Brief Announcement: The Temporal Firefighter Problem

Samuel D. Hand
School of Computing Science, University of Glasgow, UK

Jessica Enright
School of Computing Science, University of Glasgow, UK

Kitty Meeks
School of Computing Science, University of Glasgow, UK

Abstract

The Firefighter problem asks how many vertices can be saved from a fire spreading over the vertices of a graph. At timestep 0 a vertex begins burning, then on each subsequent timestep a non-burning vertex is chosen to be defended, and the fire then spreads to all undefended vertices that it neighbours. The problem is NP-Complete on arbitrary graphs, however existing work has found several graph classes for which there are polynomial time solutions. We introduce Temporal Firefighter, an extension of Firefighter to temporal graphs. We show that Temporal Firefighter is also NP-Complete, and remains so on all but one of the underlying classes of graphs on which Firefighter is known to have a polynomial-time solution. This motivates us to explore restrictions on the temporal structure of the graph, and we find that Temporal Firefighter is fixed parameter tractable with respect to the temporal graph parameter vertex-interval-membership-width.

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1 Problem Definition and Restrictions on the Underlying Graph

The Firefighter problem considers a fire spreading over a connected, undirected, loop-free, rooted graph [5]. At time 0 the fire begins burning at the root. Then, at each subsequent time, a chosen vertex is defended, and the fire then spreads to all undefended vertices adjacent to the fire. Once a vertex is burning or defended it remains so until the process is over, which happens once the fire can no longer spread. The decision problem then asks whether it is possible to save k vertices on a rooted graph (G, r). This problem is NP-Complete on arbitrary graphs, although progress has been made on identifying graph classes for which it can be solved in polynomial time [2, 3].

We introduce Temporal Firefighter, an extension of Firefighter to temporal graphs, using the definition of temporal graph first introduced by Kempe et al. [6]. Here a temporal graph is a pair (G, λ), where G is an underlying static graph and $\lambda : E(G) \to 2^{\mathbb{N}}$ is a labeling function mapping edges of the graph to the set of timesteps at which they are active. We refer to the maximum timestep at which any edge of such a graph is active as the lifetime Λ. Furthermore, we say that two vertices v_1 and v_2 are temporally adjacent at time i if there is an edge between them active at time i, that is if $i \in \lambda(v_1, v_2)$. Temporal Firefighter is then defined analogously to Firefighter except the fire can only spread to vertices to which it is temporally adjacent.

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Temporal Firefighter is NP-Complete on arbitrary graphs, as we can assign times in a rooted temporal graph $((G, \lambda), r)$ such that Temporal Firefighter simulates Firefighter for any rooted graph (G, r). This is achieved by setting $\lambda(e) = \{1, ..., |V(G)| - 1\}$ for every edge e. By time $|V(G)| - 1$ every vertex would have been defended, so the process must be over. Thus, for the entirety of the time during which the fire can spread, every edge is active, just as in Firefighter. In this respect we can view Firefighter to be a special case of Temporal Firefighter. Noting that the above reduction preserves the underlying graph class we then have that for every class $\mathcal{C}$ of graphs for which Firefighter is NP-Complete, Temporal Firefighter is NP-Complete on the class of temporal graphs with the graphs of $\mathcal{C}$ as the underlying graphs.

Firefighter has polynomial time solutions on interval graphs, permutation graphs, P_k-free graphs for $k > 5$, split graphs, cographs, and graphs of maximum degree three providing the root is of degree two [2, 3]. We find that Temporal Firefighter is NP-Complete when the underlying graph belongs to any of these classes except the last, for which there is a polynomial time algorithm. Every class except the last contains arbitrarily large cliques, and we find that Temporal Firefighter is NP-Complete when the underlying graph is a clique. This can be shown by reduction from Firefighter by assigning times to the edges in a static graph G so that they will be active at all times up until $|V(G)| - 1$, at which point the fire can certainly no longer spread. We then add further edges to make the graph a clique, and have them only active from time $|V(G)| - 1$ onwards such that they will not affect the spread of the fire. Temporal Firefighter on such a clique will then simulate Firefighter on G.

