The Impact of Marginal Cost Pricing in Resource Allocation Games *

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Abstract

We study the impact of marginal cost pricing in resource allocation games on the worst case efficiency of Nash equilibria. Resource allocation games are closely related to congestion games and model the strategic interaction of players competing over a finite set of congestible resources. Examples of resource allocation games are routing and congestion control games in networks.

For convex and differentiable marginal cost functions, we prove that marginal cost pricing leads to a worst-case efficiency loss of Nash equilibria of at most $2/(2\,n+1)$, where n is the number of players. This is the first bound that holds for resource allocation games with arbitrary convex and differentiable marginal cost functions. For polynomial marginal cost functions with non-negative coefficients, we precisely characterize the price of anarchy. We also prove that the efficiency of Nash equilibria significantly improves if all players have the same strategy space and the same utility function.

We propose a class of distributed dynamics and prove that whenever a game admits a potential function, these dynamics globally converge to a Nash equilibrium. Finally, we show that in general the *only* class of marginal cost functions that guarantees the existence of a potential function are affine linear functions.

1 Introduction

Congestion games as introduced by Rosenthal [43] are non-cooperative games in which player's strategies consist of a set of subsets of resources, and the utility of a player depends only on the number of players choosing the same or some overlapping strategy.

We consider in this paper a variant of congestion games that are known as *resource allocation games* in which each player assigns to each of its available subsets a non-negative demand. The payoff for a player is then defined by the difference of the utility associated with the sum of the demands and the associated costs on the used resources. A prominent example of such a game is the traffic routing game of Haurie and Marcotte [21], which builds on the classical model of Wardrop [54]: The arcs in a given network represent the resources, the different origin-destination pairs correspond to the players, and the

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subsets of resources are the available paths in the network between the respective origin-destination pair. The strategy of a player is a distribution of traffic flow over its avaliable paths. The cost of an arc describes the delay experienced by traffic traversing that arc as a function of the flow on that arc.

Resource allocation games also play a key role in telecommunication networks, where users want to route packets from their source node to some sink node in the network. Here, it is assumed that each user receives non-negative utility from transmitting at a certain packet rate and that each link maintains a flow dependent cost function modeling packet delay, see Kelly et al. [32].

It is well known that Nash equilibria can be *inefficient* in the sense that they need not achieve socially desirable objectives [6, 15]. In the context of resource allocation games, a Nash equilibrium in general does not maximize the social welfare; or said differently, selfish behavior may cause a performance degradation. Koutsoupias and Papadimitriou [34] initiated the systematic investigation of the efficiency loss caused by selfish behavior. They introduced a measure to quantify the inefficiency of Nash equilibria, which they termed the *price of anarchy*. The price of anarchy is defined as the worst-case ratio of the social welfare of a system optimum and that of a Nash equilibrium.

In recent years, considerable progress has been made in quantifying the degradation in performance caused by the selfish behavior of non-cooperative players. In a seminal work, Roughgarden and Tardos [46] showed that the price of anarchy for network routing games with nonatomic players and affine latency functions is 4/3; in particular, this bound holds independently of the underlying network topology.

In resource allocation games, a simple example reveals that the price of anarchy is already unbounded for games with only one resource having an affine cost function. Due to this large efficiency loss, researchers have proposed different approaches to reduce the price of anarchy. In the context of congestion and resource allocation games, one of the most prominent approaches is the use of non-negative tolls on resources. In the area of transportation networks, this concept has been called *congestion toll pricing*, see for example Knight [33] and Beckmann et al. [4]. This mechanism assigns tolls to certain arcs of the network which are charged to those users that decide to take routes through them.

The concept of congestion prices has also been applied to telecommunication networks. Here, the congestion prices are related to physical quantities such as communication delays or packet loss rates at links. As common transport protocols such as TCP determine the communication rate according to packet loss rates or perceived delay, the equilibria of congestion pricing games can be interpreted as equilibria of such transport protocols, see for instance Low and Lapsley [36], Kelly et al. [32], Kelly and Voice [31], Kunniyur and Srikant [35], or the book by Srikant [51].

A central topic in the economics literature is to derive and study pricing mechanisms for allocating the joint costs so as to align the player's incentives with social objectives. Perhaps the best studied congestion pricing mechanism is marginal cost pricing, while other popular price mechanisms include the Aumann-Shapley pricing, the average cost pricing, and the Ramsey pricing, see Tauman [53]. Chen and Zhang [10] recently presented a class of pricing mechanisms for network resource allocation games satisfying four axioms that are considered desirable. In particular, their mechanisms are characterized by the axioms *rescaling, additivity, positivity, and weak consistency*, which have been proposed by Samet and Tauman [47]. This family of price mechanisms include the marginal cost pricing, the Aumann-Shapley pricing, and the average cost pricing. The main objective of Chen and Zhang is to find among all mechanisms that satisfy the four axioms an *optimal mechanism*, i.e., one that minimizes the induced price of anarchy. Their main result states that for affine cost functions, the optimal mechanism is obtained by an affine transformation of marginal cost prices, and that marginal cost pricing itself is nearly optimal.

In light of these positive results for the marginal cost price mechanism, our goal is to further analyze this mechanism with respect to the induced price of anarchy in the context of resource allocation games. While most of the previous works focused on affine linear marginal cost functions, the present work studies *non-linear* marginal cost functions. Non-linear marginal cost functions are particularly relevant in practice, since for example link delays grow super-linearly with link flows in close-to-capacity regions, e.g., $M/M/1$ functions. In road networks, for instance, the most frequently used functions modeling delay are polynomials whose degrees and coefficients are determined from real-world data through statistical evaluation methods, see Patrikkson [41], Branston [7], or the Bureau of Public Roads [8].

We also generalize in this paper the considered strategy spaces of players. Most of the previous works, e.g., Johari and Tsitsiklis [28, 27], consider games in which the strategy space of players is defined via feasible flows in a multi-commodity flow network. Our model contains general combinatorial strategy spaces similar to congestion games. In particular, our formulation contains game variants, such as multi-commodity routing, multicast routing, routing on trees and network design.

The second goal of this paper is to design distributed dynamics that provably converge to a Nash equilibrium. This is a central problem in evolutionary game theory, where the goal is to design simple and distributed improvement dynamics for players that provably converge to an equilibrium. Classical examples of improvement dynamics are best-response dynamics, replicator dynamics, gradient descent dynamics etc., see also the book by Sandholm [49].

1.1 Our Results

We study resource allocation games with marginal cost pricing. Our contributions to the above objectives are the following:

1. For resource allocation games (players have arbitrary differentiable, non-decreasing and concave utility functions) with $n \in \mathbb{N}$ players and differentiable, non-decreasing and convex marginal cost functions, we prove a lower bound of $2/(2\,n+1)$ on the worst case efficiency of Nash equilibria. In particular, this bound carries over to practically relevant M/M/1 functions that model queuing delays with arc-capacities.

   We completely characterize the price of anarchy for polynomial marginal cost functions with non-negative coefficients (previous results only covered affine marginal costs). For resource allocation games with a single resource (known as Cournot games) with differentiable, non-decreasing, semi-convex and concave marginal cost functions, we prove that the efficiency of Nash equilibria is at least 1/2. This result holds for an arbitrary number of players.

2. For symmetric games (players have equal utility functions and equal strategy space), we present a series of results showing that the efficiency of Nash equilibria significantly improves. In particular, we prove that the worst case efficiency for differentiable, non-decreasing and convex marginal cost functions is bounded by $2\,n/(2\,n+1)$. For polynomial marginal costs we prove a tight bound of 3/4.

For Cournot games with differentiable, non-decreasing, semi-convex and concave marginal cost functions, we prove a lower bound of $n/(n+1)$ in this case.

3. We define a class of distributed dynamics that can be implemented by players. We show that this class contains, among others, the gradient method and a combination of the gradient method with replicator dynamics. We prove that dynamics from this class converge to a Nash equilibrium from any initial strategy profile if the game admits an exact potential function. We show that an exact potential function always exists if (i) marginal cost functions are linear, or (ii) all players have the same utility function and symmetric strategy spaces. We also show that without restrictions on utility functions and the underlying strategy spaces, the only marginal cost functions that guarantee the existence of a potential are affine linear functions.

1.2 Significance and Techniques

Our first results generalize the result of Johari and Tsitsiklis [28] for linear marginal cost functions. We prove a tight characterization of the price of anarchy for polynomial marginal cost functions. It is worth noting that our proof technique is quite simple and different from [28]. In [28], the authors explicitely identify the worst possible game by analytically solving a sequence of quadratic optimization problems. Hence, this approach becomes increasingly complicated if such optimization problems involve polynomial cost functions of higher degree. Our approach hinges on variational inequalities, which are used to relate the total surplus of a Nash profile with that of an optimal profile. As a consequence, this technique can be applied to derive bounds on the price of anarchy for arbitrary subclasses of semi-convex marginal cost functions, see for instance our results for concave marginal cost functions. Moreover, all our results hold for general resource allocation games, which contain games with network structure as a special case. Our formulation is based on the general notion of congestion games, which include network variants such as multi-cast routing, routing on trees, e.g., minimum spanning-trees, Steiner trees, etc. We see this as a non-trivial generalization of previous works, as, for instance, Johari and Tsitsiklis [28] explicitely used the network structure (essentially using max-flow computations) to prove their bounds on the price of anarchy.

Finally, we study distributed dynamics that can be implemented by players. Using potential theory, we derive conditions under which these dynamics globally converge to a Nash equilibrium. As a byproduct of our analysis, we establish (under mild differentiability assumptions) a characterization of the existence of potentials. Since the initiating paper of Rosenthal [43] about congestion games and potential functions, a central topic of game theory is to determine classes of games that admit a potential. Thus, we believe that our result is of independent interest as it precisely describes, which classes of resource allocation games (depending on the class of marginal cost functions) admit a potential function.

1.3 Related Work

Related to the model considered in this paper are network routing games with a finite number of players who can split the flow along available paths, see Altman et al. [2], Haurie and Marcotte [21], Hayrapetyan et al. [22] and Cominetti et al. [12]. Haurie and Marcotte presented a general framework for studying atomic splittable network games with elastic demands. This class of games contains network resource allocation games if the elastic demand functions model the equilibrium demand functions. Haurie and

Marcotte, however, do not study the efficiency of Nash equilibria with respect to an optimal solution.

