) = \frac{\mathrm{Ker}\ f_n^*}{\mathrm{Im}\ f_{n-1}^*} \oplus \frac{\mathrm{Ker}\ (f_n')^*}{\mathrm{Im}\ (f_{n-1}')^*}\ .$$

By Lemma 3.4.11,

$$\frac{\mathrm{Ker}\ f_n^* \oplus\ \mathrm{Ker}\ (f_n')^*}{\mathrm{Im}\ f_{n-1}^* \oplus\ \mathrm{Im}\ (f_{n-1}')^*} \cong \frac{\mathrm{Ker}\ f_n^*}{\mathrm{Im}\ f_{n-1}^*} \oplus \frac{\mathrm{Ker}\ (f_n')^*}{\mathrm{Im}\ (f_{n-1}')^*}$$

since F_n and F_n' are abelian. This completes the proof of (2).

3. Consider the short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_n \longrightarrow 0$, where $f(k) = nk$ and $\pi(k) = [k]_n$. Let G be an abelian group and dualize to obtain the sequence

$$0 \longleftarrow \mathrm{Hom}(\mathbb{Z},G) \xleftarrow{f^*} \mathrm{Hom}(\mathbb{Z},G) \xleftarrow{\pi^*} \mathrm{Hom}(\mathbb{Z}_n,G) \longleftarrow 0.$$

By definition, $\mathrm{Ext}(\mathbb{Z}_n,G) = H^1(\mathbb{Z}_n;G) = \dfrac{\mathrm{Ker}\ 0}{\mathrm{Im}\ f^*} = \dfrac{\mathrm{Hom}(\mathbb{Z},G)}{\mathrm{Im}\ f^*}$. We claim that $\mathrm{Im}\ f^* = n\mathrm{Hom}(\mathbb{Z},G)$. Given any $\psi \in \mathrm{Hom}(\mathbb{Z},G)$ there exists a unique $g \in G$ such that $\psi(n) = ng$. So we may express $\psi = \psi_g$, where $\psi_g(n) = ng$. Then $f^*(\psi_g)(k) = \psi_g(f(k)) = \psi_g(nk) = n\psi_g(k)$. Thus, given any $\psi = \psi_g \in \mathrm{Hom}(\mathbb{Z},G)$, $f^*(\psi_g) = n\psi_g$. Therefore, $\mathrm{Im}\ f^* = n\mathrm{Hom}(\mathbb{Z},G)$. So

$$\mathrm{Ext}(\mathbb{Z}_n,G) = \frac{\mathrm{Hom}(\mathbb{Z},G)}{n\mathrm{Hom}(\mathbb{Z},G)}.$$

To complete the proof it suffices to show $\dfrac{\mathrm{Hom}(\mathbb{Z},G)}{n\mathrm{Hom}(\mathbb{Z},G)} \cong \dfrac{G}{nG}$. Consider the map $\varphi : \mathrm{Hom}(\mathbb{Z},G) \to G/nG$ by $\varphi(\psi_g) = g + nG$. Let $\psi_g, \psi_h \in \mathrm{Hom}(\mathbb{Z},G)$, and suppose $\psi_g = \psi_h$. Then $\psi_g = \psi_h$ implies $n(g-h) = 0$. Therefore, $g - h \in nG$, which happens only if $g + nG = h + nG$. So $\varphi(\psi_g) = \varphi(\psi_h)$, and therefore, φ is well-defined. For any $\psi_g, \psi_h \in \text{Hom}(\mathbb{Z}, G)$ notice $\psi_g + \psi_h = \psi_{g+h}$, and that $\varphi(\psi_g + \psi_h) = \varphi(\psi_{g+h}) = (g+h) + nG = (g+nG) + (h+nG) = \varphi(\psi_g) + \varphi(\psi_h)$. So φ is a homomorphism. Thus, by the first isomorphism theorem for groups,

$$\frac{\text{Hom}(\mathbb{Z}, G)}{\text{Ker } \varphi} \cong \text{Im}\varphi.$$

Finally, we show φ is surjective and Ker $\varphi = n\text{Hom}(\mathbb{Z}, G)$. If both statements are true then we have

$$\frac{\text{Hom}(\mathbb{Z}, G)}{n\text{Hom}(\mathbb{Z}, G)} \cong \frac{G}{nG}, \text{ and by transitivity, } \text{Ext}(\mathbb{Z}_n, G) \cong \frac{G}{nG}.$$

To show φ is surjective, let $g + nG \in G/nG$. We seek $\psi_{g'} \in \text{Hom}(\mathbb{Z}, G)$ so that $\varphi(\psi_{g'}) = g + nG$. So find $g' \in G$ such that $g' + nG = g + nG$, which happens only if $g' = g + nh$ for some $h \in G$. So given $g + nG \in G/nG$ choose $\psi_{g'} \in \text{Hom}(\mathbb{Z}, G)$ where $g' = g + nh$ for some $h \in G$, then $\varphi(\psi_{g'}) = \varphi(\psi_{(g+nh)}) = (g + nh) + nG = g + nG$. Therefore, φ is surjective. Next notice if $\psi_g \in Ker\ \varphi$ then $\varphi(\psi_g) = g + nG = nG$, and thus, $g \in nG$. So $g = nh$ for some $h \in G$. Then

$$\psi_g(k) = \psi_{nh}(k) = n(hk) = n\psi_h(k) \in n\text{Hom}(\mathbb{Z}, G).$$

Hence, Ker $\varphi \subset n\text{Hom}(\mathbb{Z}, G)$. For the reverse inclusion, take $n\psi_h \in n\text{Hom}(\mathbb{Z}, G)$. Then $n\psi_h(k) = \psi_h(nk) = n(hk) + nG = nG$. So we have $n\psi_h \in$ Ker φ, and $\text{Hom}(\mathbb{Z}, G) \subset$ Ker φ. Therefore Ker $\varphi = n\text{Hom}(\mathbb{Z}, G)$. This completes the proof of (3).

□

We will apply the properties from Theorem 3.4.13 to help prove the following examples.

Example 3.4.14. $\text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$.

Proof. Observe that $H = \mathbb{Z}$ is a free $\mathbb{Z}$-module. Therefore, by property (1) of Theorem 3.4.13, $\text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$. □

Example 3.4.15. $\operatorname{Ext}(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$.

Proof. By property (2) of Theorem 3.4.13, $\operatorname{Ext}(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, but then $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$. Therefore, $\operatorname{Ext}(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$. □

Example 3.4.16. $\operatorname{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_d$, where $d = gcd(m, n)$.

Proof. By (3) of Theorem 3.4.13, $\operatorname{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_m/n\mathbb{Z}_m$. We show that $\mathbb{Z}_m/n\mathbb{Z}_m \cong \mathbb{Z}_d$, where $d = gcd(m, n)$, via the map $\varphi : \mathbb{Z}_m \to \mathbb{Z}_d$ by $\varphi([a]_m) = [a]_d$ for all $[a]_m \in \mathbb{Z}_m$. This completes the proof. □

Example 3.4.17. Let H be a finitely generated abelian group. We will prove that $\operatorname{Ext}(H, \mathbb{Z}) \cong T(H)$, where $T(H) = \{h \in H \mid nh = 0 \text{ for some nonzero } n \in \mathbb{Z}\}$ is the torsion submodule of the $\mathbb{Z}$-module H as defined in Example 2.1.6.

