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Analysis of optimal boundary control of the Boussinesq approximation

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Abstract. In the present paper we complement the work in [2] with presenting the analytical framework for general optimal boundary control problems of the Boussinesq approximation. We prove existence of optimal controls, use results of [6] to prove existence and uniqueness of solutions to state and the adjoint system, and derive first order necessary as well as second order sufficient optimality conditions.

Keywords: Time–dependent Boussinesq approximation, optimal boundary control, optimality conditions.

1 Introduction

In the present work we analyze the following optimal boundary control problem considered in [2];

$$(P) \qquad \begin{cases} \min J(y,\theta,u) \\ \text{s.t.} \quad u \in U_{ad} \end{cases} \quad \text{and} \tag{1}$$

$$y_t + (y\cdot\nabla)y - \nu\Delta y + \nabla p + \beta\,\theta\,\boldsymbol{g} = 0 \quad \text{in } \Omega_T, \tag{2}$$

$$-div\ y = 0 \quad \text{in } \Omega_T, \tag{3}$$

$$\theta_t + (y\cdot\nabla)\theta - \chi\Delta\theta - f = 0 \quad \text{in } \Omega_T\ , \tag{4}$$

$$\left.\begin{aligned} y = 0, &\quad \text{on } \Gamma_T, \\ \chi\tfrac{\partial\theta}{\partial\mathbf{n}} + b\theta = Bu &\quad \text{on } \Gamma_T, \end{aligned}\right\} \tag{5}$$

with initial conditions are chosen as

$$y \quad = 0 \text{ in } \Omega, \tag{6}$$

$$\theta(0) = \theta_0 \text{ in } \Omega, \tag{7}$$

with a given temperature field θ_0. Here, $\Omega \subset \mathbb{R}^2$ denotes the bounded flow domain with boundary Γ, and the time horizon is given by $[0,T]$. Further, $y = (y_1,y_2,y_3)$ denotes the flow velocity vector field, p a pseudo pressure scaled by the density[1], and θ denotes the temperature. Furthermore, the function Bu denotes

[1] pseudo pressure p here means that p has the form $\hat{p} - \rho_0(x_1+x_2+x_2)$ with the scalar pressure $\hat{p}$ and the mean density ρ_0

the temperature flux on the boundary with u serving as abstract control variable. System (2)-(7) represents a mathematical model for the thermally driven flow of a semi-conductor melt with the viscosity $\nu > 0$, thermometric conductivity $\chi > 0$, and thermal expansion coefficient $\beta > 0$, compare [2]. The force of gravity is denoted by $g \in \mathbb{R}^3$. Further $\Omega_T := \Omega \times [0,T]$, $\Gamma_T := \Gamma \times [0,T]$, and $b > 0$ denotes a coefficient which models the heat conductivity of the container walls which form the domain boundary Γ. $B : U \to L^2(0,T;H^{-1/2}(\Gamma))$ denotes the bounded, linear control operator which maps abstract controls of the Hilbert space U to feasible temperature boundary controls, and $U_{ad} \subseteq U$ the closed and convex set of admissible controls.

We consider cost functionals of separated type, i.e.

$$J(y,\theta,u) = J_1(y,\theta) + J_2(u) \ ,$$

where $J_1 : W_y \times W_\theta \to \mathbb{R}$, $J_2 : U \to \mathbb{R}$ are bounded from below, and are weakly lower semi–continuous, twice continuously Fréchet-differentiable with Lipschitz continuous second derivatives. Further, J_2 is assumed to be radially unbounded, i.e. $J_2(u) \to \infty$ for $\|u\|_U \to \infty$. The spaces W_y, W_θ are specified in Section 2.

The following frequently considered examples are covered by this setting.

Example 1. Typical cost functionals are given by

$$J_1(y,\theta) = \frac{1}{2}\int_0^T \int_\Omega [|y-\bar{y}|^2 + |\theta - \bar{\theta}|^2]\, d\Omega dt\ , \quad J_2(u) = \frac{\alpha}{2}\|u\|_U^2\ , \tag{8}$$

$$J_1(y,\theta) = \frac{1}{2}\int_0^T \int_\Omega [|curl\, y|^2 + |\nabla\theta|^2]\, d\Omega dt\ , \quad J_2(u) = \frac{\alpha}{2}\|u\|_U^2\ , \tag{9}$$

where $\bar{y} \in L^2(Q)^2$ and $\bar{\theta} \in L^2(Q)$ denote desired velocity and temperature fields respectively. The constant $\alpha > 0$ denotes a weighting factor for the control costs.