Hayrapetyan et al. [22] study congestion games with colluding players. Their goal is to investigate the price of collusion: the factor by which the quality of Nash equilibria can deteriorate when coalitions form. Altman et al. [2] and Cominetti et al. [12] studied the atomic splittable selfish routing model. Altman et al. bounded the price of anarchy for monomial latency functions (plus a constant). They also derived conditions under which a Nash equilibrium is unique. Uniqueness of Nash equilibria has been further studied by Fleischer et al. [5] and Orda et al. [39]. Cominetti et al. observed that the price of anarchy of the atomic splittable game may exceed that of the standard nonatomic selfish routing game. Based on the work of Catoni and Pallotino [9], they presented an instance with affine latency functions, where the price of anarchy is 1.34. The main difference between our model and that of Hayrapetyan et al. [22] and Cominetti et al. [12] is that our model involves *elastic* demands that are varied by players. As a result, in our model the payoff of players is a linear combination of utility (derived from sending flow) and associated costs.

Kelly, Maulloo and Tan [32] and Kelly [30] studied network resource allocation games and proposed a pricing mechanism that they termed proportionally fair pricing in which every link charges a price per unit resource equal to *marginal cost*. Despite the simplicity and scalability of this mechanism, Kelly et al. showed that an optimal solution can be achieved as an equilibrium if players are *price takers*, that is, if they do not anticipate the consequence of price change in response to a change of their flow.

Johari et al. [26, 25] studied network resource allocation games, where players submit a bid to each link in the network and each link allocates resources to players according to Kelly's proportionally fair allocation mechanism. For this mechanism they established a bounded efficiency loss of the marginal pricing scheme with fixed and elastic resource capacities. However, the proposed mechanism is not scalable since each player has to submit an individual bid to each link in the network. If, instead, players can only submit a single bid per path, Yang and Hajek [56] proved that the efficiency can be arbitrary low for the case of hard capacities and Johari [24] for the case of elastic capacities.

Perhaps closest to this paper is the work by Johari and Tsitsiklis [28, 27], who studied network resource allocation games with marginal cost pricing. On the negative side, they showed that for non-differentiable marginal cost functions, the price of anarchy is unbounded even for games with two players. For the special case of linear marginal cost functions, Johari and Tsitsiklis [28] showed that the efficiency loss is bounded by 2/3. Remarkably, this result holds for an arbitrary collection of concave utility functions and arbitrary networks. For a game with one resource and n players having equal utility functions, Johari and Tsitsiklis [27] proved a bound of $2\,n/(2\,n+1)$ for convex marginal cost functions.

Moulin [38] studied the price of anarchy for resource allocation games on a single link (Cournot games) with three different pricing mechanisms that are based on cost sharing principles. The used social welfare function, however, differs from our setting and, thus, the derived bounds are not transferable to our setting. Chen and Zhang [10] defined certain axioms for a desirable pricing mechanism for network resource allocation games and derived for quadratic cost functions (which corresponds to linear marginal cost functions) a slightly better efficiency guarantee (0.686) than the bound (2/3) proved for the marginal cost pricing. Guo and Yang [18] study the price of anarchy in Cournot oligopoly models and derived bounds on the price of anarchy depending on the market structure (market share of each player). Their model exhibits structural differences to our model since they assume explicit demand functions (instead of utility functions) and their social welfare function is different to ours.

In nonatomic network routing games, Roughgarden and Tardos [46] showed that the price of anarchy for network routing games with nonatomic players and linear latency functions is 4/3. The case of more general families of latency functions has been studied by Roughgarden [44] and Correa, Schulz, and Stier-Moses [13]. (For an overview of related results, we refer to the book by Roughgarden [45] and the survey by Altman et al. [3].) Despite these bounds for specific classes of latency functions, it is known that the price of anarchy in routing games with general latency functions is unbounded even on simple parallel-arc networks [46].

A large body of work in the area of transportation networks is concerned with congestion toll pricing, see for example Knight [33], Beckmann et al. [4], Smith [50], and Hearn and Ramana [23]. This mechanism assigns tolls to certain arcs of the network which are charged to those users that decide to take routes through them. Beckmann et al. [4] showed that for the Wardrop model with homogeneous users charging users the difference between the marginal cost and the real cost in the socially optimal solution (marginal cost pricing) leads to an equilibrium flow which is optimal. Cole et al. [11] considered the case of heterogeneous users, that is, users value latency relative to monetary cost differently. For single-commodity networks, the authors showed the existence of tolls that induce an optimal flow as Nash flow. Fleischer et al. [17], Karakostas and Kolliopoulos [29], and Yang and Huang [55] proved that there are tolls inducing an optimal flow for heterogenous users even in general networks. Swamy [52] proved the existence of optimal tolls for the atomic splittable model with fixed demands. Finally, Acemoglu and Ozdaglar [1] and Ozdaglar [40] study a model of parallel arc networks in which the arcs are owned by service providers that compete for the available traffic by setting prices. For this model they prove a tight worst-case bound for the efficiency loss of equilibria.

2 The Model

We now introduce resource allocation games as natural generalizations of variants of congestion games. For the sake of a clean mathematical definition, we first introduce the notion of a congestion model.

Definition 2.1 (Congestion Model)**.** *A tuple $\mathcal{M} = (N, R, X, (c_r)_{r\in R})$ is called a* congestion model *if $N = \{1, \dots, n\}$ is a non-empty, finite set of players, $R = \{1, \dots, m\}$ is a non-empty, finite set of resources, and $X = \times_{i\in N} X_i$ is a product space of accessible sets. For each player $i \in N$, her collection of accessible sets $X_i = \{R_{i1}, \dots, R_{i\,m_i}\}$, $m_i \in \mathbb{N}$, is a non-empty, finite set of subsets of R. Every resource $r \in R$ has a cost function $c_r : \mathbb{R}_+ \to \mathbb{R}_+$.*

Assumption 2.2. *Cost functions $c_r : \mathbb{R}_+ \to \mathbb{R}_+$, $r \in R$, are differentiable, semi-convex, non-decreasing functions, with $c_r(0) \geq 0$ and $\lim_{x\to\infty} c_r(x) = \infty$. Note that a function is semi-convex if $x \cdot c_r(x)$ is a convex function of x. Such functions are also called* standard *[44].*

Given a congestion model $\mathcal{M} = (N, R, X, (c_f)_{f\in F})$ we derive a corresponding *resource allocation model* $\mathcal{RM} = (N, R, X, \Phi, (c_r)_{r\in R})$, where $\Phi = \times_{i\in N}\Phi_i$, and $\Phi_i = \mathbb{R}_+^{m_i}$ defines the strategy space for player i. The strategy profile $\phi_i = (\phi_{i1}, \dots, \phi_{i\,m_i})$ of player i can be interpreted as a distribution of non-negative *demands* over the elements in X_i. The total demand of player i is defined by $\|\phi_i\|_1 = \sum_{j=1}^{m_i} \phi_{ij}$. We will use the short-hand notation

$M_i = \{1, \ldots, m_i\}$. For $i \in N$, $-i$ denotes the complementary set of i. Instead of $\Phi_{-\{i\}}$ we will write Φ_{-i}, and with a slight abuse of notation we will write sometimes a strategy profile as $\phi = (\phi_i, \phi_{-i})$ meaning that $\phi_i \in \Phi_i$ and $\phi_{-i} \in \Phi_{-i}$.

By choosing a strategy ϕ_i, player i receives a certain benefit measured by a utility function $U_i(\|\phi_i\|_1)$. We assume that utility functions satisfy the following conditions.

Assumption 2.3. *Each utility function $U_i : \mathbb{R}_+ \to \mathbb{R}_+$, is differentiable, strictly increasing, and concave.*

For a given profile ϕ, the load generated by player i on resource $r \in R$ is defined by $\phi_i^r = \sum_{R_{ij} \in X_i : r \in R_{ij}} \phi_{ij}$, $i \in N, r \in R$. The total load on resource r is defined by $\phi^r = \sum_{i=1}^n \phi_i^r$.

We are now ready to formally define a resource allocation game.

Definition 2.4 (Resource Allocation Game)**.** *Let $\mathcal{RM} = (N, R, X, \Phi, (c_r)_{r \in R})$ be a resource allocation model derived from $\mathcal{M}$. The corresponding* resource allocation game *is the strategic game $G(\mathcal{RM}) = (N, \Phi, \pi)$, where the payoff $\pi = (\pi_1, \ldots, \pi_n)$ is defined as*

$$\pi_i(\phi) := U_i\big(\|\phi_i\|_1\big) - \sum_{r \in R} \phi_i^r \cdot c_r(\phi^r). \tag{1}$$

In the sequel of this paper, we will write G instead of $G(\mathcal{RM})$.

Remark 2.5. *Assumptions 2.3 and 2.2 imply $\lim_{\|\phi_i\| \to \infty} \pi_i(\phi_i; \phi_{-i}) = -\infty$, hence, we can effectively restrict the strategy space for every player to a compact set. As the payoff functions are concave, a pure Nash equilibrium exists, see the result of Rosen [42].*

The total cost function for a profile ϕ is defined as

$$C(\phi) = \sum_{r \in R} \int_0^{\phi^r} c_r(z)\, dz\,.$$

Defining the total cost function as the integral over the cost functions c_r, $r \in R$, implies that the price mechanism implicitly defined in (1) corresponds to *marginal cost pricing.*

The *total surplus* of a profile ϕ is defined as

$$\mathcal{U}(\phi) := \sum_{i=1}^n U_i\big(\|\phi_i\|_1\big) - C(\phi)\,. \tag{2}$$

A profile of maximum total surplus is called *optimal.*

3 Price of Anarchy for Marginal Cost Pricing

In the following, we will study the price of anarchy with respect to a class of marginal cost functions $\mathcal{C}$ satisfying Assumption 2.2. Throughout the paper we assume that utility functions for every game satisfy Assumption 2.3.

Definition 3.1. *Let $\mathcal{C}$ be a class of marginal cost functions. Let $\mathcal{G}_n\big(\mathcal{C}\big)$ be the set of all resource allocation games with n players and marginal cost functions in $\mathcal{C}$. For $G \in \mathcal{G}_n\big(\mathcal{C}\big)$, let ψ_G be an optimal profile and let Θ_G be the set of pure Nash equilibria, respectively. Then, the* worst case efficiency *of Nash equilibria is defined by*

$$\rho_n\big(\mathcal{C}\big) = \inf_{G \in \mathcal{G}_n(\mathcal{C})} \inf_{\theta \in \Theta_G} \frac{\mathcal{U}_G(\theta)}{\mathcal{U}_G(\psi_G)}\,,$$

where $\mathcal{U}_G$ *is the total surplus function for game* G*. Conversely,* $\rho_n\big(\mathcal{C}\big)^{-1}$ *defines the* price of anarchy*.*

In order to simplify the notation, we denote the derivative of the cost of player i with respect to his demand satisfied by set R_{ij} by $\hat{c}_{ij}(\phi)$. We obtain

$$\hat{c}_{ij}(\phi) := \frac{\partial}{\partial \phi_{ij}}\Big(\sum_{r\in R}\phi_i^r \cdot c_r(\phi^r)\Big) = \sum_{r\in R_{ij}}\Big(c_r(\phi^r) + c_r'(\phi^r)\,\phi_i^r\Big)\,. \tag{3}$$

We call $\hat{c}_{ij}(\phi)$ *player-specific marginal costs* of the set R_{ij}.