Proof. By the fundamental theorem for finitely generated abelian groups,

$$H \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}.$$

Identify H as its isomorphic direct sum, and observe that $\operatorname{Ext}(H, \mathbb{Z}) \cong \operatorname{Ext}(H', \mathbb{Z})$ whenever $H \cong H'$ by Corollary 3.4.9. Notice for each copy of $\mathbb{Z}$ if $h \neq 0$ and $nh = 0$ then $n = 0$ since $\mathbb{Z}$ is an integral domain and has characteristic zero. Now observe for each $i \in \{1, 2, \ldots, k\}$, $\mathbb{Z}_{n_i}$ has characteristic n_i. Therefore, $T(H) \cong \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$. Thus, it suffices to show $\operatorname{Ext}(H, \mathbb{Z}) \cong \bigoplus_{i=1}^k \mathbb{Z}_{n_i}$. Using (2) from Theorem 3.4.13 and the previous two examples, we have

$$\begin{aligned}
\operatorname{Ext}(H, \mathbb{Z}) &\cong \operatorname{Ext}(\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}, \mathbb{Z}) \\
&\cong \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \cdots \oplus \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Ext}(\mathbb{Z}_{n_1}, \mathbb{Z}) \oplus \cdots \oplus \operatorname{Ext}(\mathbb{Z}_{n_k}, \mathbb{Z}) \\
&\cong \{0\} \oplus \{0\} \oplus \cdots \oplus \{0\} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \\
&\cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \\
&= \bigoplus_{i=1}^{k} \mathbb{Z}_{n_i} \\
&\cong T(H).
\end{aligned}$$

□

In this chapter homology and cohomology were introduced, and we saw that there are differences between the two. In particular, we discovered that cohomology of a chain complex may not be trivial despite the fact that homology is. In doing so, we brought to the forefront a limitation of the Universal Coefficient Theorem, thus showing the theorem may not apply to any chain complex, rather, it applies to chain complexes which consist of only free abelian groups. This result lead us to the definition of the Ext group. We also have proved many properties of the Ext group, which will be used in Chapter 5. We now have the mathematical machinery in place to prove the Universal Coefficient Theorem.

Chapter 4

The Universal Coefficient Theorem for Cohomology

As we have seen, given a chain complex $\mathcal{C} = \{C_n, \partial\}$, one may compute the cohomology groups of $\mathcal{C}$ by choosing an abelian group G, replacing C_n with $C_n^* = \text{Hom}(C_n, G)$, and finally, computing the homology of the resulting co-chain complex $\{C_n^*, d\}$. Since it seems there is an algebraic relationship between $H^n(\mathcal{C}; G)$ and $H_n(\mathcal{C})$, one may wonder that if $H_n(\mathcal{C})$ is known, then can $H^n(\mathcal{C}; G)$ be computed? *The Universal Coefficient Theorem for Cohomology* states that we can. The theorem allows one to express the cohomology of a chain complex, with arbitrary coefficients, in terms of its homology groups with integer(universal) coefficients.

Without the theorem one may guess that $H^n(\mathcal{C}; G) \cong \text{Hom}(H_n(\mathcal{C}), G)$. In certain instances this is true (see Chapter 5 for details); however, the statement does not hold in general. For instance, recall the chain complex from Example 3.2.2. From Examples 3.2.2 and 3.3.3, $H_2(\mathcal{C}) \cong 0$ and $H^2(\mathcal{C}, \mathbb{Z}) \cong \mathbb{Z}$. Now, $\text{Hom}(H_2(\mathcal{C}), \mathbb{Z}) \cong \text{Hom}(0, \mathbb{Z}) \cong \{0\}$ which is clearly not isomorphic to $\mathbb{Z}_2 \cong H^2(\mathcal{C}; \mathbb{Z})$. We more thoroughly investigate these differences in Chapter 5.

We have not introduced the Universal Coefficient Theorem, but if we apply it to $\mathfrak{M}$ in Example 3.3.4, we obtain that $H^1(\mathfrak{M}; \mathbb{Z}) \cong \{0\}$. But computing cohomology directly yields $H^1(\mathfrak{M}, \mathbb{Z}) \cong \mathbb{Z}_2$. This difference arises from the fact that, in $\mathfrak{M}$, $\mathbb{Z}_2$ is not a free abelian group. The Universal Coefficient Theorem only applies when the chain complex in consideration consists solely of free abelian groups. We now introduce the tool which expresses cohomology in terms of homology, known as the *Universal Coefficient Theorem.*

Theorem 4.0.18 (The Universal Coefficient Theorem for Cohomology). [Hat02] *Consider a chain complex C of free abelian groups*

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots .$$

If C has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ are determined by the split exact sequence

$$0 \longrightarrow Ext(H_{n-1}(C), G) \longrightarrow H^n(C;G) \longrightarrow Hom(H_n(C), G) \longrightarrow 0.$$

Thus,

$$H^n(C;G) \cong Ext(H_{n-1}(C), G) \oplus Hom(H_n(C), G).$$

The proof will be separated into two parts. In Section 4.1, we show there is a well-defined surjection h from $H^n(C;G)$ to $\mathrm{Hom}(H_n(C), G)$. Secondly, in Section 4.2, we investigate $\mathrm{Ext}(H_{n-1}(C), G)$ by analyzing a series of diagrams, and complete the proof of the Universal Coefficient Theorem.

4.1 Part I : Construction of h

We will first construct a homomorphism $h : H^n(C;G) \to \mathrm{Hom}(H_n(C), G)$. Then, we will demonstrate that h is well-defined. Finally, it is crucial that h be surjective because if h is surjective, then, by Theorem 2.2.2, $H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$ is exact, establishing part of Theorem 4.0.18. Before we begin, we must prove the following lemma, which states that if you have a map defined on a quotient group, then there is a way to extend it to a larger domain.

Lemma 4.1.1. *Let $\overline{\varphi} : G/N \to H$, where $N \lhd G, H$ are groups. Then there exists a map $\varphi : G \to H$ so that if $\pi : G \to G/N$ is the canonical projection, then $\varphi = \overline{\varphi} \circ \pi$.*

Proof. This follows directly from composition of functions, in particular, the composite $\varphi : G \xrightarrow{\pi} G/N \xrightarrow{\overline{\varphi}} H$. □

Theorem 4.1.2. *Adopt the notation from Theorem 4.0.18. Then the following are true:*

1. *There exists a well-defined homomorphism* $h : H^n(\mathcal{C}; G) \to Hom(H_n(\mathcal{C}), G)$*;*

2. h *is surjective.*

Proof. (1) We desire a homomorphism which takes us from $H^n(\mathcal{C}; G)$ to $\mathrm{Hom}(H_n(\mathcal{C}), G)$. Let $d = \partial_n^* : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G)$. So we wish to construct a map from $\mathrm{Ker}d/\mathrm{Im}d$ to $\mathrm{Hom}(H_n(\mathcal{C}), G)$. Now, $[\alpha] \in H^n(\mathcal{C}; G)$ means $\alpha \in \mathrm{Ker}d \subset \mathrm{Hom}(C_n, G)$. Next, let $\alpha_0 = \alpha|_{Z_n}$, where $Z_n = \mathrm{Ker}\partial_n$. Thus, $\alpha_0 : Z_n \to G$. Let $B_n = \mathrm{Im}\partial_{n+1}$. We will show that $\alpha_0(B_n) = 0$. Let $\beta \in B_n$. Then, $\beta = \partial_{n+1}\gamma$ for some $\gamma \in C_{n+1}$. Observe, $\alpha_0(\beta) = \alpha(\beta)$ since α_0 is α restricted to a smaller domain. So

$$\begin{aligned} B_n \subset \mathrm{Ker}\partial_n = Z_n \iff \alpha_0(\beta) &= \alpha(\beta) \\ &= \alpha(\partial_{n+1}\gamma) \\ &= \alpha\partial_{n+1}(\gamma) \\ &= d\alpha(\gamma) \\ &= 0. \end{aligned}$$

So, $[\alpha] \in H^n(\mathcal{C}; G)$ means α must lie in $\mathrm{Ker}d$. Moreover, $\alpha_0|_{B_n} = 0$. Since $\alpha_0(B_n) = 0$, α_0 induces a homomorphism $\overline{\alpha}_0$ from Z_n/B_n to G. From this, we will define our map $h : H^n(\mathcal{C}; G) \to \mathrm{Hom}(H_n(\mathcal{C}), G)$ by $h[\alpha] = \overline{\alpha}_0$. Thus, h is the two step process which first restricts α to Z_n, then projects the result to be a map on a quotient.