Typical control operators and control spaces are given by

1. $U = L^2(\Gamma_T)$, $B := \mathcal{I}$, where $\mathcal{I} : U \to L^2(0,T;H^{-1/2}(\Gamma))$ denotes the injection. $U_{ad} := \{u \in L^2(\Gamma_T); a \le u \le b \text{ a.e. on } \Gamma_T\}$, where $a,\ b \in L^\infty(\Gamma_T)$.
2. $U = L^2(0,T)^m$, $Bu(t) := \sum_{i=1}^m u_i(t) f_i$, where $f_i \in H^{-1/2}(\Gamma)$, $i = 1,\dots,m$. $U_{ad} := \{u \in L^2(0,T); a \le u \le b \text{ a.e. on } (0,T)\}$, where $a,\ b \in L^\infty(0,T)$.
3. $U = \mathbb{R}^m$, $Bu(t) := \sum_{i=1}^m u_i g_i(t)$, where $g_i \in L^2(0,T;H^{-1/2}(\Gamma))$, $i = 1,\dots,m$. $U_{ad} := \{u \in \mathbb{R}^m; a \le u \le b \text{ a.e. on } (0,T)\}$, where $a < b$ componentwise in $\mathbb{R}^m$.
4. $U = \{u \in H^1(0,T;L^2(\Gamma)); u(0) = 0\} \equiv H_0$, where $B := \mathcal{I}$, with $\mathcal{I} : U \to L^2(0,T;H^{-1/2}(\Gamma))$ denoting the injection. $U_{ad} = U$.

A discussion of the related literature [1,4,5,7–14] is given in [2]. Let us emphasize, that Belmiloudi in [5] discusses Robin boundary control in three spatial dimensions for a model related to the Boussinesq approximation, where he imposes certain smallness assumptions on the data and on the controls. The scope of the present paper differs from that of Belmiloudi [5] in that it focusses on a functional analytic framework which is tailored to the algorithmic approach presented in [2]. In particular more general boundary controls are allowed and first and second order necessary, and second order suffient optimality conditions are presented. Moreover a different proof technique is used which is based on Gajewski's work [6].

Above, and from now onwards, derivatives w.r.t. time are denoted by the subscript t, i.e. $v_t := \frac{\partial v}{\partial t}$.

2 Mathematical model

In order to formulate problem (P) mathematically we introduce the solenoidal spaces

$$H = \{v \in L^2(\Omega),\, div\, v = 0\}\ , \quad V = \{v \in H^1_0(\Omega),\, div\, v = 0\}\ ,$$

and set

$$W_y = \{v \in L^2(V),\ v_t \in L^2(V^*)\}\ , \quad W_\theta = \{\theta \in L^2(H^1(\Omega)),\ \theta_t \in L^2(H^1(\Omega)^*)\}\ .$$

Here, we abbreviate $L^p(Z) = L^p(0,T;Z)$ for Z denoting a Banach space.

Next, we introduce the operator

$$e : W_y \times W_\theta \times U \to L^2(V^*) \times H \times L^2(H^1(\Omega)^*) \times L^2(\Omega) =: Z^*$$

by

$$\begin{aligned} e(y,\theta,u) = (&y_t + (y\cdot\nabla)y - \nu\Delta\, y + \beta\,\theta g, y(0),\\ &\theta_t + (y\cdot\nabla)\theta - \chi\Delta\,\theta + E(b\theta - Bu), \theta(0))\,, \end{aligned}$$

where

$$E : L^2(H^{-1/2}(\Gamma)) \to L^2(H^1(\Omega)^*)$$

defines a linear bounded operator whose action is defined by

$$\langle E\, z, v\rangle_{L^2(H^1(\Omega)^*),L^2(H^1(\Omega))} = \int_0^T \langle z, \gamma v\rangle_{H^{-1/2}(\Gamma)),H^{1/2}(\Gamma)}\, dt,$$