Lemma 3.2. *Consider a resource allocation game* G*. The profiles* θ *and* ψ *are a Nash equilibrium and an optimal profile, respectively, if and only if the following conditions hold for all players* i*:*

$$\nabla\pi_i\big(\theta_i;\,\theta_{-i}\big)\cdot\big(\xi_i-\theta_i\big)\le 0, \quad \textit{for all } \xi_i \ge \mathbb{R}_+^{m_i}\,, \tag{4}$$

$$\begin{aligned} U_i'\big(\|\theta_i\|_1\big) &= \hat{c}_{ij}(\theta), \quad \textit{for all } \theta_{ij}>0,\ j\in M_i\,,\\ U_i'\big(\|\theta_i\|_1\big) &\le \hat{c}_{ij}(\theta), \quad \textit{for all } \theta_{ij}=0,\ j\in M_i\,, \end{aligned} \tag{5}$$

$$\begin{aligned} U_i'\big(\|\psi_i\|_1\big) &= c_{ij}(\psi), \quad \textit{for all } \psi_{ij}>0,\ j\in M_i\,,\\ U_i'\big(\|\psi_i\|_1\big) &\le c_{ij}(\psi), \quad \textit{for all } \psi_{ij}=0,\ j\in M_i\,. \end{aligned} \tag{6}$$

Proof. The function π_i is differentiable and concave with respect to ϕ_i. Furthermore, the set of profiles Φ is convex. Since θ is a Nash equilibrium, the strategy θ_i is a maximizer of $\pi_i(\phi_i;\,\phi_{-i})$. Thus, we can invoke the variational inequality as a necessary and sufficient optimality condition giving (4). Note that the derivative of π_i with respect to ϕ_{ij} is given by

$$\frac{\partial \pi_i}{\partial \phi_{ij}}(\phi_i;\,\phi_{-i}) = U_i'\big(\|\phi_i\|_1\big) - \hat{c}_{ij}(\phi)\,.$$

Thus, the second and third conditions follow directly from the Karush-Kuhn-Tucker conditions for maximizing $\pi_i(\phi_i;\,\phi_{-i})$ and $\mathcal{U}(\phi)$, respectively. □

The next lemma, which can be found in Johari and Tsitsiklis [28], Moulin [38], and Chen and Zhang [10], shows that for bounding the price of anarchy it is sufficient to bound the price of anarchy for games with only linear utility functions. The idea for proving the lemma is to observe that every Nash equilibrium θ for a game G can be transformed to a Nash equilibrium $\tilde{\theta}$ for a modified game $\tilde{G}$ in which linear utility functions of the form $\tilde{U}_i(x) = x\cdot U_i'\big(\|\theta_i\|_1\big)$ are used. Then, using concavity of U_i one can easily verify that the price of anarchy of the transformed game only increases.

Lemma 3.3. *[[10],[28],[38]] For bounding the price of anarchy, it is enough to consider resource allocation games in which utility functions are linear.*

In the following, we represent the total surplus of a Nash equilibrium and that of an optimal profile in terms of the involved marginal cost functions.

Lemma 3.4. *Consider a game* G *in which utility functions are linear, that is,* $U_i(x) = u_i\cdot x,\ u_i\ge 0,\ i\in N$*. Let* ψ *be an optimal profile and* θ *be a Nash equilibrium. Then,* ψ *and* θ *generate total surplus of*

$$\mathcal{U}(\psi) = \sum_{r\in R}\left(\psi^r\cdot c_r(\psi^r) - \int_0^{\psi^r} c_r(z)\,dz\right)\,,$$

$$\mathcal{U}(\theta) = \sum_{r\in R}\left(\theta^r\cdot c_r(\theta^r) + \sum_{i=1}^{n}(\theta_i^r)^2\cdot c_r'(\theta^r) - \int_0^{\theta^r} c_r(z)\,dz\right)\,.$$

Proof. Using the optimality condition (6) in Lemma 3.2 we get $u_i = c_{ij}(\psi)$ for all $i \in N$, $j \in M_i$, with $\psi_{ij} > 0$. By reordering the summation, we obtain

$$\mathcal{U}(\psi) = \sum_{i=1}^{n} u_i \cdot \|\psi_i\|_1 - \sum_{r\in R} \int_0^{\psi^r} c_r(z)\,dz = \sum_{r\in R} \left(\psi^r \cdot c_r\big(\psi^r\big) - \int_0^{\psi^r} c_r(z)\,dz \right),$$

proving the first part of the lemma. The second equation follows similarly by using the optimality condition (5), which implies $u_i = \hat{c}_{ij}(\theta)$, for all $i \in N$, $j \in M_i$, with $\theta_{ij} > 0$. □

The next lemma plays a key role in bounding the price of anarchy. It essentially provides an inequality bounding the surplus of an arbitrary Nash equilibrium in terms of the optimal surplus and an additional term that depends on both, the Nash and the optimal profile, respectively.

Lemma 3.5. *Consider a game G with n players in which utility functions are linear, .i.e., $U_i(x) = u_i \cdot x$, $u_i \geq 0$, $i \in N$. Let θ be a Nash profile and ψ be an optimal profile. For every $r \in R$, we define $\mu_r := \max_{i\in N}\left\{\frac{\theta_i^r}{\theta^r}\right\} \in \left[\frac{1}{n}, 1\right]$, if $\theta^r > 0$, and $\mu_r := 0$, otherwise. Then, for all $\lambda > 0$, the following inequality is valid:*

$$\mathcal{U}(\psi) \leq \lambda\,\mathcal{U}(\theta) + \sum_{r\in R} \left(c_r\big(\psi^r\big)\,\theta^r + \mu_r\,\theta^r\,\psi^r\,c_r'\big(\theta^r\big) - \lambda\,\tau_r(c_r,\theta^r,\mu_r) - \int_0^{\psi^r} c_r(z)\,dz \right) \quad (7)$$

where $\tau_r(c_r,\theta^r,\mu_r) := \left(c_r\big(\theta^r\big)\,\theta^r + \mu_r^2\,c_r'\big(\theta^r\big)\,\big(\theta^r\big)^2 - \int_0^{\theta_r} c_r(z)\,dz \right)$.

Proof. We first sum the variational inequality (4) in Lemma 3.2 with $\xi_i = \psi_i$ over all $i \in N$, i.e.,

$$\sum_{i=1}^{n} U_i'\big(\|\theta_i\|_1\big)\,\big(\|\psi_i\|_1 - \|\theta_i\|_1\big) - \sum_{r\in R} \left(c_r\big(\theta^r\big)\,\big(\psi^r - \theta^r\big) + \sum_{i=1}^{n} c_r'\big(\theta^r\big)\,\theta_i^r\,\big(\psi_i^r - \theta_i^r\big) \right) \leq 0.$$

Using that utility functions are linear and rewriting yields

$$\sum_{i=1}^{n} u_i\,\|\psi_i\|_1 \leq \sum_{i=1}^{n} u_i\,\|\theta_i\|_1 + \sum_{r\in R} \left(c_r\big(\theta^r\big)\,\big(\psi^r - \theta^r\big) + \sum_{i=1}^{n} c_r'\big(\theta^r\big)\,\theta_i^r\,\big(\psi_i^r - \theta_i^r\big) \right).$$

By Lemma 3.4 and the definition of μ_r we get

$$\begin{aligned} &\sum_{r\in R} c_r\big(\psi^r\big)\,\psi^r \\ \leq &\sum_{i=1}^{n} u_i\,\|\theta_i\|_1 + \sum_{r\in R} \left(c_r\big(\theta^r\big)\,\big(\psi^r - \theta^r\big) + c_r'\big(\theta^r\big)\,\mu_r\,\theta^r\,\psi^r - \sum_{i=1}^{n} c_r'\big(\theta^r\big)\,\big(\theta_i^r\big)^2 \right) \quad (8) \\ = &\sum_{r\in R} \left(c_r\big(\theta^r\big)\,\psi^r + c_r'\big(\theta^r\big)\,\mu_r\,\theta^r\,\psi^r \right). \end{aligned}$$

Subtracting $\sum_{r\in R} \int_0^{\psi^r} c_r(z)\,dz$ on both sides gives

$$\mathcal{U}(\psi) \leq \sum_{r\in R} \left(c_r\big(\theta^r\big)\,\psi^r + c_r'\big(\theta^r\big)\,\mu_r\,\theta^r\,\psi^r - \int_0^{\psi^r} c_r(z)\,dz \right).$$

Now we add and subtract $\lambda\,\mathcal{U}(\theta)$ for some $\lambda > 0$ on the right-hand side

$$\mathcal{U}(\psi) \leq \lambda\,\mathcal{U}(\theta) + \sum_{r\in R}\Big(c_r\big(\theta^r\big)\,\psi^r + c_r'\big(\theta^r\big)\,\mu_r\,\theta^r\,\psi^r - \int_0^{\psi^r} c_r(z)\,dz\Big) - \lambda\,\mathcal{U}(\theta)\,.$$

Finally, it holds that $\lambda\,\mathcal{U}(\theta) \geq \lambda\sum_{r\in R}\Big(c_r\big(\theta^r\big)\,\theta^r + \mu_r^2\,c_r'\big(\theta^r\big)\,\big(\theta^r\big)^2 - \int_0^{\theta^r} c_r(z)\,dz\Big)$. Thus, the claim is proved. □

Lemma 3.5 provides an inequality of the form $\mathcal{U}(\psi) \leq \lambda\,\mathcal{U}(\theta) + \gamma(\psi,\theta,\lambda,\mu)$, where $\gamma(\psi,\theta,\lambda,\mu) := \sum_{r\in R}\Big(c_r\big(\theta^r\big)\,\psi^r + c_r'\big(\theta^r\big)\,\mu_r\,\theta^r\,\psi^r - \lambda\,\tau_r(c_r,\theta^r,\mu_r) - \int_0^{\psi^r} c_r(z)\,dz\Big)$.

The main idea for proving bounds on the price of anarchy is to bound γ from above in terms of $\omega\,\mathcal{U}(\psi)$ for some $\omega < 1$. This would imply the inequality $\mathcal{U}(\psi) \leq \lambda\,\mathcal{U}(\theta) + \omega\,\mathcal{U}(\psi)$, which yields a bound on the worst case efficiency of $\frac{1-\omega}{\lambda}$. As a consequence, we will then optimize over λ so as to derive the best possible bound. This technique (λ-technique) has been previously applied to bounding the price of anarchy in congestion games, see Harks [19].