To show h is well-defined, suppose there are $[\alpha]$, $[\beta] \in H^n(\mathcal{C}; G)$ such that $[\alpha] = [\beta]$. We must demonstrate $h([\alpha]) = h([\beta])$. Observe $[\alpha] = [\beta]$ implies $\alpha = \beta + \mathrm{Im}d$. Now h is linear since it is the restriction of a linear map composed with a quotient map. So, showing h is well-defined is equivalent to showing $h(\mathrm{Im}d) = 0$, since if $h(\mathrm{Im}d) = 0$, then $h([\alpha]) = h([\beta + \mathrm{Im}d]) = h([\beta]) + h(\mathrm{Im}d) = h([\beta])$. If $\gamma \in \mathrm{Im}d$ there must be some $\tau \in \mathrm{Hom}(C_{n-1}, G)$ such that $\gamma = d\tau$. Now, $h(\gamma)(\sigma) = h(d\tau)(\sigma) = \tau(\partial(\sigma)) = \tau(0) = 0$ for any $\sigma \in Z_n$. So, $h([\alpha]) = h([\beta + \mathrm{Im}d]) = h([\beta]) + h(\mathrm{Im}d) = h([\beta])$. Hence, h well-defined.

(2) We show h is surjective by finding a map $\Psi : \mathrm{Hom}(H_n(\mathcal{C}), G) \to H^n(\mathcal{C}; G)$ such that $h \circ \Psi$ is the identity map on $\mathrm{Hom}(H_n(\mathcal{C}), G)$. Consider the following short exact sequence of free $\mathbb{Z}$-modules $\mathfrak{C}$, where $Z_n \to C_n$ is the inclusion of Z_n into C_n:

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0.$$

Since $\mathbb{Z}$ is a PID, and B_{n-1} is a submodule of the free module C_{n-1}, $\mathfrak{E}$ is split exact. So, by Theorem 2.2.4, there must be a projection map $p : C_n \to Z_n$ such that $i \circ p$ is the identity map on Z_n, where $i : Z_n \to C_n$ is the inclusion homomorphism. We will use the following short exact sequence to construct Ψ:

$$0 \to Z_n \overset{i}{\rightleftarrows} C_n \xrightarrow{\partial} B_{n-1} \to 0.$$

Let $\overline{\varphi} \in \mathrm{Hom}(H_n(\mathcal{C}), G)$. Then, by Lemma 4.1.1, there exists a map $\varphi : Z_n \to G$, where $\varphi = \overline{\varphi} \circ \pi$. This gives us a map β as an extension of φ by p, defined by the composite $\beta : C_n \xrightarrow{p} Z_n \xrightarrow{\varphi} G$. We must show $\beta \in \mathrm{Ker} d$, where $d : C_n^* \to C_{n+1}^*$ is the differential on cohomology with coefficients in G. Once we establish this, we set $\Psi[\varphi] = [\beta]$, the equivalence class represented by β in $H^n(\mathcal{C}; G) = \mathrm{Ker}\ d/\mathrm{Im}\ d$.

We now show $d\beta = 0$. Let $\gamma \in H_{n+1}(\mathcal{C})$, and we will apply it to $d\beta$. Notice that $B_n \subset Z_n$, so that $p|_{B_n} = id|_{Z_n}$. So,

$$d\beta(\gamma) = \beta(\partial\gamma) = \varphi(p(\partial\gamma)) = \varphi(\partial\gamma).$$

Now $\varphi : Z_n \to G$ was constructed to factor through the projection homomorphism π, as in Lemma 4.1.1. So denoting $\pi(\partial\gamma) = [\partial\gamma]$, we have

$$\varphi(\partial\gamma) = \overline{\varphi}[\partial\gamma] = \overline{\varphi}(0) = 0.$$

It is now clear that $d\beta = 0$.

We now prove assertion (2) by showing $h \circ \Psi = Id$, where Id is the identity map on $\mathrm{Hom}(H_n(\mathcal{C}), G)$ for the following reason: If $h \circ \Psi = Id$, then the map h is surjective, since it is the second in the composition of maps which is equal to a surjective map. Consider any $\overline{\varphi} \in \mathrm{Hom}(H_n(\mathcal{C}), G)$. The map $\Psi(\overline{\varphi})$ extends φ from C_n to G, and after projection, is the element $\Psi(\overline{\varphi})$. The first step in evaluating h is to choose a representative, namely β, and restrict it to be a map from Z_n to G. But, since $p \circ i$ is the identity on Z_n, extending φ to β, then restricting it gives you φ back again. Thus, upon projection, which is the final step in evaluating h, we end up with $\overline{\varphi}$ back again. Therefore, $h \circ \Psi = Id$. This completes the construction of h. □

4.2 Part II : $\mathrm{Ext}(H_{n-1}(\mathcal{C}), G)$

By Theorem 2.2.5, since h is surjective, there exists the following short exact sequence. In fact, we have demonstrated it is split exact by Theorem 2.2.4 since we have constructed a Ψ splitting in Section 4.1:

$$0 \longrightarrow \mathrm{Ker}h \xrightarrow{i} H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C),G) \longrightarrow 0.$$

To analyze Ker$h$, consider $\mathfrak{B}$ a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & B_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0.
\end{array}$$

We say the diagram above is a sequence of short exact sequences because it is a piece of the diagram from Figure 4.1. The maps from $Z_{n+1} \to Z_n$ and $B_n \to B_{n-1}$ are restrictions of the boundary map $\partial : C_n \to C_{n-1}$ to Ker∂ and Im$\partial \subset$ Ker∂ , and therefore, are 0. For example, consider $\partial : B_n \to B_{n-1}$, and let $\beta \in B_n$. There exists $c \in C_{n+1}$ such that $\beta = \partial(c)$. So for $\beta \in B_n$, $\partial\beta = \partial\partial(c) = 0$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n+2} & \longrightarrow & C_{n+2} & \xrightarrow{\partial} & B_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & B_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial} & B_{n-2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Figure 4.1: A Sequence of Short Exact Sequences

Observe, both rows of $\mathfrak{B}$ are split exact since the abelian groups B_n and B_{n-1}, being submodules of free modules, are free. Hence, dualizing will result in the following commutative diagram $\mathfrak{B}^*$, containing rows which are exact (where $\partial^* = d$) by Theorem

3.3.5:

$$
\begin{tikzcd}
0 & Z^*_{n+1} \arrow[l] & C^*_{n+1} \arrow[l] & B^*_n \arrow[l, "d"'] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l] \arrow[u, "d"] & B^*_{n-1} \arrow[l, "d"'] \arrow[u, "0"] & 0. \arrow[l]
\end{tikzcd}
\tag{4.1}
$$

Note, in $\mathfrak{B}^*$, the maps from B^*_{n-1} to B^*_n and Z^*_n to Z^*_{n+1} are $0^* = 0$ since $0^*\varphi = \varphi \circ 0 = 0$ for any φ in B^*_{n-1} and Z^*_n. Now $\mathfrak{B}^*$ is a sequence of short exact sequences, and we now view it as in Figure 4.2.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & Z^*_{n+1} \arrow[l] \arrow[u, "0"] & C^*_{n+1} \arrow[l] \arrow[u, "d"] & B^*_n \arrow[l, "d"'] \arrow[u, "0"] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l] \arrow[u, "d"] & B^*_{n-1} \arrow[l, "d"'] \arrow[u, "0"] & 0 \arrow[l] \\
0 & Z^*_{n-1} \arrow[l] \arrow[u, "0"] & C^*_{n-1} \arrow[l] \arrow[u, "d"] & B^*_{n-2} \arrow[l, "d"'] \arrow[u, "0"] & 0 \arrow[l] \\
 & \vdots \arrow[u, "0"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "0"] &
\end{tikzcd}
$$

Figure 4.2: A Dual Short Exact Sequence of Chain Complexes

Let $\tilde{C} = \{Z_i, 0\}$ and $\hat{C} = \{B_i, 0\}$ be the chain complexes in Figure 4.1. Now $\mathfrak{B}$ gives rise to the following long exact sequence on cohomology [Wei94].

