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Snort: A combinatorial game

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SNORT: A COMBINATORIAL GAME

A Thesis

Presented to the

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San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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by

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Abstract

The strategies of playing a two-person game have been analyzed using Graph Theory. This paper focuses on the game of Snort, which is a combinatorial game on graphs. Opposability is an important concept in a combinatorial game and leads to a win for the second player. This paper also explores the characteristics of opposability through examples. More fully, we obtain some necessary conditions for a graph to be opposable. Since an opposable graph guarantees a second player win, we examine graphs that result in a first player win. Specifically, we show that certain types of trees are first player wins. Finally, the value of game is introduced as a yardstick which predicts the outcome of a game. Its relation to opposability is remarked.

ACKNOWLEDGEMENTS

I would like to thank Dr. Joseph Chavez and my committee members, Dr. Gary Griffing and Dr. Rolland Trapp for all their help and support in the development of this paper. Dr. Chavez, in particular, spent much time in every week to meet with me and provide me with invaluable comments on each version of my draft. This thesis would not appear without his leadership and patience.

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I would like to thank my family. Their sincere, encouraging words and support led me to finish this program. Their knowledge and experience in multiple fields also made the completion of this thesis possible.

Most importantly, I would like to express my gratitude to my husband. I could not have finished this without his numerous and rigorous suggestion as a mathematician and help as a great husband. This thesis is dedicated to him as an expression of my appreciation for having been my husband for 30 years.

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Chapter 1

Introduction

1.1 Snort and Combinatorial Games

Snort, named after its creator S. Norton, is a game based on the story of two farmers, who rent fields in a pasture for their herds. Mr. Black raises bulls and Mr. White raises cows. Mr. Black claims a field first and Mr. White claims second, and they take turns claiming the remaining fields. Because of the nature of bulls and cows, they cannot put the two herds in the same field or in fields next to each other. Thus if the field adjacent to a field claimed by Mr. Black is empty, that field will be reserved for only Mr. Black to rent. The same is true for the field adjacent to a field claimed by Mr. White. If a field is next to Mr. Black's and Mr. White's at the same time, then neither farmer can claim that field. The farmer who is the last one able to claim an open space wins the Snort game. Since a player can gain one field in each move, if Mr. Black wins, he has more fields than Mr. White. If Mr. White wins, then he has at least as many fields as Mr. Black. Again we should note that in Snort the winner is the player who has the last move, not the player with more fields claimed.

Snort is a typical combinatorial game that has been studied in mathematics for almost 80 years. Many researchers who study Combinatorial Game Theory use *Winning Ways for Your Mathematical Plays* [BCG82] and *On Numbers and Games* [Con76] as foundational references. They participated in Proceedings of Symposia in Applied Mathematics and defined the combinatorial game in a formal and unified way in 1991. They also joined in a symposium held at the Mathematical Sciences Research Institute (MSRI) in 1994, and Richard K. Guy, one of the authors of *Winning Ways for Your Mathematical Plays,* contributes a paper titled "What is a Game?" and states the definition of a *Combinatorial Game* by giving eight basic conditions [Guy98]. We state the conditions of a combinatorial game as follows.

Definition 1.1. *A game is called a* combinatorial game *if it satisfies the following properties:*

1. *It is a two-player game.*
2. *The two players play alternately.*
3. *Specific* rules *dictate how the two players can make their move from a current position to the next position.*
4. *The winner is the player who has the last available legal move.*
5. *The players have a* starting position *and several* positions *(or* patterns*) to choose during a turn.*
6. *There is an* ending condition *to determine the win or loss for each player.*
7. *No chance device, such as dice, spinners, or cards is needed.*
8. *The players can see all the details of a position and the current state of the game on each turn. We call these details* complete information.

We explain definitions of a combinatorial game more precisely using *Snort* as an example.

1. There are two players, Black and White.
2. Black is the first player and White is the second and they take turns. Both players can claim one empty field in each turn.
3. There are two basic rules in Snort. The first rule is that an empty field adjacent to a field that a player claims is reserved for that player. The second rule is that a property adjacent to Black's and White's fields at the same time is not available to either Black or White. The field is called *piebald.*

4. The player who claims the last available field wins in Snort.

5. Each turn has a starting position that shows the condition before the move. Then there are some possible moves, regarded as strategies. After each such move a new position is obtained (Figure 1.1).

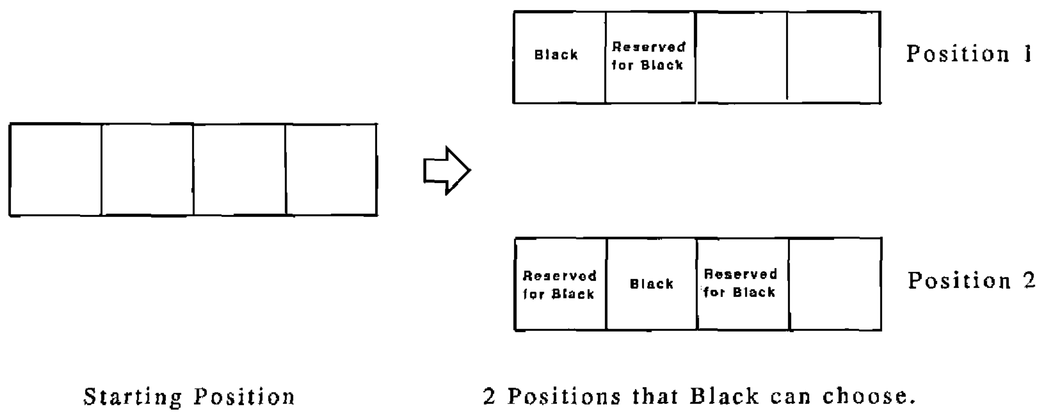


Figure 1.1: Starting position and positions after Black's first move

6. There is an ending condition to determine who wins or loses. In the above example, if Black chooses Position 2, then White continues the game to have the position given below (Figure 1.2).

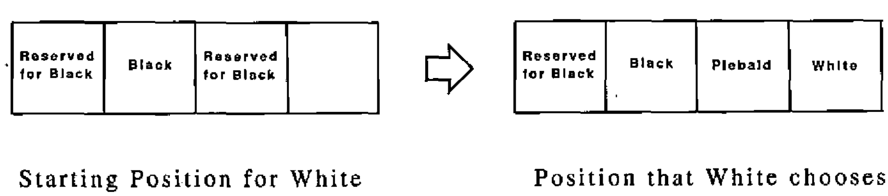


Figure 1.2: Starting position and position after White's first move

After White claims his field, Black claims the field that is reserved for him and wins the game (Figure 1.3).

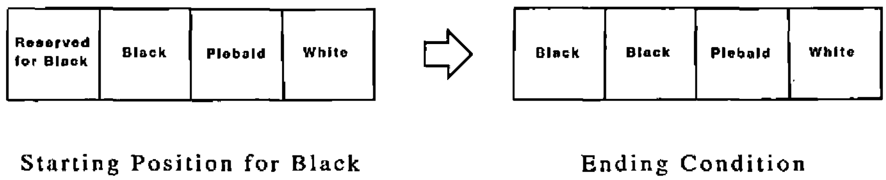


Figure 1.3: Ending condition

7. On Snort we do not use any chance device to decide the players' moves.

8. Two players can always see each state or position of the game and can consider their strategies.

As a general rule, both players must play the game so as to win, i.e., they do the best to win the game. In their book [ANW07], Albert, Nowakowski, and Wolfe call this rule as *playing perfectly,* and we define it below.

Definition 1.2. *To* play perfectly *is to satisfy one of the following conditions:*

1. A player makes a move that allows him to force a win.

2. A player performs a move that makes the opponent's life as difficult as possible.

This concept leads to a basic theorem that is also stated in the book [ANW07] more officially.

Theorem 1.3. (Fundamental Theorem of Combinatorial Games) *If a combinatorial game is played by two persons, then either the first player can force a win, or the second player can force a win, but not both. So therefore, there is no tie.*

We can determine if a game that is familiar to us is a combinatorial game. *Tic-Tac-Toe* has all of the above properties in Definition 1.1 except for the condition 4. Once a player has placed his three marks in a straight line on the board, then he wins even though there are some empty places for the opponent. Also there is a tie situation, that is, neither player can win. Thus, Tic-Tac-Toe is not a combinatorial game. The game *Domineering* can satisfy all the properties given in Definition 1.1. Two players, Mr. Verty and Mr. Horry, alternately play. They have dominos, which are formed to cover two squares on a checker-board. Mr. Verty places his dominos vertically and Mr. Horry places his dominos horizontally. The player who cannot find a place to put his domino in his turn loses the game. In each of their turns there is a starting position that is formed by several squares and also there are finitely many number of options to play. They can think about their next move since they have complete information. The card game *Poker* is not a combinatorial game since there is lack of perfect information and also *Monopoly* is not a combinatorial game because players use dice to determine how many places to move at each of their turns.

We use a graph to illustrate a pasture in Snort, so that we can clearly view the game when we discuss how to play and some strategies. We also study Snort games on various types of graphs.

We shall define the basic terminologies of graphs in the next section.

1.2 Snort and the Basic Terminologies of Graphs

We give a formal definition of a graph and an interpretation of Snort as a game on graphs.

Definition 1.4. *A* graph $G = (V, E)$ *consists of a finite set* V *of vertices and a set* E *of edges, where each edge connects a pair of distinct vertices. For vertices* $a, b \in V$ *an edge* e *connecting* a, b *is denoted by* $e = (a, b)$. *Basic terminology and definitions in graph theory can be found in [Tuc02], [Wil72].*

As we study Snort in this paper, we use a vertex to represent a field in a whole pasture and an edge connecting two vertices to show the relation that two fields are adjacent. Thus instead of considering a real Snort we can play Snort on graphs, as in Figure 1.4 below.

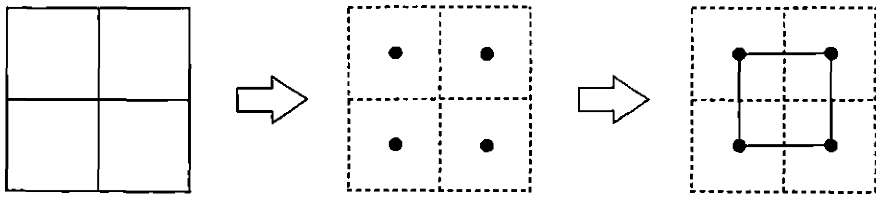


Figure 1.4: A pasture converted to a graph

Although we can use an arbitrary graph to play Snort, we mainly examine simple basic graphs in Graph Theory. We demonstrate how to play Snort on graphs in more detail in Chapter 2 using basic graphs such as paths, cycles, and bipartite graphs.

1.3 Snort and Opposability

In their paper [SS] Stokes and Schlatter introduce opposable graphs through the game of Snort as one strategy for which the second player has a chance to win. A graph $G = (V, E)$ is said to be *opposable* if there is an automorphism f on V such that $f(v)$ is not equal to v, $f(v)$ is not adjacent to v, and $f(f(v)) = v$ for any vertex v in V. On an opposable graph the second player claims the vertex that is the mirror image of the vertex claimed by the first player. Stokes and Schlatter use a hexagon-shaped graph that has a cycle and an even number of vertices as a typical example of an opposable graph. We use the words *cycle* and *even* to describe two of the main conditions for opposability. We explain more precise definitions of these terms in Chapter 3 and suggest additional conditions and nature of opposability in the rest of this paper.

1.4 Snort and Nonopposable Graphs

After we play Snort using the opposable strategy, we focus on nonopposable graphs as a next step. Since an opposable graph is a second player win, there arises a question: If a graph is a second player win, then is it opposable? We do not know the answer, however, we shall give a partial solution by showing that particular nonopposable graphs are first player win. Since we shall show that any opposable graph contains a cycle, a tree is a nonopposable graph. We prove that, for an even integer n, a complete n-nary tree is a first player win. The case where n is odd is not obvious. We can verify that a complete ternary tree is a first player win. The general odd case is unknown to us.

1.5 Snort and Values

Since any Snort game is either a first player win or a second player win, we want to determine who wins the given game. One of the ways to see this is to consider the value of a game. First we shall explain the value of a Domineering game. The values are defined inductively from simple regions (positions) to complicated ones. Some rules are introduced to calculate the values of positions. We shall consider Domineering and Snorts. Here, we should note that combinatorial games are classified into two categories according to their characteristics: partizan (or hot) and impartial (or cold). For instance, Domineering is called partizan since two players have different options to play, and Snort is called impartial since two players have the same options to play. After discussing Domineering we consider values of Snort games. We shall see that an opposable graph has a value 0, and that if the value is positive, then the graph is a first player win.

Chapter 2

Snort

In this chapter, we describe how to play Snort on graphs in more detail. Examples are given to play Snort on basic graphs such as paths, cycles, and bipartite graphs, where some strategies are suggested.

2.1 How to Play Snort

We play Snort on graphs and here investigate some of its properties. Let G be a graph represented by a set of vertices and edges. Then Snort on G is played by two players who take turns to claim a vertex to be theirs. Since the winner of Snort is the last person who can claim a vertex to be his, not the player who has more vertices, we delete the vertex that a player claims on his turn. We use the black color for the first player and the white color for the second player to reserve vertices. In this way we focus on those vertices that are either reserved or available to both players rather than those vertices that are already claimed by the players. We also assume that at the beginning all vertices are available to both players. We describe the rules and procedures of Snort again using the graphs as follows.

1. The players take turns to claim any available vertex.

2. At the beginning of the game, the first player may claim any vertex on the graph G.

3. Then, the vertex that the first player claims is removed, and the vertices that are

adjacent to that vertex are reserved for the first player. Any vertex reserved for the first player is shown as a large black dot.

4. On the second player's turn, he may claim any vertex on G that is not reserved for the first player. As above, the claimed vertex is removed from G and the vertices that are adjacent to the one claimed are reserved for the second player. Any vertex reserved for the second player is shown as a large white dot.

5. Any vertices that are claimed are omitted after each turn.

6. Any edge connecting two vertices that are reserved for the same player is omitted.

7. A *piebald spot* is a vertex that is adjacent to both a vertex claimed by the first player and a vertex claimed by the second player at the same time. A piebald spot is not available to either players and is denoted by $\otimes$.

8. The winner is the player who claims the last available vertex.

To clarify the above rules and procedures, we depict some examples in the next sections.

2.2 Paths

First we deal with paths. A path is a very simple graph and can be used to understand a simple game of Snort. The first part of the following example was used in Section 1.1. to introduce Snort.

Example 2.1. Let P_n be a *path* consisting of n vertices $v_1, v_2, \ldots, v_n$ and $n-1$ edges (v_i, v_{i+1}) for $i = 1, 2, \ldots, n-1$. Let us denote it as $P_n = (v_1, v_2, \ldots, v_n)$.

(1) Consider P_4 given in Figure 2.1.

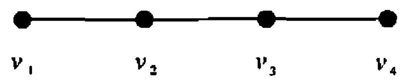


Figure 2.1: The graph of P_4

On the graph of P_4, suppose the first player claims v_2. Then v_2 is removed and vertices v_1 and v_3, adjacent to v_2, are then reserved for the first player (Figure 2.2).

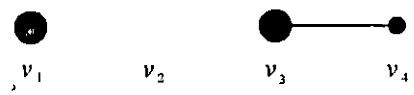


Figure 2.2: The graph of P_4 after first move

In the second player's turn he claims v_4, so that it is removed. This implies that v_3 becomes piebald (Figure 2.3).

Figure 2.3: The graph of P_4 after second move

In the third move the first player claims v_1 and wins this game.

Next consider the same graph P_4, where the first player claims v_1 at the beginning. This makes v_2, which is adjacent to v_1, reserved for the first player (Figure 2.4).

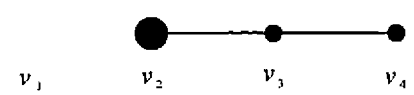


Figure 2.4: The graph of P_4 with another strategy

The second player then chooses the strategy to claim v_3, so that v_2 becomes piebald and the vertex v_4 is reserved for the second player (Figure 2.5). Thus, the second player wins.

Figure 2.5: The graph of P_4 after second move

Therefore, there are two strategies for the first player: one which leads him to win and the other which leads him to lose. Hence, the first player has to claim v_2 in his first turn because of the rule of playing perfectly.

(2) Consider P_7 given in Figure 2.6.

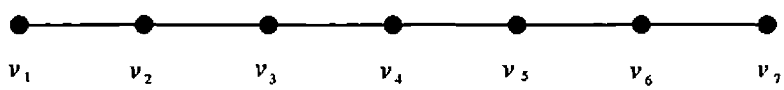


Figure 2.6: The graph of P_7

On the graph of P_7, suppose the first player claims v_4. Vertices v_3 and v_5 are now reserved for the first player (Figure 2.7).

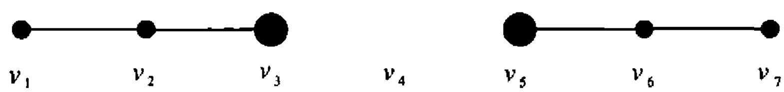


Figure 2.7: The graph of P_7 after first move

In the second player's turn he claims v_2. This implies that v_1 is reserved for the second player and v_3 becomes piebald (Figure 2.8).

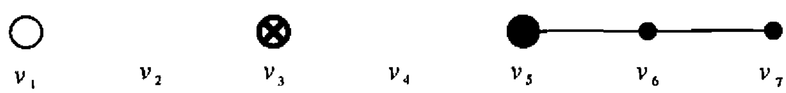


Figure 2.8: The graph of P_7 after second move

In the third move the first player claims v_7. This implies that v_6 is reserved for the first player (Figure 2.9).

Figure 2.9: The graph of P_7 after third move

In the second player's turn he can claim only v_1, which is reserved for him (Figure 2.10).

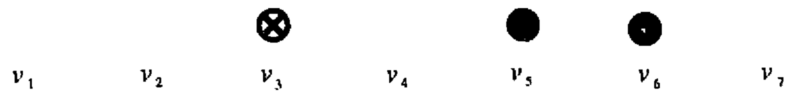


Figure 2.10: The graph of P_7 after fourth move

The final stage of the graph shows that the first player wins in this game since there remain only vertices reserved for him.

In the above example of P_7 we examine only one strategy for the first player. There are essentially four strategies for the first player. That is, the first move is to take one of v_1, v_2, v_3, and v_4.

Here arises a question: For Snort on a path is there a strategy by which the first player always wins? That is, is a path a first player win? We shall define trees that contains paths and see the answer to this question in Chapter 4.

2.3 Cycles

In this section we deals with cycles. We shall see the difference between odd cycles and even cycles.

Example 2.2. Let C_n be a *cycle* consisting of n vertices $v_1, v_2, \ldots, v_n$ and n edges $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)$.

(1) We shall start with an even cycle. So consider C_6 given in Figure 2.11.

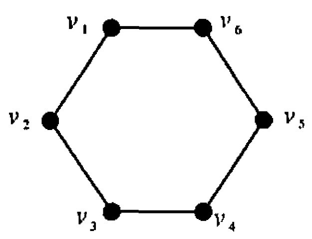


Figure 2.11: The graph of C_6

On the graph of C_6, suppose the first player claims v_1. The two vertices v_2 and v_6, adjacent to v_1, are now reserved for the first player (Figure 2.12).

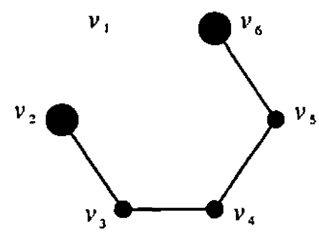


Figure 2.12: The graph of C_6 after first move

In the second player's turn, he considers symmetry of the figure and claims v_4 that is symmetric to v_1. This implies that v_3 and v_5 are reserved for the second player (Figure 2.13).

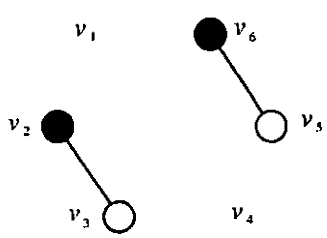


Figure 2.13: The graph of C_6 after second move

Next there are two equivalent choices for the first player and he claims v_2 to make v_3 piebald (Figure 2.14).

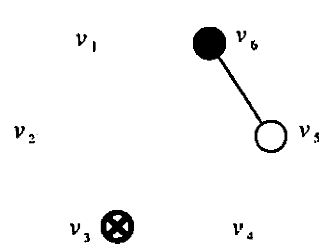


Figure 2.14: The graph of C_6 after third move

Then the second player necessarily claims v_5 and v_6 becomes piebald. Therefore, the second player wins. (Figure 2.15).

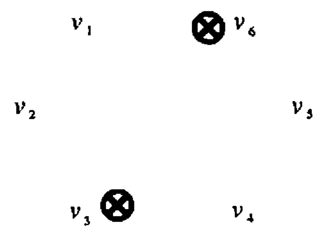


Figure 2.15: The graph of C_6 after fourth move

The above example C_6 is an even cycle graph that Stoker and Schlatter claim as opposable [SS]. Now, we add one more vertex to obtain an odd cycle graph C_7 to play Snort.

(2) Consider an odd cycle C_7 given in Figure 2.16.

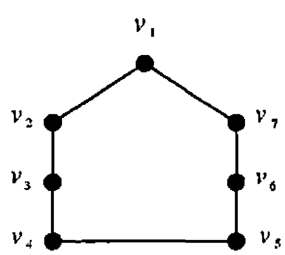


Figure 2.16: The graph of C_7

On the graph of C_7, the first player claims v_1. Vertices v_2 and v_7 are then reserved for the first player (Figure 2.17).

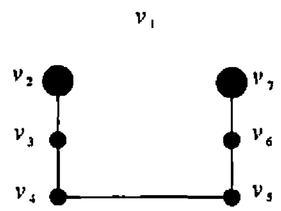


Figure 2.17: The graph of C_7 after first move

In the second player's turn, he cannot find a suitable vertex corresponding to v_1 as was done in C_6 and claims v_5 instead, so that the vertices v_4 and v_6 are reserved for the second player (Figure 2.18).

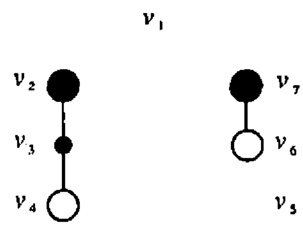


Figure 2.18: The graph of C_7 after second move

Next the first player claims v_3, and v_4 becomes piebald (Figure 2.19).

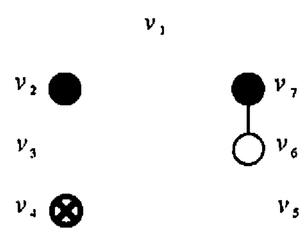


Figure 2.19: The graph of C_7 after third move

In Figure 2.19, we see that the second player has only v_6 to claim. After the second player claims v_6, the vertex v_7 becomes piebald and the vertex v_2 is available for the first player, so that the first player wins.

2.4 Bipartite Graphs

The third example we consider in this chapter is a bipartite graph. Two types of bipartite graphs are studied: *complete* and *incomplete*. At first we shall start with complete bipartite graphs.

Example 2.3. Let $K_{m,n}$ be a *complete bipartite* graph consisting of two sets of vertices $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_n\}$ such that every edge joins a vertex in V with a vertex in W, and there is an edge between every $v \in V$ and $w \in W$. For each vertex v let $deg(v)$ denote the *degree* of v, i.e., the number of edges stemming from v. In $K_{m,n}$ we have that $deg(v_i) = n$ and $deg(w_j) = m$.

Within complete bipartite graphs, we consider three cases where $m = 2$ and $n = 3$, $m = 3$ and $n = 3$, and $m = 2$ and $n = 4$.

(1) Consider $K_{2,3}$ given in Figure 2.20.

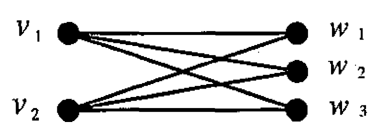


Figure 2.20: The graph of $K_{2,3}$

On the graph of $K_{2,3}$, suppose the first player claims w_2. The vertices v_1 and v_2 become reserved for the first player (Figure 2.21).

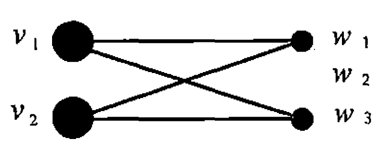


Figure 2.21: The graph of $K_{2,3}$ after first move

In the second player's turn, he claims w_3, making, v_1 and v_2 piebald. Thus the first player wins (Figure 2.22).

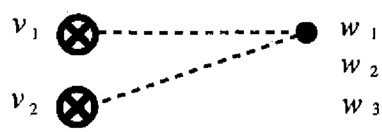


Figure 2.22: The graph of $K_{2,3}$ after second move

(2) Consider $K_{3,3}$ given in Figure 2.23.

This complete bipartite graph has cycles and even number (six) of vertices. Recall that the cycle C_6 has six vertices and is a second player win. We shall see if this is the case for $K_{3,3}$.