To this end, we define for a cost function c and parameters $\lambda > 0$ and $\mu \in \{0\} \cup [\frac{1}{n}, 1]$ the following value:

$$\omega_n(c;\lambda) := \sup_{\mu\in\{0\}\cup[\frac{1}{n},1]}\ \sup_{(x,y)\in\mathbb{R}_+^2} \frac{c(x)\,y + c'(x)\,\mu\,x\,y - \lambda\,\tau(c,x,\mu) - \int_0^y c(z)\,dz}{c(y)\,y - \int_0^y c(z)\,dz}\,. \tag{9}$$

For a given class of functions $\mathcal{C}$, we further define $\omega_n(\mathcal{C};\lambda) := \sup_{c\in\mathcal{C}} \omega_n(c;\lambda)$. Given a class of marginal cost functions $\mathcal{C}$ that satisfies Assumption 2.2, we define the feasible λ-region as $\Lambda_n(\mathcal{C}) := \{\lambda > 0 \,|\, \omega_n(\mathcal{C};\lambda) < 1\}$.

Theorem 3.6. *Let $\mathcal{C}$ be a class of marginal cost functions. Consider the set $\mathcal{G}_n(\mathcal{C})$ of games with at most $n \in \mathbb{N}$ players. Then, the worst case efficiency is at least*

$$\rho_n(\mathcal{C}) \geq \sup_{\lambda\in\Lambda_n(\mathcal{C})}\left[\frac{1-\omega_n(\mathcal{C};\lambda)}{\lambda}\right].$$

Proof. The proof follows directly from Lemma 3.3, Lemma 3.5 and the definition of $\omega_n(\mathcal{C};\lambda)$. □

Notice that Theorem 3.6 can now be used to derive bounds on the price of anarchy for arbitrary classes of cost functions (satisfying Assumption 2.2). The challenge is to calculate the function $\omega_n(\mathcal{C};\lambda)$ for a given class $\mathcal{C}$.

3.1 Convex Marginal Cost Functions

We start with applying Theorem 3.6 for convex marginal cost functions.

Theorem 3.7. *Let $\mathcal{C}^{conv}$ be a class of convex marginal cost function. Consider the set $\mathcal{G}_n(\mathcal{C}^{conv})$ of games with at most $n \in \mathbb{N}$ players. Then, $\rho_n(\mathcal{C}^{conv}) \geq \frac{2}{2n+1}$.*

Proof. We define $\lambda = \frac{1}{2} + n$ and prove the claim by showing $\omega_n(c;\lambda) \leq 0$ for $c \in \mathcal{C}^{conv}$. We proceed by a case distinction. First, we assume $x \geq y$. Then, the nominator of (9) can be bounded from above as follows.

$$\begin{aligned} c(x)\,y + c'(x)\,\mu\,x\,y - \lambda\,\tau(c,x,\mu) - \int_0^y c(z)\,dz &\leq\ c'(x)\,\big(\mu\,x\,y - \lambda\,\mu^2\,x^2\big)\\ &\leq\ c'(x)\,x^2\big(\mu - \lambda\,\mu^2\big)\,. \end{aligned}$$

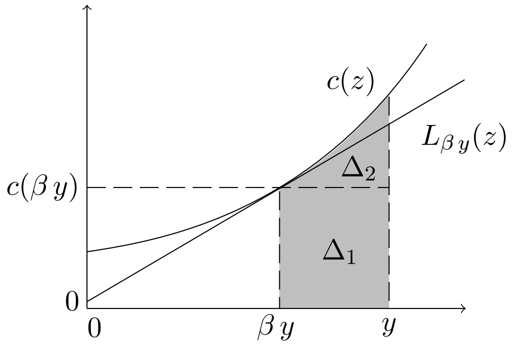

Figure 1: Illustration of the inequality (10) in the proof of Theorem (3.7). The gray-shaded area illustrates the term $\int_0^y c(z)\,dz - \int_0^{\beta\,y} c(z)\,dz = \Delta_1 + \Delta_2$. The linear approximation $L_{\beta\,y}(\cdot)$ of the convex function $c(\cdot)$ bounds $c(z)$ from below, i.e., $L_{\beta\,y}(z) \le c(z)$. Then, we have $\Delta_1 = (y - \beta\,y)\,c(\beta\,y)$ and $\Delta_2 \ge \frac{(y-\beta\,y)^2}{2}\,c'(\beta\,y)$.

For the first inequality, we used that

$$c(x)\,y - \lambda\,c(x)\,x + \lambda\int_0^x c(z)\,dz - \int_0^y c(z)\,dz \le 0\,,$$

since $y \le x$ and $\lambda \ge 1$. The second inequality follows from $y \le x$ and $c'(x) \ge 0$.

Then, $\lambda = \frac{1}{2} + n$ yields $\omega_n(c;\lambda) \le 0$ as $\max_{\mu\in\{0\}\cup[\frac{1}{n},1]} \mu - (\frac{1}{2}+n)\,\mu^2 \le 0$.

Now, we consider the case $x < y$. We define $\beta := \frac{x}{y} \in [0,1)$.

We now observe that

$$\int_0^y c(z)\,dz - \lambda\int_0^{\beta\,y} c(z)\,dz = \int_0^y c(z)\,dz - \int_0^{\beta\,y} c(z)\,dz - (\lambda-1)\int_0^{\beta\,y} c(z)\,dz\,.$$

Then, we use the following inequality, which is illustrated in Fig. 1.

$$\int_0^y c(z)\,dz - \int_0^{\beta\,y} c(z)\,dz \ge (y-\beta\,y)\,c(\beta\,y) + \frac{(y-\beta\,y)^2}{2}\,c'(\beta\,y)\,. \tag{10}$$

Together with

$$(\lambda-1)\int_0^{\beta\,y} c(z)\,dz \le (\lambda-1)\,c(\beta\,y)\beta\,y\,,$$

we obtain

$$\omega_n(c;\lambda) \le \sup_{\substack{\beta\in[0,1),\,y\in\mathbb{R}_+\\ \mu\in\left\{\{0\}\cup[\frac{1}{n},1]\right\}}} \frac{c'(\beta\,y)\,y^2\big(\beta\,\mu - \lambda\,\mu^2\,\beta^2 - \frac{(1-\beta)^2}{2}\big)}{c(y)\,y - \int_0^y c(z)\,dz}.$$

We use

$$\max_{\beta\in[0,1)}\Big(\beta\,\mu - \lambda\,\mu^2\,\beta^2 - \frac{(1-\beta)^2}{2}\Big) \le \frac{\mu\,(2+\mu-2\,\lambda\,\mu)}{2\,(2\,\lambda\,\mu^2+1)},$$

where $\beta^* = \frac{\mu+1}{2\,\lambda\,\mu^2+1}$ is the unique maximizer. Thus, since $\lambda = \frac{1}{2}+n$ and using that $\mu \ge \frac{1}{n}$ we obtain $\omega_n(c;\lambda) \le 0$. Notice that also $\mu = 0$ implies $\omega_n(c;\lambda) \le 0$.

Applying Theorem 3.6 for both cases proves the claim. □

The above result gives a bound on the efficiency loss for convex marginal cost functions that scales with the number of players. Compared to the negative result of Johari and Tsitsiklis [27] for two-player games with non-differentiable marginal cost functions, our result shows that differentiability of marginal cost functions is enough to obtain a bounded efficiency loss.

Remark 3.8. *We conjecture that the lower bound for the efficiency loss is tight, that is,* $\rho_n(\mathcal{C}^{conv}) = \frac{2}{2n+1}$. *In the next section (Proposition 3.13), we present an upper bound for the efficiency loss of* $\rho_n(\mathcal{C}^{conv}) \leq \frac{2(n-\sqrt{n})}{\sqrt{n}(n-1)}$.

3.2 Polynomial Marginal Cost Functions

In practice, the most frequently used functions modeling delay are polynomials whose degrees and coefficients are determined from real-world data through statistical evaluation methods, see Patrikkson [41] and Branston [7]. Thus, we will explicitly calculate the price of anarchy for the class

$$\mathcal{C}_d := \Big\{ c(z) = \sum_{j=1}^{d} a_j \, z^j, \; a_j \geq 0, \; j = 0, \dots, d \Big\}.$$

To simplify the analysis, we focus on the general case $n \in \mathbb{N} \cup \{\infty\}$. Let us define $\omega_\infty(c;\lambda) := \lim_{n\to\infty} \omega_n(c;\lambda)$. Then, it is easy to see that $\omega_\infty(c;\lambda) \geq \omega_n(c;\lambda)$ for any $n \in \mathbb{N}$, implying $\rho_\infty(c) \leq \rho_n(c)$.

Remark 3.9. *We observe that for polynomial marginal cost functions the total cost function* $C(\phi)$ *is linear in each of the marginal cost functions* $c_r(\cdot)$. *We can therefore reduce the analysis to monomial price functions. For this, we subdivide each resource* r *into* d *resources* $r_0, \dots, r_d$ *with monomial price functions* $c_{r_s}(\phi^r) = c_{r_s} \cdot \left(\phi^r\right)^s$ *for* $s \in \{0, \dots, d\}$. *By extending the accessible sets of every player accordingly, we obtain a transformed game in which the set of Nash equilibria, optimal profiles and corresponding surplus values coincide.*

Lemma 3.10. *Consider the class* $\mathcal{M}_j := \Big\{ c(z) = a_j \, z^j, \; a_j \geq 0, \; j \in \mathbb{N} \Big\}$. *Then, it holds that*

$$\omega_\infty(\mathcal{M}_j;\lambda) \leq \Big(\Big(\frac{1+\mu(j)\, j}{\lambda\,(1+\mu(j)^2\, j + \mu(j)^2)} \Big)^j \left(\mu(j)\, j + 1\right) - 1 \Big) / \, j \,,$$

where $\mu(j) = \frac{1}{\sqrt{j+1}}$.

