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Short Report

A Numerical Method for Determining the Parameters of Schwarz-Christoffel Transformation

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Abstract

In the present paper we construct a numerical method for determining the unknown parameters that appear in the Schwarz-Christoffel transformation. For the polygon on the upper half domain with two cuts, we introduce some integrals and their derivatives and obtain the integral formulae which can be easily computed by the numerical quadratures. We show the numerical computations in the cases of d = 1/4, 1/8, 1/12, 1/16 ($2d$ is the distance between two cuts).

Introduction

In the paper [1] we established a method of the finite element approximation by which we can construct the parallel slit mappings. Parallel slit mapping is a kind of conformal mapping [2, 3] by which we can obtain the exact solutions of partial differential equations expressing physical phenomenons like fluid flows and electric fields in the two-dimensional problems. The purpose of the present paper is to show how to find the exact solutions which can confirm the precision of finite element approximations for the problem having a restricted domain.

Let z_j ($j = 1, \cdots, n$) be the vertices of polygon Π on the extended z-plane and let $\theta_j\pi$ ($0 < \theta_j \leq 2$; $j = 1, \cdots, n$) be the interior angles of Π. Let $w = f(z)$ be the conformal map of the interior of the polygon Π on z-plane onto the upper half of w-plane: Im $w > 0$, and let w_j ($j = 1, \cdots, n$) be the points on the real-axis of the w-plane satisfying $w_j = f(z_j)$ ($j = 1, \cdots, n$). Then we can state the Schwarz-Christoffel transformation:

If $w_j \neq \infty$ ($j = 1, \cdots, n$), it follows that

$$z = f^{-1}(w) = C\int_{w_0}^{w}(w-w_1)^{\theta_1-1}(w-w_2)^{\theta_2-1}\cdots(w-w_n)^{\theta_n-1}\,dw + C';$$

and if $w_n = \infty$, it follows that

$$z = f^{-1}(w) = C\int_{w_0}^{w}(w-w_1)^{\theta_1-1}(w-w_2)^{\theta_2-1}\cdots(w-w_{n-1})^{\theta_{n-1}-1}\,dw + C';$$

where $C \neq 0$ and C' are complex constants, w_0 is a fixed point.

We note that the Schwarz-Christoffel transformation is written in terms of w_j and θ_j ($j = 1, \cdots, n$). The parameters C, C' and w_j ($j = 1, \cdots, n$) of Schwarz-Christoffel transformation can be determined analytically

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for some restricted polygons. In the present paper we construct a numerical method for determining the parameters for the polygon lying on the upper half domain with two cuts. We apply our result to numerical calculations of
d=1/4, 1/8, 1/12, 1/16 (2d is the distance between two cuts).

1. Polygon Π_d and conformal mapping

Let D be the upper half domain with two cuts:

$$D=\{z \mid y>0\}-\{z \mid x=-d,\ 0<y\leq 1\}\cup\{z \mid x=d,\ 0<y\leq 1\}$$

on the z-plane($z=x+iy,\ d>0$), and let ∂D be the oriented boundary of D (see Figure 1). Let Π_d be the polygon $D+\partial D$.