with $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ denoting the trace operator. The action of e applied to an element
$z = (z_1, z_2, z_3, z_4) \in Z = L^2(V) \times H \times L^2(H^1(\Omega)) \times L^2(\Omega)$ is given by

$$\begin{aligned} \langle e(y,\theta,u), z\rangle_{Z^*Z} = &\int_0^T \langle y_t, z_1\rangle_{V^*V}\, dt + \int_0^T\int_\Omega \nu\nabla y\nabla z_1 + (y\cdot\nabla)y z_1\, dxdt+\\ &+ (y(0), z_2)_H + \int_0^T \langle\theta_t, z_3\rangle_{H^{1*}H^1}\, dt + \int_0^T\int_\Omega \chi\nabla\theta\nabla z_3 + (y\cdot\nabla)\theta z_3\, dxdt+\\ &+ \int_0^T [\int_\Gamma b\theta z_3\, ds - \langle Bu, z_3\rangle_{H^{-1/2}(\Gamma)H^{1/2}(\Gamma)}]\, dt + (\theta(0), z_4)_{L^2(\Omega)}\\ &=: \langle(e^1, e^2, e^3, e^4)(y,\theta,u), z\rangle_{Z^*Z}\ . \end{aligned}$$

Using e, the system of state equations can be written in the form

$$e(y,\theta,u) = 0 \text{ in } Z^* \ . \tag{10}$$

Considering [6, Bemerkung 2.1] together with [6, Satz 3.1,Folgerung 5.1] gives

Theorem 1. *Let $f \in L^2(V^*)$, $q \in L^2(H^1(\Omega)^*)$, $y_0 \in H$, $\theta_0 \in L^2(\Omega)$ and $u \in U$. Then*

$$e(y,\theta,u) = (f, y_0, q, \theta_0) \text{ in } Z^*$$

admits a unique solution $(y,\theta) \in W_y \times W_\theta$.

For the differentiability of the operator e we prove

Theorem 2. *The operator e is infinitely often Fréchet-differentiable with Lipschitz continuous second derivatives and vanishing derivatives of third and higher order. For the first and second derivatives there holds*

$$\begin{aligned}
e_y(y,\theta,u)\,v &= (v_t + (v\cdot\nabla)y + (y\cdot\nabla)v - \nu\Delta\, v, v(0), (v\cdot\nabla)\theta, 0)\\
e_\theta(y,\theta,u)\,s &= (\beta\, sg, 0, s_t + (y\cdot\nabla)s - \chi\Delta\, s + E(bs), s(0))\\
e_u(y,\theta,u)\,\tilde{u} &= (0,0,E(-B\tilde{u}),0)\\
e_{y\theta}[v,s] &= (0,0,(v\cdot\nabla)s,0)\\
e_{\theta y}[s,v] &= (0,0,(v\cdot\nabla)s,0)\\
e_{yy}[v_1,v_2] &= ((v_2\cdot\nabla)v_1 + (v_1\cdot\nabla)v_2, 0,0,0)\ ,
\end{aligned}$$

all other derivatives vanish.

Proof. Since e^2, e^4 represent linear operators, it is sufficient to consider e^1 and e^3. Let

$$b_\theta(y,\theta,\chi) := \langle (y\cdot\nabla)\theta, \chi\rangle_{H^{1*}H^1}$$

and

$$b_y(y,v,\phi) := \langle (y\cdot\nabla)v, \phi\rangle_{V^*V}\ .$$

Since we only consider two-dimensional spatial domains, it follows from [15, p. 293] that

$$|b_\theta(y,\theta,\chi)|^2 \le 2\|y\|_H\|y\|_V\|\theta\|_{L^2}\|\theta\|_{H^1}\|\chi\|^2_{H^1} \tag{11}$$