Proof. Using the definition of $\omega_\infty(c;\lambda)$ for $c \in \mathcal{M}_j$ we get

$$\begin{aligned} \omega_\infty(c;\lambda) &= \sup_{\mu\in[0,1]} \; \sup_{(x,y)\in\mathbb{R}^2_+} \frac{x^j\, y + j\, x^j\, \mu\, y - \lambda\left(1+\mu^2\, j - \frac{1}{j+1}\right) x^{j+1} - \frac{1}{j+1}\, y^{j+1}}{\left(1 - \frac{1}{j+1}\right) y^{j+1}} \\ &= \sup_{\mu\in[0,1]} \; \sup_{\beta\geq 0} \frac{\beta^j \left(1 + j\,\mu\right) - \lambda\left(1+\mu^2\, j - \frac{1}{j+1}\right)\beta^{j+1} - \frac{1}{j+1}}{\left(1 - \frac{1}{j+1}\right)}\,, \end{aligned}$$

where $\beta := \frac{x}{y}$ (the case $y = 0$ can be excluded since then ω_∞ becomes non-positive). The supremum with respect to β is a strictly convex program with the unique global maximizer

$$\beta^* = \frac{1+\mu\, j}{\lambda\left(1+\mu^2\, j + \mu^2\right)}.$$

Thus, since $c \in \mathcal{M}_j$ was arbitrary, we get

$$\omega_\infty(\mathcal{M}_j;\lambda) \le \sup_{\mu\in[0,1]} \Big(\Big(\frac{1+\mu\, j}{\lambda\,(1+\mu^2\, j+\mu^2)}\Big)^j\,(\mu\, j+1)-1\Big)/\, j.$$

The unique maximizer for this supremum is given by $\mu(j) = \frac{1}{\sqrt{j}+1}$. ☐

Theorem 3.11. *Let $\mathcal{C}_d$ be the class of polynomial cost functions with non-negative coefficients and maximum degree $d \in \mathbb{N}$. Then,*

$$\rho(\mathcal{C}_d) = \frac{1+\mu(d)^2\, d+\mu(d)^2}{\Big(1+\mu(d)\, d\Big)^{1+\frac{1}{d}}},\ \text{where } \mu(d) = \frac{1}{\sqrt{d}+1}.$$

Proof. We define

$$\lambda = \frac{\Big(1+\mu(d)\, d\Big)^{1+\frac{1}{d}}}{1+\mu(d)^2\, d+\mu(d)^2}.$$

Then, Lemma 3.10 implies $\omega_\infty(\mathcal{M}_j;\lambda) \le 0$ for all $j < d$ and $\omega_\infty(\mathcal{M}_j;\lambda) = 0$ for $j = d$. Thus, using Theorem 3.6, we have

$$\lambda\, \mathcal{U}(\theta) \ge \mathcal{U}(\psi)$$

for an arbitrary Nash equilibrium θ and an optimal profile ψ.

Now we prove the upper bound. Consider a game with one resource having marginal cost function $c(x) = x^d$ for some $d \in \mathbb{N}$. Assume we have n players, where player 1 has the utility function $U_1(\phi_1) = \phi_1$, while the remaining $n-1$ players have utility functions $U_k(\phi_k) = b\,\phi_k$ for some $b \in [0,1]$ specified later. The total load of the resource is given by $\|\phi\|_1$.

Consider a Nash equilibrium $\theta(n)$ in this game. The necessary and sufficient Nash condition for player 1 yields

$$1 = \|\theta(n)\|_1^d + d\,\|\theta(n)\|_1^{d-1}\,\theta_1(n)\,.$$

Thus, we have

$$\theta_1(n) = \frac{1-\|\theta(n)\|_1^d}{d\,\|\theta(n)\|_1^{d-1}}\,.$$

The conditions for players $k = 2,\dots,n$ yield

$$\theta_k(n) = \frac{b-\|\theta(n)\|_1^d}{d\,\|\theta(n)\|_1^{d-1}}\,.$$

Summing all demands we get

$$\|\theta(n)\|_1 = \frac{1-\|\theta(n)\|_1^d}{d\,\|\theta(n)\|_1^{d-1}} + (n-1)\,\frac{b-\|\theta(n)\|_1^d}{d\,\|\theta(n)\|_1^{d-1}} \quad\Leftrightarrow\quad \|\theta(n)\|_1 = \left(\frac{1+(n-1)\,b}{d+n}\right)^{\frac{1}{d}}.$$

For large n we get

$$\lim_{n\to\infty} \|\theta(n)\|_1 = b^{\frac{1}{d}}$$

$$\lim_{n\to\infty} \theta_1(n) = \frac{b^{\frac{1}{d}}\,(1-b)}{d\, b}$$

$$\lim_{n\to\infty} b\,(n-1)\,\theta_k(n) = \frac{b^{\frac{1}{d}}\,(b\, d-1+b)}{d}\,.$$

Thus, we get in the limit for the total surplus of the Nash equilibrium $\theta(n)$

$$\lim_{n\to\infty} \mathcal{U}\big(\theta(n)\big) = \frac{b^{\frac{1}{d}}\,(1-b)}{d\,b} + \frac{b^{\frac{1}{d}}\,(b\,d-1+b)}{d} - \frac{b^{\frac{1}{d}}\,b}{d+1}\,.$$

An optimal solution is given by $\psi = (1,0,\dots,0)$ with total surplus of

$$\mathcal{U}(\psi) = 1 - \frac{1}{d+1}.$$

Now choosing $b = \frac{1+d^{\frac{3}{2}}}{d^2+d+1}$ one can check that the ratio $\frac{\mathcal{U}(\theta)}{\mathcal{U}(\psi)}$ coincides with the lower bound of the theorem. □

Remark 3.12. *The worst case efficiency for marginal cost functions in $\mathcal{C}_d$ is asymptotically bounded from below by $\Omega\Big(1/\sqrt{d}\Big)$.*

Note that the example we used during the proof to construct the upper bound can also be used to construct an upper bound for $\rho_n(\mathcal{C}^{conv})$ complementing Theorem 3.7. This is subject of the following Proposition.

Proposition 3.13. *Let $\mathcal{C}^{conv}$ be a class of convex marginal cost functions. Consider the set $\mathcal{G}_n(\mathcal{C}^{conv})$ of games with at most $n \in \mathbb{N}$ players. Then, $\rho_n(\mathcal{C}^{conv}) \le \frac{2(n-\sqrt{n})}{\sqrt{n}(n-1)}$.*

Proof. Consider the example in the proof of Theorem 3.7. Let $\theta(n)$ again be the Nash equilibrium in the game with n players. Further, let ψ be the optimum profile. Straightforward calculation shows that ψ yields a total surplus of $\frac{d}{d+1}$, which does not depend on n. We obtain

$$\rho_n(\mathcal{C}^{conv}) \le \lim_{d\to\infty} \frac{\mathcal{U}(\theta(n))}{\mathcal{U}(\psi)} = \frac{1+n\,b^2-b^2}{1+n\,b-b}\ ,\ \forall b \in [0,1]\,.$$

Since $\frac{1+n\,b^2-b^2}{1+n\,b-b}$ has a global minimum with respect to b with value $\frac{2(n-\sqrt{n})}{\sqrt{n}(n-1)}$, the lemma is proved. □

3.3 Concave Cournot Games

In the following, we analyze the efficiency loss of Nash equilibria resource allocation games with only one resource. This special case is known as a Cournot game [14] in which multiple producers strategically determine quantities they will produce. The cost of a producer is given by her offered quantity multiplied with the market price, which is usually a decreasing function of the total quantity offered by all producers. We will establish bounds on the price of anarchy for Cournot games involving concave marginal cost functions. These functions are of particular interest for Cournot competition as they model the effect of economy of scale.

Theorem 3.14. *Let $\mathcal{C}^{conc}$ be a class of concave marginal cost functions. Consider the set $\mathcal{G}(\mathcal{C}^{conc})$ of resource allocation games with one resource. Then, $\rho(\mathcal{C}^{conc}) \ge \frac{1}{2}$.*

Proof. Let θ and ψ be a Nash and an optimal strategy, respectively. Let $\beta := \frac{\|\theta\|_1}{\|\psi\|_1}$. Note that Lemma 3.4 implies that $\beta \le 1$. Then, on the one hand side, the variational inequality and the subsequent analysis (essentially using (8)) gives

$$c\big(\|\psi\|_1\big) \le c\big(\beta\,\|\psi\|_1\big) + c'\big(\beta\,\|\psi\|\big)\,\beta\,\mu\,\|\psi\|_1\,.$$

On the other hand, since c is concave,

$$c\big(\|\psi\|_1\big) \le c\big(\beta\,\|\psi\|_1\big) + c'\big(\beta\,\|\psi\|_1\big)\,(1-\beta)\,\|\psi\|_1\,.$$

Thus, by subtracting both inequalities and using $c'(\cdot) \ge 0$ it follows that $\beta \ge \frac{1}{1+\mu}$ must hold. Then, since $\mu \le 1$ we have $\beta \in [1/2, 1]$ and since $\mathcal{U}(z)$ is concave: $\mathcal{U}(\beta\,\psi) \ge \beta\mathcal{U}(\psi)$. □

4 Symmetric Games

In this section, we consider *symmetric games* in which all players have the same utility function $U(\cdot)$ and the same strategy space, that is, $R_i = R_j$ for all $i, j \in N$. In this case, we get improved bounds on the price of anarchy.

Consider a symmetric game with n players. Then, there exists a symmetric optimal profile ψ such that $\psi_i^r = \frac{1}{n}\psi^r$ for all $i \in N$. Using an adapted version of Lemma 3.5, we get the following variational inequality relating any Nash equilibrium θ to a symmetric optimal profile ψ, i.e.,

$$\sum_{r\in R} c_r\big(\psi^r\big)\,\psi^r \le \sum_{r\in R}\left(c_r\big(\theta^r\big)\,\psi^r + c'\big(\theta^r\big)\,\theta^r\,\frac{\psi^r}{n}\right)\,. \tag{11}$$

Furthermore, Lemma 3.4 implies

$$\mathcal{U}(\theta) \ge \sum_{r\in R}\left(c_r\big(\theta^r\big)\,\theta^r + \sum_{i=1}^{n} c_r'\big(\theta^r\big)\,\frac{\big(\theta^r\big)^2}{n} - \int_0^{\theta^r} c_r(z)\,dz\right)\,.$$

In the following, we evaluate the efficiency of Nash equilibria for symmetric games and several classes of marginal cost functions using a similar technique as in the general case. For a cost function c and parameters $\lambda > 0$ and $n \in \mathbb{N}$ we define the following value:

$$\delta_n(c;\lambda) := \sup_{(x,y)\in\mathbb{R}_+^2} \frac{c(x)\,y + c'(x)\,\frac{x\,y}{n} - \lambda\left(c(x)\,x + c'(x)\,\frac{x^2}{n} - \int_0^x c(z)\,dz\right) - \int_0^y c(z)\,dz}{c(y)\,y - \int_0^y c(z)\,dz}. \tag{12}$$

For a given class of functions $\mathcal{C}$, we further define $\delta_n(\mathcal{C};\lambda) := \sup_{c\in\mathcal{C}} \delta_n(c;\lambda)$. Given a class of marginal cost functions $\mathcal{C}$ that satisfies Assumption 2.2, we define the feasible λ-region as $\Delta_n(\mathcal{C}) := \{\lambda > 0 \,|\, \delta_n(\mathcal{C};\lambda) < 1\}$.