$$\cdots \leftarrow H^n(\hat{C}; G) \xleftarrow{\delta} H^n(\tilde{C}; G) \leftarrow H^n(C; G) \leftarrow H^{n-1}(\hat{C}; G) \xleftarrow{\delta} H^{n-1}(\tilde{C}; G) \leftarrow \cdots$$

The fact that a sequence of short exact sequences gives rise to a long exact sequence on cohomology is not immediately obvious, so we will give a brief discussion. Consider the sequence of short exact sequences $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, where $A = \{A_n, \partial\}$, $B = \{B_n, \partial\}$, and $C = \{C_n, \partial\}$ are chain complexes. Let G be an abelian group and dualize the sequence of short exact sequences, and represent the dual maps to f and g by f' and g', respectively. Then $0 \leftarrow A^* \xleftarrow{f'} B^* \xleftarrow{g'} C^* \leftarrow 0$ is the dual sequence which is short exact by Theorem 3.3.5. Now, by Lemma 3.4.4, we know f' and g' induce homomorphisms $f'_* : H^n(B) \to H^n(A)$ and $g'_* : H^n(C) \to H^n(C)$ on homology. The "connecting homomorphism" $\delta : H^n(A) \to H^{n+1}(C)$ [HS71], and we presently discuss its construction. Let $\alpha \in \mathrm{Ker} d$. Then there is $\beta \in B_n^*$ such that $f'(\beta) = \alpha$. Notice $d\beta \in \mathrm{Ker} f'$ since $f'(d\beta) = df'(\beta) = d(\alpha) = 0$. So, by exactness, $d\beta \in \mathrm{Img}'$. Thus, there exists $\gamma \in C_{n+1}^*$ such that $g'(\gamma) = d\beta$. So set $\delta(\alpha) = \gamma$. We leave the details of showing δ is well-defined and that the resulting sequence in Equation (4.2) below is, in fact, exact to [Wei94]. This concludes our discussion on how the connecting homomorphism is constructed, and now we apply its construction to our proof.

The map δ, as labeled below, is the connecting homomorphism and we follow [Wei94] below to describe this map, after some preliminary remarks. Since the maps $d : B_n^* \to B_{n+1}^*$ and $d : Z_n^* \to Z_{n+1}^*$ are the zero map, we have $H^n(\bar{C}; G) \cong Z_n^*$ and $H^n(\hat{C}; G) \cong B_n^*$ for each n. Therefore, we may rewrite the long exact sequence on the cohomology of $\mathfrak{B}$ as follows:

$$\cdots \leftarrow H^{n+1}(C; G) \leftarrow B_n^* \xleftarrow{\delta} Z_n^* \leftarrow H^n(C; G) \leftarrow B_{n-1}^* \xleftarrow{\delta} Z_{n-1}^* \leftarrow \cdots . \tag{4.2}$$

The connecting homomorphism δ takes us from Z_n^* to B_n^*. We refer to Equation (4.1) to compute an output of δ. Now δ maps $\alpha_0 \in Z_n^*$ into B_n^* in three steps.

1. Extend the domain of α_0 to C_n to get an element of C_n^*, and denote this extension by α.

2. Apply d to α to get a member of C_{n+1}^*, that is, $d\alpha : C_{n+1} \to G$.

3. Finally, restrict $d\alpha$ to B_n to obtain a member of B_n^*.

Theorem 4.2.1. *The connecting homomorphism δ, as labeled in Equation (4.2), is i_n^* for all n, where i_n^* is the dual map to the inclusion homomorphism $i_n : B_n \to Z_n$.*

Proof. Let $i_n : B_n \to Z_n$ be the inclusions of boundaries into cycles. We will analyze the map δ, and wish to deduce that it is $i_n^* : Z_n^* \to B_n^*$. To compute an output of δ, one needs to refer to diagram $\mathfrak{B}^*$ in Equation (4.1) and the process just mentioned. Take any $\alpha_0 \in Z_n^*$. Since $B_n \subset Z_n$ the net effect of applying δ to any $\alpha_0 \in Z_n^*$ is to restrict α_0 to B_n. Now, $i_n^* : Z_n^* \to B_n^*$. Take any $\gamma \in Z_n^*$, then $\gamma : Z_n \to G$. By definition, $i_n^*(\gamma) = \gamma \circ i_n : B_n \to G$. So, i_n^* takes homomorphisms from Z_n to G and restricts them to B_n, which is exactly how δ affects elements of Z_n^*. Therefore, $\delta = i_n^*$. $\square$

Referencing the discussion after Theorem 2.2.5, any long exact sequence may be broken into short exact sequences. So one may break up the long exact sequence in Equation (4.2) at $H^n(\mathcal{C};G)$ into the short exact sequences below, where we call $\delta = i_n^*$ for all n:

$$0 \leftarrow \text{Ker } i_n^* \leftarrow H^n(\mathcal{C};G) \leftarrow \text{Coker } i_{n-1}^* \leftarrow 0, \tag{4.3}$$

and Coker $i_n^* = \frac{B_{n-1}^*}{\text{Im } i_{n-1}^*}$.

Next we will show that Ker $i_n^* \cong \text{Hom}(H_n(\mathcal{C}),G)$ for each n. Consider the map $\Phi : \text{Ker } i_n^* \to \text{Hom}(H_n(\mathcal{C}),G)$ by $\Phi(\varphi) = \overline{\varphi}$ for $\varphi \in \text{Ker } i_n^*$, where $\overline{\varphi} : \frac{Z_n}{B_n} \to G$. Observe, if $\gamma \in \text{Ker } i_n^*$ then for any $z \in Z_n$, $i_n^*(\gamma)(z) = \gamma(i_n z) = \gamma(z) = 0$. So, elements of Ker i_n^* vanish on B_n since the range of i_n is B_n. Clearly, Φ is a homomorphism since it restricts homomorphisms to quotient maps, which are homomorphisms. To prove the isomorphism we show Φ has a two sided inverse. Let $A = \text{Ker } i_n^*$ and $B = \text{Hom}(H_n(\mathcal{C}),G)$. We seek a map $\Psi : B \to A$ so that $\Psi \circ \Phi = 1_A$ and $\Phi \circ \Psi = 1_B$. Let $\pi : Z_n \to \frac{Z_n}{B_n}$ be the coset projection. Then $\varphi = \overline{\varphi} \circ \pi$ is a map from Z_n to G as in Lemma 4.1.1, and hence, an element of Z_n^*. Now for $\bar{\varphi} \in B$, define $\Psi : B \to A$ by $\Psi(\overline{\varphi}) = \overline{\varphi} \circ \pi$. Let $\varphi \in A$ and $z \in Z_n$. Then, $\Psi \circ \Phi(\varphi)(z) = \Psi(\overline{\varphi})(z) = (\overline{\varphi} \circ \pi)(z) = \varphi(z)$. So, $\Psi \circ \Phi = 1_A$. Take any $\overline{\varphi} \in B$ and $[\alpha] \in Z_n/B_n$. Then, $\Phi \circ \Psi(\overline{\varphi})([\alpha]) = \Phi(\overline{\varphi} \circ \pi)([\alpha]) = \Phi(\varphi)([\alpha]) = \overline{\varphi}([\alpha])$. So, $\Phi \circ \Psi = 1_B$. Therefore, Φ is an isomorphism.