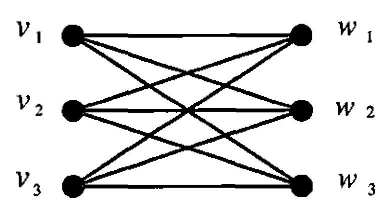


Figure 2.23: The graph of $K_{3,3}$

On the graph of $K_{3,3}$, suppose the first player claims v_2. Then three vertices w_1, w_2, and w_3, adjacent to v_2, become reserved for the first player (Figure 2.24).

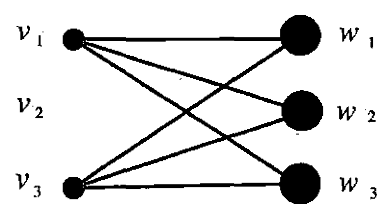


Figure 2.24: The graph of $K_{3,3}$ after first move

The second player claims v_1, and the vertices w_1, w_2, and w_3 become piebald (Figure 2.25).

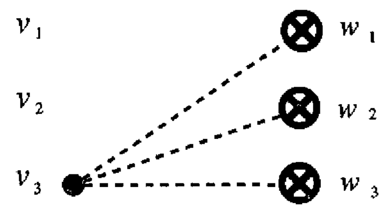


Figure 2.25: The graph of $K_{3,3}$ after second move

Figure 2.25 shows that the first player, who has the remaining option, wins. Hence, $K_{3,3}$ is not opposable.

(3) Consider $K_{2,4}$ given in Figure 2.26.

This complete bipartite graph also has cycles and six vertices, and there are even number of vertices on both sides, which is different from $K_{3,3}$.

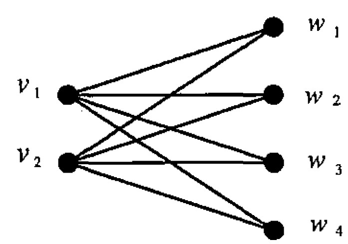


Figure 2.26: The graph of $K_{2,4}$

On the graph of $K_{2,4}$, suppose the first player claims v_1, so that the four vertices w_1, w_2, w_3, and w_4, adjacent to v_1, become reserved for the first player (Figure 2.27).

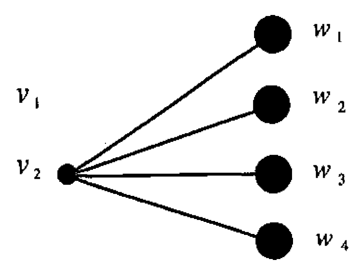


Figure 2.27: The graph of $K_{2,4}$ after first move

In the second player's turn, he claims v_2 and the rest of the four vertices become piebald (Figure 2.28).

Figure 2.28: The graph of $K_{2,4}$ after second move

Figure 2.28 shows that the second player wins.

There is another strategy for the first player. Suppose that the first player claims w_1 on his first move, so that vertices v_1 and v_2 are reserved for him. Then, the second player claims w_2, which makes v_1 and v_2 piebald. After that, each player claims w_3 or w_4, and hence the second player wins. Thus, we can conclude that $K_{2,4}$ is a second player win.

The general complete bipartite graphs $K_{m,n}$ are classified as opposable and non-opposable according to the parities of m and n in Chapter 3.

Now we move to incomplete bipartite graphes.

Example 2.4. Let G be a bipartite graph with two sets V and W of vertices, where every edge joins a vertex in V and a vertex in W. Here we assume that G is *incomplete,* i.e., there exist two vertices $u, v \in V$ or $u, v \in W$ such that $deg(u) < deg(v)$. Let us consider the following bipartite graph G.

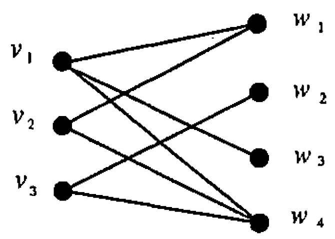


Figure 2.29: An incomplete bipartite graph

On the graph of G (Figure 2.29), suppose the first player claims v_1, and three vertices w_1, w_3, and w_4 that are adjacent to v_1, become reserved for the first player (Figure 2.30).

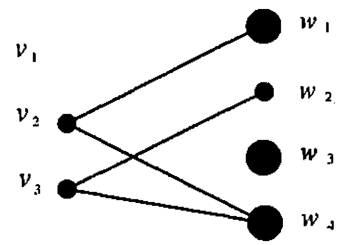


Figure 2.30: The incomplete bipartite graph after first move

In the second player's turn, he claims v_3, so w_2 becomes reserved for the second player and w_4 becomes piebald (Figure 2.31).

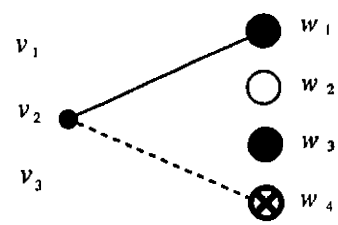


Figure 2.31: The incomplete bipartite graph after second move

In the first player's turn, he claims w_1 and the vertex v_2 is reserved for the first player. Then we can see that the first player wins, since there are two vertices reserved for the first player and there is only one vertex reserved for the second player (Figure 2.32).

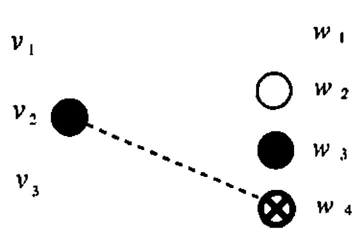


Figure 2.32: The incomplete bipartite graph after third move

Remark 2.5. At the end of this chapter we can suggest some basic strategies to win a Snort game.

1. Try to increase the number of vertices that are reserved for one's own. That is, claim a vertex of a higher degree.

2. If a vertex is reserved for the opponent, try to make it piebald by claiming a vertex adjacent to it.

3. If the graph has a symmetry in some sense (that is related to an automorphism defined later), in each turn the second player can choose the symmetric vertex corresponding to the one chosen by the first player.

4. If the graph has some symmetry after the first player's move, then the first player can take the same strategy as above.

Chapter 3

Opposable Graphs

In this chapter, we introduce opposable graphs. When a graph is connected, we obtain some necessary conditions and sufficient conditions for opposability. Examples of two and three dimensional grid graphs are considered.

3.1 Definitions and Examples

In Chapter 2, we found that if a graph is symmetrical in a certain sense, the second player can make strategies to win, i.e., the graph represents a second player win. This type of graph is called opposable whose definition is given in Section 1.3. In this chapter we consider this notion in more detail by giving examples and obtaining structures of opposable graphs.

We begin with some basic definitions from graph theory.

Definition 3.1. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Then a mapping $f : G \to G'$ is called an* isomorphism *if $f : V \to V'$ is one-to-one and onto, and if $a, b \in V$ are adjacent in G, then $f(a), f(b) \in V'$ are adjacent in G', and vice versa. In this case, the graphs G and G' are said to be isomorphic and is denoted by $G \cong G'$. When $G = G'$, then an isomorphism $f : G \to G$ is called an* automorphism.

Definition 3.2. *A graph $G = (V, E)$ is said to be* opposable *if there is an automorphism $f : G \to G$ such that, for any $v \in V$, $f(v)$ is neither adjacent to nor equal to v, and $f(f(v)) = v$.*

Remark 3.3. A consequence of the above definition is that any opposable graph has an even number of vertices and hence any graph with an odd number of vertices is not opposable. Thus, an odd cycle and a bipartite graph $K_{m,n}$ with m, n of opposite parity are not opposable. We prove this fact as a proposition below.

Proposition 3.4. *Graphs with an odd number of vertices are not opposable.*

Proof. Let G be an opposable graph with $(2n+1)$ vertices where n is a positive integer. Let f be a required automorphism on G, so that G and f meet the conditions of Definition 3.2. For any $v_1 \in G$, we have $f(v_1) \neq v_1$ and $f(f(v_1)) = v_1$. Hence, we can make a pair $(v_1, f(v_1))$, and for any $v_2\,(\neq v_1) \in V$, we can make another pair $(v_2, f(v_2))$. Continuing this process we can make n pairs

$$(v_1, f(v_1)), ..., (v_n, f(v_n))$$

of vertices and there is one vertex v left. Since f is onto, there is some i $(1 \leq i \leq n)$ such that $f(v) = v_i$ or $f(v) = f(v_i)$. If $f(v) = v_i$, then $v = f(f(v)) = f(v_i)$, a contradiction. If $f(v) = f(v_i)$, then $v = v_i$ since f is one-to-one, a contradiction. Thus, G is not opposable. □

One basic example of an opposable graph is given.

Proposition 3.5. *An even cycle C_{2n} with $n \geq 2$ is opposable.*

Proof. Let C_{2n} be a cycle consisting of $2n$ vertices $v_1, v_2, \ldots, v_{2n}$ with $2n$ edges (v_1, v_2), (v_2, v_3), $\ldots$, (v_{2n-1}, v_{2n}), (v_{2n}, v_1).

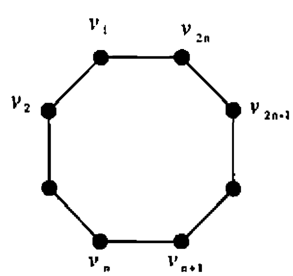


Figure 3.1: The graph of C_{2n}

Define $f : V \to V$ by

$$f(v_i) = \begin{cases} v_{i+n}, & \text{if } i = 1, 2, \ldots, n \\ v_{i-n}, & \text{if } i = n+1, n+2, \ldots, 2n \end{cases}$$

Then, clearly f is one-to-one and onto. For $1 \leq i \leq n-1$, we have that v_i and v_{i+1} are adjacent and that $f(v_i) = v_{i+n}$ and $f(v_{i+1}) = v_{i+1+n}$ are adjacent. For $i = n$, we have that v_n and v_{n+1} are adjacent and that $f(v_n) = v_{2n}$ and $f(v_{n+1}) = v_1$ are adjacent. For $n+1 \leq i \leq 2n-1$, we have that v_i and v_{i+1} are adjacent and that $f(v_i) = v_{i-n}$ and $f(v_{i+1}) = v_{i+1-n}$ are adjacent. For $i = 2n$, we have that v_{2n} and v_1 are adjacent and that $f(v_{2n}) = v_n$ and $f(v_1) = v_{n+1}$ are adjacent. Thus, f is an automorphism of G.

Moreover, we have that

$$f(f(v_i)) = f(v_{i+n}) = v_{(i+n)-n} = v_i, \qquad \text{for } 1 \leq i \leq n$$

$$f(f(v_i)) = f(v_{i-n}) = v_{(i-n)+n} = v_i, \qquad \text{for } n+1 \leq i \leq 2n$$

Finally, $f(v)$ is not adjacent to v and not equal to v. Therefore, f satisfies the conditions of opposability of G. □

It follows from the definition and examples above that any opposable graph is a second player win, which is shown in [SS].

Proposition 3.6 [SS]. *Any opposable graph is a second player win. Hence, if a graph is a first player win, then it is not opposable.*

Proof. Let G be opposable with an automorphism f satisfying the condition of Definition 3.2. We can assume that G has $2n$ vertices $v_1, v_2, \ldots, v_{2n}$. Then, a strategy for the second player is as follows:

If the first player claims v_i, then claim $f(v_i)$.

This is possible for the first stage because v_i and $f(v_i)$ are not equal or adjacent by definition. After some moves, we have to show that if the first player picks v_j, then $f(v_j)$ is available for the second player. We consider some cases. If $f(v_j)$ is used by the first player, then $v_j = f(f(v_j))$ must have been already used by the second player, which is a contradiction. If $f(v_j)$ is used by the second player, then v_j has been used by the first player, which is also a contradiction. Finally, if $f(v_j)$ is reserved for the first player, then there is a vertex v_k that is played by the first player and is adjacent to $f(v_j)$. But then, the second player must have picked $f(v_k)$ that is adjacent to v_j since v_k and $f(v_j)$ are adjacent and they are mapped by f. This implies that the first player cannot pick v_j since it is reserved for the second player or it is piebald, which is a contradiction. So, if the first player picks v_j, then $f(v_j)$ is available for the second player.

Therefore, the second player always finishes the last move and wins. □

Recall that we showed two bipartite graphs $K_{2,4}$ and $K_{3,3}$ in Section 2.4. $K_{2,4}$ has an even number of vertices in both sides and is an opposable graph. $K_{3,3}$ has an odd number of vertices in both sides and is not an opposable graph though it has an even number of vertices in total. We shall generalize this result in the following.

Proposition 3.7. *Let m and n be even positive integers. Then a complete bipartite graph $K_{m,n} = (V, E)$ is opposable.*

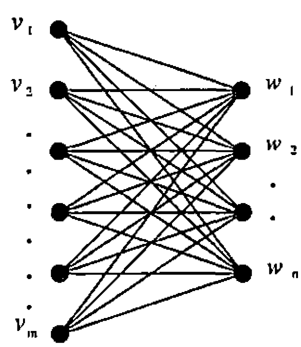


Figure 3.2: The complete bipartite graph $K_{m,n}$ with even m, n

Proof. As in the figure above, let $v_1, v_2, \ldots, v_m$ be the set of vertices on the left side and $w_1, w_2, \ldots, w_n$ be the set of vertices on the right side. Define a mapping $f : V \to V$ by

$$
\begin{aligned}
f(v_1) &= v_2 & \quad f(w_1) &= w_2 \\
f(v_2) &= v_1 & \quad f(w_2) &= w_1 \\
f(v_3) &= v_4 & \quad f(w_3) &= w_4 \\
f(v_4) &= v_3 & \quad f(w_4) &= w_3 \\
&\vdots & &\vdots \\
f(v_{m-1}) &= v_m & \quad f(w_{n-1}) &= w_n \\
f(v_m) &= v_{m-1} & \quad f(w_n) &= w_{n-1}
\end{aligned}
$$

Clearly, f is one-to-one and onto. If v and w are adjacent, then $v = v_i$ and $w = w_j$ for some i $(1 \leq i \leq m)$ and j $(1 \leq j \leq n)$ (or $v = w_j$ and $w = v_i$ for some i and j), since the graph is a complete bipartite graph. Then, $f(v)$ and $f(w)$ are obviously adjacent by the definition of f. Thus, f is an automorphism. The condition $f(f(v)) = v$ is clear from the definition. Moreover, $f(v) \neq v$, and $f(v)$ is not adjacent to v from the definition. Therefore, $K_{m,n}$ is opposable. □

We know that any opposable graph has an even number of vertices. However, this is not sufficient for opposability as we see below that $K_{m,n}$ with odd m, n is not opposable.

Proposition 3.8. *Let m and n be odd positive integers. Then a complete bipartite graph $K_{m,n}$ is not opposable.*

Proof. As in the figure below, let $v_1, \ldots, v_m$ be the set of vertices on the left and $w_1, \ldots, w_n$ be the set of vertices on the right.

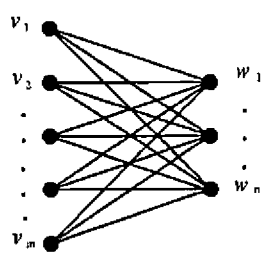


Figure 3.3: The complete bipartite graph $K_{m,n}$ with odd m,n

Assume that $K_{m,n}$ is opposable and let $f : V \to V$ be an automorphism that satisfies the opposability conditions. Since $f(v_1)$ is not v_1 and not adjacent to v_1, $f(v_1)$ must be one of $v_2, \ldots, v_m$. Let $f(v_1) = v_{i_1}$, $i_1 \neq 1$. By the same reason, we have that $f(v_2) = v_{i_2}$ for some $i_2 \neq 1, 2, i_1$. If we continue this process, then we have $\frac{n-1}{2}$ pairs of vertices from $v_1, \ldots, v_m$, and there will be one vertex v_k left. Then, $f(v_k)$ cannot be defined since all other vertices on the left side are exhausted. This is a contradiction. Therefore, $K_{m,n}$ is not opposable. □

3.2 Properties

Now we investigate some more properties of opposable graphs. The first one is that opposability is isomorphism invariant.

Proposition 3.9. *If two graphs are isomorphic and one is opposable, then the other is also opposable.*

Proof. Suppose two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic and G_1 is opposable. Let $g : G_1 \to G_2$ be an isomorphism and $f_1 : G_1 \to G_1$ be an automorphism that makes G_1 opposable. We show that G_2 is opposable. Define $f_2 : V_2 \to V_2$ by

$$f_2(v) = g\Big[f_1\big(g^{-1}(v)\big)\Big], \qquad v \in G_2.$$

We show that f_2 is an automorphism of G_2 that satisfies the opposability condition for G_2.

f_2 is one-to-one since g^{-1}, f_1 and g are one-to-one.

f_2 is onto since $g^{-1} : G_2 \to G_1$ is onto, $f_1 : G_1 \to G_1$ is onto, and $g : G_1 \to G_2$ is onto.

Let $v_1, v_2 \in G_2$ be adjacent. Then $g^{-1}(v_1), g^{-1}(v_2) \in G_1$ are adjacent since g and hence g^{-1} are isomorphisms. Hence, $f_1(g^{-1}(v_1)), f_1(g^{-1}(v_2)) \in G_1$ are adjacent since f_1 is an automorphism. Thus, $g(f_1(g^{-1}(v_1))), g(f_1(g^{-1}(v_2)) \in G_2$ are adjacent since g is an isomorphism, so that $f_2(v_1), f_2(v_2) \in G_2$ are adjacent. Therefore, f_2 is an automorphism.

For $v \in G_2$ it holds that

$$\begin{aligned} f_2(f_2(v)) &= g(f_1(g^{-1}(g(f_1(g^{-1}(v)))))) \\ &= g(f_1(f_1(g^{-1}(v)))) \\ &= g(g^{-1}(v)) \\ &= v. \end{aligned}$$

Moreover, $f_2(v) \neq v$ for $v \in G_2$. In fact, suppose that $f_2(v) = v$. Then, we have that $v = g(f_1(g^{-1}(v)))$ and hence $g^{-1}(v) = f_1(g^{-1}(v))$, which is a contradiction since $f_1(u) \neq u$ for all $u \in G_1$.

Finally, for $v \in G_2$, $f_2(v)$ and v are not adjacent. For, if $f_2(v)$ and v are adjacent, then $g(f_1(g^{-1}(v)))$ and v are adjacent, which implies that $f_1(g^{-1}(v))$ and $g^{-1}(v)$ are adjacent. This is a contradiction since $f_1(u)$ and u are not adjacent for any $u \in G_1$.

Thus, f_2 is the required automorphism to make G_2 opposable. □

Proposition 3.10. *If two graphs G_1 and G_2 are isomorphic, then the graph $G_1 \cup G_2$ is opposable.*

Proof. Let $G_i = (V_i, E_i)$, $i = 1, 2$ be isomorphic graphs, where $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Suppose that a mapping $f : V_1 \to V_2$ is an isomorphism given by $f(u_i) = v_i$ for $i = 1, 2, \ldots, n$. Then, define $\tilde{f} : V_1 \cup V_2 \to V_1 \cup V_2$ by

$$\tilde{f}(u_i) = f(u_i) = v_i, \qquad i = 1, 2, \ldots, n,$$

$$\tilde{f}(v_i) = f^{-1}(v_i) = u_i, \qquad i = 1, 2, \ldots, n.$$

Clearly, $\tilde{f}$ is one-to-one and onto. Note that if u_i and u_j $(i \neq j)$ are adjacent, then $\tilde{f}(u_i) = v_i$ and $\tilde{f}(u_j) = v_j$ are adjacent, since G_1 and G_2 are isomorphic. Similarly, if v_i and v_j $(i \neq j)$ are adjacent, then $\tilde{f}(v_i) = u_i$ and $\tilde{f}(v_j) = u_j$ are adjacent. Thus, $\tilde{f}$ is an automorphism. The condition $\tilde{f}(\tilde{f}(u)) = u$ for any $u \in V_1 \cup V_2$ is clear from the definition. Moreover, for any $u \in V_1 \cup V_2$, $\tilde{f}(u) \neq u$, and $\tilde{f}(u)$ is not adjacent to u from the definition. Therefore, $G = G_1 \cup G_2$ is opposable. □

We list the necessary conditions for opposability as follows.

Theorem 3.11. *Let $G = (V, E)$ be an opposable graph with an automorphism $f : V \to V$. Then the following statements are true.*

(1) *The number of vertices in V is even.*
(2) *The number of edges in E is even.*
(3) *For any $v \in V$ we have $deg(v) = deg\, f(v)$. Hence, if we let $V_k = \{v \in V : deg(v) = k\}$ for $k \geq 1$, then V_k is f-invariant.*
(4) *For any $k \geq 1$ the number of vertices in V_k is even or zero.*
(5) *For any $k \geq 1$, if $v_1, v_2, \ldots, v_m \in V_k$ and $m \geq 2$, then for any v_i there is some v_j with $i \neq j$ such that v_i and v_j are not adjacent.*

Proof. (1) follows from Proposition 3.4.

(2) Let $e = (u, v) \in E$ be an edge. Then, since G is opposable, $f(e) = (f(u), f(v))$ is an edge different from e. Hence we have a pair of edges $\{e, f(e)\}$. Suppose we have an odd number of edges, then one edge e^* is left out. Then, $f(e^*) = e$ for some $e\,(\neq e^*) \in E$ and $\{e, f(e^*)\}$ is a pair, which implies that $e = f(f(e^*)) = e^*$, a contradiction. Thus, G has an even number of edges.

(3) Let $v \in V$ and suppose $deg(v) = k$. Then, there are k distinct vertices $v_1, \ldots, v_k$ that are adjacent to v. This implies that $f(v_1), \ldots, f(v_k)$ are distinct and adjacent to $f(v)$ since f is an automorphism. Hence, $deg\, f(v) \geq k$. By a similar argument we have $deg(v) = deg\, f(f(v)) \geq deg\, f(v)$. Thus, $deg(v) = deg\, f(v)$.

(4) By (3), if $v \in V_k$, then $f(v) \in V_k$ and hence $\{v, f(v)\}$ is a pair in V_k.

(5) Follows from (3) and that v and $f(v)$ are not adjacent for all $v \in V$. □

Let m and n be odd integers. We saw that $K_{m,n}$ is not opposable in Proposition 3.8. This also follows from Theorem 3.11(2) since $K_{m,n}$ has mn edges and mn is odd. Note that $K_{m,n}$ satisfies the conditions (1), (3), and (5) (but not (4)) of Theorem 3.11.

Example 3.12. We show that the following graph $G = (V, E)$ given in Figure 3.4 is opposable and satisfies conditions (1) – (5) of the above theorem.

Let $f : V \to V$ be defined by

$$f(v_1) = v_5, \quad f(v_2) = v_6, \quad f(v_3) = v_7, \quad f(v_4) = v_8, \quad f(v_5) = v_1,$$

$$f(v_6) = v_2, \quad f(v_7) = v_3, \quad f(v_8) = v_4, \quad f(v_9) = v_{10}, \quad f(v_{10}) = v_9.$$

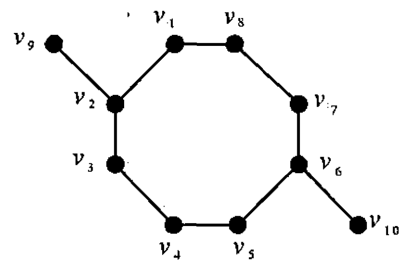


Figure 3.4: One example of an opposable graph

It is easily verified that f is an automorphism that satisfies the opposability conditions for the graph G. Moreover, (1) of Theorem 3.11 holds since there are 10 vertices. (2) also holds since there are 10 edges. As to (3), we see that $V_1 = \{v_9, v_{10}\}$, $V_2 = \{v_1, v_3, v_4, v_5, v_7, v_8\}$, and $V_3 = \{v_2, v_6\}$ are f-invariant. Clearly, (4) holds since V_1, V_2, and V_3 have even numbers of vertices. Finally, (5) can be verified easily.

Example 3.13. We show that the following graph (Figure 3.5) is not opposable even though there is an automorphism that satisfies the conditions (1) – (5) of Theorem 3.11.

Let $f : V \to V$ be defined by

$$f(v_1) = v_8, \quad f(v_2) = v_7, \quad f(v_3) = v_6, \quad f(v_4) = v_5, \quad f(v_5) = v_4,$$

$$f(v_6) = v_3, \quad f(v_7) = v_2, \quad f(v_8) = v_1, \quad f(v_9) = v_{10}, \quad f(v_{10}) = v_9.$$

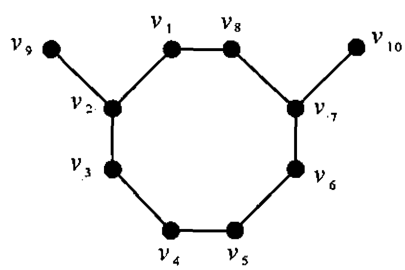


Figure 3.5: One example of nonopposable graph

It is verified that f is an automorphism that satisfies the conditions (1) – (5) of Theorem 3.11. But $f(v_1) = v_8$ with v_1, v_8 adjacent. Hence f does not satisfy the opposability condition. If we make another automorphism f', then f': $V_2 \to V_2 = \{v_1, v_3, v_4, v_5, v_6, v_8\}$ is onto and one-to-one. But we cannot have f' such that $f'(v)$ and v being not adjacent for any $v \in V_2$. This leads to a conclusion that G is not opposable. When we consider a difference between the graphs given in Examples 3.12 and 3.13, we can come up with Theorem 3.15 below where more necessary conditions for opposability are obtained.