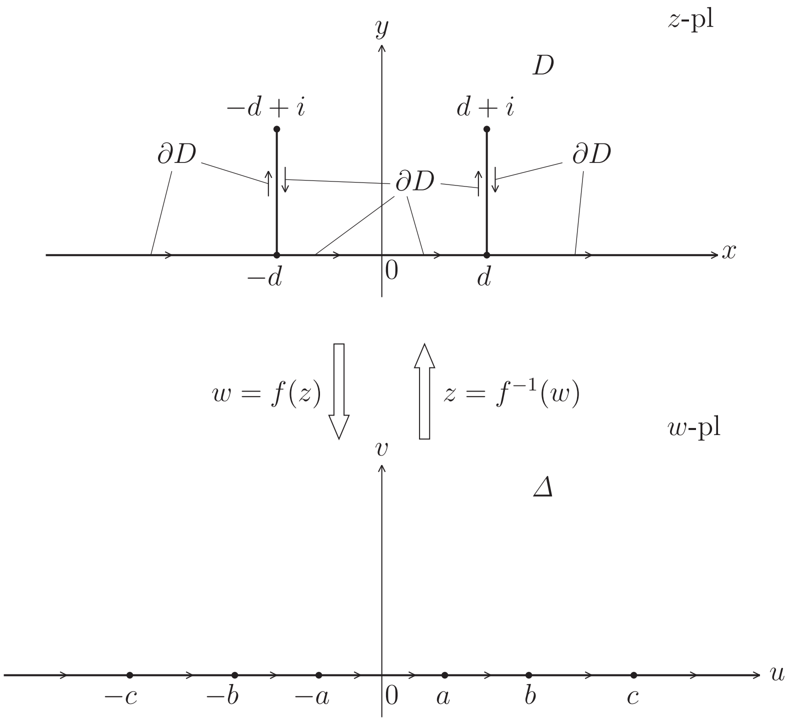

Fig. 1 Conformal map $w=f(z): D \to \Delta$

Let Δ be the upper half domain $\Delta=\{w \mid v>0\}$ on the w-plane ($w=u+iv$). For suitably chosen values a, b and c ($0<a<b<c$), the domain D can be conformally mapped onto Δ so that the points: $-d-0,\ -d+i,\ -d+0,\ d-0,\ d+i,\ d+0$ and ∞ on the z-plane are mapped to the points: $-c,\ -b,\ -a,\ a,\ b,\ c$ and ∞ on the w-plane respectively as shown in Figure 1.

Let $w=f(z)$ be the conformal map: $D \to \Delta$. The conformal mapping of Δ onto D is given by

$$z=f^{-1}(w)\int_0^w \frac{w^2-b^2}{\sqrt{(w^2-a^2)(w^2-c^2)}}dw.$$

The unknown parameters a, b and c are obtained by solving the following simultaneous equations for a given constant d:

$$\varphi_1(a,\ b,\ c) \equiv \int_0^a \frac{b^2-w^2}{\sqrt{(a^2-w^2)(c^2-w^2)}}dw = d,$$

$$\varphi_2(a,\ b,\ c) \equiv \int_a^b \frac{b^2-w^2}{\sqrt{(w^2-a^2)(c^2-w^2)}}dw = 1 \quad \text{and}$$

$$\varphi_3(a,\ b,\ c) \equiv \int_b^c \frac{w^2-b^2}{\sqrt{(w^2-a^2)(c^2-w^2)}}dw = 1.$$

The simultaneous equations can be solved numerically by adopting the following iterative procedure:
(i) Assume the initial values a_0, b_0 and c_0 ($0 < a_0 < b_0 < c_0$);
(ii) Obtain Δa_j, Δb_j and Δc_j ($j = 1, 2, \cdots$) by solving the simultaneous linear equations.

$$\frac{\partial \varphi_1}{\partial a}\Delta a_j + \frac{\partial \varphi_1}{\partial b}\Delta b_j + \frac{\partial \varphi_1}{\partial c}\Delta c_j = d - \varphi_1(a_{j-1},\ b_{j-1},\ c_{j-1}),$$

$$\frac{\partial \varphi_2}{\partial a}\Delta a_j + \frac{\partial \varphi_2}{\partial b}\Delta b_j + \frac{\partial \varphi_2}{\partial c}\Delta c_j = 1 - \varphi_2(a_{j-1},\ b_{j-1},\ c_{j-1}),$$

$$\frac{\partial \varphi_1}{\partial a}\Delta a_j + \frac{\partial \varphi_1}{\partial b}\Delta b_j + \frac{\partial \varphi_1}{\partial c}\Delta c_j = 1 - \varphi_3(a_{j-1},\ b_{j-1},\ c_{j-1}),$$
$$(j = 1,2,\cdots);$$

(iii) Compute the modified values a_j, b_j and c_j ($j = 1,2,\cdots$) by

$$a_j = a_{j-1} + \Delta a_j,\ b_j = b_{j-1} + \Delta b_j,\ c_j = c_{j-1} + \Delta c_j (j = 1,\ 2,\ \cdots);$$

(iv) Repeat the steps (ii) and (iii) until the following norm becomes smaller than a sufficiently small number $\varepsilon > 0$:

$$\text{norm} = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2} < \varepsilon,$$

where

$$\varepsilon_1 = \varphi_1(a_j,\ b_j,\ c_j) - d,\ \varepsilon_2 = \varphi_2(a_j,\ b_j,\ c_j) - 1,\ \varepsilon_3 = \varphi_3(a_j,\ b_j,\ c_j) - 1$$
$$(j = 1,\ 2,\ \cdots).$$

The values of $\frac{\partial \varphi_i}{\partial a}$, $\frac{\partial \varphi_i}{\partial b}$ and $\frac{\partial \varphi_i}{\partial c}$ ($i = 1, 2, 3$) can be obtained by using the numerical integrations after the singularities in $\frac{\partial \varphi_i}{\partial a}$, $\frac{\partial \varphi_i}{\partial b}$ and $\frac{\partial \varphi_i}{\partial c}$ ($i = 1, 2, 3$) are removed. The details of removing the singularities are shown in the following section.