Using the result Ker $i_n^* \cong \text{Hom}(H_n(\mathcal{C}),G)$, one may rewrite the sequence of Equation (4.3) as:

$$0 \leftarrow \text{Hom}(H_n(\mathcal{C}),G) \leftarrow H^n(\mathcal{C};G) \leftarrow \text{Coker } i_{n-1}^* \leftarrow 0.$$

Finally, we consider Coker i_{n-1}^*, and claim Coker i_{n-1}^* is none other than $\text{Ext}(H_{n-1}(\mathcal{C}),G)$. Consider the following short exact sequence:

$$0 \to B_n \overset{i_n}{\hookrightarrow} Z_n \to \frac{Z_n}{B_n} \to 0.$$

This sequence is a free resolution for $H_n(\mathcal{C}) = \frac{Z_n}{B_n}$ since Z_n and B_n, being submodules of the free $\mathbb{Z}$-module C_n, are free. Let G be an abelian group, and dualize the sequence to obtain the co-chain complex:

$$0 \leftarrow B_n^* \overset{i_n^*}{\leftarrow} Z_n^* \leftarrow Hom(H_n(\mathcal{C}),G) \leftarrow 0.$$

This co-chain complex is not necessarily exact since $H_n(\mathcal{C})$ may not be free. By definition,

$$\text{Ext}(H_n(\mathcal{C}),G) = \frac{\text{Ker } 0}{\text{Im } i_n^*} = \frac{B_n^*}{\text{Im } i_n^*} := \text{Coker } i_n^*.$$

To summarize these results, we have just shown that $\mathrm{Ext}(H_n(\mathcal{C}), G) = \mathrm{Coker}\ i_n^*$ and $\mathrm{Ker}\ i_n^* \cong \mathrm{Hom}(H_n(\mathcal{C}), G)$. Finally, consider the commutative diagram:

$$\begin{tikzcd}
0 & \mathrm{Ker}\ i_n^* \arrow[l] & H^n(\mathcal{C};G) \arrow[l, "j^*"'] & \mathrm{Coker}\ i_{n-1}^* \arrow[l] & 0 \arrow[l] \\
0 & \mathrm{Hom}(H_n(\mathcal{C}),G) \arrow[l] \arrow[u, "\cong"', "\Phi"] & H^n(\mathcal{C};G) \arrow[l, "h"'] \arrow[u, "id"'] & \mathrm{Coker}\ i_{n-1}^* \arrow[l] \arrow[u, "id"'] & 0 \arrow[l] \\
0 & \mathrm{Hom}(H_n(\mathcal{C}),G) \arrow[l] \arrow[u, "id"'] & H^n(\mathcal{C};G) \arrow[l, "h"'] \arrow[u, "id"'] & \mathrm{Ext}(H_{n-1}(\mathcal{C}),G) \arrow[l] \arrow[u, equal] & 0. \arrow[l]
\end{tikzcd}$$

Let $j : Z_n \hookrightarrow C_n$ be the inclusion homomorphism. Then $j^* : C_n^* \to Z_n^*$ is the induced homomorphism in our original long exact sequence in Equation (4.2), and put into short exact sequences in Equation (4.3). We examine $j^* : H^n(\mathcal{C};G) \to \mathrm{Ker}\ i_n^*$ in the top left corner of the proceding diagram, and claim that $h = \Phi^{-1} \circ j^*$, that is, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Ker}\ i_n^* & H^n(\mathcal{C};G) \arrow[l, "j^*"'] \\
\mathrm{Hom}(H_n(\mathcal{C}),G) \arrow[u, "\cong"', "\Phi"] & H^n(\mathcal{C};G) \arrow[l, "h"'] \arrow[u, "id"']
\end{tikzcd}$$

Take any $[\alpha] \in H^n(\mathcal{C};G)$. Then the representative $\alpha : C_n \to G$ and $d\alpha = 0$. Next, for $z \in C_n$, $j^*(\alpha(z)) = \alpha(j(z))$. So, since $j(z) \in Z_n$, deduce that j^* restricts α to Z_n. Let $\partial\beta \in B_n$. Then $j^*(\alpha)(\partial\beta) = \alpha(j\partial\beta) = \alpha(\partial\beta) = d\alpha(\beta) = 0(\beta) = 0$. Thus, $j^*\alpha(B_n) = 0$, and $j^*\alpha \in \mathrm{Ker}\ i_n^*$. Now, since Φ is an isomorphism, there is a unique $\varphi \in \mathrm{Hom}(H_n(\mathcal{C}), G)$ with $\Phi(\varphi) = j^*(\alpha)$, which implies $\varphi = \Phi^{-1}j^*(\alpha) \in \mathrm{Hom}(H_n(\mathcal{C}), G)$. Since h is surjective, there exists $[\beta] \in H^n(\mathcal{C};G)$ such that $h([\beta]) = \varphi = \Phi^{-1}j^*(\alpha)$, but since $id : H^n(\mathcal{C};G) \to H^n(\mathcal{C};G)$ is the identity map, $[\beta] = [\alpha]$. Thus, the diagram commutes.

From Section 4.1, there is a surjective homomorphism h from $H^n(\mathcal{C};G)$ to $\mathrm{Hom}(H_n(\mathcal{C}), G)$. Thus, $H^n(\mathcal{C};G) \xrightarrow{h} \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$ is exact. Now, by Theorem 2.2.6, the sequence $0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \to H^n(\mathcal{C};G) \xrightarrow{h} \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$ is exact. Also, from the construction of h, we know the short exact sequence is split exact. Therefore, $H^n(\mathcal{C};G) \cong \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \oplus \mathrm{Hom}(H_n(\mathcal{C}), G)$. This completes the proof of the Universal Coefficient Theorem.

To summarize, the Universal Coefficient Theorem for Cohomology allows one to express cohomology with arbitrary coefficients in terms of homology with integer (universal) coefficients. In the next chapter we apply this theorem to a number of topological spaces. Also, we investigate how the theorem can be rewritten for homology groups which are finitely generated.

Chapter 5

Applications of the Universal Coefficient Theorem

This chapter is devoted to illustrating the usefulness of the Universal Coefficient Theorem. In Section 5.1 we prove a result of the theorem for finitely generated homology groups. We apply the theorem to calculate the cohomology groups of the Klein bottle, the real projective space $\mathbb{R}P^5$ (with a description of the process in general for $\mathbb{R}P^k$ when k is odd), and the torus $\mathcal{T}$ with integer coefficients in Section 5.2. Finally, in Section 5.3, we demonstrate how to change coefficients from integer homology to cohomology with coefficients other that $\mathbb{Z}$.

5.1 Finitely Generated Homology Groups

The Universal Coefficient Theorem for Cohomology allows one to express the cohomology groups of a chain complex with arbitrary abelian coefficients in terms of homology groups with integer coefficients. The following theorem explains how to compute $H^n(C;\mathbb{Z})$ when $H_n(C)$ are finitely generated.

Theorem 5.1.1. *Let C be a chain complex where $H_n(C) = \bigoplus_1^{r_n} \mathbb{Z} \oplus T_n$ are finitely generated for all n, where $T_n = T\left(H_n(C)\right)$ is the torsion submodule of the $\mathbb{Z}$-module $H_n(C)$. Then*

$$H^n(C;\mathbb{Z}) \cong \mathbb{Z}^{r_n} \oplus T_{n-1},$$

where $\mathbb{Z}^{r_n} = \bigoplus_1^{r_n} \mathbb{Z}$.

Proof. By the Universal Coefficient Theorem, $H^n(C;\mathbb{Z})$ is determined by the following split exact sequence

$$0 \to \operatorname{Ext}(H_{n-1}(C),\mathbb{Z}) \to H^n(C;\mathbb{Z}) \to \operatorname{Hom}(H_n(C),\mathbb{Z}) \to 0.$$

That is, $H^n(C;\mathbb{Z}) \cong \operatorname{Ext}(H_{n-1}(C)) \oplus \operatorname{Hom}(H_n(C),\mathbb{Z})$. By Example 3.4.17, for each n, $\operatorname{Ext}(H_n(C),\mathbb{Z}) \cong T_n$. To complete the proof we need to show $\operatorname{Hom}(H_n(C),\mathbb{Z}) \cong \mathbb{Z}^{r_n}$. By Theorem 2.4.4 and Examples 2.4.2 and 2.4.3,

$$\begin{aligned}
\operatorname{Hom}(H_n(C),\mathbb{Z}) &\cong \operatorname{Hom}\left(\bigoplus_{i=1}^{r_n}\mathbb{Z} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \mathbb{Z}_{n_k}, \mathbb{Z}\right)\\
&\cong \bigoplus_{1}^{r_n} \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \oplus \bigoplus_{j=1}^{k} \operatorname{Hom}(\mathbb{Z}_{n_j},\mathbb{Z})\\
&\cong \bigoplus_{1}^{r_n}\mathbb{Z} \oplus \bigoplus_{j=1}^{k}\{0\}\\
&\cong \bigoplus_{1}^{r_n}\mathbb{Z}\\
&= \mathbb{Z}^{r_n}.
\end{aligned}$$

This completes the proof. □

In retrospect, Theorem 5.1 says if $H_n(C)$ are finitely generated, then $H^n(C,\mathbb{Z})$ is the free part of $H_n(C)$ plus the torsion submodule of $H_{n-1}(C)$. So one could say that to compute cohomology take the free part of homology at the same dimension and add torsion from homology one dimension lower. In the following examples we apply this process.