Before we show other conditions for opposability, we define connected graphs.

Definition 3.14. *A graph $G = (V, E)$ is said to be* connected *if for any two distinct vertices $u, v \in V$ there exists a path from u to v.*

Theorem 3.15. *Let $G = (V, E)$ be a connected opposable graph with an automorphism $f : V \to V$. Then the following assertions hold.*

(1) *For any vertex $v \in V$ there exists a pair of two distinct paths P_1 and P_2 of the same length from v to $f(v)$. More fully, if $P_1 = (v, v_1, v_2, \ldots, v_n, f(v))$, then $P_2 = (v, f(v_n), f(v_{n-1}), \ldots, f(v_1), f(v))$ and*

$$f\left(v_{\frac{n}{2}}\right) \neq v_{\frac{n}{2}+1}, \quad f\left(v_{\frac{n}{2}+1}\right) \neq v_{\frac{n}{2}}, \qquad \text{if } n \text{ is even,} \tag{3.1}$$

$$f\left(v_{\frac{n+1}{2}}\right) \neq v_{\frac{n+1}{2}}, \qquad \text{if } n \text{ is odd.} \tag{3.2}$$

(2) *For any $v \in V$ there is an even number of distinct paths from v to $f(v)$.*

(3) *G contains a cycle.*

Proof. (1) Let $v \in V$. Since G is connected there is a path P_1 from v to $f(v)$. Let $P_1 = (v, v_1, v_2, \ldots, v_n, f(v))$. Since v_i and v_{i+1} are adjacent, so are $f(v_i)$ and $f(v_{i+1})$ for $1 \leq i \leq n$. Hence, $P_2 = (v, f(v_n), f(v_{n-1}), \ldots, f(v_1), f(v))$ is a path from v to $f(v)$ of the same length as P_1. We claim that $P_1 \neq P_2$. For, if $P_1 = P_2$, then we have that

$$\begin{cases} \text{if } n \text{ is even, then } f\left(v_{\frac{n}{2}}\right) = v_{\frac{n}{2}+1} \text{ and } f\left(v_{\frac{n}{2}+1}\right) = v_{\frac{n}{2}}, \\ \\ \text{if } n \text{ is odd, then } f\left(v_{\frac{n+1}{2}}\right) = v_{\frac{n+1}{2}}. \end{cases}$$

Both of these conditions are against the opposability of G. Thus, we have shown that $P_1 \neq P_2$. Moreover, the statements (3.1) and (3.2) are true.

(2) and (3) follow from (1). □

At this point we can state that the graph G in Example 3.13 is not opposable since if G is opposable with an automorphism f, then $f(v_2) = v_7$ since $deg(v_2) = deg(v_7) = 3$. However, there are only two paths from v_2 to v_7 of different lengths. Thus, by Theorem 3.15, G is not opposable.

We can give sufficient conditions for opposability of connected graphs as follows.

Theorem 3.16. *Let $G = (V, E)$ be a connected graph with four or more vertices. If for any vertex $u \in V$ there exists a unique vertex $v \in V$ such that*

(1) *$deg(u) = deg(v)$;*

(2) *There are two distinct paths from u to v of the same length greater than or equal to 2;*

(3) *If $P_1 = (u, u_1, u_2, \ldots, u_n, v)$ and $P_2 = (u, v_1, v_2, \ldots, v_n, v)$ are such paths from u to v, then it holds that*

$$deg(u_i) = deg(v_{n+i-1}), \quad 1 \le i \le n,$$

$$u_{\frac{n}{2}} \neq v_{\frac{n}{2}+1} \text{ and } u_{\frac{n}{2}+1} \neq v_{\frac{n}{2}} \qquad \text{if } n \text{ is even},$$

$$u_{\frac{n+1}{2}} \neq v_{\frac{n+1}{2}} \qquad \text{if } n \text{ is odd}.$$

(4) *For $u_1, u_2 \in V$ let $v_1, v_2 \in V$ be the unique vertices corresponding to u_1 and u_2, respectively. If u_1 and u_2 are adjacent, then v_1 and v_2 are also adjacent.*

Then G is opposable.

Proof. For each $u \in V$ define $f(u)$ to be the unique $v \in V$ mentioned in the theorem. Then, $f : V \to V$ is well-defined.

Suppose that $u_1 \neq u_2$ and $f(u_1) = f(u_2) = v$. Then for v there are two distinct vertices u_1, u_2 satisfying the assumption of the theorem. This is a contradiction. Hence, f is one-to-one. Since V is finite and f is one-to-one, it follows that f is onto. Now the conditions (1) and (4) imply that f is an automorphism.

Let $u \in V$. For the vertex $v = f(u)$ the vertex u satisfies the assumption of the theorem. Hence, $f(v) = f(f(u)) = u$. It is clear from the conditions (2) and (3) that $f(u) \neq u$ and that $f(u)$ and u are not adjacent.

Therefore, G is opposable. ∎

We depict some examples of an opposable graph that are extensions of three basic graphs: the Square (C_4), Hexagon (C_6), and Octagon (C_8). We see that the conditions given in Theorems 3.11, 3.15 and 3.16 are satisfied by these examples.

[1] Square

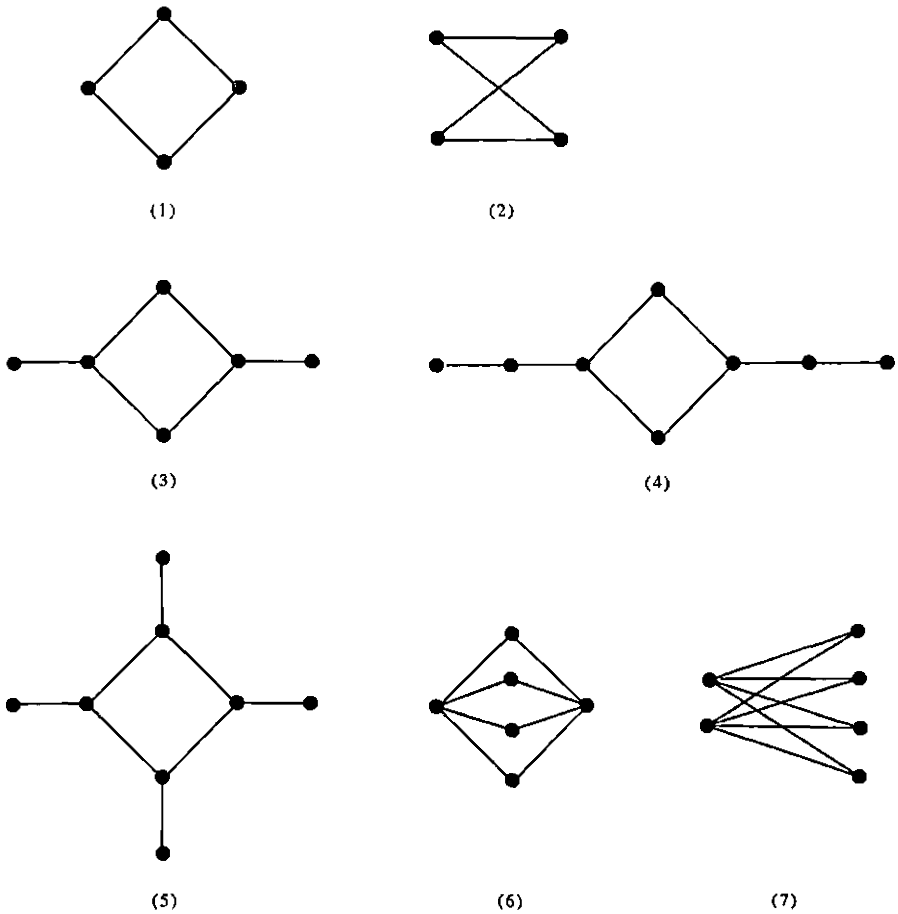


Figure 3.6: Examples of opposable graphs based on the square

In the above figure, (1) indicates the basic square. The figures (2) through (7) are obtained by modifying the previous ones along with the conditions given in Theorems 3.11 and 3.15 as follows.

(1) to (2): Twisting ($C_4 \cong K_{2,2}$).

(1) to (3): Adding 2 vertices and 2 edges outside.

(3) to (4): Adding 2 vertices and 2 edges outside.

(3) to (5): Adding 2 vertices and 2 edges outside.

(1) to (6): Adding 2 vertices and 4 edges inside.

(6) to (7): Twisting ((6) $\cong K_{2,4}$)

[2] Hexagon

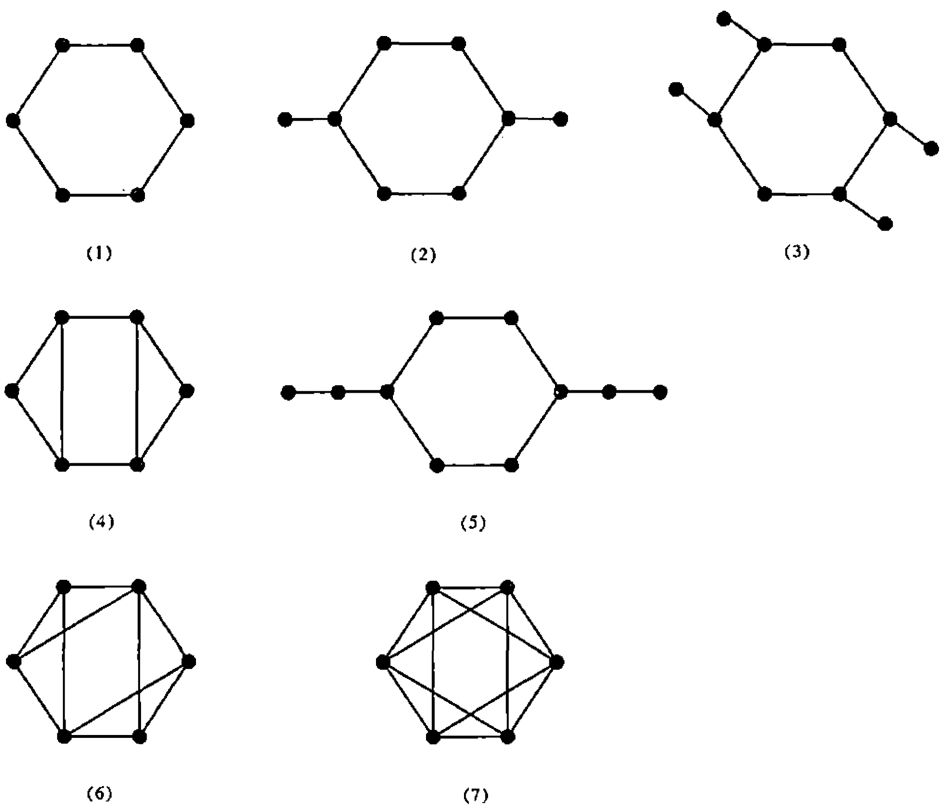


Figure 3.7: Examples of opposable graphs based on the hexagon

In the above figure, (1) indicates the basic hexagon. The figures (2) through (7) are obtained by modifying the previous ones along with the conditions given in Theorems 3.11 and 3.15 as follows.

(1) to (2): Adding 2 vertices and 2 edges outside.

(2) to (3): Adding 2 vertices and 2 edges outside.

(1) to (4): Adding 2 edges inside.

(2) to (5): Adding 2 vertices and 2 edges outside.

(4) to (6): Adding 2 edges inside.

(6) to (7): Adding 2 edges inside.

[3] Octagon

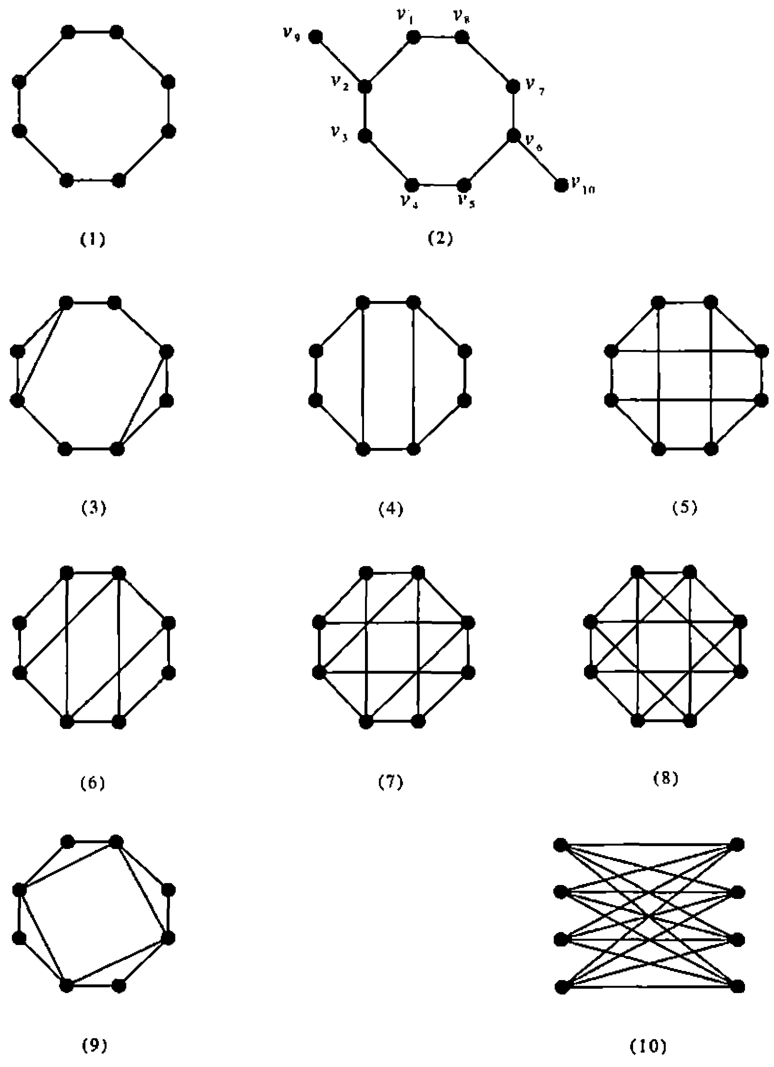


Figure 3.8: Examples of opposable graphs based on the octagon

In the above figure, (1) indicates the basic octagon. The figures (2) through (10) are obtained by modifying the previous ones along with the conditions given in Theorems 3.11 and 3.15 as follows.

(1) to (2): Adding 2 vertices and 2 edges outside.

(1) to (3): Adding 2 edges inside.

(1) to (4): Adding 2 edges inside.

(4) to (5): Adding 2 edges inside.

(4) to (6): Adding 2 edges inside.

(6) to (7): Adding 2 edges inside.

(7) to (8): Adding 2 edges inside.

(3) to (9): Adding 2 edges inside.

(8) to (10): Twisting ((8) $\cong K_{4,4}$)

3.3 Examples in Two Dimensional Grid Graphs

Since we studied opposable graphs and their properties, we now apply these ideas to consider Snort on two-dimensional grid graphs and their extension to three-dimensional figures. We shall play on three types of two-dimensional grids: (odd)×(odd) number of vertices, (even)×(even) number of vertices, and (even)×(odd) number of vertices.

Example 3.17. For a pair of positive integers (m, n) let $G_{m,n}$ be a grid in a coordinate plane with the set of vertices $V = \{v_{i,j} = (i, j) : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and the set of edges $E = \{(v_{i,j}, v_{i,j+1}),\ (v_{i,j}, v_{i+1,j}),\ (v_{m,j}, v_{m,j+1}),\ (v_{i,n}, v_{i+1,n}) : 1 \leq i \leq m-1,\ 1 \leq j \leq n-1\}$, so that it has n rows and m columns.

(1) Consider the graph $G_{3,3}$ given in Figure 3.9.

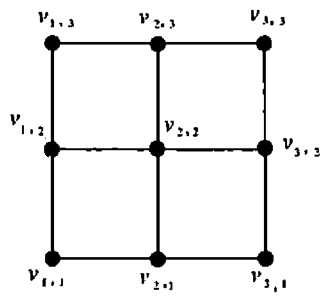


Figure 3.9: The graph of $G_{3,3}$

On $G_{3,3}$, suppose the first player claims $v_{2,2}$ following Remark 2.5 (1), so that the four vertices $v_{1,2}, v_{2,1}, v_{2,3}$, and $v_{3,2}$ become reserved for the first player (Figure 3.10).

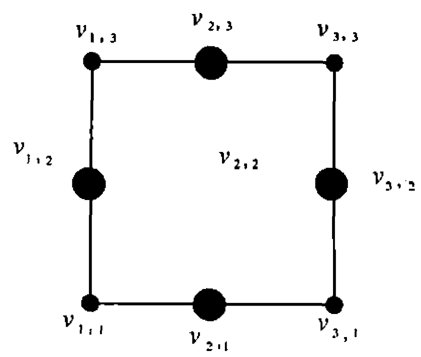


Figure 3.10: Snort on $G_{3,3}$ after first move

The second player claims $v_{1,3}$ and then the two vertices $v_{1,2}$ and $v_{2,3}$ become piebald (Figure 3.11).

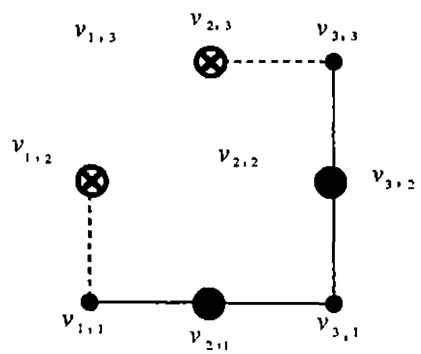


Figure 3.11: Snort on $G_{3,3}$ after second move

On the next turn there are two choices for the first player: $v_{3,1}$ and $v_{3,2}$. To claim $v_{3,1}$ is the ordinary opposable strategy since the graph in Figure 3.10 is opposable. And then the first player wins. To claim $v_{3,2}$ is a little better since the first player can increase the number of reserved vertices. If the first player claims $v_{3,2}$ and then two vertices $v_{3,3}$ and $v_{3,1}$ become reserved for the first player (Figure 3.12).

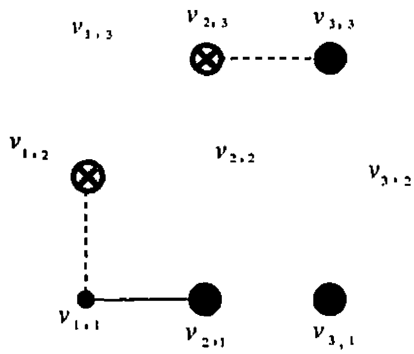


Figure 3.12: Snort on $G_{3,3}$ after third move

Now we can see that the first player wins since there is only one vertex available for the second player while there are two vertices available for the first player after the second player's move.

(2) Consider $G_{4,4}$ given in Figure 3.13.

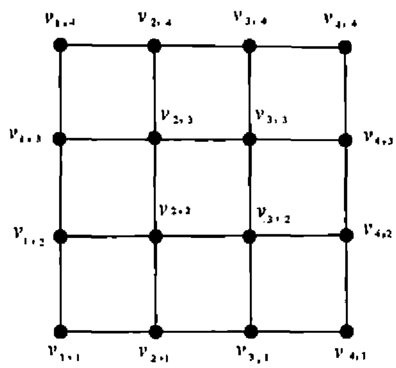


Figure 3.13: The graph of $G_{4,4}$

The graph of $G_{4,4}$ seems to satisfy the conditions (1) – (5) of Theorem 3.11. Check (1): The number of vertices is 16. Check (2): The number of edges is 24. Check (4): The number of degree 2 vertices is 4, the number of degree 3 vertices is 8, and the number of degree 4 vertices is 4. Now we shall construct an automorphism f that satisfies the opposability condition and hence conditions (3) and (5) of Theorem 3.11. Define f by $f(v_{i,j}) = v_{5-i,5-j}$ for $i, j = 1, 2, 3, 4$. For instance, $f(v_{1,2}) = v_{4,3}$. Then, we see that f is one-to-one, onto, and $f(f(v)) = v \neq f(v)$ for all v. Moreover, if v and w are adjacent, then so are $f(v)$ and $f(w)$. Thus, $G_{4,4}$ is opposable.

We shall show a few of the first steps of the game as follows. The first player may choose $v_{2,3}$ since $v_{2,3}$ has the largest degree 4. Then, four vertices $v_{1,3}, v_{2,2}, v_{2,4}$ and $v_{3,3}$ that are adjacent to $v_{2,3}$ are reserved for the first player (Figure 3.14).

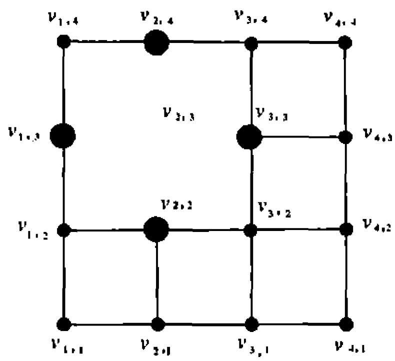


Figure 3.14: Snort on $G_{4,4}$ after first move

The second player claims $f(v_{2,3}) = v_{3,2}$. He has two vertices $v_{3,1}$ and $v_{4,2}$ reserved for him, and two vertices $v_{2,2}$ and $v_{3,3}$ become piebald (Figure 3.15). Note that the degree of $v_{3,2}$ is four (Figure 3.15).

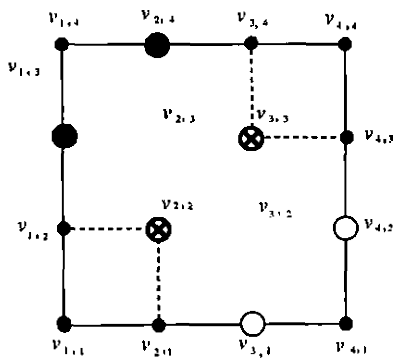


Figure 3.15: Snort on $G_{4,4}$ after second move

On the first player's turn he can claim $v_{4,4}$ and then the two vertices $v_{3,4}$ and $v_{4,3}$ become reserved for the first player (Figure 3.16).

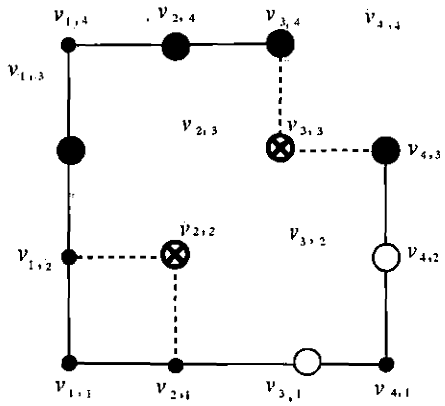


Figure 3.16: Snort on $G_{4,4}$ after third move

On the second player's turn, he claims $f(v_{4,4}) = v_{1,1}$ to make $v_{1,2}$ and $v_{2,1}$ reserved for the second player (Figure 3.17).

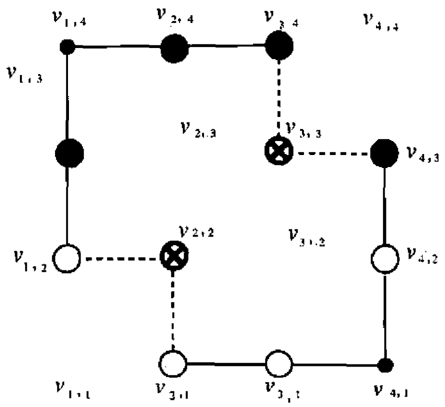


Figure 3.17: Snort on $G_{4,4}$ after fourth move

Now it is the first player's turn. We can see the next step of the first player is to claim either $v_{1,4}$ or $v_{4,1}$, and then the second player follows to claim the remaining vertex of the two and wins.

We have examined two graphs $G_{m,n}$ with odd m, n or even m, n, and now we consider the graph $G_{m,n}$ in which m and n are of opposite parity.

(3) Consider $G_{4,3}$ given in Figure 3.18.

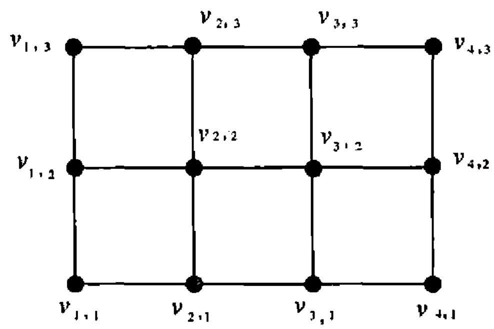


Figure 3.18: The graph of $G_{4,3}$

The first player may claim $v_{2,2}$ and four vertices, $v_{1,2}, v_{2,1}, v_{2,3}$, and $v_{3,2}$, become reserved for the first player (Figure 3.19).

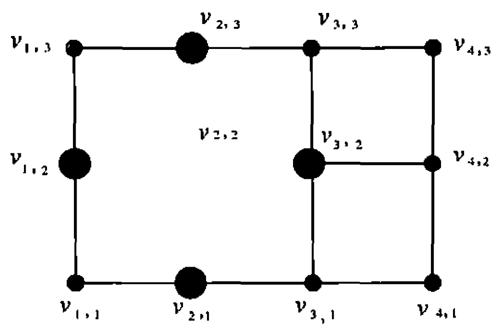


Figure 3.19: Snort on $G_{4,3}$ after first move

The second player claims $v_{4,2}$, which implies that $v_{3,2}$ becomes piebald and $v_{4,1}$ and $v_{4,3}$ become reserved for the second player (Figure 3.20).