Theorem 4.1. *Let $\mathcal{C}$ be a class of marginal cost functions. Consider the set $\mathcal{G}_n(\mathcal{C})$ of symmetric games with at most $n \in \mathbb{N}$ players. Then, the worst case efficiency is at least*

$$\rho_n(\mathcal{C}) \ge \sup_{\lambda\in\Delta_n(\mathcal{C})}\left[\frac{1-\delta_n(\mathcal{C};\lambda)}{\lambda}\right].$$

Proof. The proof follows directly from Lemma 3.5, the representation of $\mathcal{U}(\theta)$ and the definition of $\delta_n(\mathcal{C};\lambda)$. □

4.1 Convex Marginal Cost Functions

The following result for convex marginal cost functions has been previously obtained by Johari and Tsitsiklis [27] for the special case of a game with a single resource. We present here a more general result (arbitrary symmetric strategy space) with a simpler proof.

Proposition 4.2. *Let $\mathcal{C}^{conv}$ be the class of convex marginal cost function. Consider the set $\mathcal{G}_n(\mathcal{C}^{conv})$ of symmetric games with at most $n \in \mathbb{N}$ players. Then, $\rho_n(\mathcal{C}^{conv}) \geq \frac{2\,n}{2\,n+1}$.*

Proof. The proof proceeds along the lines of the proof of Theorem 3.7, except that $\lambda = \frac{1+2\,n}{2\,n}$ and the values μ and μ^2 are replaced by $\frac{1}{n}$. Then, the only interesting difference occurs for the case $x < y$ in evaluating the following maximum:

$$\max_{\beta\in[0,1)} \left(\frac{\beta}{n} - \frac{\lambda\,\beta^2}{n} - \frac{(1-\beta)^2}{2}\right) \;\leq\; \frac{1+2\,n-2\,n\,\lambda}{2\,n\,(2\,\lambda+n)}\,.$$

Thus, since $\lambda = \frac{1+2\,n}{2\,n}$, the claim is proven. □

4.2 Polynomial Marginal Cost Functions

For polynomials with non-negative coefficients and arbitrary degree $d \in \mathbb{N} \cup \{\infty\}$, we prove the following.

Theorem 4.3. *Let $\mathcal{C}_\infty$ be the class of polynomial marginal cost function with non-negative coefficients and arbitrary degree $d \in \mathbb{N} \cup \{\infty\}$. Consider the set $\mathcal{G}_\infty(\mathcal{C}_\infty)$ of symmetric games with an arbitrary number of players and marginal cost functions in $\mathcal{C}_\infty$. Then, $\rho_\infty(\mathcal{C}_\infty) = \frac{3}{4}$.*

Proof. Let θ be a Nash equilibrium profile and ψ the system optimum. Using Remark 3.9 it is sufficient to consider monomial marginal cost function $c(z) = a_j\,z^j$, $a_j \geq 0$, for some $j \in \mathbb{N} \cup \{\infty\}$. Then, the value $\delta_n(c_j;\lambda)$ is given by

$$\delta_n(c_j;\lambda) = \sup_{(x,y)\in\mathbb{R}^2_+} \frac{x^j\,y + j\,x^{j-1}\,\frac{x\,y}{n} - \lambda\left(x^j\,x + j\,x^{j-1}\,\frac{x^2}{n} - \frac{1}{j+1}\,x^{j+1}\right)}{y^j\,y - \frac{1}{j+1}\,y^{j+1}}.$$

Defining $\beta := \frac{x}{y}$ (we can exclude the case $y = 0$ since then the expression becomes negative) and rewriting yields

$$\delta_n(c_j;\lambda) \leq \sup_{\beta\geq 0} \frac{\left(1+\frac{j}{n}\right)\beta^j - \lambda\left(1-\frac{j}{n}+\frac{1}{j+1}\right)\beta^{j+1} - \frac{1}{j+1}}{\frac{j}{j+1}}\,.$$

The above problem is a strictly convex program with unique solution $\beta^* = \frac{n+j}{\lambda\,(n+j+1)}$. Thus, we get

$$\delta_n(c_j;\lambda) \leq \left(\frac{n+j}{\lambda\,(n+j+1)}\right)^j \left(\frac{n+j}{n\,j}\right) - \frac{1}{j}\,.$$

We define $\lambda = \lambda(j,n) = \left(\frac{n+j}{n+j+1}\right)\left(\frac{n}{n+j}\right)^{-1/j}$ implying $\delta_n(c_j;\lambda(j,n)) = 0$. Thus, applying Theorem 4.1 yields

$$\mathcal{U}(\psi) \leq \lambda(j,n)\,\mathcal{U}(\theta)$$

for a Nash equilibrium θ and optimal profile ψ. We now observe that $\lambda(j,n)$ is a decreasing function in j and n. Hence, the worst case occurs for $j = 1$ and $n = 1$ leading to the desired bound of 3/4.

To prove that the bound is tight, we consider a single resource game with cost function $c(z) = z$. We consider n players with utility functions $U(\phi_i) = \phi_i$. Then, the following conditions hold for a Nash equilibrium θ:

$$1 - (\|\theta\|_1 + \theta_i) = 0 \quad \Rightarrow \quad \theta_i = 1 - \|\theta\|_1 .$$

Hence, we have

$$\|\theta\|_1 = n\,\theta_i = n\left(1 - \|\theta\|_1\right) \quad \Rightarrow \quad \|\theta\|_1 = \frac{n}{n+1}.$$

The total surplus evaluates to $\mathcal{U}(\theta) = \frac{n}{n+1} - \frac{1}{2}\frac{n^2}{(n+1)^2}$. The optimal profile ψ has value 1 and its total surplus evaluates to $\mathcal{U}(\psi) = \frac{1}{2}$. Evaluating the ratio $\frac{\mathcal{U}(\theta)}{\mathcal{U}(\psi)}$ proves the claim. □

4.3 Symmetric Concave Cournot Games

Similar to the previous section, we will also provide a bound that holds for symmetric Cournot games with concave marginal cost functions.

Theorem 4.4. *Let $\mathcal{C}^{conc}$ be the class of concave marginal cost functions. Consider the set $\mathcal{G}_n(\mathcal{C})$ of symmetric games with at most $n \in \mathbb{N}$ players. Then, $\rho(\mathcal{C}^{conc}) \geq \frac{n}{n+1}$.*

Proof. Let θ and ψ be Nash and optimal strategies, respectively. Let $\beta := \frac{\|\theta\|_1}{\|\psi\|_1}$. Then, on the one hand side, the variational inequality gives

$$c\big(\|\psi\|_1\big) \leq c\big(\beta\,\|\psi\|_1\big) + c'\big(\beta\,\|\psi\|_1\big)\,\frac{\beta\,\|\psi\|_1}{n}.$$

Following a similar argumentation as in Theorem 3.14 it holds that $\beta \geq \frac{n}{n+1}$ proving the claim. □

5 Distributed Dynamics and Potential Functions

In this section, we study distributed dynamics for a resource allocation game G. Similar to population dynamics introduced by Sandholm [49], we define a class $\mathcal{D}_G$ of algorithms and show that all dynamics from this class converge to a Nash equilibrium from any initial profile, provided that G admits an exact potential function. We then point out two well-known representatives of this class. The first natural dynamic is the well-known gradient descent method. The second dynamic is a combination of the gradient method with replicator dynamics known from evolutionary game theory, see Fischer et al. [16]. Our work differs from [49] in the following aspect. In [49], the total size (also termed *mass*) of a population is fixed, whereas we allow players to strategically vary their demands. This demand elasticity has also an effect on the structure of the payoff functions, which involves a utility function that reflects the benefit for a certain demand.

As we link the stability of a class of dynamics with a potential function argument, we consequently study necessary and sufficient conditions for a resource allocation game to possess a potential function. We show that a game G with affine linear marginal costs always admits an exact potential function. We also show that affine linear cost functions are the *only* functions that always guarantee the existence of an exact potential. For non-linear marginal costs, we show that if we restrict the set of profiles Φ to symmetric profiles, then an exact potential function exists. As an example, where such a restriction is reasonable, we consider a game, where all players have the same accessible sets and the same utility function. In this case, a symmetric Nash equilibrium always exists.

We remark that all following arguments still hold if we replace exact potentials by ordinal ones. We do, however, not know a useful description of a class of resource allocation games that admits an ordinal but no exact potential.

5.1 Stability of Distributed Dynamics

We will now define a class $\mathcal{D}_G$ of dynamics that are stable if game G admits an (exact) potential function. A similar class of dynamics has been previously defined for population games by Sandholm [48].

Definition 5.1. *Given a game G, we say that a dynamic described by a differential equation $\dot{\phi}_{ij} = f^G_{ij}(\phi)$, $i \in N$, $j \in M_i$, belongs to the class $\mathcal{D}_G$ if*

1. *ϕ being a Nash equilibrium of G implies $f^G_{ij}(\phi) = 0$, $\forall i \in N$, $j \in M_i$,*
2. *$\sum_{j=1}^{m_i} \frac{\partial \pi_i(\phi)}{\partial \phi_{ij}} \cdot f^G_{ij}(\phi) \geq 0$, for all profiles $\phi \in \Phi$, $i \in N$,*
3. *$\sum_{j=1}^{m_i} \frac{\partial \pi_i(\phi)}{\partial \phi_{ij}} \cdot f^G_{ij}(\phi) = 0$ if and only if $\phi \in \Phi$ is a Nash equilibrium.*

Before we study stability of dynamics in $\mathcal{D}_G$, we define the notion of an exact potential function for the game G and present necessary and sufficient conditions for a game to admit an exact potential.

Definition 5.2 (Monderer and Shapley [37])**.** *A function $\Psi : \Phi \to \mathbb{R}$ is an exact potential function for the game G if and only if*

$$\Psi(\phi_i, \phi_{-i}) - \Psi(\tilde{\phi}_i, \phi_{-i}) = \pi_i(\phi_i, \phi_{-i}) - \pi_i(\tilde{\phi}_i, \phi_{-i}),$$

for all $\phi \in \Phi$, $\tilde{\phi}_i \in \Phi_i$, $i \in N$.

In other words, an exact potential function for game G is a real-valued function on the profile space, which exactly tracks the difference in the payoff that occurs if one player unilaterally deviates.