5.2 Examples

Example 5.2.1. Compute the cohomology groups for the Klein bottle K.

To suppress our calculations we refer to the homology groups for the Klein bottle K [Dur08]. A topological description of the Klein bottle (and our other topological examples) is not needed, rather, a description of its homology $H_n(K)$, which we give here.

For a further description of topological applications to this subject, see [Hat02, KSW89, Wei94]. The homology groups for the Klein bottle can be found in Equation (5.1):

$$H_n(K) \cong \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2, & \text{for } n = 1 \\ 0, & \text{for } n > 1. \end{cases} \tag{5.1}$$

By Theorem 5.1, since $H_0(K) = \mathbb{Z}$, $H^0(K;\mathbb{Z})$ is one copy of $\mathbb{Z}$ plus the torsion submodule from homology one dimension lower which is $\{0\}$ by definition. So we obtain $H^0(K;\mathbb{Z}) \cong \mathbb{Z}$. Next, since $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $T(H_0(K)) \cong \{0\}$, $H^1(K;\mathbb{Z}) = \mathbb{Z}$. Finally, $H^2(K;\mathbb{Z})$ is one copy of $\{0\} \cong H_2(K)$ plus the torsion of $H_1(K)$. From this, $H^2(K;\mathbb{Z}) = \{0\} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_2$. These results have been placed in Table 5.1. In Table 5.1, notice that torsion ($\mathbb{Z}_2$) of homology has been shifted up one dimension in cohomology and the free factors remain.

n	$H_n(K)$	$H^n(K;\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$
2	0	$\mathbb{Z}_2$
≥ 3	0	0

Table 5.1: Cohomology of the Klein Bottle K

In the next example we compute the cohomology groups for real projective space $\mathbb{R}P^5$. Note that the same process in Exampe 5.2.2 may be applied to compute $\mathbb{R}P^k$ (for odd k) by considering the following cases: $n = 0$ and $n = k$, $0 < n < k$ for odd n, and when $n > k$. Referring to Table 5.2, its seems as though for even n, $0 < n < k$, $H^n(\mathbb{R}P^k;\mathbb{Z}) \cong \mathbb{Z}_2$, and $H^n(\mathbb{R}P^k;\mathbb{Z}) \cong \{0\}$ for $n \neq k$ and odd.

Example 5.2.2. Compute the cohomology groups for $\mathbb{R}P^k$.

We will compute the cohomology groups of $\mathbb{R}P^5$, and note that the process for computing cohomology of $\mathbb{R}P^k$ will the same as computing cohomology of $\mathbb{R}P^5$. The homology groups for $\mathbb{R}P^5$ are [Hat02]:

$$H_n(\mathbb{R}P^5) \cong \begin{cases} \mathbb{Z}, & \text{for } n = 0 \text{ and } n = 5 \text{ odd} \\ \mathbb{Z}_2, & \text{for } n \text{ odd, } 0 < n < 5 \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Since the homology groups of $\mathbb{R}P^5$ are finitely generated we may apply Theorem 5.1. The same process described before will be applied here; however, we are more explicit in this example. We compute the cohomology for $\mathbb{R}P^5$ below.

$\underline{H^0(\mathbb{R}P^5;\mathbb{Z})}$:

$H^0(\mathbb{R}P^5) \cong T(H_{-1}(\mathbb{R}P^5)) \oplus \mathrm{Hom}(H_0(\mathbb{R}P^5;\mathbb{Z}),\mathbb{Z}) \cong \{0\} \oplus \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$.

$\underline{H^1(\mathbb{R}P^5;\mathbb{Z})}$:

$H^1(\mathbb{R}P^5;\mathbb{Z}) \cong T(H_0(\mathbb{R}P^5)) \oplus \mathrm{Hom}(H_1(\mathbb{R}P^5),\mathbb{Z}) \cong \{0\} \oplus \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) \cong \{0\} \oplus \{0\} \cong \{0\}$.

$\underline{H^2(\mathbb{R}P^5;\mathbb{Z})}$:

$H^2(\mathbb{R}P^5;\mathbb{Z}) \cong T(H_1(\mathbb{R}P^5)) \oplus \mathrm{Hom}(H_2(\mathbb{R}P^5),\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathrm{Hom}(\{0\},\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \{0\} \cong \mathbb{Z}_2$.

$\underline{H^3(\mathbb{R}P^5;\mathbb{Z})}$:

$H^3(\mathbb{R}P^5;\mathbb{Z}) \cong T(H_2(\mathbb{R}P^5)) \oplus \mathrm{Hom}(H_3(\mathbb{R}P^5),\mathbb{Z}) \cong \{0\} \oplus \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) \cong \{0\} \oplus \{0\} \cong \{0\}$.

$\underline{H^4(\mathbb{R}P^5;\mathbb{Z})}$:

$H^4(\mathbb{R}P^5;\mathbb{Z}) \cong T(H_3(\mathbb{R}P^5)) \oplus \mathrm{Hom}(H_4(\mathbb{R}P^5),\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathrm{Hom}(\{0\},\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \{0\} \cong \mathbb{Z}_2$.

$\underline{H^5(\mathbb{R}P^5;\mathbb{Z})}$:

$H^5(\mathbb{R}P^5;\mathbb{Z}) \cong T(H_4(\mathbb{R}P^5)) \oplus \mathrm{Hom}(H_5(\mathbb{R}P^5),\mathbb{Z}) \cong \{0\} \oplus \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \{0\} \oplus \mathbb{Z} \cong \mathbb{Z}$.

The results have been placed in Table 5.2, and once again, observe that torsion of homology has been shifted up one dimension in cohomology.

The previous examples feature topological spaces which have torsion in their homology, and upon calculating cohomology, we see that the torsion of homology is shifted up by one dimension in cohomology. Clearly, from Examples 5.2.1 and 5.2.2, $H^n(C,\mathbb{Z})$ is not isomorphic to $\mathrm{Hom}(H_n(C),\mathbb{Z})$ for all n. For example, refer to Table 5.2 for homology and cohomology of $\mathbb{R}P^5$ when $n = 2$. Now $\text{Hom}(H_2(\mathbb{R}P^5), \mathbb{Z}) \cong \text{Hom}(\{0\}, \mathbb{Z}) \cong \{0\}$, but $H^2(\mathbb{R}P^5; \mathbb{Z}) \cong \mathbb{Z}_2$. Contrast these results with the following example.

n	$H_n(\mathbb{R}P^5)$	$H^n(\mathbb{R}P^5; \mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}_2$	0
2	0	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$	0
4	0	$\mathbb{Z}_2$
5	$\mathbb{Z}$	$\mathbb{Z}$
≥ 6	0	0

Table 5.2: Cohomology of $\mathbb{R}P^5$

Example 5.2.3. Compute the cohomology groups of the Torus $\mathcal{T}$.

We refer to Equation (5.3) for the homology groups of the Torus [Hat02].

$$H_n(\mathcal{T}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{for } n = 1 \\ \mathbb{Z}, & \text{for } n = 0, 2 \\ 0, & \text{for } n \geq 3 \end{cases} \tag{5.3}$$

We calculate the cohomology groups of $\mathcal{T}$ below by applying the Universal Coefficient Theorem.