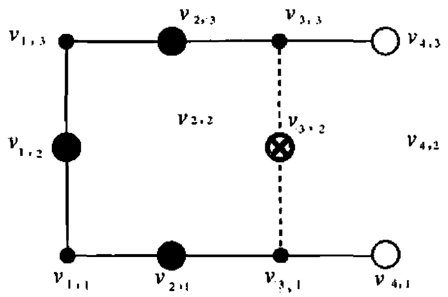


Figure 3.20: Snort on $G_{4,3}$ after second move

The first player claims $v_{1,2}$, so now the vertices reserved for the first player are $v_{1,1}, v_{1,3}, v_{2,1}$, and $v_{2,3}$ (Figure 3.21).

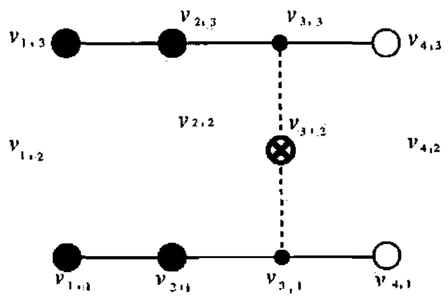


Figure 3.21: Snort on $G_{4,3}$ after third move

On the next turn the second player claims $v_{3,1}$, and $v_{2,1}$ becomes piebald (Figure 3.22).

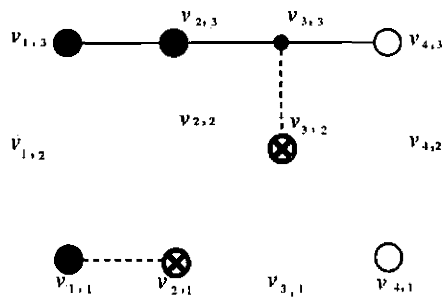


Figure 3.22: Snort on $G_{4,3}$ after fourth move

On the first player's turn, he claims $v_{3,3}$. Then, he still has three vertices reserved for him and decreases the number of vertices reserved for the second player to one by making $v_{4,3}$ piebald. Therefore, we can confirm that the first player wins.

Now we can summarize the above discussion into the following proposition.

Proposition 3.18. *Snort on the graph $G_{m,n}$ is a first player win if at least one of m, n is odd, and is a second player win otherwise.*

Proof. Let $G_{m,n}$ be a graph in a coordinate plane with the set of vertices $V = \{v_{i,j} = (i,j) : i = 1, 2, \dots, m,\ j = 1, 2, \dots, n\}$.

Case (1). m, n are both odd so that $m = 2k - 1$ and $n = 2\ell - 1$ for some positive integers k and ℓ.

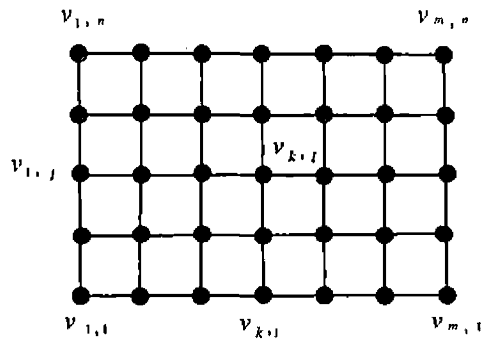


Figure 3.23: The graph of $G_{m,n}$; m, n both odd

Note that the vertex $v_{k,\ell}$ is the center of this graph. The first player claims $v_{k,\ell}$. After the first player claims $v_{k,\ell}$, the graph $G_1 = G \backslash \{v_{k,\ell}\}$ becomes opposable, where an automorphism $f : V \to V$ is defined by $f(v_{i,j}) = v_{m+1-i,\,n+1-j}$. Thus, G is a first player win.

Case (2). m, n are of opposite parity so that $m = 2k$ and $n = 2\ell - 1$ for some positive integers k and ℓ.

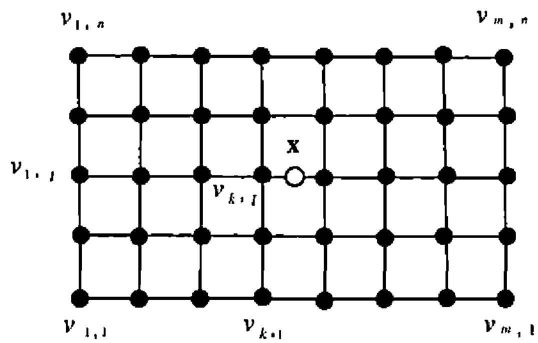


Figure 3.24: The graph of $G_{m,n}$; m even, n odd

Let x$= \left(k+\frac{1}{2},\ell\right)$ be the center of the graph, that is not a vertex of the graph. The first player claims $v_{k,\ell}$. Note that the second player cannot claim $v_{k,\ell+1}$, which is reserved for the first player and is symmetric to $v_{k,\ell}$ about the point **x**. After the first player chooses that specific vertex, the graph $G_1 = G\backslash\{v_{k,\ell},\, v_{k+1,\ell}\}$ becomes opposable, where an automorphism $f : V \to V$ is defined by $f(v_{i,j}) = v_{m+1-i,n+1-j}$. Thus, G is a first player win.

Case (3). m, n are both even so that $m = 2k$ and $n = 2\ell$ for some positive integers k and ℓ.

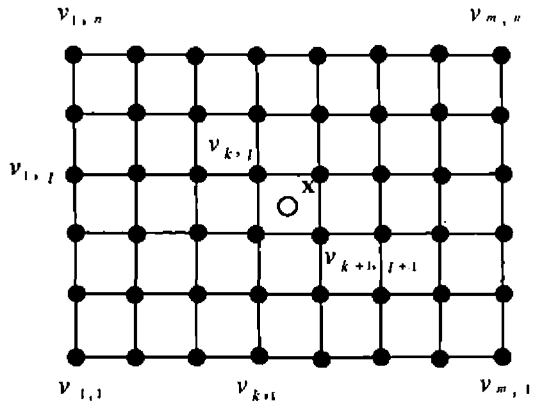


Figure 3.25: The graph of $G_{m,n}$; m, n both even

This graph is opposable since **x**$= \left(k+\frac{1}{2},\ell+\frac{1}{2}\right)$ is the center of the graph and an automorphism $f : V \to V$ can be defined by $f(v_{i,j}) = v_{m+1-i,\,n+1-j}$. Thus, G is a second player win. □

3.4 Examples in Three Dimensional Grid Graphs

So far we have examined two-dimensional grid graphs. Now we can predict who wins on three-dimensional grid graphs based on the strategies played on two-dimensional grid graphs. Thus, for a triplet of positive integers (ℓ, m, n) let $G_{\ell,m,n}$ be a three dimensional graph with a set of vertices $V = \{v_{i,j,k} = (i,j,k) : 1 \le i \le \ell,\ 1 \le j \le m,\ 1 \le k \le n\}$ and a set of edges $E = \{(v_{i,j,k}, v_{i+1,j,k}) : 1 \le i \le \ell - 1, 1 \le j \le m, 1 \le k \le n\} \cup \{(v_{i,j,k}, v_{i,j+1,k}) : 1 \le i \le \ell, 1 \le j \le m-1, 1 \le k \le n\} \cup \{(v_{i,j,k}, v_{i,j,k+1}) : 1 \le i \le \ell,\ 1 \le j \le m,\ 1 \le k \le n-1\}$. There are four patterns on three-dimensional grid graphs $G_{\ell,m,n}$, as shown in the following.

Proposition 3.19. *The graph $G_{\ell,m,n}$ is a first player win if at least two of ℓ, m, n are odd and is a second player win otherwise.*

Proof. Consider four cases.

Case (1). All of ℓ, m, n are odd, so that $\ell = 2p-1, m = 2q-1$ and $n = 2r-1$ for some positive integers p, q and r.

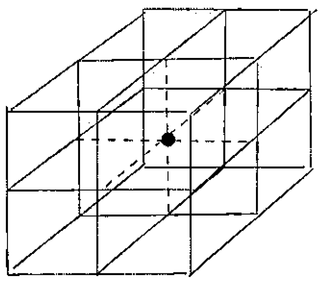


Figure 3.26: The 3-D graph of $G_{\ell,m,n}$; l, m, n all odd

Note that the vertex $v_{p,q,r}$ is the geometric center of this graph. The first player claims the vertex $v_{p,q,r}$. After the first player claims $v_{p,q,r}$, the graph $G_1 = G \backslash \{v_{p,q,r}\}$ becomes opposable, where an automorphism $f : V \to V$ is defined by $f(v_{i,j,k}) = v_{\ell+1-i,\,m+1-j,\,n+1-k}$.

Therefore, the first player plays as the second player in G_1 and G is a first player win.

Case (2). ℓ, m are odd and n is even, so that $\ell = 2p-1$, $m = 2q-1$ and $n = 2r$ for some positive integers p, q and r.

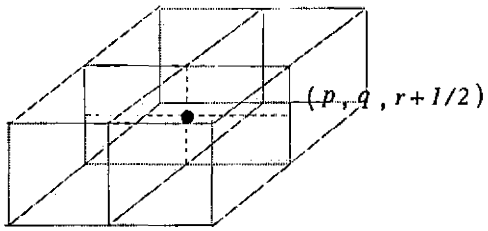


Figure 3.27: The 3-D graph of $G_{\ell,m,n}$; l, m odd, n even

Observe that $\mathbf{x} = (p, q, r+\frac{1}{2})$ is the geometric center of the graph that is not a vertex of the graph. The first player claims $v_{p,q,r}$. Note that the second player cannot claim $v_{p,q,r+1}$, which is reserved for the first player and is symmetric to $v_{p,q,r}$ about the point $\mathbf{x}$. After the first player chooses that specific vertex, the graph $G_1 = G \backslash \{v_{p,q,r}, v_{p,q,r+1}\}$ becomes opposable, where an automorphism $f : V \to V$ is given by $f(v_{i,j,k}) = v_{\ell+1-i,m+1-j,n+1-k}$. Thus, G is a first player win.

Case (3). ℓ is odd and m, n are even, so that $\ell = 2p-1$, $m = 2q$, and $n = 2r$ for some positive integers p, q and r.

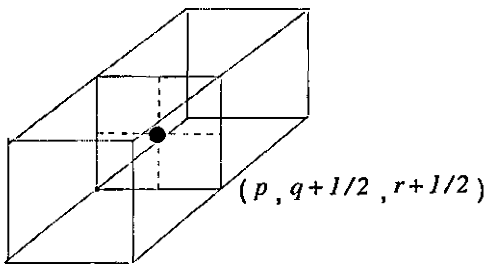


Figure 3.28: The 3-D graph of $G_{\ell,m,n}$; l odd, m, n even

Note that $\mathbf{x} = (p, q+\frac{1}{2}, r+\frac{1}{2})$ is the geometric center of the graph that is not a vertex of the graph. Define $f : V \to V$ by $f(v_{i,j,k}) = v_{\ell+1-i,m+1-j,n+1-k}$. Then, it is easy to see that f is an automorphism of G and makes G opposable. Thus, G is a second player win.

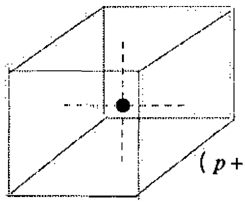


Figure 3.29: The 3-D graph of $G_{\ell,m,n}$; l, m, n even

Case (4). All of ℓ, m, n are even, so that $\ell = 2p$, $m = 2q$, and $n = 2r$ for some positive integers p, q, r.

This graph is opposable since $\mathbf{x} = (p + \frac{1}{2}, q + \frac{1}{2}, r + \frac{1}{2})$ is the geometric center of the graph and an automorphism $f : V \to V$ can be defined by $f(v_{i,j,k}) = v_{\ell+1-i,\, m+1-j,\, n+1-k}$. Thus, G is a second player win. □

Remark 3.20. From the above propositions and those in Section 3.3 we can summarize Snort on a two- or three- dimensional grid graph G as follows.

1. We look for the geometric center of G.

2. When the center is a vertex of the graph, then the first player should claim that vertex to win (Figure 3.26).

3. There are two cases when the center is not a vertex of the graph.

   a. The first case is when any two nearest vertices to the center are adjacent (Figure 3.27). In this case, the first player colors one of these nearest vertices and wins.

   b. The second case is when there are four or eight vertices that are nearest to the center (Figures 3.28 and 3.29). In this case, the graph G is opposable and the second player wins.

4. We can rephrase the item 3 as follows. If the set of closest vertices to the center and the edges among them is an opposable graph, then G is a second player win. Otherwise, G is a first player win.

Chapter 4

Nonopposable Graphs: Trees

In this chapter, we pursue whether the first player always wins on a nonopposable graph. Since trees are nonopposable, we play on complete and incomplete trees and see if the first player can win. When n is an even integer, it is easy to see that a complete n-nary tree is a first player win. When n is an odd integer, we shall prove the case where $n = 3$. That is, any complete ternary tree is a first player win. Some examples are shown in detail to develop the ideas for the proof of this result.

4.1 Nonopposable Graphs

We see in Proposition 3.6 that any opposable graph is a second player win. Is the converse true? Hence, we state this as a conjecture.

Conjecture 4.1. *A graph is opposable if and only if it is a second player win. In other words, a graph is nonopposable if and only if it is a first player win.*

To examine this conjecture, we need to collect some examples of a nonopposable graphs. As we see below a typical nonopposable graph is a tree.

Definition 4.2. *A* tree *is a connected graph that contains no cycles. We assume that any tree has a unique* root *that is at* level *0. Any vertex adjacent to the root is at level 1, any vertex adjacent to a vertex at level 1 (other than the root) is at level 2, and so forth. If p is the largest level of the tree, then p is called the* height *of the tree.*

Theorem 4.3. *A tree is a nonopposable graph.*

Proof. This immediately follows from Theorem 3.15 (3) and Definition 4.2. □

A typical example of a tree is illustrated in the following.

Example 4.4. Let n, p be positive integers such that $n \geq 2$. Then, let $T_{n,p}$ denote a *complete n-nary tree of height p.* That is, the root has degree n, any vertex of level i, $1 \leq i \leq p-1$ has degree $n+1$, and any vertex of level p has degree 1. There are n^i vertices at level i for $i = 1, 2, \ldots, p$. The graph of $T_{2,3}$, the complete binary tree of height 3, is shown below.

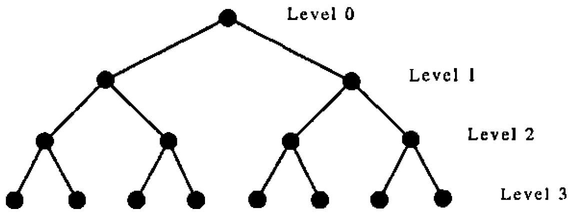


Figure 4.1: The complete binary tree $T_{2,3}$

Before we play Snort on complete trees, we introduce a new strategy called "suicide." In their paper [SS] Stokes and Schlatter describe suicide as a move that is efficient for tree graphs.

Definition 4.5. *A* suicide *is a move in which a player claims a vertex which is reserved for himself.*

We need the following lemmas to prove Proposition 4.8 in the next section.

Lemma 4.6. *If two graphs G_1 and G_2 are opposable, then their union $G = G_1 \cup G_2$ is also opposable.*

Proof. Let $G_i = (V_i, E_i)$ and f_i be an automorphism of G_i that makes G_i opposable, $i = 1, 2$. Then define a mapping f on G by $f(v) = f_i(v)$ if $v \in V_i$. Clearly, f is an automorphism of G that makes G opposable. □

Lemma 4.7. *Let G' and G'' be identical graphs. Then, the union $G = G' \cup G''$ is opposable.*

Proof. This follows from Proposition 3.10. Or let $G' = (V', E')$ and $G'' = (V'', E'')$ be two identical graphs, where two corresponding vertices in V' and V'' are denoted by v' and v'', i.e., $v' = v''$, $v' \in V'$ and $v'' \in V''$. Define a mapping f on G by $f(v') = v''$ and $f(v'') = v'$. Then, f is an automorphism of $G = G' \cup G''$ and makes it opposable. □

4.2 Complete Trees

Let us fix a notation for a complete n-nary tree $G = T_{n,p}$ of height p, where $n \geq 2$ and $p \geq 1$. The vertices of $T_{n,p}$ are denoted as follows. In the level zero, v_0 is the root; in the level 1 from left to right, v_i, $1 \leq i \leq n$; in the level 2 from left to right, $v_{i,j}$, $1 \leq i, j \leq n$, where, for a fixed i, $v_{i,j}$ is connected to v_i $(1 \leq j \leq n)$; in the level 3 from left to right, $v_{i,j,k}$, $1 \leq i, j, k \leq n$, where, for fixed i, j, $v_{i,j,k}$ is connected to $v_{i,j}$ $(1 \leq k \leq n)$; and so on. A subgraph of G with a root v_i $(1 \leq i \leq n)$ is denoted by G_i that is isomorphic to $T_{n,p-1}$. Similarly, a subgraph of G with a root $v_{i,j}$ $(1 \leq i, j \leq n)$ is denoted by $G_{i,j}$ that is isomorphic to $T_{n,p-2}$, and so on. Note that G_0, the subgraph of $T_{n,p}$ with the root v_0, is exactly the graph $T_{n,p}$ itself.

Proposition 4.8. *For any even integer $n \geq 2$ a complete n-nary tree is a first player win.*

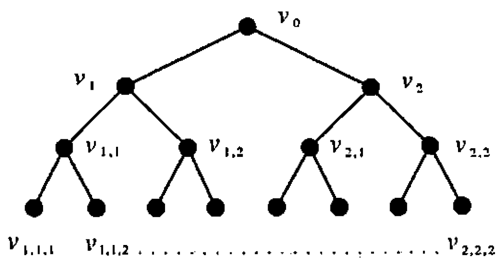


Figure 4.2: A complete binary tree

Proof. Since n is even, we have $n = 2m$ for some positive integer m. For example a complete binary tree of height 3, $T_{2,3}$, is given in Figure 4.2 above. The first player takes the root v_0 at the first move. Then, vertices v_i $(1 \leq i \leq 2m)$ are reserved for the first player. There are $2m$ subgraphs $G_1, G_2, \ldots, G_{2m}$ that are complete $2m$-nary trees with the roots $v_1, v_2, \ldots, v_{2m}$, respectively. Let us put $G = G_1 \cup G_2 \cup \cdots \cup G_{2m} = T_{2m,p} \backslash \{v_0\}$, $G_* = G_1 \cup \cdots \cup G_m$ and $G_{**} = G_{m+1} \cup \cdots \cup G_{2m}$. Then, G_* and G_{**} are identical graphs, and G is opposable by Lemma 4.7. Hence, $T_{n,p}$ is a first player win since the first player plays as a second player in the game G. In the graph of $T_{2,3}$, $G_* = G_1$ and $G_{**} = G_2$ as shown in Figure 4.3. □

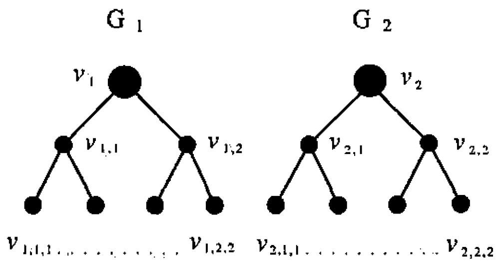


Figure 4.3: A complete binary tree after first move

For an odd integer $n \geq 3$ it is not obvious that $T_{n,p}$ is a first player win. In the rest of this section, we consider complete ternary trees $T_{3,p}$ for $p = 1, 2, 3, 4$ and 5. At the end of this section we shall prove that $T_{3,p}$ is a first player win for any $p \geq 1$.

The following terminology may be helpful.

Definition 4.9. *A vertex is said to be* absolutely reserved *for a player if it is reserved for him and cannot be made piebald by his opponent. In the rest of this paper we denote absolutely reserved vertices in boldface letters.*

Example 4.10. Consider complete ternary trees $T_{3,p}$ for $p = 1, 2$, and 3.

(1) If $p = 1$, then the first player wins by claiming the root at the first move.

(2) If $p = 2$, we introduce two strategies by which the first player can win, where the graph $T_{3,2}$ is shown in Figure 4.4 next page.

Strategy (1). Suppose the first player claims the root, so that v_1, v_2, v_3 are reserved for the first player. The second player has 9 equal choices, i.e., vertices of level 2. Suppose he claims $v_{1,1}$, making v_1 piebald. Then the first player claims v_2 or v_3 of level

1, say v_3, making three vertices $v_{3,j}$ for $j = 1,2,3$ absolutely reserved for him. Then the second player may claim $v_{2,1}$ to make v_2 piebald. At this stage, $v_{1,2}, v_{1,3}, v_{2,2}$, and $v_{2,3}$ are available for both players and $v_{3,j}$, $j = 1,2,3$ are available only for the first player. Thus, it is clear that this game is a first player win by saving two absolutely reserved vertices at the end of this game. We summarize the procedure in Table 4.1. Recall that the absolutely reserved vertices are denoted in boldface letters.

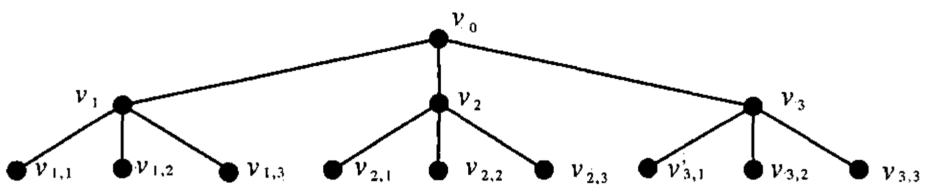


Figure 4.4: A complete ternary tree $T_{3,2}$

Round	First player	Second player	Reserved for first	Reserved for second	Piebald
1	v_0		v_1, v_2, v_3		
1		$v_{1,1}$			v_1
2	v_3		$\mathbf{v_{3,1}}, \mathbf{v_{3,2}}, \mathbf{v_{3,3}}$		
2		$v_{2,1}$			v_2
3	$v_{2,3}$				
3		$v_{1,2}$			
4	$v_{2,2}$				
4		$v_{1,3}$			
5	$\mathbf{v_{3,1}}$				

Table 4.1: Strategy (1) applied to $T_{3,2}$

Strategy (2). Suppose the first player claims v_1 for the first move to make v_0 reserved and $v_{1,1}, v_{1,2}$, and $v_{1,3}$ absolutely reserved for him. Then the second player claims v_2 to make v_0 piebald and $v_{2,1}, v_{2,2}, v_{2,3}$ absolutely reserved for him. Then, the first player claims v_3 and it is obvious that this game is a first player win. In this case, the first player can save three vertices for him at the end of the game. A detailed procedure is given in Table 4.2 below.

Round	First player	Second player	Reserved for first	Reserved for second	Piebald
1	v_1		v_0, $\boldsymbol{v_{1,1}}$, $\boldsymbol{v_{1,2}}$, $\boldsymbol{v_{1,3}}$		
1		v_2		$\boldsymbol{v_{2,1}}$, $\boldsymbol{v_{2,2}}$, $\boldsymbol{v_{2.3}}$	v_0
2	v_3		$\boldsymbol{v_{3,1}}$, $\boldsymbol{v_{3,2}}$, $\boldsymbol{v_{3,3}}$		

Table 4.2: Strategy (2) applied to $T_{3,2}$

(3) If $p = 3$, we will try three possible strategies for the first player where he wins each case.

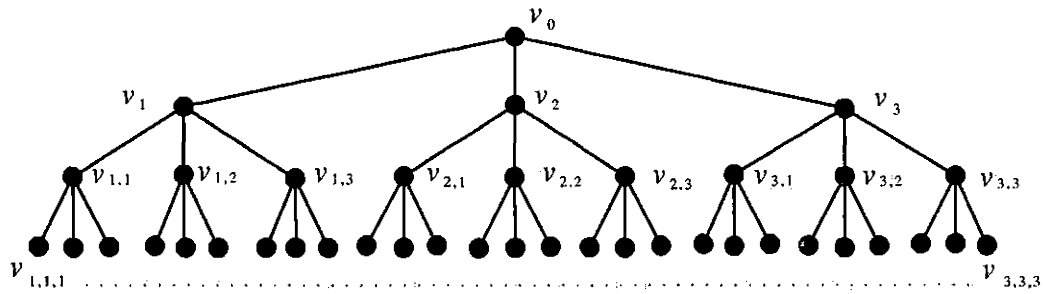


Figure 4.5: A complete ternary tree $T_{3,3}$

Strategy (1). Figure 4.5 shows $T_{3,3}$. Suppose the first player claims the root v_0 on the first move, so that v_1, v_2, and v_3 of level 1 are reserved for him. Then, the vertices of levels 2 and 3 are available for the second player. Evidently it is more efficient for the second player to pick a vertex of level 2 than to pick one of level 3. So the second player claims $v_{1,1}$, making three vertices $v_{1,1,k}$, $k = 1, 2, 3$ absolutely reserved for him and v_1 piebald. Next the first player claims v_3 (a suicide move) and makes $v_{3,j}$, $j = 1, 2, 3$ reserved for him. For his turn the second player claims $v_{2,1}$ to make v_2 piebald and $v_{2,1,k}$, $k = 1, 2, 3$ absolutely reserved for him. Now the rest of this process is apparent. After two players share the four vertices $v_{1,2}, v_{1,3}, v_{2,2}$, and $v_{2,3}$ evenly, the first player claims $v_{3,3}$ (a suicide move), the second player claims $v_{3,1,1}$ to make $v_{3,1}$ piebald, and the first claims $v_{3,2}$ (a suicide move) to make $v_{3,2,k}$, $k = 1, 2, 3$ absolutely reserved for him. At this stage, both players have the same number of vertices absolutely reserved for each and there are two vertices $v_{3,1,2}$ and $v_{3,1,3}$ available for both. Since the second player starts the rest of the game, the first player wins. Note that all the vertices are exhausted and there are no vertices absolutely reserved for the first player.