2. Computation of partial derivatives

We constructed the formulae for the partial derivatives of $\varphi_j(a, b, c)$ ($j = 1, 2, 3$) with respect to a, b and c. We only show here the formulae for $j = 1$ because the another cases ($j = 2, 3$) are similar to those.

Firstly, we define integrals $F_1(\lambda, k)$ and $G_1(k)$ for real parameters k, λ ($0 < k < \lambda < 1$), and modify them to the forms in which singularities are removed.

$$F_1(\lambda,\ k) \equiv \int_0^1 \frac{1-\lambda^2 x^2}{\sqrt{(1-x^2)(1-k^2 x^2)}}dx$$
$$= \int_0^1 \frac{f_1(x;\ \lambda,\ k) - f_1(1;\ \lambda,\ k)}{\sqrt{1-x^2}}dx + \frac{\pi}{2}f_1(1;\ \lambda,\ k),$$

$$G_1(k) \equiv \int_0^1 \frac{x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}dx$$

$$= \int_0^1 \frac{g_1(x;\ k)-g_1(1;\ k)}{\sqrt{1-x^2}}dx + \frac{\pi}{2}g_1(1;\ k),$$

where

$$f_1(x;\ \lambda,\ k) \equiv \frac{1-\lambda^2x^2}{\sqrt{1-k^2x^2}}, \qquad g_1(x;\ k) \equiv \frac{x^2}{\sqrt{1-k^2x^2}},$$

We note that the integrands in these modified forms are finite in the whole range of integration. Thus the values of integrals in F_1 and G_1 can be obtained precisely by using the numerical quadratures.

For real parameters k, λ ($0 < k < \lambda < 1$), we define the following partial differentials:

$$\dot{F}_1(\lambda,\ k) \equiv \frac{\partial}{\partial k}F_1(\lambda,\ k) \equiv \int_0^1 \frac{(1-\lambda^2x^2)kx^2}{\sqrt{1-x^2}(1-k^2x^2)^{3/2}}dx,$$

Then we have the following formulae for the partial derivatives of $\varphi_1(a, b, c)$ with respect to a, b and c.

$$\frac{\partial\varphi_1}{\partial a} = \frac{b}{c}\left(-2\frac{a}{b}G_1\left(\frac{a}{c}\right) + \frac{b}{c}\ \dot{F}_1\left(\frac{a}{b},\ \frac{a}{c}\right)\right),$$

$$\frac{\partial\varphi_1}{\partial b} = 2\frac{b}{c}F_1\left(\frac{a}{b},\ \frac{a}{c}\right) + 2\frac{a}{b}\frac{a}{c}\ G_1\left(\frac{a}{c}\right),$$

$$\frac{\partial\varphi_1}{\partial c} = -\left(\frac{b}{c}\right)^2\left(F_1\left(\frac{a}{b},\ \frac{a}{c}\right) + \frac{a}{c}\ \dot{F}_1\left(\frac{a}{b},\ \frac{a}{c}\right)\right).$$

Table 1 Computational results of the polygon Π_d

$d = \frac{1}{4}$	$a = 0.106921 \times 10^{-2}$ $b = 0.464919$ $c = 1.358104$ norm* $< 10^{-99}$
$d = \frac{1}{8}$	$a = 0.131958 \times 10^{-5}$ $b = 0.309317$ $c = 1.202315$ norm* $< 10^{-33}$
$d = \frac{1}{12}$	$a = 0.195804 \times 10^{-8}$ $b = 0.246405$ $c = 1.144459$ norm* $< 10^{-33}$
$d = \frac{1}{16}$	$a = 0.311939 \times 10^{-11}$ $b = 0.210494$ $c = 1.113578$ norm* $< 10^{-32}$

* norm $= \sqrt{(\varphi_1 - d)^2 + (\varphi_2 - 1)^2 + (\varphi_3 - 1)^2}$

3. Numerical examples

We obtain a, b and c for the cases of d = 1/4, 1/8, 1/12 and 1/16 of the polygon Π_d defined in §1. We adopted the Gauss method of 128 integration points for obtaining the quadratures appearing in §2. The computational results are shown in Table 1, and the contour lines are shown in Figure 2 (d = 1/4) and Figure 3 (d = 1/16). Since the norms in Table 1 are less than 10^{-32}, we can say that our computational results are very good.