$\underline{H^0(\mathcal{T}; \mathbb{Z})}$:
$H^0(\mathcal{T}; \mathbb{Z}) \cong \text{Hom}(H_0(\mathcal{T}), \mathbb{Z}) \oplus T(H_{-1}(\mathcal{T})) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \{0\} \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$.

$\underline{H^1(\mathcal{T}; \mathbb{Z})}$:
$H^1(\mathcal{T}; \mathbb{Z}) \cong \text{Hom}(H_1(\mathcal{T}), \mathbb{Z}) \oplus T(H_0(\mathcal{T})) \cong \text{Hom}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}) \oplus \{0\} \cong \mathbb{Z} \oplus \mathbb{Z}$.

$\underline{H^2(\mathcal{T}; \mathbb{Z})}$:
$H^2(\mathcal{T}; \mathbb{Z}) \cong \text{Hom}(H_2(\mathcal{T}), \mathbb{Z}) \oplus T(H_1(\mathcal{T})) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \{0\} \cong \mathbb{Z} \oplus \{0\} \cong \mathbb{Z}$.

$\underline{H^n(\mathcal{T}; \mathbb{Z}), n \geq 3}$:
$H^n(\mathcal{T}; \mathbb{Z}) \cong \text{Hom}(H_n(\mathcal{T}), \mathbb{Z}) \oplus T(H_{n-1}(\mathcal{T})) \cong \text{Hom}(0, \mathbb{Z}) \oplus \{0\} \cong \{0\} \oplus \{0\} \cong \{0\}$.

These calculations have been placed in Table 5.3.

n	$H_n(\mathcal{T})$	$H^n(\mathcal{T};\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$
2	$\mathbb{Z}$	$\mathbb{Z}$
≥ 3	0	0

Table 5.3: Cohomology of the Torus $\mathcal{T}$

Examining Table 5.3, we notice that $H^n(\mathcal{T};\mathbb{Z}) \cong \mathrm{Hom}(H_n(\mathcal{T}),\mathbb{Z})$ for all n. Observe that $H_n(\mathcal{T})$ is torsion free, that is, $T_n = T(H_n(\mathcal{T})) = 0$ for all n. Now substituting $T_n = 0$ for each n in the calculations above one sees that $H^n(\mathcal{T};\mathbb{Z}) \cong \mathrm{Hom}(H_n(\mathcal{T}),\mathbb{Z})$. The results lead to the next corollary.

Corollary 5.2.4. *Let $\mathcal{C} = \{C_n, \partial\}$ be a chain complex. If $H_n(\mathcal{C})$ is finitely generated and $T_n = T(H_n(\mathcal{C})) = 0$ for all n, then $H^n(\mathcal{C},\mathbb{Z}) \cong Hom(H_n(\mathcal{C}),\mathbb{Z})$.*

Proof. By the Universal Coefficient Theorem,

$$H^n(\mathcal{C};\mathbb{Z}) \cong \mathrm{Ext}(H_{n-1}(\mathcal{C}),\mathbb{Z}) \oplus \mathrm{Hom}(H_n(\mathcal{C}),\mathbb{Z}).$$

Now, by Example 3.4.17, $\mathrm{Ext}(H_{n-1}(\mathcal{C}),\mathbb{Z}) \cong T_{n-1} = T(H_{n-1}(\mathcal{C}))$. Since $T_n = 0$ for all n, it follows directly that $H^n(\mathcal{C};\mathbb{Z}) \cong \mathrm{Hom}(H_n(\mathcal{C}),\mathbb{Z})$. □

Recall, in Chapter 4, we showed that in general $H^n(\mathcal{C};G) \ncong \mathrm{Hom}(H_n(\mathcal{C}),G)$. Now Corollary 5.2.4 shows that $H^n(\mathcal{C};G) \cong \mathrm{Hom}(H_n(\mathcal{C}),G)$ whenever $H_n(\mathcal{C})$ is free and finitely generated, and the coefficient group is taken to be $G = \mathbb{Z}$.

Example 5.2.5. Let X be a path connected space, where $H_1(X)$ is finitely generated. Then $H^1(X;\mathbb{Z})$ is free.

First observe that since X is path connected $H_0(X) \cong \mathbb{Z}$ [Hat02]. By the Universal Coefficient Theorem, $H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X),\mathbb{Z}) \oplus \mathrm{Ext}(H_0(X),\mathbb{Z})$. Now $H_0(X)$ is free, so by Theorem 3.4.13(1), $\mathrm{Ext}(H_0(X),\mathbb{Z}) = 0$. Thus, $H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X),\mathbb{Z})$. Now, $H_1(X) \cong \mathbb{Z}^m \oplus T(H_1(X))$. From the proof of Theorem 5.1, $\mathrm{Hom}(H_1(K),\mathbb{Z}) \cong \mathbb{Z}^m$. Therefore, $H^1(X;\mathbb{Z}) \cong \mathbb{Z}^m$ is free since it is a direct sum of free $\mathbb{Z}$-modules.

5.3 Changing Coefficients

In this section we demonstrate how to change cohomology coefficients for the Klein Bottle K and the Torus T. The ability to change coefficient groups is important because one coefficient group may illuminate more geometrical data for a topological space than another coefficient group.

Example 5.3.1. Compute the cohomology groups of the Klein Bottle with coefficients in $\mathbb{Z}_2$.

First note that Theorem 5.1.1 does not apply here since it requires cohomology with coefficients in $\mathbb{Z}$. Nonetheless, by the Universal Coefficient Theorem where $G = \mathbb{Z}_2$, $H^n(K;\mathbb{Z}_2) \cong \mathrm{Hom}(H_n(K),\mathbb{Z}_2) \oplus \mathrm{Ext}(H_{n-1}(K),\mathbb{Z}_2)$. This relation will allow us to translate homology with coefficients in $\mathbb{Z}$ to cohomology with coefficients in $\mathbb{Z}_2$. Refer to Equation (5.1) for the homology of K with integer coefficients. We perform the desired calculations below.

<u>$H^0(K;\mathbb{Z}_2)$:</u>

$$\begin{aligned} H^0(K;\mathbb{Z}_2) &\cong \mathrm{Hom}(H_0(K),\mathbb{Z}_2) \oplus \mathrm{Ext}(H_{-1}(K)\mathbb{Z}_2) \\ &\cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2) \oplus \{0\} \\ &\cong \mathbb{Z}_2 \oplus \{0\} \\ &\cong \mathbb{Z}_2. \end{aligned}$$

<u>$H^1(K;\mathbb{Z}_2)$:</u>

$$\begin{aligned} H^1(K;\mathbb{Z}_2) &\cong \mathrm{Hom}(H_1(K),\mathbb{Z}_2) \oplus \mathrm{Ext}(H_0(K),\mathbb{Z}_2) \\ &\cong \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}_2,\mathbb{Z}_2) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) \\ &\cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2) \oplus \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \{0\} \text{ by Theorem 3.4.13} \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

$\underline{H^2(K;\mathbb{Z}_2)}$:

$$\begin{aligned}
H^2(K;\mathbb{Z}_2) &\cong \operatorname{Hom}(H_2(K),\mathbb{Z}_2) \oplus \operatorname{Ext}(H_1(K),\mathbb{Z}_2) \\
&\cong \operatorname{Hom}(0,\mathbb{Z}_2) \oplus \operatorname{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2,\mathbb{Z}_2) \\
&\cong \{0\} \oplus \operatorname{Ext}(\mathbb{Z},\mathbb{Z}_2) \oplus \operatorname{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) \\
&\cong \{0\} \oplus \{0\} \oplus \operatorname{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) \text{ by Theorem 3.4.13 (1) since } \mathbb{Z} \text{ is free} \\
&\cong \{0\} \oplus \{0\} \oplus \mathbb{Z}_2 \text{ by Example 3.4.16} \\
&\cong \mathbb{Z}_2
\end{aligned}$$