and

$$|b_y(y,v,\phi)|^2 \le 2\|y\|_H\|y\|_V\|v\|_H\|v\|_V\|\phi\|^2_V\ . \tag{12}$$

hold. In order to argue Lipschitz continuity of e^1, we estimate the difference

$$\begin{aligned}
&\langle e^1(y,\theta,u) - e^1(\tilde{y},\tilde{\theta},\tilde{u}), \phi\rangle_{L^2(V^*)L^2(V)} = \\
&\qquad \langle (y-\tilde{y})_t + [(y-\tilde{y})\cdot\nabla]\tilde{y} + (y\cdot\nabla)(y-\tilde{y}), \phi\rangle_{L^2(V^*)L^2(V)} \\
&\qquad + \nu\int_0^T (\nabla(y-\tilde{y}),\nabla\phi)_{(L^2)^2}\,dt - \int_0^T ((\theta-\tilde{\theta})g,\phi)_{L^2}\,dt \\
&\le \sqrt{2}\int_0^T \|y-\tilde{y}\|_H^{1/2}\|y-\tilde{y}\|_V^{1/2}(\|\tilde{y}\|_H^{1/2}\|\tilde{y}\|_V^{1/2} + \|y\|_H^{1/2}\|y\|_V^{1/2})\|\phi\|_V\,dt \\
&\qquad + C\{\|\theta-\tilde{\theta}\|_{W_\theta} + \|y-\tilde{y}\|_{W_y}\}\|\phi\|_{L^2(V)} \\
&\le C\left\{\|y-\tilde{y}\|_{C(H)}^{1/2}\left(\|\tilde{y}\|_{C(H)}^{1/2} + \|y\|_{C(H)}^{1/2}\right)\left(\int_0^T \|y-\tilde{y}\|_V(\|\tilde{y}\|_V + \|y\|_V)\,dt\right)^{1/2}\right. \\
&\qquad \left. + \|\theta-\tilde{\theta}\|_{W_\theta} + \|y-\tilde{y}\|_{W_y}\right\}\|\phi\|_{L^2(V)} \\
&\le C\left\{(\|y\|_{W_y} + \|\tilde{y}\|_{W_y} + 1)\|y-\tilde{y}\|_{W_y} + \|\theta-\tilde{\theta}\|_{W_\theta}\right\}\|\phi\|_{L^2(V)}\ .
\end{aligned}$$

Here, we have used the continuous embeddings

$$W_y, W_\theta \hookrightarrow C([0,T];H), C([0,T];L^2(\Omega))\ .$$

A similar estimate holds for e^3. In order to argue Fréchet differentiability, it is sufficient to consider component e^3, since component e^1 admits a similar structure. We obtain

$$e^3(y,\theta,u) - e^3(\tilde{y},\tilde{\theta},\tilde{u}) - e^3_{(y,\theta,u)}(y,\theta,u)(y-\tilde{y},\theta-\tilde{\theta},u-\tilde{u}) = [(y-\tilde{y})\cdot\nabla](\theta-\tilde{\theta})\ ,$$

so that estimation similar as above yields

$$\begin{aligned}
&\|e^3(y,\theta,u) - e^3(\tilde{y},\tilde{\theta},\tilde{u}) - e^3_{(y,\theta,u)}(y,\theta,u)(y-\tilde{y},\theta-\tilde{\theta},u-\tilde{u})\|_{L^2(H^1)} \\
&\qquad = \sup_{\|\chi\|_{L^2(H^1)}=1}\int_0^T |b_\theta(y-\tilde{y},\theta-\tilde{\theta},\chi)|\,dt \\
&\qquad\qquad \le C\|y-\tilde{y}\|_{W_y}\|\theta-\tilde{\theta}\|_{W_\theta}\ .
\end{aligned}$$

The expression for the second derivative can be verified by an estimate analogous to the one for the first derivative. The second derivative is independent of the point at which it is taken, and thus it is necessarily Lipschitz continuous.

From here onwards, it is appropriate to set $x = (y,\theta)$, $W = W_y \times W_\theta$, and to denote derivatives with respect to (y,θ) accordingly.