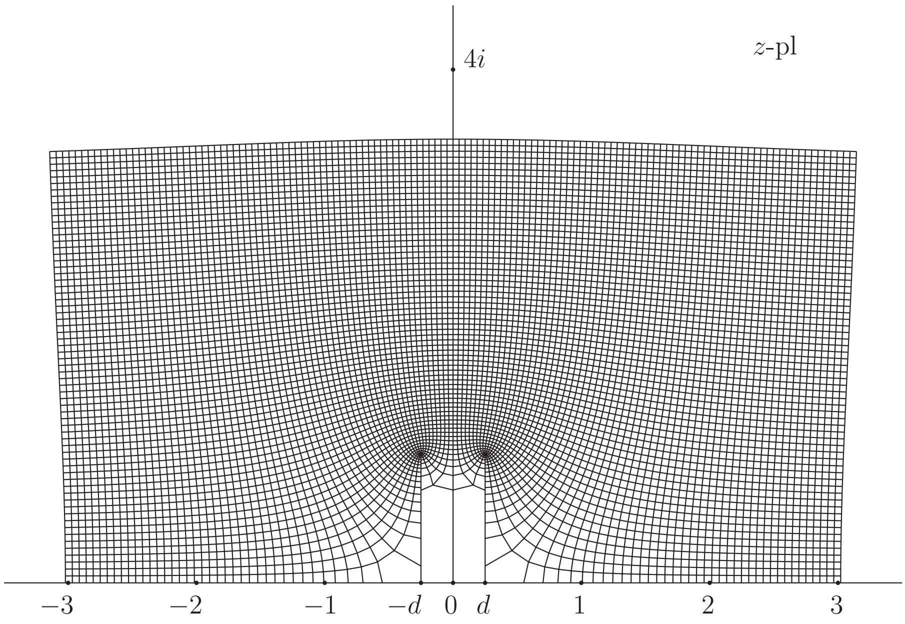

Fig. 2 Contour lines in the case of d = 1/4; $u = \pm 0.05n$, $v = 0.05n$ (n = 0,1,2,⋯)

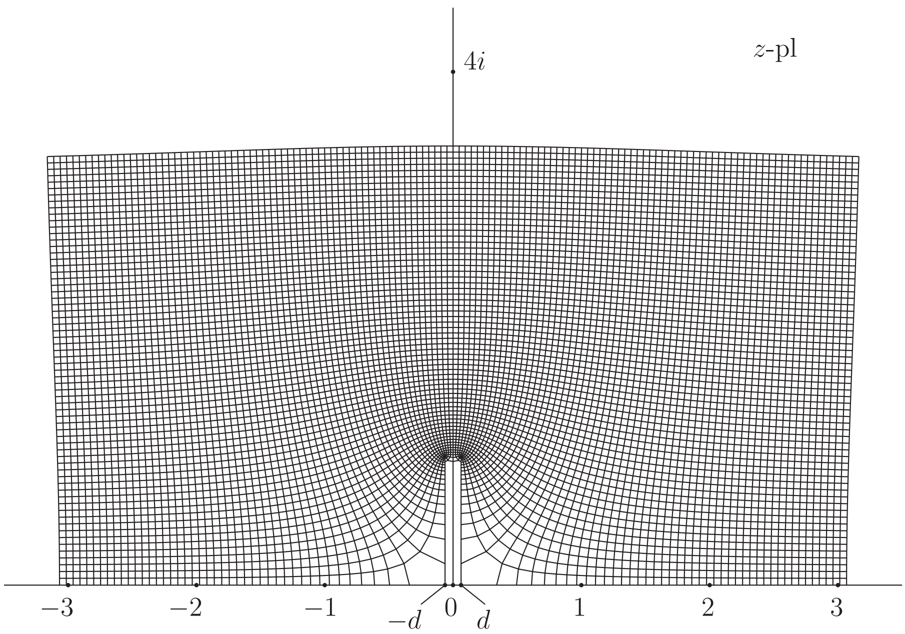

Fig. 3 Contour lines in the case of d = 1/16; $u = \pm 0.05n$, $v = 0.05n$ (n = 0,1,2,⋯)

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