For a reference we list the moves of each player as follows.

Round	First player	Second player	Reserved for first	Reserved for second	Piebald
1	v_0		v_1, v_2, v_3		
1		$v_{1,1}$		$\boldsymbol{v_{1,1,1}}$, $\boldsymbol{v_{1,1,2}}$, $\boldsymbol{v_{1,1,3}}$	v_1
2	v_3		$v_{3,1}, v_{3,2}, v_{3,3}$		
2		$v_{2,1}$		$\boldsymbol{v_{2,1,1}}$, $\boldsymbol{v_{2,1,2}}$, $\boldsymbol{v_{2,1,3}}$	v_2
3	$v_{2,2}$		$\boldsymbol{v_{2,2,1}}$, $\boldsymbol{v_{2,2,2}}$, $\boldsymbol{v_{2,2,3}}$		
3		$v_{1,2}$		$\boldsymbol{v_{1,2,1}}$, $\boldsymbol{v_{1,2,2}}$, $\boldsymbol{v_{1,2,3}}$	
4	$v_{2,3}$		$\boldsymbol{v_{2,3,1}}$, $\boldsymbol{v_{2,3,2}}$, $\boldsymbol{v_{2,3,3}}$		
4		$v_{1,3}$		$\boldsymbol{v_{1,3,1}}$, $\boldsymbol{v_{1,3,2}}$, $\boldsymbol{v_{1,3,3}}$	
5	$v_{3,3}$		$\boldsymbol{v_{3,3,1}}$, $\boldsymbol{v_{3,3,2}}$, $\boldsymbol{v_{3,3,3}}$		
5		$v_{3,1,1}$			$v_{3,1}$
6	$v_{3,2}$		$\boldsymbol{v_{3,2,1}}$, $\boldsymbol{v_{3,2,2}}$, $\boldsymbol{v_{3,2,3}}$		
6		$v_{3,1,2}$			

Table 4.3: Strategy (1) applied to $T_{3,3}$

Strategy (2). Suppose the first player claims v_1, making $v_0, v_{1,1}, v_{1,2}$ and $v_{1,3}$ reserved for him. There are some choices for the second player. Reasonable choices are v_2, v_3 and $v_{2,j}, v_{3,j}$ for $j = 1, 2, 3$. We list the moves of each player up to round 6 in Table 4.4 when the second player claims v_2.

Round	First player	Second player	Reserved for first	Reserved for second	Piebald
1	v_1		$v_0, v_{1,1}, v_{1,2}, v_{1,3}$		
1		v_2		$v_{2,1}$, $v_{2,2}$, $v_{2,3}$	v_0
2	v_3		$v_{3,1}, v_{3,2}, v_{3,3}$		
2		$v_{2,1}$		$\boldsymbol{v_{2,1,1}}$, $\boldsymbol{v_{2,1,2}}$, $\boldsymbol{v_{2,1,3}}$	
3	$v_{1,1}$		$\boldsymbol{v_{1,1,1}}$, $\boldsymbol{v_{1,1,2}}$, $\boldsymbol{v_{1,1,3}}$		
3		$v_{2,2}$		$\boldsymbol{v_{2,2,1}}$, $\boldsymbol{v_{2,2,2}}$, $\boldsymbol{v_{2,2,3}}$	
4	$v_{1,2}$		$\boldsymbol{v_{1,2,1}}$, $\boldsymbol{v_{1,2,2}}$, $\boldsymbol{v_{1,2,3}}$		
4		$v_{2,3}$		$\boldsymbol{v_{2,3,1}}$, $\boldsymbol{v_{2,3,2}}$, $\boldsymbol{v_{2,3,3}}$	
5	$v_{1,3}$		$\boldsymbol{v_{1,3,1}}$, $\boldsymbol{v_{1,3,2}}$, $\boldsymbol{v_{1,3,3}}$		
5		$v_{3,1,1}$			$v_{3,1}$
6	$v_{3,3}$		$\boldsymbol{v_{3,3,1}}$, $\boldsymbol{v_{3,3,2}}$, $\boldsymbol{v_{3,3,3}}$		
6		$v_{3,2,1}$			$v_{3,2}$

Table 4.4: Strategy (2) applied to $T_{3,3}$

At the last stage in Table 4.4, $v_{3,1,2}, v_{3,1,3}, v_{3,2,2}$, and $v_{3,2,3}$ are available for both players and $v_{1,1,i}, v_{1,2,i},\ v_{1,3,i}, v_{3,3,i},\ i = 1, 2, 3$ are available only for the first player and $v_{2,1,j}, v_{2,2,j}, v_{2,3,j},\ j = 1, 2, 3$ are available only for the second player. Thus, it is clear that this game is a first player win with two vertices absolutely reserved for him at the end of this game.

If we note that the graph $G = G_2 \cup G_3$ is opposable, then the first player (as a second player in G) can employ the opposable method to win the game G. Obviously in the graph G_1 the first player wins. Thus, the first player wins the entire game $T_{3,3}$. This opposable method will be used again in strategy (3) below.

Strategy (3). Suppose the first player claims $v_{1,1}$, so that $v_1, v_{1,1,1}, v_{1,1,2}$, and $v_{1,1,3}$ are reserved for him. Now we can make $G_0 = T_{3,3}$ be decomposed into three graphs: the set of three vertices $v_{1,1,1}, v_{1,1,2}, v_{1,1,3}$ that are absolutely reserved for the first player, $G_1 \backslash G_{1,1}$ and $G_0 \backslash G_1$. Those subgraphs are defined before Proposition 4.8. Let $H_1 = G_1 \backslash G_{1,1}$ and $H_0 = G_0 \backslash G_1$. Now we need to be aware in which subgraphs the second player begins to play.

After the first player claims $v_{1,1}$, if the second player claims $v \in H_1$ and plays the game in H_1, then the first player can win by playing as the second player on the opposable graph H_1. The first player still has three vertices absolutely reserved.

Then, when the second player claims $v_0 \in H_0$, we need to see the game on H_0 carefully. We consider two examples to play the game H_0. One is a second player win and the other is a first player win. The first example is using an opposable strategy to win for the second player. After the second player claims v_0, there will be six graphs of $T_{3,1}$ in the graph $H_0 \backslash \{v_0, v_2, v_3\}$ that is opposable. Hence, the second player wins the game H_0. Even though the first player loses the game H_0, if all the vertices on H_0 are exhausted, the first player can win the whole game.

The second example is that the second player does not use the opposability of the graph. At this point we show the consecutive moves of each player after the second player claims v_0 as follows.

Round	First player	Second player	Reserved for first	Reserved for second	Piebald
1		v_0		v_2, v_3	v_1
2	$v_{2,1}$		$\mathbf{v_{2,1,1}, v_{2,1,2}, v_{2,1,3}}$		v_2
2		v_3		$v_{3,1}, v_{3,2}, v_{3,3}$	
3	$v_{2,2}$		$\mathbf{v_{2,2,1}, v_{2,2,2}, v_{2,2,3}}$		
3		$v_{2,3}$		$\mathbf{v_{2,3,1}, v_{2,3,2}, v_{2,3,3}}$	
4	$v_{3,1,1}$				$v_{3,1}$
4		$v_{3,3}$		$\mathbf{v_{3,3,1}, v_{3,3,2}, v_{3,3,3}}$	
5	$v_{3,2,1}$				$v_{3,2}$

Table 4.5: Strategy (3) applied to $T_{3,3}$

Note that for the second player the only chance not to use the opposable strategy is to claim v_3 after the first player claims $v_{2,1}$. At this stage, $v_{3,1,2}, v_{3,1,3}, v_{3,2,2}$, and $v_{3,2,3}$ are available for both players, $v_{2,1,i}, v_{2,2,i}$, $i = 1,2,3$ are available only for the first player, and $v_{2,3,j}, v_{3,3,j}$, $j = 1,2,3$ are available only for the second player. Thus, the game in the subgraph H_0 is a first player win since the first player finishes the game. Therefore, the first player also wins the whole game in this example.

In the reverse order, if the second player claims v_0 on his first move, then he wins the game H_0 using the opposable method, where all the vertices are exhausted. Then, the first player claims $v_{1,1,1}$. Hence, the second player begins to play the game H_1 by claiming any vertex there other than v_1 and the first player wins by the opposable method, exhausting all the vertices there. Therefore, the first player wins the whole game with two vertices absolutely reserved for him.

From the above examples we can summarize the following.

1. Up to $p = 3$ the first player can win by claiming the root v_0 on his first move.

2. In $T_{3,2}$, there are two choices on the first move for the first player, i.e., v_0 and v_1. In each case, at the end of the game there are a few vertices absolutely reserved for the first player.

3. In $T_{3,3}$, there are three choices on the first move for the first player, i.e., v_0, v_1, and $v_{1,1}$. If the first move is v_0, then all the vertices are used. However, if the first move is v_1 or $v_{1,1}$, then there are two vertices absolutely reserved for the first player at the end of the game. That is, he can save two vertices.

Now we note some clues for the first player to win that we can consider from the above examples.

1. Try to save vertices as absolutely reserved for one's own.
2. Decompose the graph G_0 into some subgraphs in each of which we can easily set up the strategy.
3. Play on opposable graphs as a second player if it is possible.
4. On each subgraph the first player should make all vertices exhausted before finally he returns to the vertices of absolutely reserved for him.

We extend these ideas to play the complete ternary trees $T_{3,p}$ for $p = 4$ and $p = 5$ in the following.

Example 4.11. We consider the complete ternary tree $T_{3,4}$.

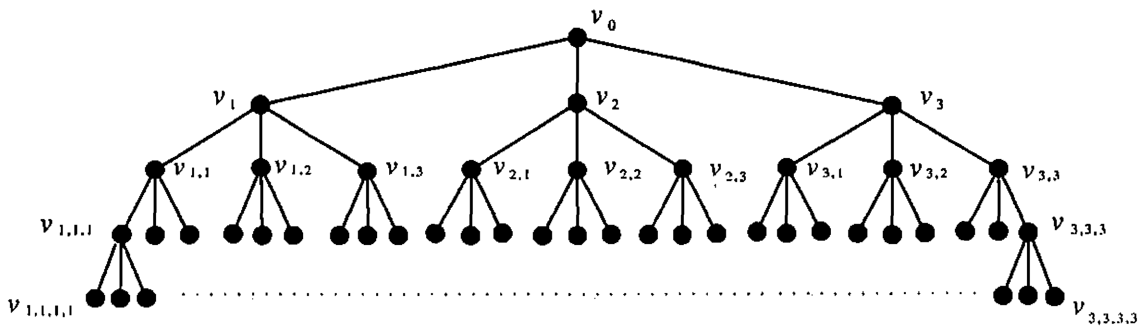


Figure 4.6: A complete ternary tree $T_{3,4}$

(1) In the case where $p = 4$, at the beginning we will show that the first player cannot always win if he claims v_0 at the first move. Suppose the first player employs the same strategy (1) as given in the case $p = 3$ in Example 4.10.

After the point when the second player claims $v_{3,1,2}$ in Table 4.3, the first player claims $v_{3,1,3}$, and he has three vertices $v_{3,1,3,\ell}$, $\ell = 1, 2, 3$ absolutely reserved and 12 vertices of level 3 reserved for him. The second player has six vertices $v_{3,1,k,\ell}$, $k = 1, 2$, $\ell = 1, 2, 3$ absolutely reserved and 12 vertices of level 3 reserved for him. Thus, it is clear that the second player wins this game.

(2) Instead of claiming v_0, suppose the first player claims v_1 at the first move, so that $v_0, v_{1,1}, v_{1,2}$, and $v_{1,3}$ are reserved for him. Then, the original graph is decomposed into two subgraphs: $T_{3,4} = G_1 \cup \{T_{3,4} \backslash G_1\}$, where we ignore the edge (v_0, v_1) and G_1 is the subgraph of G_0 isomorphic to $T_{3,3}$ with root v_1. Let $H = T_{3,4} \backslash G_1$. If the second player starts by claiming a vertex $v \in G_1$, then the first player can claim vertices as was done in the case of $T_{3,3}$ to win the game on $T_{3,3}$. If the second player starts by claiming $v \in H$, then since $v \neq v_0$ and the graph $H \backslash \{v_0\}$ is opposable the first player can win the game on H. Therefore, the first player wins the whole game.

Although we need to proceed to the case where the first player claims $v_{1,1,1}$ to have three vertices absolutely reserved for him, we would like to perform it on the graph $T_{3.5}$ with an introduction of a definition and a lemma.

Example 4.12. Now we consider the complete ternary tree $T_{3,5}$.

The case $p = 5$ is a little bit more complicated. We know that the first player does not want to claim v_0 at his first turn because of the experience of $T_{3,4}$. If the first player claims v_1, then the graph G_1 is isomorphic to $T_{3,4}$. Since v_0 is reserved for the first player, the graph $G_0 \backslash G_1$ is opposable. Thus, if the second player begins to play on $G_0 \backslash G_1$, then he loses that game and starts to play the game on G_1. Then the second player wins the game G_1, as was shown in Example 4.11, and hence the whole game $G_0 = T_{3,5}$. Therefore, we would like to make the first player claim $v_{1,1}$ on his first move. However, we shall show that the second player also wins $T_{3,5}$ in this case. To see this, let

$$H_0 = G_0 \backslash G_1, \quad H_1 = G_1 \backslash G_{1,1}, \quad H_2 = G_{1,1}.$$

The following is a way to win for the second player. H_2 consists of three graphs of $T_{3,2}$ where the roots are reserved for the first player. If the second player claims any $v \in H_2$ and both players play on H_2. Then the first player wins and all the vertices are exhausted. Then, the second player claims $v_{1,2} \in H_1$ to make v_1 piebald. Then, on H_1 the first player wins, all the vertices are exhausted, and the second player moves on to claim $v_0 \in H_0$. Since $H_0 \backslash \{v_0\}$ is now opposable, the second player wins on H_0 and hence wins the whole game G_0. As a consequence, in order to win the game $T_{3,5}$ the first player has to claim $v_{1,1,1}$ or $v_{1,1,1,1}$.

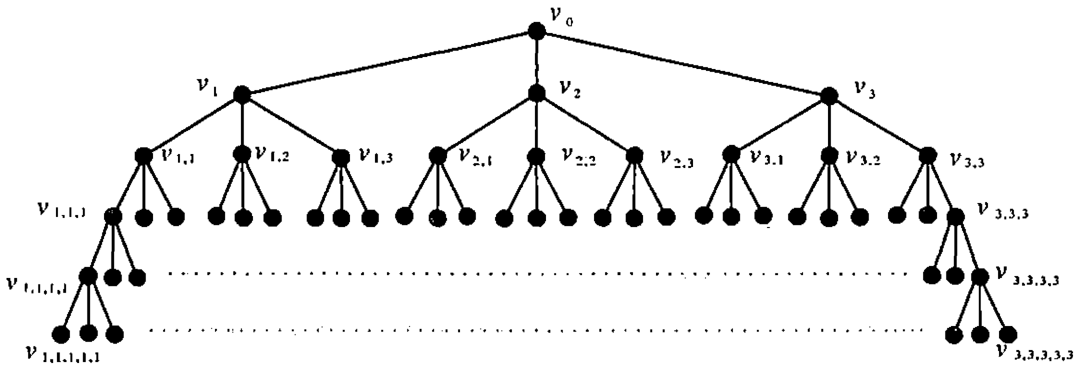


Figure 4.7: A complete ternary tree $T_{3,5}$

Let us consider the case where the first player claims $v_{1,1,1,1}$ to start the game on $T_{3,5}$ by following the clues that we previously suggested right before Example 4.11. We explain how to play in $T_{3,5}$ along the clues.

(1) Try to save vertices as absolutely reserved for one's own: The first player's initial step makes $v_{1,1,1}$ reserved and $v_{1,1,1,1,i}$ for $i = 1, 2, 3$ absolutely reserved for the first player.

(2) Decompose the graph G_0 into some subgraphs in each of which we can easily set up the strategy: Now G_0 is decomposed into $G_0\backslash G_1$, $G_1\backslash G_{1,1}$, $G_{1,1}\backslash G_{1,1,1}$, $G_{1,1,1}\backslash G_{1,1,1,1}$ and three absolutely reserved vertices by ignoring the edges (v_0, v_1), $(v_1, v_{1,1})$, $(v_{1,1}, v_{1,1,1})$, and $(v_{1,1,1}, v_{1,1,1,1})$. Let

$$H_0 = G_0\backslash G_1, \quad H_1 = G_1\backslash G_{1,1}, \quad H_{1,1} = G_{1,1}\backslash G_{1,1,1}, \quad H_{1,1,1} = G_{1,1,1}\backslash G_{1,1,1,1}.$$

(3) Play on opposable graphs as a second player if it is possible: The subgraph H_0 consists of two graphs of $T_{3,4}$ combined to the root v_0, H_1 consists of two graphs of $T_{3,3}$ combined to the root v_1, $H_{1,1}$ consists of two graphs of $T_{3,2}$ combined to the root $v_{1,1}$, and $H_{1,1,1}$ consists of two graphs of $T_{3,1}$ combined to the root $v_{1,1,1}$. If the second player claims the root of any subgraph, that subgraph becomes opposable, advantageous for the second player. Only $H_{1,1,1}$ is opposable and advantageous for the first player.

(4) In each subgraph the first player should make all vertices exhausted before he returns to the vertices that are absolutely reserved: The winner is the person who claims the last available vertex in Snort. Therefore, if both players consume all the vertices, the first player can win since he has three absolutely reserved vertices.

It follows from the strategy (3) in Example 4.10 (3) for the case $p = 3$ that the graphs H_0 and H_1 are of interest and should be examined more in order to prove our main theorem below. It would be convenient to have the following definition.

Definition 4.13. *For $p \geq 2$ let K_p denote a tree of height p that consists of two graphs of $T_{3,p-1}$ whose roots are connected to the root of K_p. Thus, we name the vertices of K_p as follows. In the level zero, the root u_0; in the level 1 from left to right u_1, u_2; in the level 2 from left to right $u_{1,j}$, $u_{2,j}$ for $j = 1, 2, 3$; in the level 3 from left to right $u_{1,j,k}$, $u_{2,j,k}$ for $j, k = 1, 2, 3$; and so on. Figure 4.8 shows the graph K_3.*

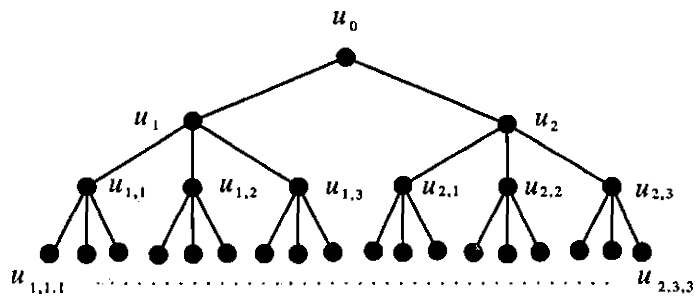


Figure 4.8: A tree K_3

As is easily seen, the graph $K_p \backslash \{u_0\}$ is opposable, and hence K_p is a first player win if he claims the root and employs the opposable method. In this case all the vertices are used. However, this is the best possible result for the first player in the sense that he cannot leave any vertices absolutely reserved for him, which will be proved below.

Lemma 4.14. *In the game of K_p for $p \geq 3$ the first player cannot win with leaving any vertices absolutely reserved for him. More fully,*

(1) *The graph K_p is a first player win if the first player claims the root at his first move and employs the ordinary opposable method. In this case, all the vertices are exhausted.*

(2) *In the graph K_p, if the first player claims the root at his first move and does not employ the ordinary opposable method, then all the vertices are exhausted or the second player wins the game.*

Proof. (1) The first player claims the root u_0 at his first move. Then, since the graph $K_p \backslash \{u_0\}$ is opposable, the first player (as the second player in the game $K_p \backslash \{u_0\}$) wins by the ordinary opposable method and all the vertices are exhausted. Here, one choice of the automorphism $f : K_p \backslash \{u_0\} \to K_p \backslash \{u_0\}$ that makes $K_p \backslash \{u_0\}$ opposable is given by

$$f(u_{i_1,i_2,\ldots,i_k}) = u_{i_1',i_2,\ldots,i_k}, \tag{4.1}$$

where $i_1' = 1$ if $i_1 = 2$ and $i_1' = 2$ if $i_1 = 1$. Note that there are more automorphisms that make $K_p \backslash \{u_0\}$ opposable. For instance, if f_1 is an automorphism making $K_p \backslash \{u_0\}$ opposable, then it can satisfy

$$f_1(u_{1,1}) = u_{2,2}, \quad f_1(u_{1,2}) = u_{2,3}, \quad f_1(u_{1,3}) = u_{2,1},$$

and it must satisfy that $f_1(u)$ is a vertex of level $k \geq 1$ for any vertex u of level k, and that $f_1(u) \in L_1$ for any $u \in L_2$ and $f_1(u) \in L_2$ for any $u \in L_1$, where L_i is a subgraph of K_p with root u_i such that $L_i \cong T_{3,p-1}$ for $i = 1, 2$.

(2) Let $I(i)$ (or $II(i)$) denote the vertex that the first (or second) player claims on the ith round. Then,

$$I(1) = u_0, \qquad II(1) = u_{1,1},$$

so that u_1 becomes piebald. If the first player does not employ the ordinary opposable method, then

$$I(2) \neq u_{2,j}, \quad j = 1, 2, 3. \tag{4.2}$$

In this case, the basic ideas are that, if possible, the second player employs the ordinary opposable method and that he makes vertices reserved for the first player piebald. Given the condition (4.2), we consider some cases below.

If $I(2) = u_{1,2}$ or $u_{1,3}$, then $II(2) = u_{2,1}$ to make u_2 piebald. Then, in order for the first player to win this game, he must follow the ordinary opposable method to claim $I(3) = u_{2,2}$ or $u_{2,3}$, since otherwise the second player uses that method. Thus all the vertices are exhausted.

If $I(2)$ is a vertex of level k with $3 \leq k \leq p-1$, then $II(2) = f(I(2))$, where f is an automorphism defined by (4.1). Note that $I(2)$ must not be a vertex of level p since it is better to claim a vertex of level $p-1$ to make three vertices of level p absolutely reserved for the first player.

If $I(2) = u_2$, then first we consider the graph K_3. The procedure looks as follows.

$$I(1) = u_0, \quad II(1) = u_{1,1}, \qquad I(2) = u_2, \quad II(2) = u_{1,2},$$

$$I(3) = u_{1,3}, \quad II(3) = u_{2,1,1}, \qquad I(4) = u_{2,3}, \qquad II(4) = u_{2,2,1}.$$

Note that the vertices $u_{2,1}$ and $u_{2,2}$ are piebald. At this moment, six vertices of level 3 are absolutely reserved for the first player and also for the second player, and four vertices of level 3 are available for both players. Since the first player starts the rest of the game, the second player wins and all the vertices are exhausted. Now in K_p for $p \geq 3$, if we proceed as above up to 4th round, then the rest of the graph is opposable and the first player starts the game. Thus we conclude that the second player wins and all the vertices are exhausted. □

Remark 4.15. (1) One important consequence of the above lemma is that the first player can win the game K_p for $p \geq 2$, but he cannot leave any vertices absolutely reserved for him at the end of the game since all the vertices are used.

(2) The first player must take the root at his first move. This is simply because if he does so, then he wins. Both players must do their best to win the game. If the first player claims other than the root on his first move, $u_{1,1}$ say. Then, the second player uses the opposable method, so that he claims $u_{2,1}$. If they continue, then eventually the second player wins and this is against the idea of "perfect play" (cf. Definition 1.2 in Section 1.1).

Now we can state and prove our main theorem of this chapter.

Theorem 4.16. *Any complete ternary tree is a first player win.*

Proof. The cases where $p = 1, 2$ are shown in Example 4.10.

Let $p \geq 3$. First we divide $T_{3,p}$ into p subgraphs $G_{1,1,\dots,1}$, $K_2, \dots, K_{p-1}$, K_p with the roots $v_{1,1,\dots,1}$ (of level $p-1$), $v_{1,\dots,1}, \dots, v_1$, v_0, respectively, where we ignore the edges (v_0, v_1), $(v_1, v_{1,1})$, etc and $G_{1,1,\dots,1} \cong T_{3,1}$. The first player begins the game by claiming the vertex $v_{1,1,\dots,1}$ of level $p-1$, leaving the vertex $v_{1,\dots,1}$ of level $p-2$ reserved for him and three vertices of level p absolutely reserved for him.