Lemma 1. *Let $(x,u) \in W \times U$. Then $e_x(x,u) : W \to Z^*$ is a homeomorphism, and thus also $e_x^* : Z \to W^*$.*

Proof. Let $(f, v_0, h, s_0) \in Z^*$. It suffices to prove that the system

$$v_t - \nu\Delta v + (y\cdot\nabla)v + (v\cdot\nabla)y + \nabla p_v + \beta\, sg = f \tag{13}$$
$$v(0) = v_0 \tag{14}$$
$$s_t - \chi\Delta s + (y\cdot\nabla)s + (v\cdot\nabla)\theta = h \tag{15}$$
$$s(0) = s_0 \tag{16}$$

admits a unique solution $u = (v, s)$. With $a = (a^1, a^2, a^3)$, $b = (b^1, b^2, b^3)$ we set

$$B(a,b) = \begin{pmatrix} (a^{1,2}\cdot\nabla)b^{1,2} \\ (a^{1,2}\cdot\nabla)b^3 \end{pmatrix}\ , \quad C(b) = \begin{pmatrix} \beta\, b^3 g \\ 0 \end{pmatrix}$$

and for $u = (v_1, v_2, s)$ we define

$$Au = \begin{pmatrix} \nu\Delta v_1 \\ \nu\Delta v_2 \\ \chi\Delta s \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} f \\ h \end{pmatrix}\ .$$

Then system (13) may be written as initial value problem in the form

$$u' + Au + B(x,u) + B(u,x) + C(u) = F\ , \quad u \in W\ , \quad u(0) = (v_0, s_0)\ ,$$

and admits exactly the form of the system [6, 2.15] if there $B_r(u,u)$ is replaced by $B(x,u)+B(u,x)+C(u)$. This completes the proof since the analysis presented in [6] also applies to this slightly modified situation. $\square$

The action of the adjoint of the operator e_x applied to $z \in Z$ as an element of W^* is defined as

$$\langle e_x(x,u)^* z, \tilde{x}\rangle_{W^*W} = \langle e_x(x,u)\tilde{x}, z\rangle_{Z^*Z}\ .$$

From lemma 1 we have

$$\|e_x(x,u)^*\|_{\mathcal{L}(Z,W^*)} = \|e_x(x,u)\|_{\mathcal{L}(W,Z^*)}\ ,$$

and for $g \in W^*$, the unique solution $w \in Z$ of $e_x(x,u)w = g$ in W^* satisfies

$$\|w\|_Z \le C\|g\|_{W^*}\ . \tag{17}$$

The constant C depends on u through $x(u)$. Due to theorem 1, it is meaningful to define the reduced functional

$$\hat{J}(u) = J(x(u), u)\ ,$$

where for given $u \in U$ the function $x(u)$ denotes the unique solution of $e(x,u) = 0$. Our optimization problem (1) then can be rewritten in the form

$$(\hat{P}) \qquad \min_{u\in U_{ad}} \hat{J}(u)\ .$$

Theorem 3. *Problem $(\hat{P})$ admits a solution.*

Proof. Since $\hat{J}$ is bounded from below, we have $d := \inf_{u \in U_{ad}} \hat{J} \geq -\infty$. Let (u^n) denote a minimizing sequence, i.e. $\hat{J}(u^n) \to d$ for $n \to \infty$. Since J_2 is radially unbounded we infer $\|u^n\|_U \leq M$ uniformly in n, so that $u^n \rightharpoonup u$ for a subsequence. Since U_{ad} is convex and closed, it is weakly closed so that $u \in U_{ad}$ holds. For all u^n exists a unique x^n satisfying $e(x^n, u^n) = 0$ and $\|x^n\|_W \leq M$ for all n. Since W is a Hilbert space, we have $x^n \rightharpoonup x$ for a further subsequence. Because of the compact embedding $W \hookrightarrow L^2([0,T]; H) \times L^2(Q)$, we also have $x^n \to x \in W$ in $L^2(H) \times L^2(Q)$ for a further subsequence. Moreover, $W \hookrightarrow C(H)$, so that $x^n \rightharpoonup x$ weak-$*$ in $L^\infty(H) \times L^\infty(L^2)$ for a further subsequence. Since B is bounded and linear it is weakly continuous so that we finally can proceed as in [15, chapter 3] to pass to the limit in the equation $0 = e(x^n, u^n) \to e(x, u)$ for $n \to \infty$, i.e. $x = x(u)$.

Finally, u is a solution to $(\hat{P})$ since the cost functional is weakly lower semi-continuous, so that $\hat{J}(u) \leq \liminf_{n \to \infty} \hat{J}(u^n) = d$.

As a consequence of the previous theorems, the implicit function theorem, and the suppositions on the functional J, the functional $\hat{J}$ is twice Fréchet differentiable with Lipschitz continuous second derivative.