Lemma 5.3. *Let G be a game with continuously differentiable payoff functions. Then, G possesses an exact potential $\Psi : \Phi \to \mathbb{R}$, if and only if*

$$\frac{\partial \Psi(\phi)}{\partial \phi_{ij}} = \frac{\partial \pi_i(\phi)}{\partial \phi_{ij}}, \ \forall i \in N, \ \forall j \in M_i, \ \forall \phi \in \Phi. \tag{13}$$

Again, similar to [37], given that the payoffs are twice continuously differentiable, we obtain the following characterization of games admitting an exact potential function.

Lemma 5.4. *Let G be a game with twice continuously differentiable payoff functions. Then, G possesses an exact potential $\Psi : \Phi \to \mathbb{R}$, if and only if*

$$\frac{\partial^2 \pi_i}{\partial \phi_{ik} \, \partial \phi_{jl}} = \frac{\partial^2 \pi_j}{\partial \phi_{ik} \, \partial \phi_{jl}}, \ \forall i, j \in N, \ \forall k \in M_i, \ l \in M_j.$$

Now, the following theorem establishes a stability result for dynamics in $\mathcal{D}_G$.

Theorem 5.5. *Let G admit an exact potential function Ψ_G. Then, all dynamics in $\mathcal{D}_G$ converge to a Nash equilibrium from any initial profile $\phi \in \Phi$.*

Proof. We will show that Ψ_G is a Lyapunov function for an arbitrary dynamic in $\mathcal{D}_G$ defined by $\dot{\phi}_{ij} = f^G_{ij}(\phi)$, $i \in N$, $j \in M_i$. According to the definition of a Lyapunov function, we need to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{\partial \Psi_G(\phi)}{\partial \phi_{ij}} \cdot f^G_{ij}(\phi) \geq 0,\ \forall \phi \in \Phi\,,$$

and that

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{\partial \Psi_G(\phi)}{\partial \phi_{ij}} \cdot f^G_{ij}(\phi) = 0 \quad \Leftrightarrow \quad x \in R \ \text{ is a Nash equilibrium.}$$

Both these conditions, however, follow from the definition of $\mathcal{D}_G$ and Lemma 5.3. □

Note that the first condition in Definition 5.1 is not required for the proof. Indeed, what we show in Theorem 5.5 is that any trajectory converges to the set of Nash equilibria. However, Theorem 5.5 does not exclude the case, where the trajectory continues to oscillate within this set. Condition 1 is a sufficient condition to preclude such oscillations.

5.2 Distributed Dynamics for Resource Allocation Games

From Theorem 5.5 it immediately follows that the gradient method is asymptotically stable for all games G that admit an exact potential function.

Corollary 5.6. *Let G admit an exact potential function. Then, the gradient method*

$$\dot{\phi}_{ij} = \kappa_i\big(\|\phi_i\|_1\big)\Big[U_i'\big(\|\phi_i\|_1\big) - \hat{c}_{ij}(\phi)\Big]^+_{\phi_{ij}}, \quad \forall i \in N,\ \forall j \in M_i\,, \tag{14}$$

where $\kappa_i\big(\|\phi_i\|_1\big)$ is a parameter determining the step size along the gradient, and

$$[a]^+_b = \begin{cases} 0 & \text{if } b = 0 \text{ and } a < 0\,, \\ a & \text{otherwise}\,, \end{cases}$$

converges to a Nash equilibrium of game G from any initial value $\phi \in \Phi$.

Proof. All we need to show is that the gradient method is in $\mathcal{D}_G$. First, note that condition 1 in Definition 5.1 follows from equation (5) and the definition of the gradient method. Next, observe that

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{\partial \pi_i(\phi)}{\partial \phi_{ij}} \cdot f^G_{ij}(\phi) = \kappa_i\left(\|\phi_i\|_1\right) \cdot \sum_{i=1}^{n}\sum_{j=1}^{m_i} \left(\left[U_i'\left(\|\phi_i\|_1\right) - \hat{c}_{ij}(\phi)\right]^+_{\phi_{ij}}\right)^2 \geq 0\,,$$

and that due to equation (5) in Lemma 3.2 the equality holds if and only if x is a Nash equilibrium. □

In the following, we present a distributed dynamic, which is a combination of the gradient method with replicator dynamics known from evolutionary game theory, see Fischer et al. [16]. We show that this dynamic is in $\mathcal{D}_G$ so that it is asymptotically stable whenever G admits a potential function.

We interprete the strategy profile of each player as a population of agents each controlling a very small fraction of the demand of the respective player. An agent continuously samples alternative accessible sets and switches with a probability depending on the difference of costs of the own and the sampled set. We apply this model in the fluid limit so that these two logical steps result in a single dynamic expressed by a differential equation. Our setting generalizes the model of [16] as we allow players to strategically vary their demands. Furthermore, our population model describes a replicator dynamic for *finitely* many players that can fractionally distribute their demands over their accessible sets. This in contrast to the nonatomic flow setting of Fischer et al. [16], where each agent has an negligible impact on the others. This structural difference becomes explicit by observing that in our model agents minimize player-specific costs (defined in (3)) instead of minimizing the standard cost of the chosen accessible set as in a Wardrop equilibrium.

The update of the distribution of the total demand over the accessible sets R_i of player i consists of two *logical* steps: sampling and migration. During the sampling step, each agent using accessible set $j \in M_i$ samples a set $k \in M_i$ with probability σ_{ijk}. During the migration step, each agent switches to the sampled set k with probability $\mu_{ijk}\big(\hat{c}_{ij} - \hat{c}_{ik}\big)$ depending on the difference of set costs. An example for a sampling policy is uniform sampling with $\sigma_{ijk} = \frac{1}{m_i}$ for $j, k \in M_i$, where each set is sampled with an equal probability. An example for a migration policy is linear migration policy $\mu_{ijk}\big(\hat{c}_{ij} - \hat{c}_{ik}\big) = \max\{\frac{\hat{c}_{ij}-\hat{c}_{ik}}{\hat{c}_{\max}}, 0\}$. We restrict the class of considered sampling policies and migration probabilities by the following Assumption.

Assumption 5.7.

1. *Sampling policies are assumed to be strictly positive:* $\sigma_{ijk} > 0,\ \forall\, j, k \in M_i,\ \forall i \in N$.
2. *Migration probability functions* $\mu_{ijk}(\cdot)$, $\forall\, j, k \in M_i$, $\forall i \in N$, *are assumed to be continuous and strictly increasing with* $\mu_{ijk}(0) = 0$. *For brevity, we will write* μ_{ijk} *instead of* $\mu_{ijk}\big(\hat{c}_{ij} - \hat{c}_{ik}\big)$.

Let us denote by γ_{ij} the fraction of total demand of player i that is served by set R_{ij} so that $\sum_{j=1}^{m_i} \gamma_{ij} = 1$, $\forall i \in N$. Then, the sample and migration probabilities induce a migration rate $r_{ijk} = \gamma_{ij} \cdot \sigma_{ijk} \cdot \mu_{ijk}$. The growth rate of the fraction of load on set j is then $\dot{\gamma}_{ij} = \sum_{k=1}^{m_i} \big(r_{ikj} - r_{ijk}\big)$. Note that this dynamic is a pure redistribution of the total demand, it does not change the total demand of player i since $\sum_{j=1}^{m_i} \dot{\gamma}_{ij} = 0$. Combining this with a gradient method to update the total demand of a player, we obtain the following dynamics (here and in the following, we denote the total demand of player i by $d_i := \|\phi_i\|_1$):

$$\dot{d}_i = \kappa_i\big(d_i\big)\left[U_i'\big(d_i\big) - \hat{c}_i(\phi)\right]^+_{d_i}, \quad \forall i \in N, \tag{15}$$

$$\dot{\gamma}_{ij} = \sum_{k=1}^{m_i} \big(r_{ikj} - r_{ijk}\big), \qquad \forall j \in M_i,\ i \in N\,. \tag{16}$$

Here, $\hat{c}_i(\phi) = \sum_{j=1}^{m_i} \gamma_{ij} \cdot \hat{c}_{ij}(\phi)$ are the average player-specific costs of player i. Note that this dynamic can also be expressed in terms of the demand as follows.

$$\dot{\phi}_{ij} = \dot{\gamma}_{ij}\, d_i + \gamma_{ij}\, \dot{d}_i\,, \quad \forall j \in M_i,\ i \in N\,,$$

where $\dot\gamma$ and $\dot d$ are as in (15), (16).

Similar to a feasible profile $\phi \geq 0$, we define a feasible tuple (d, γ).

Definition 5.8. *A tuple* (d, γ) *is* feasible *if and only if* $d, \gamma \geq 0$ *and* $\sum_{j=1}^{m_i} \gamma_{ij} = 1,\ \forall i \in N$.

Obviously, each feasible profile $\phi \geq 0$ corresponds to a unique feasible tuple (d, γ) and vice versa. Now we are ready to prove the following Theorem.

Theorem 5.9. *For a game G, the dynamic defined by* (15), (16) *is contained in* $\mathcal{D}_G$ *and thus converges to a Nash equilibrium whenever G admits an exact potential.*

Proof. The proof consists of three steps. First, we show that a profile $\phi \geq 0$ is a steady state of equation (16) if and only if for each player i

$$\hat c_{ij}(\phi) \leq \hat c_{ik}(\phi),\ \forall j, k \in M_i \text{ with } \phi_{ij} > 0\,. \tag{17}$$

The above condition can be interpreted as a Wardrop equilibrium with respect to player-specific set costs $\hat c_{ij}$. Using this fact, we then show in Step 2 that the set of feasible steady states of the system (15), (16) equals the set of Nash equilibria, which implies Condition 1 of Definition 5.1 and allows us in Step 3 to prove Conditions 2 and 3

Step 1: To see the "if" condition, assume that for a profile ϕ condition (17) holds. Assumption 5.7 then implies that $r_{ijk} = r_{ikj} = 0,\ \forall j, k \in M_i,\ \forall i \in N$. Thus, ϕ is a steady state of (16).

Now we show the "only if" condition. First, we show that for a set $k \in M_i$ with $\gamma_{ik} = 0$ it holds $\hat c_{ij} \leq \hat c_{ik}$ for all $j \in M_i$ with $\gamma_{ij} > 0$. (Note that if a player has a positive demand d_i, then $\gamma_{ij} > 0 \Leftrightarrow \phi_{ij} > 0$.) Consider a set $k \in M_i$ with $\gamma_{ik} = 0$. Then, ϕ being a steady state of equation (16) implies $\sum_{j=1}^{m_i} (r_{ijk} - r_{rkj}) = 0$. From the definition of r_{ikj} we know that $\gamma_{ik} = 0$ implies $r_{ikj} = 0,\ \forall j \in M_i$. We thus obtain $\sum_{i=1}^{m_i} r_{ijk} = 0$. Since the migration rates are always non-negative, we conclude $r_{ijk} = 0,\ \forall j \in M_i$. Now observe that for a set $j \in M_i$ with $\gamma_{ij} > 0$, it holds $\mu_{ijk} = 0$ since we assumed the sampling probabilities σ_{ijk} to be strictly positive. Assumption 5.7 then implies $\hat c_{ij} \leq \hat c_{ik}$.