Suppose that the second player claims a vertex $u \in K_i$.

If u is the root of K_i, then by Lemma 4.14 the second player wins the game K_i and all the vertices of K_i are exhausted. And then the first player starts a new game on K_j $(j \neq i)$ by claiming its root to win that game. If he cannot claim the root, then he claims any vertex of level 2, loses the game on K_j and starts a new game.

If u is not the root of K_i, then the first player claims $f(u) \in K_i$, where f is an automorphism of $K_i \backslash \{\text{the root}\}$ that makes it opposable. If this procedure continues, then regardless of the winner all the vertices of K_i are exhausted by Lemma 4.14. If the second player loses the game, then he will start a new game on K_j $(j \neq i)$.

Eventually, two players play on each K_i for $i = 2, 3, \ldots, p$ and all the vertices in these subgraphs are exhausted. Now there are three vertices absolutely reserved for the first player and hence he wins the whole game $T_{3,p}$. □

So far we have considered the complete ternary trees $T_{3,p}$ and showed that they are first player wins. For an odd integer $n > 3$ it is conjectured that any n-nary tree $T_{n,p}$ is a first player win. For instance, a method applied to $T_{3,p}$ might work for $T_{5,p}$ and this will be a future study.

4.3 Incomplete Trees

We have been playing only on a complete n-nary tree to investigate nonopposable graphs. We shall extend to inspect other types of trees that are not complete n-nary trees in this section. However, because there are infinitely many types of subtrees of complete n-nary trees, we briefly explain the wining strategies for the first player in simple such graphs.

Example 4.17. We consider a simple subtree of a complete ternary tree given in Figure 4.9.

The first player has two choices: taking the root of graph or trying to save vertices as absolutely reserved. We will examine the strategy that the first player chooses the root first.

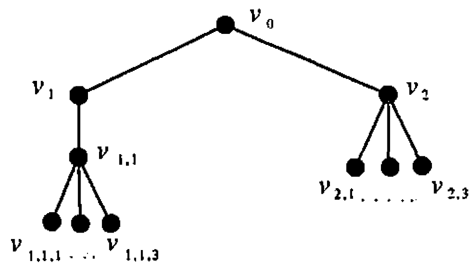


Figure 4.9: Example **(1)** of a subtree of complete ternary tree

Strategy (1). Suppose the first player claims the root v_0. This makes that v_1 and v_2 are reserved for the first player (Figure 4.10). Recall that these vertices are marked by big black dots.

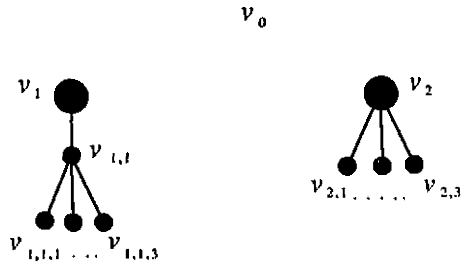


Figure 4.10: Example **(1)**, strategy (1) after first move

The second player claims $v_{1,1}$, so that $v_{1,1,i}$, $i = 1, 2, 3$ are absolutely reserved for the second player, and v_1 becomes piebald (Figure 4.11).

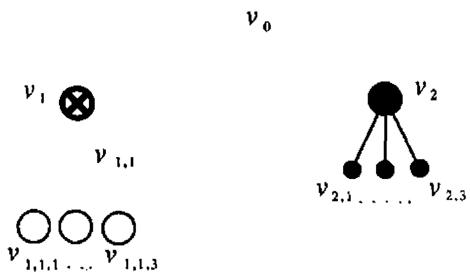


Figure 4.11: Example **(1)**, strategy (1) after second move

The first player claims v_2 in his turn and $v_{2,i}$, $i = 1, 2, 3$ are absolutely reserved for the first player (Figure 4.12).

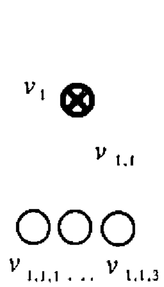


Figure 4.12: Example (1), strategy (1) after third move

We can see that the first player wins since there are the same number of absolutely reserved vertices for each player, but the second player starts the rest of the game.

Though we find a winning strategy for the first player of the game given in Figure 4.9, we consider another strategy to figure out general winning strategies.

Strategy (2). Suppose the first player claims $v_{1,1}$ to make $v_{1,1,i}$, $i = 1, 2, 3$ absolutely reserved and v_1 reserved for him (Figure 4.13).

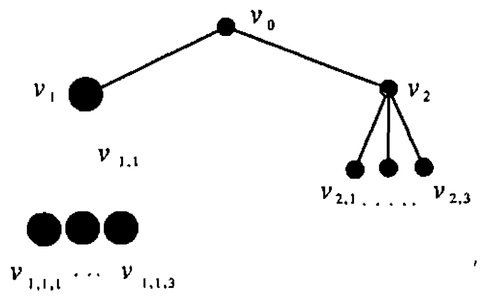


Figure 4.13: Example (1), strategy (2) after first move

The Second player claims v_2, so that $v_{2,i}$, $i = 1, 2, 3$ are absolutely reserved and also v_0 reserved for him (Figure 4.14).

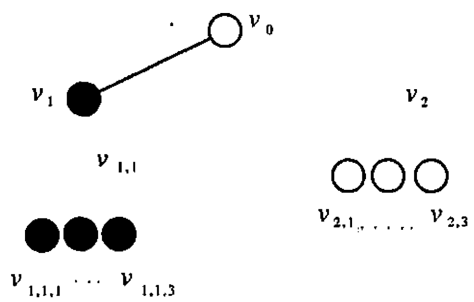


Figure 4.14: Example **(1)**, strategy (2) after second move

The first player claims v_1 to make the vertex v_0 piebald and he wins (Figure 4.15).

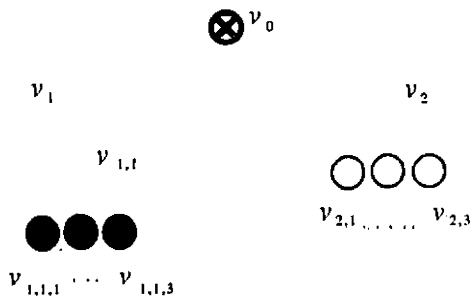


Figure 4.15: Example **(1)**, strategy (2) after third move

Example 4.18. Next we consider the graph where four more vertices $(v_{1,2}, v_{1,2,i}, i = 1,2,3)$ are added to the graph in Figure 4.9 (Figure 4.16).

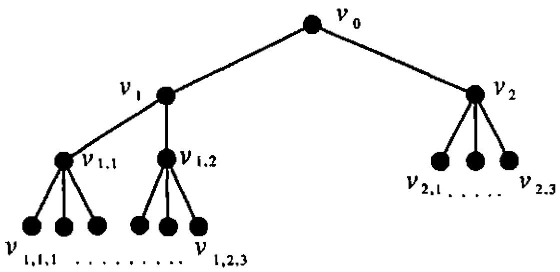


Figure 4.16: Example **(2)** of a subtree of complete ternary tree

Strategy (1). Suppose, on the above graph, the first player claims $v_{1,1}$ to save vertices as absolutely reserved for him. The three vertices $v_{1,1,i}$, $i = 1,2,3$ are absolutely reserved for the first player and the vertex v_1 is also reserved for him (Figure 4.17).

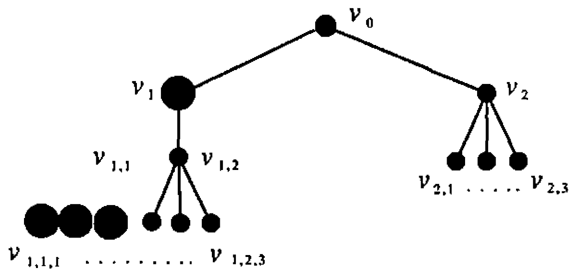


Figure 4.17: Example (**2**), strategy (1) after first move

Then the second player claims v_2. The vertices $v_{2,i}$, $i = 1,2,3$ adjacent to v_2, are then absolutely reserved for the second player, and v_0 becomes reserved for the second player (Figure 4.18).

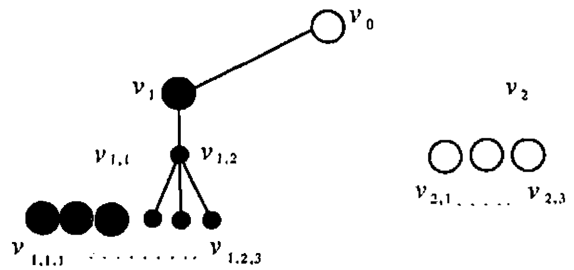


Figure 4.18: Example (**2**), strategy (1) after second move

Now we can see that the first player wins since he claims $v_{1,2}$ and he has six vertices $v_{1,1,i}$ and $v_{1,2,i}$, $i = 1,2,3$ absolutely reserved for him with v_1 reserved for him, and three vertices are available only for the second player with v_0 reserved for him.

There is another move for the second player after the first player's move of claiming $v_{1,1}$. The second player claims $v_{1,2}$ making v_1 piebald and $v_{1,2,i}$, $i = 1,2,3$ absolutely reserved for him. Then the first player claims v_2. Now it is obvious that the first player wins.

Strategy (2). In the next strategy on Figure 4.16, suppose the first player claims v_2. The idea of this strategy is similar to that of strategy (1) and is to save vertices as absolutely reserved for the first player. Now the first player has three absolutely reserved vertices $v_{2,i}$, $i = 1, 2, 3$ and one more reserved vertex v_0 (Figure 4.19).

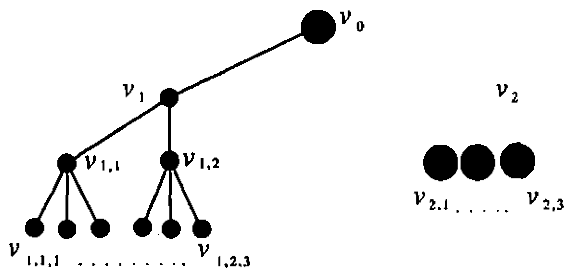


Figure 4.19: Example **(2)**, strategy (2) after first move

In the second player's turn, he claims v_1 and he has two reserved vertices $v_{1,1}$ and $v_{1,2}$. The root v_0 now becomes piebald. (Figure 4.20).

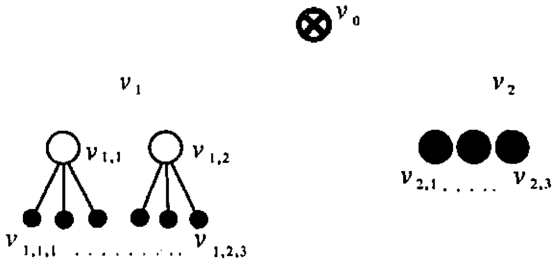


Figure 4.20: Example **(2)**, strategy (2) after second move

The first player can claim any one of vertices from $v_{1,1,i}$ and $v_{1,2,i}$, $i = 1, 2, 3$. For instance, he claims $v_{1,1,1}$ to make $v_{1,1}$ piebald but does not add any reserved vertex in his turn (Figure 4.21).

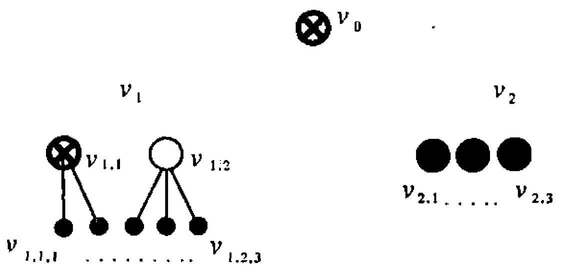


Figure 4.21: Example (2), strategy (2) after third move

The second player claims $v_{1,2}$ and he add three vertices as absolutely reserved. (Figure 4.22).

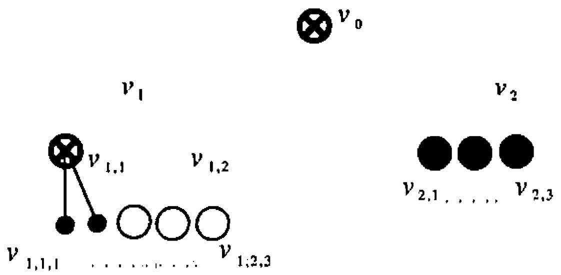


Figure 4.22: Example (2), strategy (2) after fourth move

Now we realize that the first player is going to lose this game since there are two vertices available for two players and each player has equally three vertices absolutely reserved.

From the above example we learn that the first player needs to choose the vertex carefully to save vertices as absolutely reserved for his own. We use another idea. We choose a vertex (v_1 in this example) that reserves vertices for which if we claim them then we can reserve some vertices absolutely. We show this method in strategy (3) below.

Strategy (3). Suppose the first player claims v_1 to make v_0 and $v_{1,j}$, $j = 1, 2$ reserved for him (Figure 4.23).

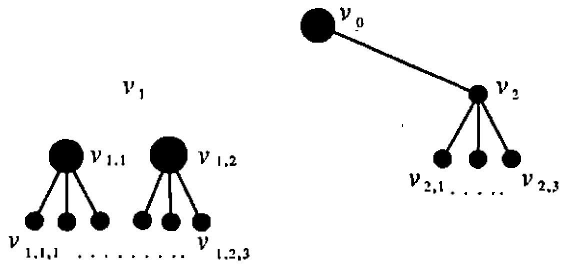


Figure 4.23: Example **(2)**, strategy (3) after first move

The second player claims v_2, which makes that $v_{2,i}$ $i = 1, 2, 3$ are absolutory reserved for him and v_0 is piebald (Figure 4.24).

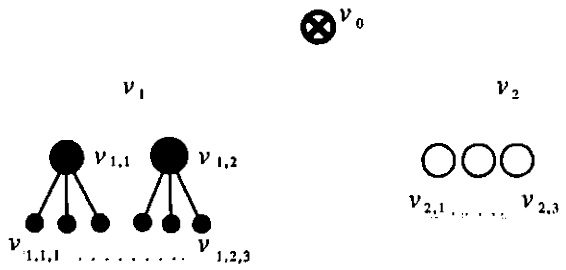


Figure 4.24: Example **(2)**, strategy (3) after second move

Next, the first player claims $v_{1,1}$ to have three absolutely reserved vertices for him (Figure 4.25).

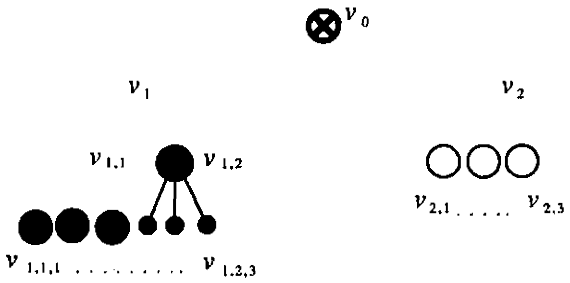


Figure 4.25: Example **(2)**, strategy (3) after third move

The second player claims one of $v_{1,2,i}$, $i = 1, 2, 3$, and makes $v_{1,2}$ piebald (Figure 4.26).

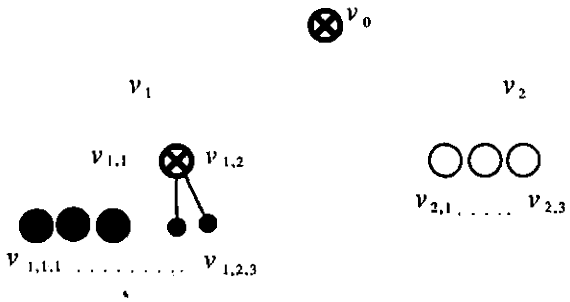


Figure 4.26: Example (2), strategy (3) after fourth move

We then predict that the first player loses this game since there are two vertices available for both players, three vertices absolutely reserved for each player, and next turn is the first player.

Example 4.19. We play one more game to conclude the suggestions that help the first player to win (Figure 4.27). We will examine three strategies in this graph.

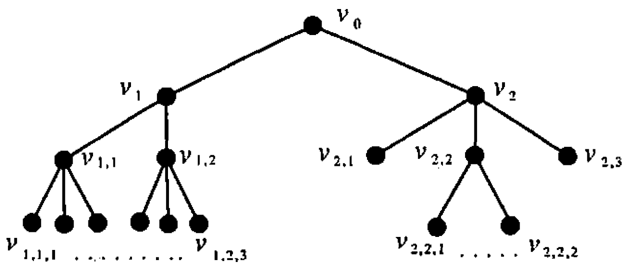


Figure 4.27: Example (3) of a subtree of complete ternary tree

Strategy (1). Suppose the first player claims v_1, which makes three vertices $v_{1,j}$, $j = 1, 2$ and v_0 reserved for him. Especially, $v_{1,j}$, $j = 1, 2$ are going to make six vertices as absolutely reserved for him if claimed (Figure 4.28).

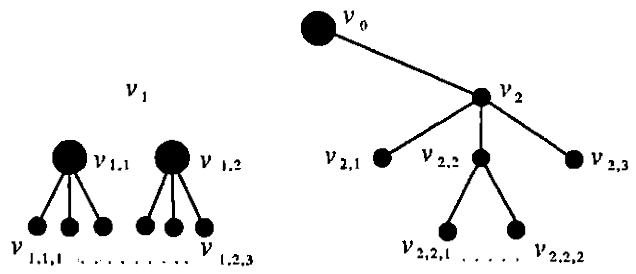


Figure 4.28: Example **(3)**, strategy (1) after first move

The second player claims v_2. It creates two absolutely reserved vertices and one vertex that leads to make two absolutely reserved vertices for him if it is claimed. Also v_0 is piebald now (Figure 4.29).

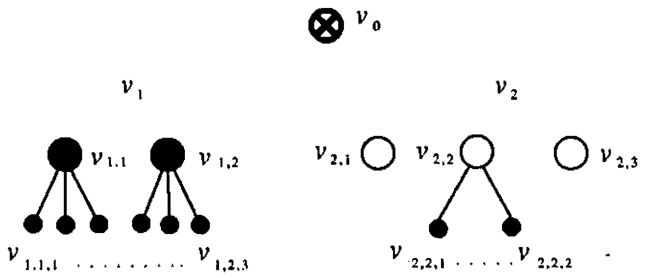


Figure 4.29: Example **(3)**, strategy (1) after second move

The first player claims $v_{1,1}$ and has three absolutely reserved vertices (Figure 4.30).

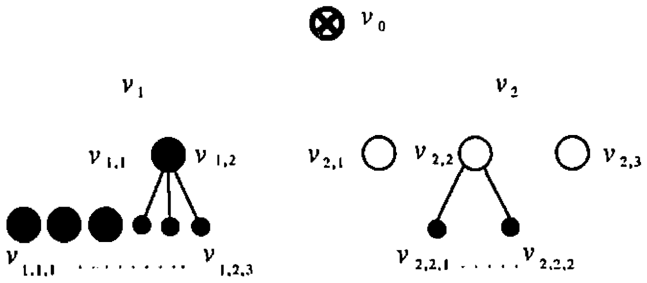


Figure 4.30: Example **(3)**, strategy (1) after third move

The second player has two choices, one of $v_{1,2,i}$, $i = 1,2,3$ or $v_{2,2}$ to claim. At this time, the second player claims $v_{1,2,1}$ and the vertex $v_{1,2}$ became piebald (Figure 4.31).

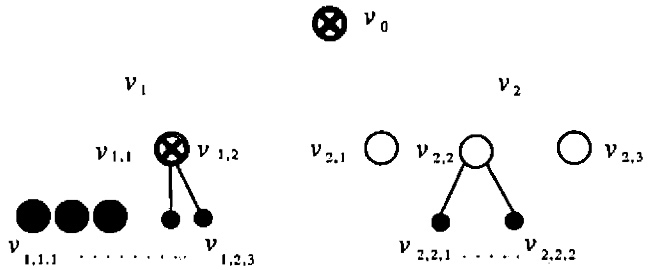


Figure 4.31: Example **(3)**, strategy (1) after fourth move

In the first player's turn, he also claims $v_{2,2,1}$ to make $v_{2,2}$ piebald (Figure 4.32).

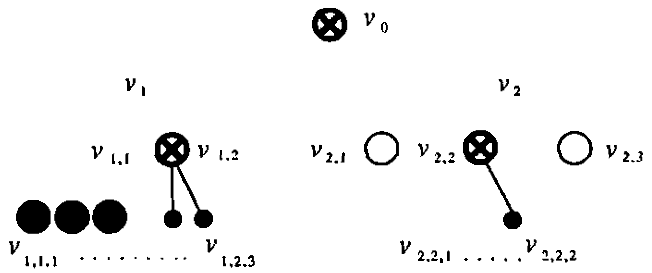


Figure 4.32: Example **(3)**, strategy (1) after fifth move

Now we can judge that the first player wins since there are three vertices available for both players, three only for the first, and two only for the second (Figure 4.32). Even though the second player claims $v_{2,2}$ in his second turn, the result is same as when he claims $v_{1,2,1}$.

Strategy (2). In this strategy, suppose the first player claims $v_{1,1}$ to maintain three absolutely reserved vertices $v_{1,1,i}$, $i = 1,2,3$ and one reserved vertex v_1 (Figure 4.33).

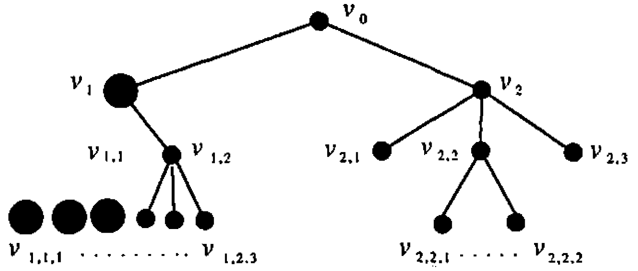


Figure 4.33: Example (3), strategy (2) after first move

Now the second player has three vertices to claim: v_2, $v_{1,2}$ and $v_{2,2}$. If the second player claims $v_{1,2}$, then the first player claims v_2, and if the second player claims $v_{2,2}$, then the first player claims $v_{1,2}$. In either case, it is easy to see that the first player wins. We show the case where the second player claims v_2 in detail below.

Suppose that the second player claims v_2 to have two absolutely reserved vertices $v_{2,1}$, and $v_{2,3}$. He also has two more reserved vertices v_0 and $v_{2,2}$ (Figure 4.34).

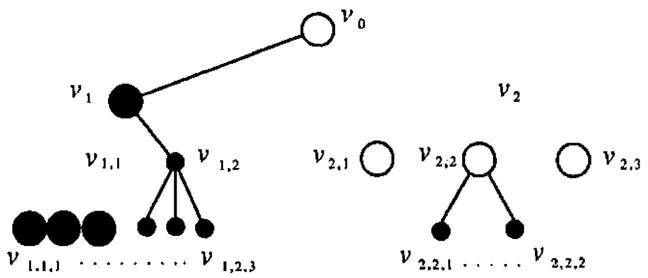


Figure 4.34: Example (3), strategy (2) after second move

In the first player's turn, he claims $v_{1,2}$. Now he has six absolutely reserved vertices and one reserved vertex that connects with the vertex v_0 that is reserved for the second player (Figure 4.35).

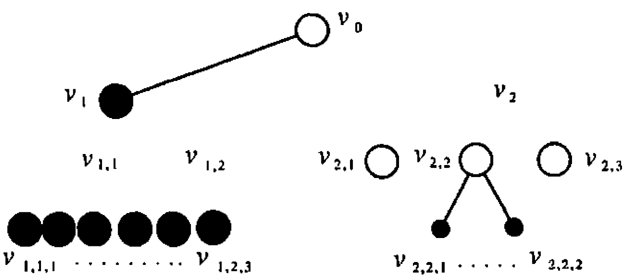


Figure 4.35: Example (**3**), strategy (2) after third move

In the second player's turn, he claims $v_{2,2}$ (Figure 3.36).

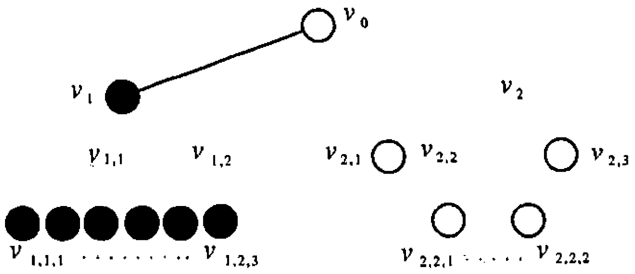


Figure 4.36: Example (**3**), strategy (2) after fourth move

Now the second player has four absolutely reserved vertices and the first player has six absolutely reserved vertices, so the first player wins.

Strategy (3). Here, suppose that the first player starts by claiming v_2 to make $v_{2,1}$ and $v_{2,3}$ absolutely reserved and v_0 and $v_{2,2}$ reserved for him (Figure 4.37).