We also find the stronger result, that Temporal Firefighter is NP-Complete when the underlying graph is a clique of size n, even when the lifetime of the graph is upper bounded by $n^{\frac{1}{c}}$ for any constant $c \in \mathbb{N}$. This can be shown by a similar reduction, in which $|G|^c - |G|$ vertices are added to the static graph G in such a way that they will all burn on the first timestep. All defences then take place on a clique constructed in the manner described above.

In the positive direction we find that there is a polynomial time algorithm for Temporal Firefighter on temporal graphs where the underlying graph has maximum degree three and the root has degree two. This algorithm works in a very similar manner to that for Firefighter as given by Finbow et al. [2].

2 Restrictions on the Temporal Structure

Our analysis of the complexity of Temporal Firefighter when restricting the underlying graph class shows that for most known graph classes $\mathcal{C}$ where Firefighter is polytime solvable, Temporal Firefighter is NP-Complete on the class of temporal graphs $\{(G, \lambda) : G \in \mathcal{C}\}$. This naturally leads us to consider whether the problem might be tractable when restricting the temporal structure of the graph.

We show that Temporal Firefighter is fixed parameter tractable when parameterised by vertex-interval-membership-width, a temporal graph parameter defined by Bumpus and Meeks [1]. Begin by letting $\text{mintime}(v)$ denote the minimum timestep upon which an incident edge of v is active for all vertices v. Define maxtime equivalently for the maximum timestep. Vertex-interval-membership-width is then defined as follows.

▶ **Definition 1** (Vertex Interval Membership Width). *The vertex interval membership sequence of a temporal graph (G, λ) is the sequence $(F_t)_{t \in [\Lambda]}$ of vertex-subsets of G where $F_t = \{v \in V(G) : \text{mintime}(v) \leq t \leq \text{maxtime}(v)\}$ and Λ is the lifetime of (G, λ). The vertex-interval-membership-width of a temporal graph (G, λ) is then the integer $\omega = \max_{t \in [\Lambda]} |F_t|$.*

To simplify our analysis in showing that Temporal Firefighter is in FPT, we actually use the related problem Temporal Firefighter Reserve. This is the temporal extension of the Firefighter Reserve problem described by Fomin et al. [3]. In Temporal Firefighter Reserve, it is not required to make a defensive move every timestep. Rather, defences may be delayed, adding to a budget that can be used on future timesteps. Allowing the defence to build up a reserve in this manner does not affect the number of vertices that can be saved, and the proof for this fact works identically to that for the static case as given by Fomin et al. [3]. We note that in Temporal Firefighter Reserve there is always an optimal strategy that only defends temporally adjacent to the fire, as any defence can be delayed until the turn upon which the defended vertex would burn.

The algorithm for Temporal Firefighter Reserve iterates over the vertex interval membership sequence of the input graph, and considers every possible set of defences adjacent to the fire for each timestep. In particular it recursively computes a sequence of sets $L_i \in \mathcal{P}(F_i) \times \mathcal{P}(F_i) \times \{1, 2, ..., \Lambda\} \times \{1, 2, ..., n\}$. An element of L_i is a 4-tuple (D, B, g, c) where D is a set of defended vertices in F_i, B is a set of burnt vertices in F_i, g is the budget that will be available on timestep $i+1$, and c is the total count of vertices that have burnt at time i. The problem can then be answered by checking if there is any entry $(D, B, g, c) \in L_\Lambda$ where Λ is the lifetime of the graph, such that $|V(G)| - c \geq k$.

We observe that $|L_i| = O(4^\omega \omega \Lambda^2)$, and that there at most 2^ω defences to consider on each timestep. The overall complexity can then be seen to be $O(8^\omega \omega \Lambda^3)$.

▶ **Theorem 2.** *It is possible to solve Temporal Firefighter in time $O(8^\omega \omega \Lambda^3)$ for a rooted temporal graph $((G, \lambda), r)$ where Λ is the lifetime of the graph, and ω is the vertex-interval-membership-width.*

We suggest investigating the complexity of Temporal Firefighter when parameterised by similar temporal parameters as future work. For example, it would be worthwhile investigating parameterising by interval-membership-width, which is also defined by Bumpus and Meeks [1].

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