In order to formulate necessary and sufficient optimality conditions we next specify the first and the second derivative of $\hat{J}$. For the first derivative we obtain

$$\langle \hat{J}'(u), \delta u \rangle_{U^* U} = \langle J_x(x, u), x'(u)\delta u \rangle_{W^* W} + \langle J_u(x, u), \delta u \rangle_{U^* U} \ .$$

Differentiation of the state equation $e(x, u) = 0$ yields

$$e_x(x, u)x'(u) + e_u(x, u) = 0 \quad \text{in} \quad Z^*$$

and thus

$$x'(u)\delta u = -e_x(x, u)^{-1} e_u(x, u)\delta u \ .$$

Introducing the adjoint variable $\lambda = (\mu, \mu_0, \kappa, \kappa_0) \in Z$ by

$$\lambda = e_x(x, u)^{-*} J_x(x, u)$$

we obtain

$$\hat{J}'(u) = J_u(x, u) - e_u(x, u)^* \lambda \ .$$

Note that in our setting $e_u(x, u^*) = (0, 0, -B^*, 0)$ holds.

Analogously, we obtain the second derivative of $\hat{J}$ by differentiating $e_x(x, u) = 0$ one more time to obtain

$$x''(u)(\delta u, \delta v) = -e_x^{-1} e_{xx}(x, u)(x'(u)\delta u, x'(u)\delta v) \ .$$

Using this, we find

$$\begin{aligned} \langle \hat{J}''(u)\delta u, \delta v \rangle_{U^* U} = {} & \langle J_{xx}(x, u)x'(u)\delta u, x'(u)\delta v \rangle_{W^* W} - \\ & \langle \lambda, e_{xx}(x, u)(x'(u)\delta u, x'(u)\delta v) \rangle_{Z, Z^*} + \langle J_{uu}(x, u)\delta u, \delta v \rangle_{U^*, U} \ . \end{aligned}$$

For example 1 we have

- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to L^2(\Gamma_T)^* \equiv L^2(\Gamma_T)$, so that B^* denotes the injection.
- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to L^2(0,T)^m,\ v \mapsto (B^*v)_i(t) = \langle f_i, v\rangle_{H^{-1/2}(\Gamma)H^{1/2}(\Gamma)}$, for $i = 1,\dots,m$.
- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to \mathbb{R}^m\ v \mapsto (B^*v)_i = \int\limits_0^T \langle g_i(t), v(t)\rangle_{H^{-1/2}(\Gamma)H^{1/2}(\Gamma)} dt$, for $i = 1,\dots,m$.
- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to H_0^*$ denotes the injection and we have

$$\begin{aligned}\langle Bu, f\rangle_{L^2(0,T;H^{-1/2}(\Gamma))L^2(0,T;H^{1/2}(\Gamma))} &= \\ \langle u, B^*f\rangle_{H_0H_0^*} = \langle u, f\rangle_{H_0H_0^*} &= \\ (u, Rf)_{H_0} &= \int_0^T \int_\Gamma [uRf + u_t(Rf)_t]dt\ ,\end{aligned}$$

where $R : H_0^* \to H_0$ denotes the Riesz operator, whose action in the present situation is defined through

$$w = Rf \iff \int_0^T \int_\Gamma [vw + v_tw_t]d\Gamma dt = \langle v, f\rangle_{H_0H_0^*}\ \forall\, v \in H_0\ .$$

Thus

$$B^*f = -w_{tt} + w\ .$$

For the cost functionals of example 1 we find

$$\langle J_x(x,u), \tilde{x}\rangle_{W^*W} = \begin{cases} \int_Q [(y-\bar{y})\tilde{y} + (\theta-\bar{\theta})\tilde{\theta}dxdt \\ \int_Q [-curl\, y\; curl\tilde{y} + \nabla\theta\,\nabla\tilde{\theta}]dxdt, \end{cases}$$