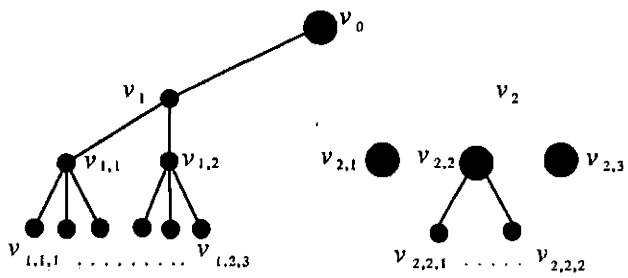


Figure 4.37: Example (**3**), strategy (3) after first move

Then the second player claims $v_{1,1}$, so that vertices $v_{1,1,i}$ for $i = 1, 2, 3$ are absolutely reserved for him and v_1 reserved for him (Figure 4.38).

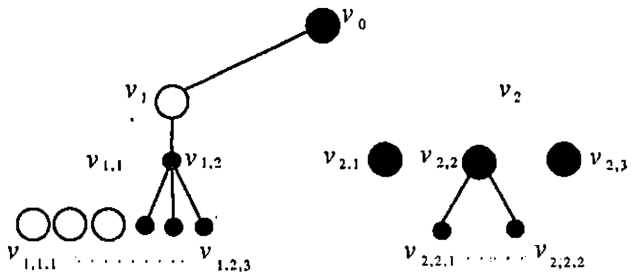


Figure 4.38: Example (**3**), strategy (3) after second move

Now it is clear that the first player wins if he claims $v_{1,2}$ (Figure 4.39).

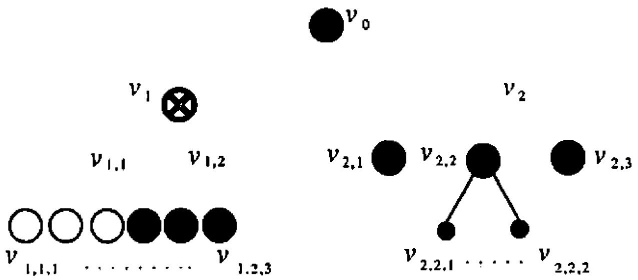


Figure 4.39: Example (**3**), strategy (3) after third move

At an end of this section, we restate and modify the clues that we previously suggested right before Example 4.11 for the first player to win.

1. Try to save vertices as absolutely reserved for one's own.

2. Try to claim a vertex of highest degree.

3. Decompose the graph G_0 into some subgraphs in each of which we can easily set up the strategy.

4. Try to claim a vertex such that, if it is claimed, then it reserves vertices that make some absolutely reserved vertices if they are claimed.

5. Play on opposable graphs as a second player if it is possible.

6. On each subgraph the first player should make all vertices exhausted before finally he returns to the vertices of absolutely reserved for him.

Chapter 5

Snort and Values

So far we have talked about Snort on simple graphs and the opposability of a game which gives the second player a strategy to win. However, if we play it on a larger or more complicated graph, is there any convenient way to predict who can win the game? J. H. Conway introduces the theory of number in a game in his early book [Con76]. He instructs us to decompose a game into sub-games and to find the value of each sub-game or region. This theory is based on four ideas. The first is that it is easy to think of an immediate strategy on a small region which does not affect the other regions. The second is that the worth of a strategy is determined by the condition after a player applies his strategy. The third is that we predict who wins the game by collecting those strategies as the game proceeds. The forth is that those strategies can be converted to numbers by certain rules. Note that numbers are easily handled. In this chapter, first we explain how to calculate the values of sub-games of the Domineering game. Then we move on to Snort games on simple graphs and evaluate the values of some games of Snort to see who wins.

5.1 Values in Simple Games

Since we used the terminology of "position" in item 5 of Definition 1.1 in Section 1.1, we shall use positions instead of regions. The value of a game is calculated based on how much worth each position has for each player. Both players may have one or more strategies to play on the position. After a player chooses a movement, there arises a new position. We define the value of a position by judging from the new position. That is, we count how much the movement contributed to the game for the player after choosing his strategy. To illustrate this idea we use the Domineering game that is also one of the combinatorial games. We first explain how to play the Domineering game below.

1. A chessboard and 1×2 dominos are used for Domineering.

2. Two players, Verty and Horry, play alternately. Verty places a domino vertically on two adjacent squares. Horry places a domino horizontally on two adjacent squares. Their dominos cannot be overlapped.

3. They place their dominos alternately until one of the players cannot put a domino in his turn.

4. The winner is the one who places the domino in the last move.

While a Domineering game is played, two players distinguish some small positions that do not connect each other. That is, in each small position two players can play the game, not affecting other regions. When a player defines the value of each position, he can make the list of values. So he calculates the sum of those values and he can predict the result of the game.

In the following figure we depict an example of Domineering, where we intentionally make a 5×5 chessboard decomposed into some positions to find their values. We can think that after some moves we come up with the following position consisting of eight positions.

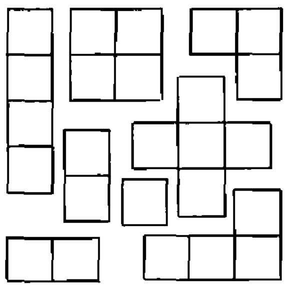


Figure 5.1: Decomposition of chessboard

There are eight basic positions that are dragged from the above chessboard (Figure 5.2).

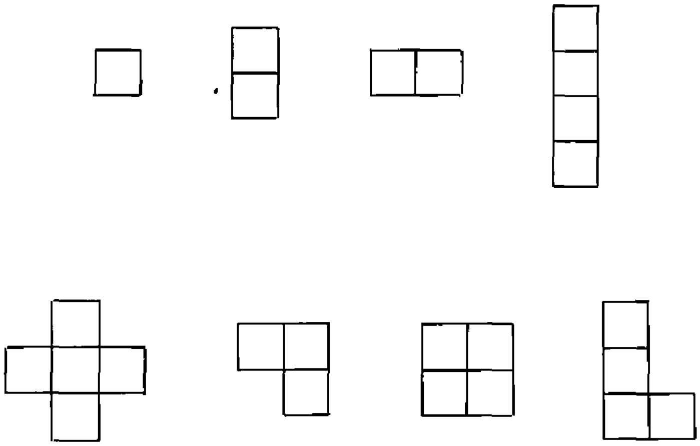


Figure 5.2: Eight positions

We will illustrate how to determine the value of each position using the bracket notation. In each game position G, two players have zero or more strategies to claim on their turns. We display all possible choices of movement inside of the brackets. Verty has choices of V_i, $i = 1, ..., n$, which are displayed on the left side of the brackets and Horry has choices of H_j, $j = 1, ..., m$, which are displayed on the right side of the brackets, where they are separated by a vertical bar. Then a typical position is denoted by $G = \{V_1, \ldots, V_n \,|\, H_1, \ldots, H_m\}$. For example, in Figure 5.3, there are three choices V_1, V_2 and V_3 for Verty and two choices H_1 and H_2 for Horry.

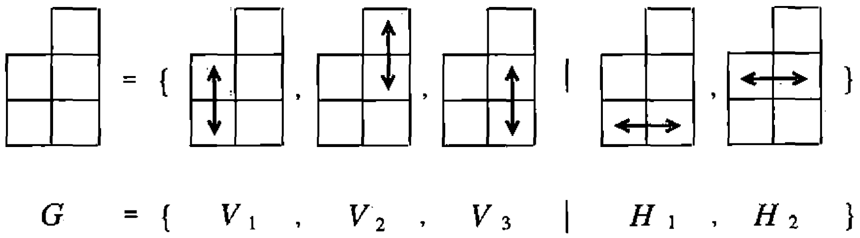


Figure 5.3: The choices of movement

When we describe a position, we show only the positions that are obtained after their movements. Here, squares where dominos are placed are removed from the positions each time. We also use $\emptyset$ to indicate that there is no legal movement for the player. Hence, the above G is expressed as

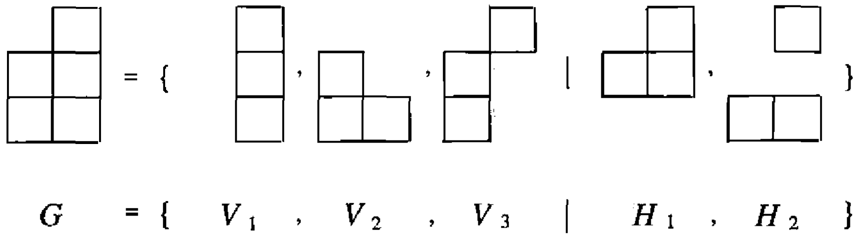


Figure 5.4: The positions after the movement

Now we define the value of positions. This is done inductively. First we define the value of simple positions. Then using them we define the value for more complicated ones. For instance, the value of a position $G = \{V_1, V_2, V_3 \,|\, H_1, H_2\}$ is defined using the values of positions V_i 's and H_j 's. Another thing is that we replace V_i 's and H_j 's by their values, so that we are identifying the positions with the numbers. Basically we use the nonnegative numbers for the values of Verty's positions V_j 's and the nonpositive numbers for Horry's positions H_j 's. For example, we may have $G = \{0, 1, 2 \,|\, -1, 0\}$, where $0, 1, 2$ are values of V_1, V_2, V_3, respectively, and $-1, 0$ are those of H_1, H_2, respectively. We shall figure out how to proceed more to obtain the value of G. In addition to numbers, there are values $\emptyset$ and $*$ for some positions, which will be explained later. Let us start to calculate the value of positions in the following.

The simplest position of a game is of value 0. This is an *endgame* where there are no legal moves for either player, so that the position for each player is denoted by $\emptyset$ as before and we indicate the value of the position $\emptyset$ by the same symbol $\emptyset$. We define this position to be of value 0 and call it the position 0 (Figure 5.5).

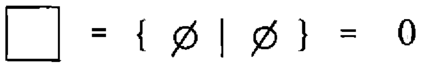
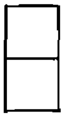


Figure 5.5: Position of value 0

The next position is of value 1 where Verty has an advantage of one movement, but Horry has no legal movement (Figure 5.6). Here, we replaced the position after Verty's move by the number 0. This whole position is called 1.



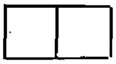
$$= \{\, 0 \mid \emptyset \,\} = 1$$

Figure 5.6: Position of value 1

We extend the idea of this example as a rule: if there is a nonnegative number n for Verty and $\emptyset$ for Horry, then the value of the position is $n+1$, i.e.,

$$\{n \mid \emptyset\} = n + 1.$$

Similarly, if Horry has an advantage of one movement and Verty has no legal movement, the position is of value -1 (Figure 5.7).



$$= \{\, \emptyset \mid 0 \,\} = -1$$

Figure 5.7: Position of value -1

This example can be generalized to a rule: if there is a nonpositive number n for Horry and $\emptyset$ for Verty, then the value of the position is $n-1$, i.e.,

$$\{\emptyset \mid n\} = n - 1.$$

In Figure 5.8 below, there are two choices for Verty, 0 and 1. In order to play better, Verty would choose the value 1. Also since there is no legal movement for Horry, the value of this position is $\{1 \mid \emptyset\} = 1 + 1 = 2$.

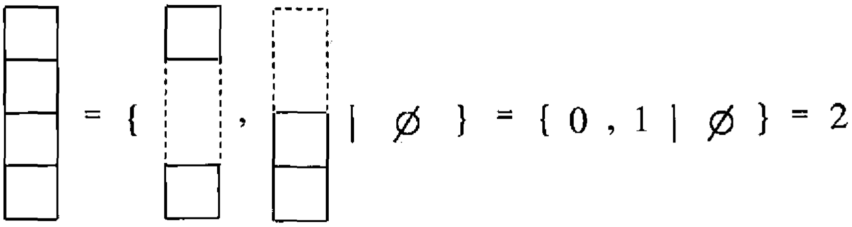


Figure 5.8: Position of value 2

It follows from this example that if we have a position $G = \{a_1, \ldots, a_n \,|\, b_1, \ldots, b_m\}$, then Verty would choose the largest number among $a_1, \ldots, a_n$ to play best. That is, for Verty the value is the larger, the better.

In Figure 5.9, there are two choices for Horry, 0 and -1. In order to play better Horry would choose the value -1. Also since there is no legal movement for Verty, the value of this position is $\{\emptyset \,|\, 0, -1\} = (-1) + (-1) = -2$.

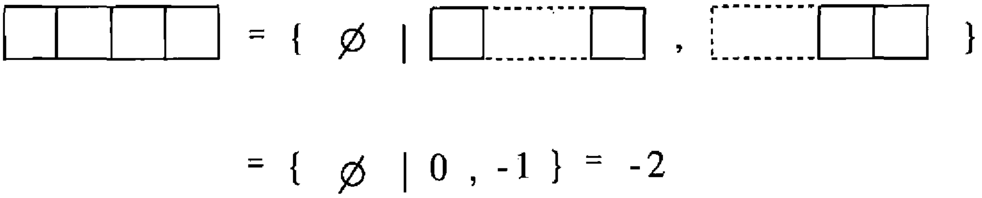


Figure 5.9: Position of value -2

It follows from this example that if we have a position $G = \{a_1, \ldots, a_n \,|\, b_1, \ldots, b_m\}$, then Horry would choose the smallest number among $b_1, \ldots, b_m$ to play best. Hence, we have that

$$G = \{a_1, \ldots, a_n \,|\, b_1, \ldots, b_m\} = \{a_i \,|\, b_j\},$$

$$\text{where } a_i = \max\{a_1, \ldots, a_n\} \text{ and } b_j = \min\{b_1, \ldots, b_m\}.$$

If a position of a game becomes an endgame after the first player's move, the value of the position is $\{0|0\}$. Recall that a position is said to be an *endgame* if it has no legal move for both players, and we say that the position has value 0 in this case. We give simple examples in Figures 5.10 and 5.11. In this situation, we use $*$ to indicate the value of the position and call this position *star*. Note that in a star position the first player always wins.

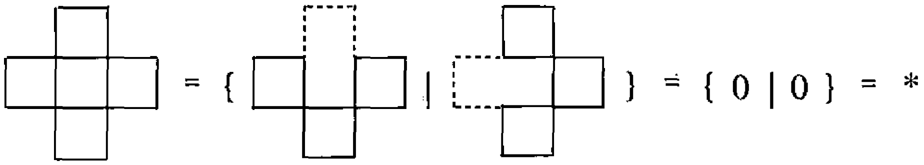


Figure 5.10: Position of value $*$

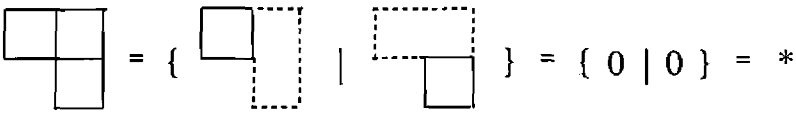


Figure 5.11: Position of value $*$

The value of Figure 5.12 is $\{1 \,|\, -1\}$ because Verty and Horry both have one advantage after their own movements. We cannot reduce this value to a single number. Although this position is a first player win, the value of this position is not $*$ since after each player's move there still is a legal move for one of the players. Also we'll use the value $\{1 \,|\, -1\}$ to calculate the value of the whole game.

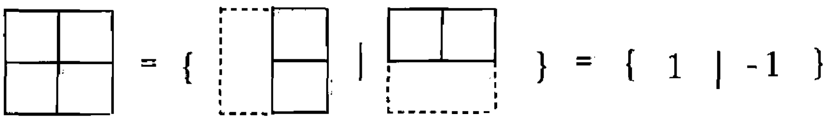
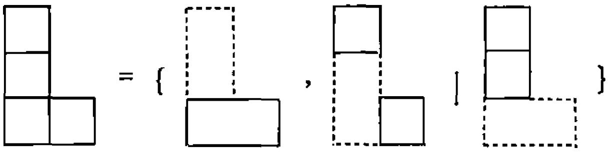


Figure 5.12: Position of value $\{1 \,|\, -1\}$

Next we need some experiments to understand the value of a position in Figure 5.13 below.



$$= \{ -1 , 0 \mid 1 \} = \{ 0 \mid 1 \}$$

Figure 5.13: Position for experiment

Before we proceed, we give two notions. First, the *union* of two positions G_1 and G_2 is defined to be their disjoint union, where we consider $G_1 \cap G_2 = \emptyset$, an empty set (Figure 5.14).

Second, we consider a position of "even-chance." In Figure 5.14, we have a union of two positions with values 1 and -1, respectively. It is evident that this is a second player win. Hence, we regard that the advantage of the left position for Verty is as same as that of the right position for Horry.

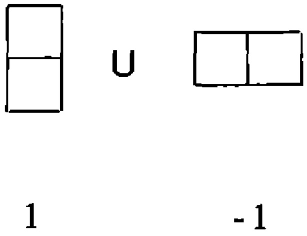


Figure 5.14: Example of even-chance

In general, if a union of game positions is a second player win, then Verty and Horry are considered to have the same advantage for winning and the union of these positions is called an *even-chance game*.

Now we go back to Figure 5.13.

Experiment (1). Estimating the value of the position in Figure 5.13.

We estimate the value of this game as 1 since Verty can win in both ways that he plays first or second and we feel that Verty has a positive advantage.

Experiment (2). Making sure of Verty's advantage.

When we set the game positions with Verty's advantage (Figure 5.13) and Horry's advantage (Figure 5.7) together in one game, the advantage of them should be equal (Figure 5.15). It should make them to have *Even-Chance*.

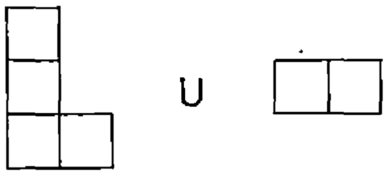


Figure 5.15: Equal advantage?

Here, we have defined a "union" of two positions, denoted as above, to be the disjoint union of two positions. When it is one player's turn, he can make a single move in a summand of his choice. As usual, the last person to move wins. Similarly, we can define a union of three or more positions to be their disjoint union. Now we see that Horry always wins in the game of Figure 5.15 no matter who starts. This results means the position of Figure 5.13 does not have the value 1.

Experiment (3). Estimating the value of the position of Figure 5.13 to be 1/2.

In the next experiment, we estimate that its value is 1/2 and so we add one copy of the position in Figure 5.13 to Figure 5.15 (Figure 5.16).

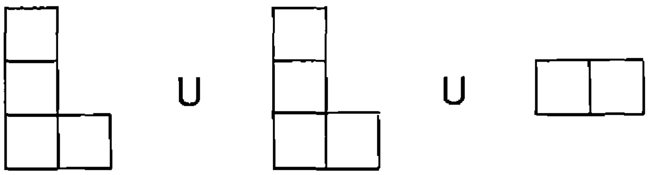


Figure 5.16: Is the position of value 1/2 ? (1)

When Verty is the first player, Horry wins (Figure 5.17).

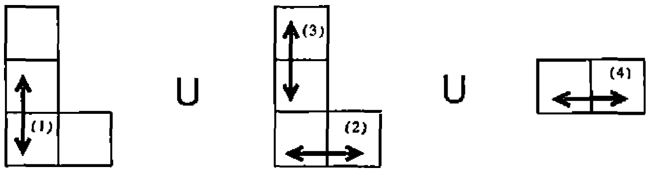


Figure 5.17: Is the position of value 1/2 ? (2)

When Horry is the first player, Verty wins (Figure 5.18).

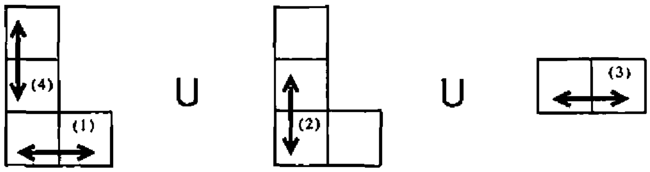


Figure 5.18: Is the position of value 1/2 ? (3)

Now we conclude that two positions in Figure 5.13 have the same advantage as the position in Figure 5.7. This result implies that the value of Figure 5.13 is 1/2.

To conclude this section we summarize the description of positions and basic rules and formulas for values of positions in the following.

1. A position is described by those after one move of each player. For example, let G be a position. Assume that there are three strategies for Verty and two for Horry. Then, after one move of Verty there are resulting positions V_1, V_2 and V_3, and after one move of Horry there are resulting positions H_1 and H_2. Then we denote the position G as
$$G = \{V_1, V_2, V_3 \mid H_1, H_2\}. \tag{5.1}$$

2. The value of a position is defined inductively from the simplest ones to more complicated ones.

3. Once the value of a position is determined, we replace the position by its value. For example, if a_i is a value of V_i and b_j is a value of H_j in (5.1), then we write
$$G = \{a_1, a_2, a_3 \mid b_1, b_2\}.$$

4. The value of a position measures the advantage of each player.

5. Basically, we use nonnegative numbers for the value of Verty and nonpositive numbers for the value of Horry. For example, we have a value of a certain position such as $\{1 \mid -1\}$. In this position, Verty has an advantage of one movement after his move and so does Horry. Note that 0 is the value of a position for a player that after one move of the player there is no legal move for him.

6. The symbol $\emptyset$ indicates the value of a position for a player if that player has no legal move in the position at the beginning.

7. The value 0 is given to a position that has no legal move for both players. That is,

$$\{\emptyset \mid \emptyset\} = 0.$$

8. The symbol $*$ indicates the value of a position given by $\{0 \mid 0\}$. That is,

$$\{0 \mid 0\} = *.$$

9. If there are several numerical values for a player, we select the value with the largest absolute value. For example, we have

$$\{1, 2, 3 \mid 0, -1\} = \{3 \mid -1\}.$$

10. If there is a nonnegative number n for Verty and $\emptyset$ for Horry, then the value of the position is $n+1$. Similarly, if there is a nonpositive number $-n$ for Horry and $\emptyset$ for Verty, the the value of the position is $-n-1$. That is,

$$\{n \mid \emptyset\} = n+1, \qquad \{\emptyset \mid -n\} = -n-1.$$

11. A union of finite positions $G_1, \ldots, G_n$ is defined to be their disjoint union $G_1 \cup \cdots \cup G_n$.

12. If p and q are nonnegative integers, then we define

$$\frac{2p+1}{2^{q+1}} = \left\{ \frac{p}{2^q} \,\middle|\, \frac{p+1}{2^q} \right\},$$

which can be justified in a same way as Figure 5.13.

5.2 Rules of Values

We have learned how to calculate the value of many positions. In this section we will study how to obtain the total value of a whole game while we are describing some rules as definitions. We use the Domineering game of Figure 5.2 as an example.

In the previous section, we used the symbol $*$, called star, for the value of a position in which after the first player claims a move, the game becomes an endgame and the first player wins. *Fuzzy* for the whole game means that there is always a winning strategy for the first player. Fuzzy is one of the categories that tells whoever has a strategy to win. Here we clearly categorize each game into four *outcome classes* depending on who has a strategy to win (see [Con76]).

1. A game is called *positive* if Verty always wins.

2. A game is called *negative* if Horry always wins.

3. A game is called *zero* if the second player always wins.

4. A game is called *fuzzy* if the first player always wins.

When we consider a sum of game positions $G = G_1 + G_2 + \cdots + G_k$, each G_i is considered independently of others. Now we define a sum of two positions using the value of each position inside the brackets as follows.

Recall that we denoted a game position by $G = \{V_1, \ldots, V_n \mid H_1, \ldots, H_m\}$, and then by $G = \{a_1, \ldots, a_n \mid b_1, \ldots, b_m\}$, where a_i is the value of V_i $(1 \leq i \leq n)$ and b_j is the value of H_j $(1 \leq j \leq m)$.

Definition 5.1. *Let* $G = \{a_1, ..., a_n \mid b_1, ..., b_m\}$ *be a game position. If there is another game position* $K = \{a'_1, \ldots, a'_p \mid b'_1, \ldots, b'_q\}$, *the* sum *of two game positions* $G + K$ *is defined by*

$$G + K = \{a_i + K, G + a'_k, 1 \leq i \leq n, 1 \leq k \leq p$$
$$\mid b_j + K, G + b'_\ell, 1 \leq j \leq m, 1 \leq \ell \leq q\}.$$

Definition 5.2. *it If x, y and z are numbers, and $x \geq y$, then*

$$\{x \mid y\} + z = \{x + z \mid y + z\}.$$

After the sum is calculated, we read values as follows.

1. The left side value in the brackets is the value that Verty plays first.
2. The right side value in the brackets is the value that Horry plays first.

Example 5.3. Let us add two simple games whose values are $\{1 \mid -1\}$ (Figure 5.12) and 1/2 (Figure 5.13). Then we have

$$\{1 \mid -1\} + \frac{1}{2} = \left\{1\frac{1}{2} \mid -\frac{1}{2}\right\}.$$

The left side value in the brackets is positive, so if Verty starts, he can win. The right side value in the brackets is negative, so if Horry starts, he can win (Figure 5.19).