Finally, we show that $\hat c_{ij} = \hat c_{ik},\ \forall j, k \in M_i$ with $\gamma_{ij},\ \gamma_{ik} > 0,\ \forall i \in N$. Assume, this does not hold for a player i. Let $j \in M_i$ be the set with highest cost $\hat c_{ij}$ among all sets in M_i. Assumption 5.7 then implies $\sum_{k=1}^{m_i} r_{ijk} > 0$ and $\sum_{k=1}^{m_i} r_{ikj} = 0$. Thus, $\dot\gamma_{ij} < 0$, which contradicts the steady state assumption.

Step 2: First, observe that condition (17) implies $\hat c_i = \hat c_{ij},\ \forall j \in M_i,\ i \in N$, with $\phi_{ij} > 0$. With this observation, necessary and sufficient conditions for ϕ to be a Nash equilibrium, as established in Lemma 3.2, are equivalent to the fact that ϕ fulfills condition (17) and that it is a steady state of equation (15). Given this and using the result of Step 1, the claim is proven.

Step 3: Note that the forward trajectory of the dynamics (15), (16) is feasible provided that the initial values are feasible. To see this, note that (i) equation (15) guarantees that $d_i,\ i \in N$, remains non-negative, and (ii) it holds that $\sum_{i=1}^{m_i} \dot\gamma_{ij} = 0$ implying $\sum_{i=1}^{m_i} \gamma_{ij} = \text{const}$. Finally, from the definition of r_{ijk} it follows that $\gamma_{ij} = 0$ implies $\dot\gamma_{ij} \geq 0$, which guarantees non-negativity of γ_{ij}.

Now observe that

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\partial \pi_i(\phi)}{\partial \phi_{ij}} \cdot f_{ij}^G(\phi) = \sum_{i=1}^{n} \kappa_i \big(\big[U_i'\big(d_i\big) - \hat c_i(\phi)\big]_{d_i}^{+}\big)^2 + \sum_{i=1}^{n} d_i \sum_{j=1}^{m_i} \sum_{k=1}^{m_i} r_{ijk}\big(\hat c_{ij} - \hat c_{ik}\big).$$

To complete the proof, we have to show that the above expression is always non-negative and equality holds if and only if x is a Nash equilibrium. The first summand is non-negative and it equals zero if and only if the dynamic (15) is in steady state. Since Assumption 5.7 implies that $r_{ijk} \geq 0$ and $r_{ijk} = 0$ if $\hat{c}_{ij} - \hat{c}_{ik} \leq 0$, we conclude that the second summand is always non-negative. Further, observe that $r_{ijk} > 0$ implies $\hat{c}_{ij} - \hat{c}_{ik} > 0$, thus, the second summand equals zero if and only if $r_{ijk} = 0$ for all $j, k \in M_i$, $i \in N$. The definition of r_{ijk} implies that $r_{ijk} = 0$ either if the demand satisfied by set R_{ij} is zero or if the player-specific costs of set R_{ik} is not strictly lower than that of R_{ij}. We conclude that $r_{ijk} = 0$ for all $j, k \in M_i$, $i \in N$, if and only if condition (17) is fulfilled. Given the result of Step 2, the claim is proved. □

5.3 Potential Functions

So far, we defined a class of dynamics that are stable whenever a game G admits an exact potential function. In the following, we study necessary and sufficient conditions for the existence of an exact potential function. The next theorem shows that without any restriction on the accessible sets R_{ij} and the class of utility functions, the *only* class of twice continuously differentiable marginal cost functions that always admits an exact potential is the class of affine linear functions.

Theorem 5.10. *Let $\mathcal{G}(\bar{\mathcal{C}})$ be the set of games with marginal cost functions in $\bar{\mathcal{C}}$ such that payoff functions are twice continuously differentiable. Then, the following statements are equivalent:*

1. *Every $G \in \mathcal{G}(\bar{\mathcal{C}})$ admits an exact potential*
2. *$\bar{\mathcal{C}}$ contains only affine linear functions on $\mathbb{R}_+$.*

Proof. Calculating the corresponding second derivatives, Lemma 5.4 implies that the game G possesses an exact potential if and only if for all profiles $\phi \in \Phi$

$$\sum_{r \in R_{ij} \cap R_{kl}} c_r''(\phi^r)\,(\phi_i^r - \phi_j^r) = 0 \ \text{ for all } i, j \in N,\ k \in M_i,\ l \in M_j\,. \tag{18}$$

The direction 2. ⇒ 1. is proved by simply verifying that affine linear marginal cost functions satisfy the above condition. For proving 1. ⇒ 2., we will assume that marginal cost functions are non-linear and then construct a game that violates condition (18). First, observe that c is affine linear on $\mathbb{R}_+$ if and only if $c''(z) = 0$ for all $z \in \mathbb{R}_+$. Assume by contradiction that $\bar{\mathcal{C}}$ contains a function c that is not affine linear on $\mathbb{R}_+$. Then, there exists a $z_0 \in \mathbb{R}_+$ such that $c''(z_0) \neq 0$. W.l.o.g., we assume $c''(z_0) > 0$. We further assume w.l.o.g. that $z_0 > 0$, since if $z_0 = 0$, by continuity of $c''(z)$, there exists $\tilde{z}_0 > 0$ with $c''(\tilde{z}_0) > 0$. Now consider a game G with two players that share a single resource. Let the demand of player 1 be $\phi_1 = 1/3\, z_0$ and the demand of player 2 be $\phi_2 = 2/3\, z_0$. It is easy to see that for these values condition (18) fails as $c''(\phi_1 + \phi_2)\,(\phi_2 - \phi_1) = c''(z_0)\, 1/3\, z_0 > 0$. □

Remark 5.11. *Note that an exact potential function for the case of affine linear marginal cost functions $c_r(\phi^r) = \xi_r \cdot \phi^r + \zeta_r$, $\xi_r \geq 0$, $\zeta_r \geq 0$ is given by*

$$\Psi(\phi) := \sum_{i=1}^{n} U_i(d_i) - \sum_{r \in R} \left(\frac{1}{2}\, \xi_r\, (\phi^r)^2 + \frac{1}{2}\, \xi_r \sum_{i=1}^{n} (\phi_i^r)^2 + \zeta_r\, \phi^r \right).$$

In a recent follow-up paper, Harks et al. [20] proved that the above characterization is even valid if one restricts the class $\mathcal{C}$ to locally bounded functions that may even be discontinuous.

Theorem 5.10 implies that if all marginal cost functions are affine linear then all dynamics in $\mathcal{D}_G$ converge to a Nash equilibrium from any initial profile $\phi \in \Phi$. Further, it implies that in order to find an exact potential function for games with non-linear marginal costs, we have to restrict the accessible sets, the utility functions, or the set of profiles Φ. Condition (18) implies another sufficient condition for a game G to possess a potential, which is based on the notion of a symmetric profile.

Definition 5.12. *A profile $\phi \geq 0$ is called symmetric, if it holds $\phi_i^r = \phi_j^r$, $\forall r \in R$, and $\forall i, j \in N$, such that $\exists\, k \in M_i$ and $\exists\, l \in M_j$ with $r \in R_{ik} \cap R_{jl}$.*

Theorem 5.13. *Let G be a game with twice continuously payoff functions. Let the set of profiles Φ be restricted to symmetric profiles. Then, G admits an exact potential function.*

Proof. The proof follows from condition (18) in Lemma 5.4 and the definition of a symmetric profile. □

In general, it is not reasonable to restrict all profiles to be symmetric, since a game G might not have a symmetric Nash equilibrium. Moreover, we would have to assure that there exists a trajectory from each symmetric initial value to a symmetric Nash equilibrium, and that there is a dynamic that produces a symmetric forward trajectory for a symmetric initial value. The following corollary shows that these conditions are satisfied for the special case when all players have the same accessible sets, the same utility function, and the considered dynamics are either the gradient method (14) or the combination of gradient method with replicator dynamics, i.e., (15) and (16).

Corollary 5.14. *Let G be a game with n players sharing the same accessible sets, have the same utility functions. Moreover, let all marginal cost functions be twice continuously differentiable. Then, the dynamics* (14) *and* (15), (16) *converge to a Nash equilibrium from any symmetric initial profile.*

Proof. From Theorem 5.13 we know that G admits an exact potential function. Given Theorem 5.5, it remains to note that starting with a symmetric initial value, both dynamics generates a symmetric forward trajectory due to the symmetry of set costs and utility functions. □

Remark 5.15. *An exact potential function for this setting is given by*

$$\Psi(\phi) = \sum_{i=1}^{n} U\big(d_i\big) - \sum_{r \in R} \Big(c_r(\phi^r)\, \frac{\phi^r}{n} + \frac{n-1}{n} \int_0^{\phi^r} c_r(z)\, dz \Big)\,. \tag{19}$$

Note that due to symmetry, it holds $\frac{\phi^r}{n} = \phi_i^r$. We remark that a similar potential function without the first term involving the utility functions has been found by Cominetti et al. [12] for the case of atomic splittable routing games with fixed demands.

Finally note that although the existence of a potential function is sufficient for dynamics in $\mathcal{D}_G$ to be stable, it is not a necessary condition. Thus, they might be stable even if a game possess no exact potential. Possible candidate Lyapunov functions are ordinal potentials, see Monderer and Shapley [37]. We do, however, not know a useful description of a resource allocation game that does not possesses an exact but an ordinal potential.

6 Conclusions and Future Work

In this work, we studied the efficiency and stability of Nash equilibria in resource allocation games for the marginal cost pricing mechanism. We derived various results about the price of anarchy depending on the structure of allowable marginal cost functions. In particular, we were able to provide the first bound for arbitrary non-decreasing, differentiable and convex cost functions. Moreover, we were able to prove tight bounds for the price of anarchy for polynomial marginal link costs. As this class of functions is quite rich and widely used for modeling for instance queuing delays at resources, we see our results as an important contribution towards the applicability of the marginal pricing mechanism in practice.

The second contribution of this paper concerns the design of a class of distributed dynamics that converge towards a Nash equilibrium. We identified conditions under which global stability of the proposed dynamics can be proved. An open issue is the stability of the proposed class of algorithms if delays are considered. The stability of distributed delayed differential equations for the resource allocation games considered in this paper is largely open.

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