and $\langle J_u(x,u), v\rangle_{U^*,U} = \alpha\langle u, v\rangle_{U^*,U}$, so that in fact $J_x(x,u)$ is an element of $L^2(Q)^2 \times L^2(Q)$, or $L^2(V^*) \times L^2(0,T;H^1(\Omega)^*)$, respectively. In these cases the adjoint variable has the form $\lambda = (\mu, \mu_0, \kappa, \kappa_0)$ with $\mu \in L^2(V)$, $\mu_t \in L^{4/3}(V^*) \cap W_y^*$, $\kappa \in L^2(H^1(\Omega))$, $\kappa_t \in L^{4/3}(H^1(\Omega))^* \cap W_\theta^*$ (compare [11]),

$$\left.\begin{aligned}
-\mu_t - \nu\Delta\mu + (\nabla y)^t\mu - (y\cdot\nabla)\mu + \nabla\xi &= -\kappa\nabla\theta + \begin{cases}(y-\bar{y}) \\ -\text{curl curl}\, y\end{cases} && \text{in } \Omega_T,\\
-\text{div}\ \mu &= 0 && \text{in } \Omega_T,\\
\mu &= 0 && \text{on } \Gamma_T,\\
\mu(T) &= 0 && \text{in } \Omega,\\
\\
-\kappa_t - \chi\Delta\kappa - y\cdot\nabla\kappa &= -\beta\, g\cdot\mu + \begin{cases}(\theta-\bar{\theta}) \\ -\Delta\theta\end{cases} && \text{in } \Omega_T,\\
\chi\tfrac{\partial\kappa}{\partial n} + b\kappa &= 0 && \text{on } \Gamma,\\
\kappa(T) &= 0 && \text{in } \Omega,
\end{aligned}\right\} \tag{18}$$

and $\mu_0 = \mu(0), \kappa_0 = \kappa(0)$. We are now in the position to specify the first order necessary optimality condition for problem $\hat{P}$. Since $\hat{J}$ is Fréchet differentiable it reads

$$\langle \hat{J}'(u), v-u\rangle_{U^*U} \geq 0 \quad \text{for all } v \in U_{ad}\ .$$

Finally let us specify a second order sufficient condition for a solution u of our control problem.

Theorem 4. *Let u denote a solution of $(\hat{P})$, such that $J_x(x,u)$ is sufficiently small, where x denotes the state associated to u. Furthermore let us assume that $J_{uu}(x,u)$ is positive definite, i.e.*

$$\langle J_{uu}(x,u)v, v\rangle_{U^*U} \geq C\|v\|_U^2 \ ,$$

holds with some positive constant C, and $J_{xx}(x,u)$ is positive semi definite. Then $\hat{J}''(u)$ is positive definite.

Proof. We have

$$\begin{aligned}\langle \hat{J}''(u)v, v\rangle_{U^*U} = {}& \langle J_{xx}(x,u)x'(u)v, x'(u)v\rangle_{W^*W} - \\ & \langle \lambda, e_{xx}(x,u)(x'(u)v, x'(u)v)\rangle_{Z,Z^*} + \langle J_{uu}(x,u)v, v\rangle_{U^*,U} \\ & \geq C\|v\|_U^2 - \langle \lambda, e_{xx}(x,u)(x'(u)v, x'(u)v)\rangle_{Z,Z^*} \\ \geq C\|v\|_U^2 - c(u)\|v\|_U^2\|\lambda\|_Z \geq {}& C\|v\|_U^2 - c(u)\|v\|_U^2\|J_x(x,u)\|_{W^*} \geq \frac{C}{2}\|v\|_U^2 \ ,\end{aligned}$$

if $c(u)\|J_x(x,u)\|_{W^*} \leq \frac{C}{2}$.

Let us finally comment on the smallness assumption on J_x. For the tracking-type functional of Example 1 this assumption is satisfied if in the optimal solution the flowfield y and the temperature field θ are close to the desired fields $\bar{y}$ and $\bar{\theta}$, say. In the case of minimizing the *curl* and the temperature stresses J_x is small if these quantities are small in the optimal solution, which is an realistic assumption.

3 Conclusion

In the present work we provide an analytical framework for optimal boundary control of instationary Boussinesq systems in two spatial dimensions. Among other things, we use the results of [6] to prove existence and uniqueness of solutions to the adjoint of the Boussinesq approximation. Furthermore we derive a first order necessary optimality condition and prove a second order sufficient optimality condition under some smallness assumptions on the derivatives of the cost functional.

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