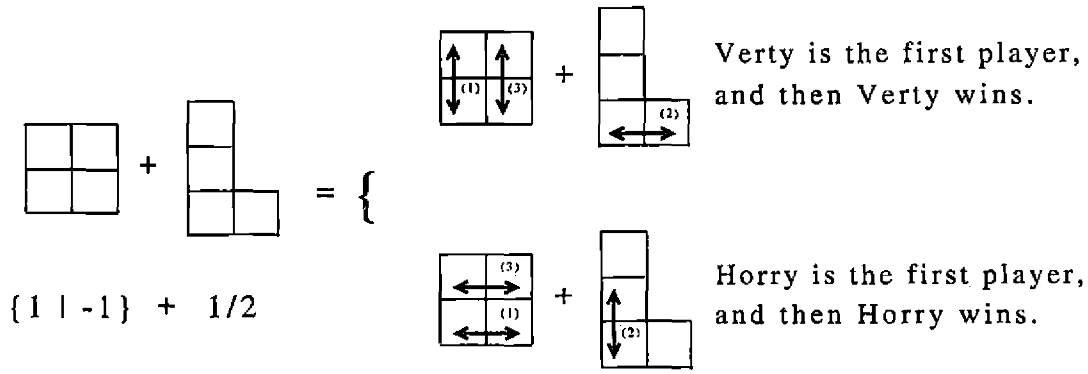


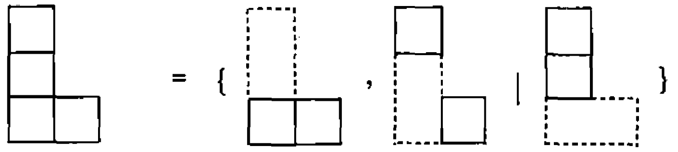
Figure 5.19: Example of $\{x \mid y\} + z = \{x + z \mid y + z\}$

Definition 5.4. *Let $G = \{a_1, \ldots, a_n \mid b_1, \ldots, b_m\}$. Then, the game $-G$ is defined by*

$$-G = \{-b_1, \ldots, -b_m \mid -a_1, \ldots, -a_n\}.$$

This definition is easily understood if we see Figures 5.20 and 5.21 in Example 5.5.

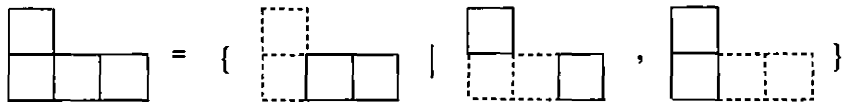
Example 5.5. Let G be the position of Figure 5.20 and see that its value is $\frac{1}{2}$.



$$= \{ -1 , 0 \mid 1 \} = 1/2$$

Figure 5.20: Example of $-G$, part 1

Then $-G = -\frac{1}{2} = \{ -1 \mid 0, 1 \}$ (Figure 5.21).



$$= \{ -1 \mid 0 , 1 \} = -1/2$$

Figure 5.21: Example of $-G$, part 2

Definition 5.6. *If x is a number, then*

$$* + * = 0,$$

$$x + * = \{x \mid x\}.$$

Example 5.7. For the first part of Definition 5.6, we set a sum of two identical positions given in Figure 5.11. Verty as a first player places his domino on one of the positions and that position becomes an endgame. Then, Horry places as a second player his domino on the remaining position and Horry wins. Therefore, the second player always wins (Figure 5.22).

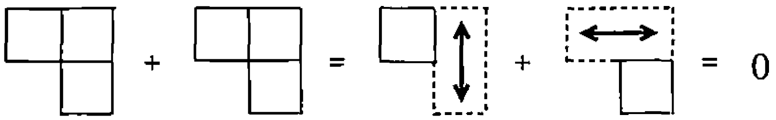


Figure 5.22: Example of $* + * = 0$

For the second part of Definition 5.6, we set a sum of two positions taken from Figure 5.13 and Figure 5.11. The values of the two positions are $\frac{1}{2} = \{0|1\}$ and $* = \{0|0\}$. Verty has two options that are $0 + *$ and $\{\frac{1}{2}\} + 0$ and Horry also has two options that are $1 + *$ and $\{\frac{1}{2}\} + 0$. The best option for Verty is $\{\frac{1}{2}\} + 0$ and the best option for Horry is $\{\frac{1}{2}\} + 0$. Therefore, $\frac{1}{2} + * = \{\frac{1}{2}|\frac{1}{2}\}$.

Finally, we again depict the values of various positions that are obtained in the previous section as follows (Figure 5.23).

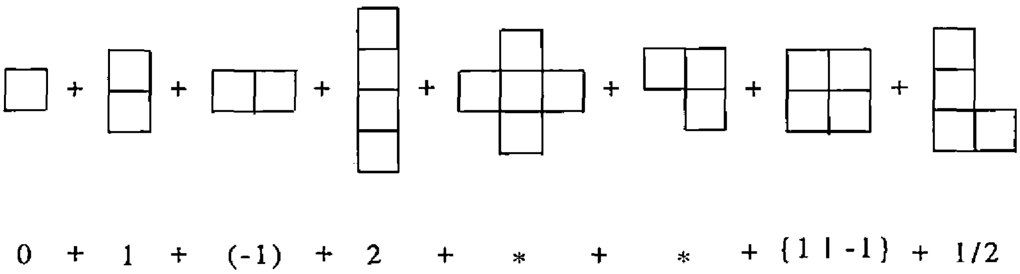


Figure 5.23: Values of positions

We can proceed to calculate the value of the whole game as

$$\begin{aligned}
0+1 \;&+\; (-1)+2+*+*+\frac{1}{2}+\{1|-1\} \\
&= 1+(-1)+2+0+\frac{1}{2}+\{1|-1\}, && \text{since } 0+1=1 \text{ and } *+*=0, \\
&= \{1|-1\} + 2\frac{1}{2}, && \text{since } 1+(-1)+2+0+\frac{1}{2}=2\frac{1}{2}, \\
&= \left\{3\frac{1}{2}\,|\,1\frac{1}{2}\right\}, && \text{by Definition 5.2.}
\end{aligned}$$

Now we predict that Verty can win on the game of Figure 5.1 no matter who starts, since the value of the whole game is positive.

5.3 Values on Snort

We have a foundation of how to calculate the values of each position and also a whole game. In this section, we evaluate the position value of Snort games and display a brief list of them. Before we start, we recall some rules shown in Section 2.1.

1. After each player claims a vertex, the vertex is removed.
2. Any adjacent vertex to the vertex that the first player claimed is reserved for him and shown as a large black dot.
3. Any vertex adjacent to the vertex that the second player claimed is reserved for him and shown as a large white dot.
4. A piebald spot that is reserved for both players is shown as $\otimes$.
5. Any edge connecting two vertices that are reserved for the same player is omitted.

We note some differences between Domineering and Snort. Verty and Horry have one restriction in a Domineering game. That is Verty places his dominos only vertically and Horry places his dominos only horizontally. So we have to consider the shape of regions or positions for Domineering game. In Snort game, there are vertices available to both players and they can claim any one of them. Also there are vertices reserved for one of the players and the opponent cannot claim them.

We show some basic examples below (Figure 5.24).

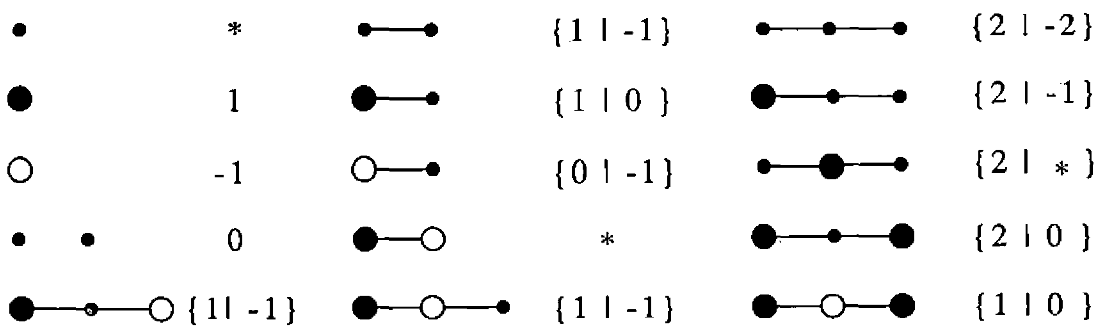


Figure 5.24: Values of Snort games

The value of a Snort game is also considered in a small position or graph. Now we instruct procedures to calculate the value of a graph.

1. List all options of movement on the graph of Snort game for the first player and display them on the left side of brackets.
2. List all options of movement on the graph of Snort game for the second player and display them on the right side of brackets.
3. When we display options, we show only the positions that are obtained after their movements.
4. Evaluate values for each option and display them as numbers inside of the brackets.
5. Select the best value for each player.
6. If it is possible, make them into one value.

Let us try to calculate the value of the graph on Figure 5.25.

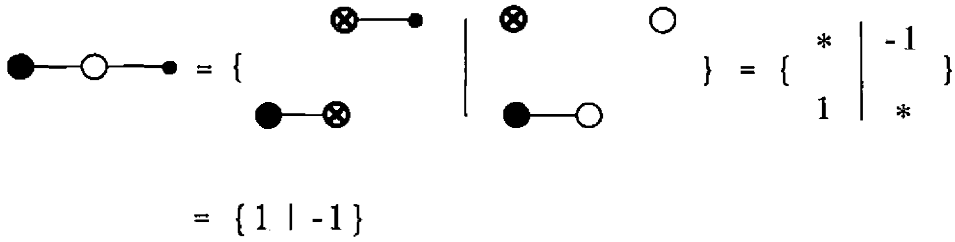


Figure 5.25: Values of Snort game (1)

1. The first player has two options: One is claiming a reserved vertex and the other is claiming the right end vertex.
2. The second player also has two options: One is claiming a reserved vertex and the other is claiming the right end vertex.
3. The values for the first player are $*$ and 1, and values for the second player are -1 and $*$.

4. The best value for the first player is 1 and the best value for the second player is -1.

5. The final value for the graph in Figure 5.25 is $\{1 \mid -1\}$.

We can evaluate the following two examples, Figure 5.26 and Figure 5.27, in a similar fashion as above.

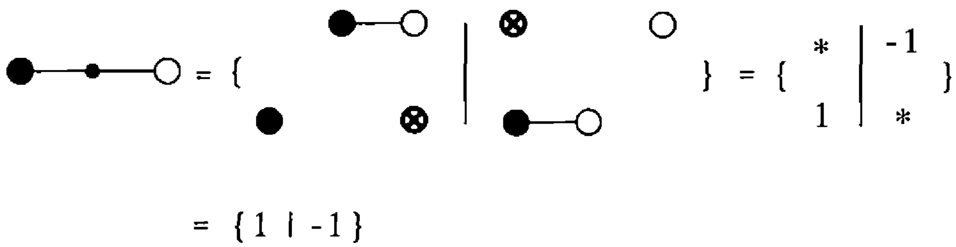


Figure 5.26: Values of Snort game (2)

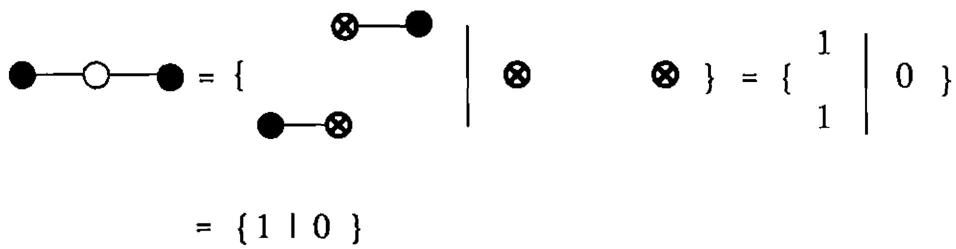


Figure 5.27: Values of Snort game (3)

In the rest of this section we will display a brief list of Snort games that is taken from [BCG82]. Since the values of positions are defined inductively, we use the following notations. If, in a certain position, we have a value $\{2 \mid 1\}$ for Verty and a value $\{-1 \mid -2\}$ for Horry, then we denote the position's value by $\{2 \mid 1 \parallel -1 \mid -2\}$ using a $\parallel$ to separate the values of both players as in the first example in Figure 5.28.

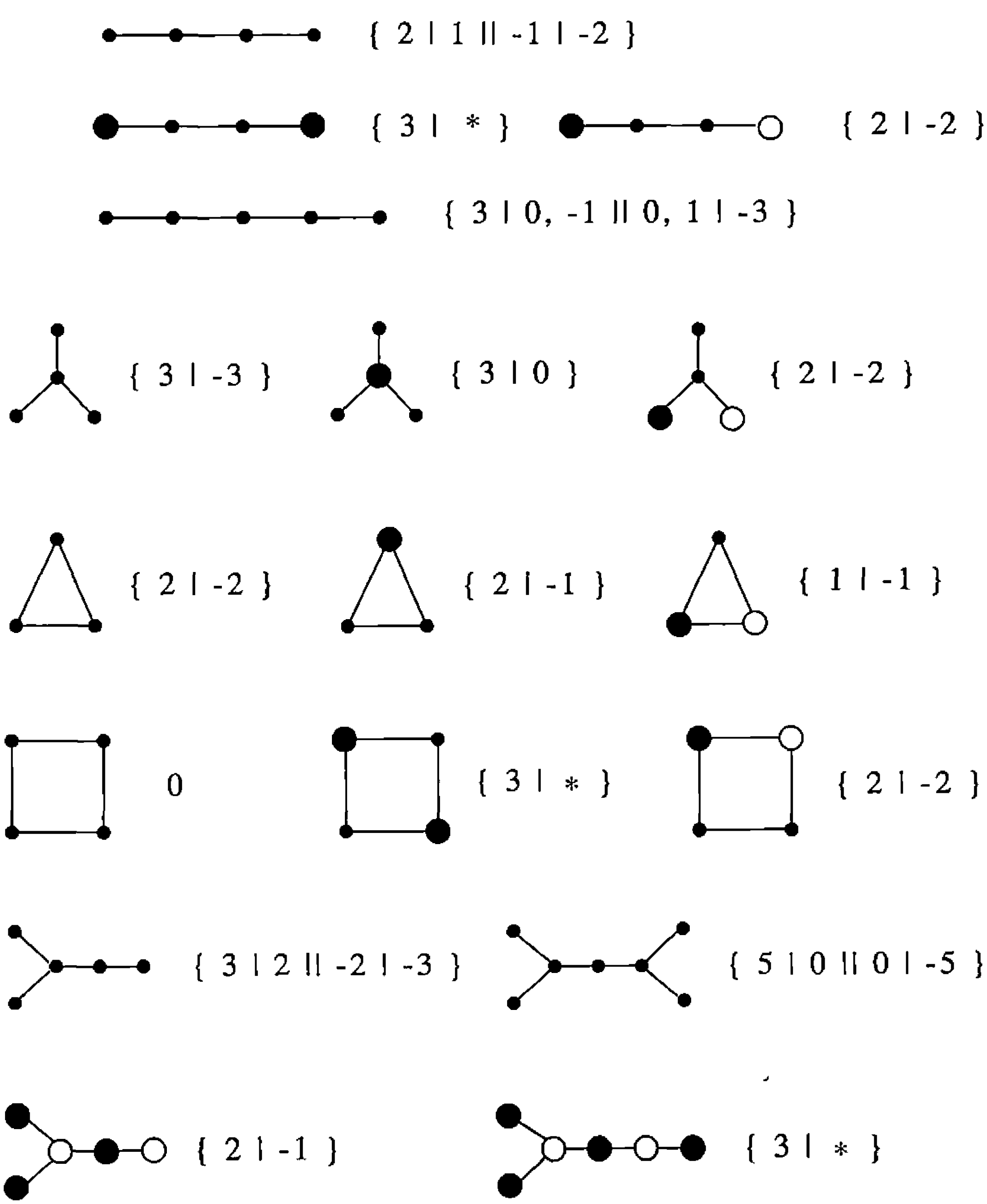


Figure 5.28: Value dictionary of Snort games

5.4 More Examples

In this section, we try to understand more about the theory of values by predicting a winner of some games, which we have examined in the previous chapters.

Example 5.8. We consider two strategies on complete ternary trees $T_{3,2}$ from Section 4.2. We have known the first player wins by either strategy. We re-examine this result by evaluating the value of the game each case (Figure 5.29).

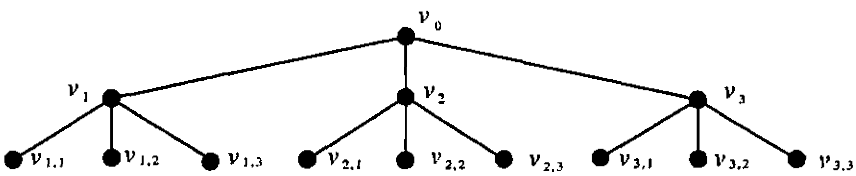


Figure 5.29: A complete ternary tree $T_{3,2}$

Strategy (1). The first player claims the root, so that v_1, v_2, v_3 are reserved for him. Now the whole game $G = T_{3,2}$ is decomposed into three of $T_{3,1}$ where each root is reserved for the first player (Figure 5.30).

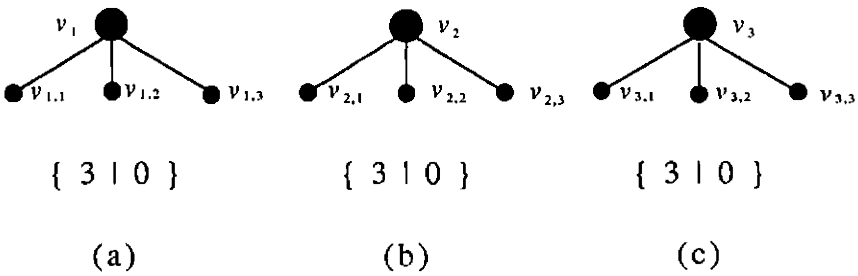


Figure 5.30: After decomposed into three $T_{3,1}$'s in strategy (1)

We have learned that the value of $T_{3,1}$ for which the root is reserved for the first player is $\{3 \mid 0\}$ from Figure 5.28. Since the next turn is the second player's, we pick the value 0 from Figure 5.30 (a). Then, for the first player, the value 3 is picked from Figure 5.30 (b), and finally the value 0 is picked from Figure 5.30 (c) for the second player. Therefore, the total value is $0+3+0=3$. The value is positive and this means that the first player wins on this game.

Strategy (2). The first player claims v_1 to make v_0 reserved for him, and also $v_{1,1}, v_{1,2}$ and $v_{1,3}$ absolutely reserved for him (Figure 5.31). Although the whole game $G = T_{3,2}$ is decomposed into two parts, it is still difficult to predict who is going to win. Then we would like to see the possible options after the second player claims some vertex.

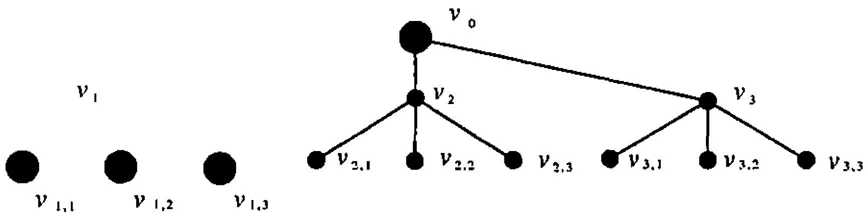


Figure 5.31: After decomposed into two parts in strategy (2)

The second player claims v_2 and makes v_0 piebald. The vertices $v_{2,1}, v_{2,2}$, and $v_{2,3}$ became absolutely reserved for the second player (Figure 5.32).

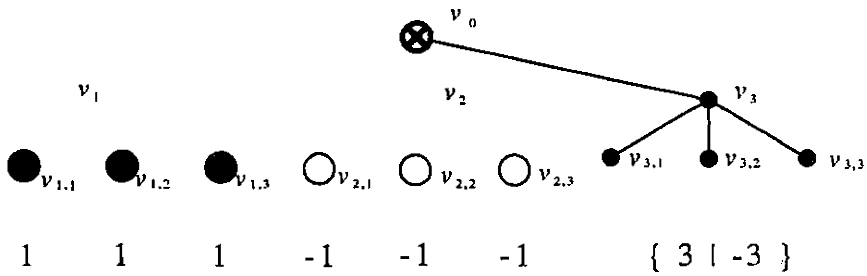


Figure 5.32: After decomposed into three parts in strategy (2)

The original $T_{3,2}$ is decomposed and the next turn is the first player's. The total value is $3+(-3)+3=3$ based on the information of values in Figure 5.24 and Figure 5.28. The value is positive, and therefore the first player wins this game.

Example 5.9. Next we examine C_6 from Section 2.3 given in Figure 5.33. We have learned that the second player uses an opposable strategy to win in this graph of Snort game.

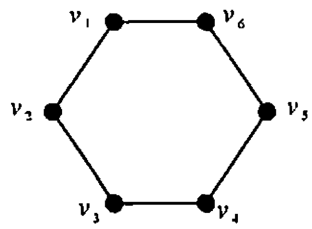


Figure 5.33: The graph of C_6

The first player claims v_1. Then the vertices v_2 and v_6 became reserved vertices for the first player (Figure 3.34).

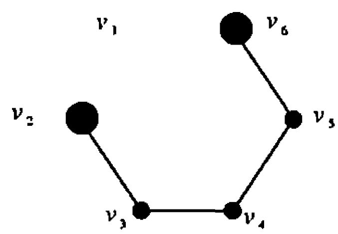


Figure 5.34: The graph of C_6 after the first player's move

We evaluate the Figure 5.34. There are essentially three choices for the first player (claiming v_4, v_3, and v_2), and two choices for the second player (claiming v_4 and v_3). We calculate the values based on the value lists in Figure 5.24 and Figure 5.28.

If the first player claims v_4, then the value is 4.

If the first player claims v_3, then the value is $1 + \{2|0\} = \{3|1\}$.

If the first player claims v_2, then the value is $\{3|*\}$.

If the second player claims v_4, then the value is $* + * = 0$.

If the second player claims v_3, then the value is $\{1\,|\,\text{-}1\}$.

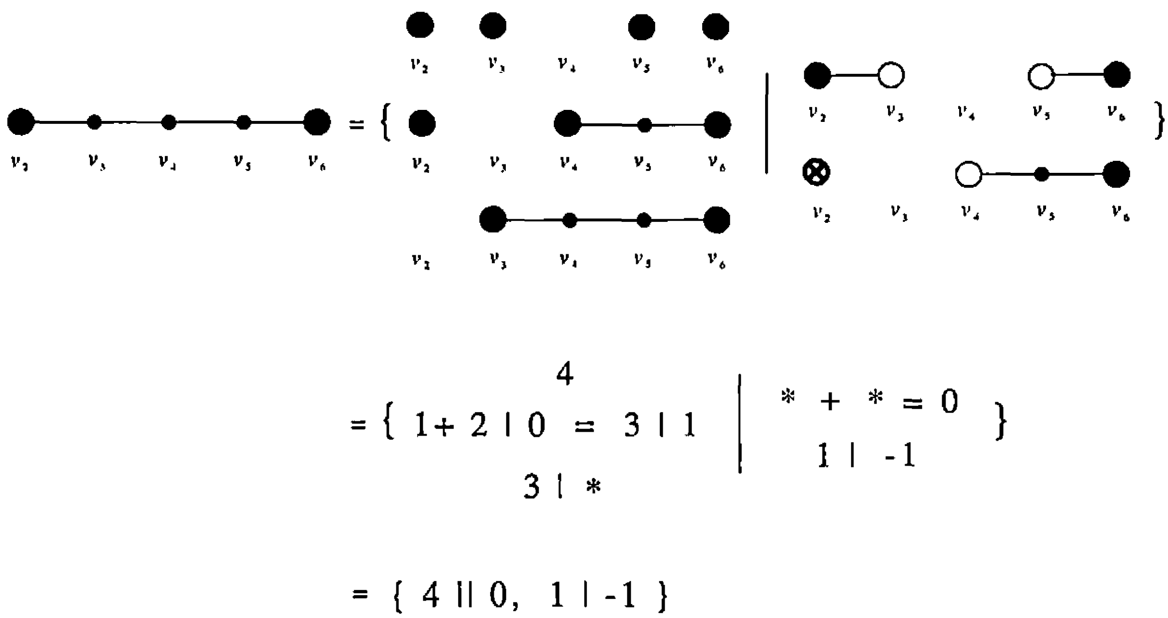
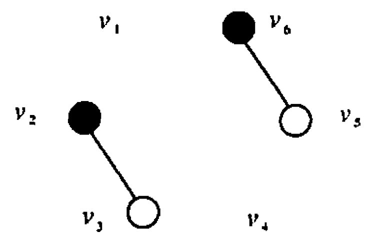


Figure 5.35: The evaluation of Figure 5.34

The best choice for the first player is 4, and for the second player is 0 since, if the second player chooses $\{1 \mid -1\}$, the first player wins by claiming v_5. Then, we let the second player claim v_4 to play with an opposable strategy, and we see the value of it (Figure 5.36).



$$* + * = 0$$

Figure 5.36: The graph of C_6 after the second player claims v_4

If a graph of Snort game has the conditions of an opposable graph, the value of the graph is 0, since the second player always wins with the opposable strategy. This is led by one of four outcome classes. Albert, et al, introduced this idea as a theorem in their book [ANW07].

Theorem 5.10. *The value of the game is zero if and only if the game is a second player win.*

Now we know the following implications for a graph G:

$$\begin{aligned} G \text{ is opposable} &\implies G \text{ is a second player win} \\ &\iff \text{The value of } G \text{ is } 0. \end{aligned}$$

We still do not know whether opposability of a graph is a necessary and sufficient condition for the graph to be a second player win. This is an open question and expected to be resolved near future.

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