Finally, for $n \geq 3$,

$$\begin{aligned}
H^n(K;\mathbb{Z}_2) &\cong \operatorname{Hom}(H_n(K),\mathbb{Z}_2) \oplus \operatorname{Ext}(H_{n-1}(K),\mathbb{Z}_2) \\
&\cong \operatorname{Hom}(0,\mathbb{Z}_2) \oplus \operatorname{Ext}(0,\mathbb{Z}_2) \\
&\cong \{0\} \oplus \{0\} \text{ by Theorem 3.4.13 (1)} \\
&\cong \{0\}.
\end{aligned}$$

We will contrast these results in Table 5.4 to $H^n(K;\mathbb{Z})$. Observe that $H^n(K;\mathbb{Z}_2)$ has copies of $\mathbb{Z}_2$ for $n = 0, 1$, and 2, whereas $H^n(K;\mathbb{Z})$ has only one copy of $\mathbb{Z}_2$ at $n = 2$. In addition, in this case, $H^n(K;\mathbb{Z}_2) \cong \operatorname{Hom}(H_n(K),\mathbb{Z}_2)$, even though $\operatorname{Ext}(H_n(K),\mathbb{Z}_2)$ is not always 0 (compare with the Torus in Example 5.2.3, where $\operatorname{Ext}(H_n(T),\mathbb{Z}) = 0$).

n	$H_n(K)$	$H^n(K;\mathbb{Z})$	$H^n(K;\mathbb{Z}_2)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
2	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
≥ 3	0	0	0

Table 5.4: The Klein Bottle K with $\mathbb{Z}_2$ Coefficients

Example 5.3.2. Compute the cohomology groups for the Torus T with coefficients in $\mathbb{Z}_m$ for $m \geq 2$.

<u>$H^0(\mathcal{T};\mathbb{Z}_m)$</u>:

$$\begin{aligned}
H^0(\mathcal{T};\mathbb{Z}_m) &\cong \mathrm{Hom}(H_0(\mathcal{T}),\mathbb{Z}_m)\oplus \mathrm{Ext}(H_{-1}(\mathcal{T}),\mathbb{Z}_m)) \\
&\cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m)\oplus \mathrm{Ext}(0,\mathbb{Z}_m) \\
&\cong \mathbb{Z}_m \oplus \{0\} \text{ by Theorem 3.4.13 (1)} \\
&\cong \mathbb{Z}_m.
\end{aligned}$$

<u>$H^1(\mathcal{T};\mathbb{Z}_m)$</u>:

$$\begin{aligned}
H^1(\mathcal{T};\mathbb{Z}_m) &\cong \mathrm{Hom}(H_1(\mathcal{T}),\mathbb{Z}_m)\oplus \mathrm{Ext}(H_0(\mathcal{T}),\mathbb{Z}_m) \\
&\cong \mathrm{Hom}(\mathbb{Z}\oplus\mathbb{Z},\mathbb{Z}_m)\oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_m) \\
&\cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m)\oplus \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m)\oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_m) \\
&\cong \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \{0\} \text{ by Theorem 3.4.13 (1)} \\
&\cong \mathbb{Z}_m \oplus \mathbb{Z}_m.
\end{aligned}$$

<u>$H^2(\mathcal{T};\mathbb{Z}_m)$</u>:

$$\begin{aligned}
H^2(\mathcal{T};\mathbb{Z}_m) &\cong \mathrm{Hom}(H_2(\mathcal{T}),\mathbb{Z}_m)\oplus \mathrm{Ext}(H_1(\mathcal{T}),\mathbb{Z}_m) \\
&\cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m)\oplus \mathrm{Ext}(\mathbb{Z}\oplus\mathbb{Z},\mathbb{Z}_m) \\
&\cong \mathbb{Z}_m \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_m)\oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_m) \\
&\cong \mathbb{Z}_m \oplus \{0\} \oplus \{0\} \text{ by Theorem 3.4.13 (1)} \\
&\cong \mathbb{Z}_m.
\end{aligned}$$

<u>$H^3(\mathcal{T};\mathbb{Z}_m)$</u>:

$$\begin{aligned}
H^3(\mathcal{T};\mathbb{Z}_m) &\cong \mathrm{Hom}(H_3(\mathcal{T}),\mathbb{Z}_m)\oplus \mathrm{Ext}(H_2(\mathcal{T}),\mathbb{Z}_m) \\
&\cong \mathrm{Hom}(0,\mathbb{Z}_m)\oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_m) \\
&\cong \{0\} \oplus \{0\} \text{ since } \mathbb{Z} \text{ is free} \\
&\cong \{0\}.
\end{aligned}$$

Finally, for $n > 3$,

$$\begin{aligned}
H^n(\mathcal{T};\mathbb{Z}_m) &\cong \mathrm{Hom}(H_n(\mathcal{T}),\mathbb{Z}_m)\oplus \mathrm{Ext}(H_{n-1}(\mathcal{T}),\mathbb{Z}_m) \\
&\cong \mathrm{Hom}(0,\mathbb{Z}_m)\oplus \mathrm{Ext}(0,\mathbb{Z}_m) \\
&\cong \{0\} \oplus \{0\} \\
&\cong \{0\}.
\end{aligned}$$

Although these calculations seem lengthy, we now have all cohomology groups of the Torus $\mathcal{T}$ with coefficients in $\mathbb{Z}_m$ translated from integer homology. The data has been placed in Table 5.5.

n	$H^n(\mathcal{T};\mathbb{Z})$	$H^n(\mathcal{T};\mathbb{Z}_m)$
0	$\mathbb{Z}$	$\mathbb{Z}_m$
1	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}_m\oplus\mathbb{Z}_m$
2	$\mathbb{Z}$	$\mathbb{Z}_m$
≥ 3	0	0

Table 5.5: The Torus $\mathcal{T}$ with $\mathbb{Z}_m$ Coefficients

Viewing Table 5.5, one notices that the cohomology groups of $\mathcal{T}$ with $\mathbb{Z}_m$ coefficients are strikingly similar to the cohomology groups with integer coefficients. Also $H^n(\mathcal{T};\mathbb{Z}) \cong \text{Hom}(H_n(\mathcal{T}),\mathbb{Z})$, but unlike Example 5.3.1, $\text{Ext}(H_n(\mathcal{T}),\mathbb{Z}) = \{0\}$ for all n.

Chapter 6

Conclusion

In this thesis we proved the Universal Coefficient Theorem for Cohomology. Along the way we examined collections of preliminary notions that were pertinent to not only proving and applying the theorem, but to also understanding its importance in homology. We studied how these preliminaries were related and lead up the the proof of the theorem.

In Chapter 2, we studied many properties associated with R-modules. Also we introduced duality and Hom groups, which allowed us to define cohomology of a chain complex. The Hom group and many of its properties are studied further in homological algebra.

In Chapter 3, we defined homology (with integer coefficients) and cohomology (with arbitrary abelian coefficients) of a given chain complex. We exhibited the necessity of the Universal Coefficient Theorem by elucidating that, in general, cohomology is not the dual to homology, although it is the case when the homology groups are finitely generated and torsion free, and we are calculating integer cohomology. In doing so, we introduced the Ext group, and showed that $\text{Ext}(H, G)$ does not depend on the free resolution used to construct it, but depends only on H and the coefficient group G. We investigated many properties of the Ext group which aided us when applying the Universal Coefficient Theorem.

Chapter 4 featured the proof of the Universal Coefficient Theorem. The proof was separated into two parts. In Part I, we showed that there was a well-defined surjection from $H^n(C; G)$ to $\text{Hom}(H_n(C), G)$. In Part II, we investigated $\text{Ext}(H_{n-1}(C), G)$ through a diagram chasing procedure, and concluded the proof of the theorem.

In Chapter 5, we applied the Universal Coefficient Theorem to a number of topological spaces, and proved results for spaces with finitely generated homology groups. Finally, since the theorem holds for any arbitrary abelian group G, we showed how one may express cohomology with coefficients in $\mathbb{Z}_m$ in terms of integer homology.

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