Discrete Models for the Cube–Root Differential Equation

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Abstract

Our main purpose is to construct one standard and three nonstandard finite difference schemes for the cube–root differential equation. After an analysis of the general qualitative features of the solutions to this equation and a calculation of the exact period, we study the stability of the numerical solutions for the four discretization schemes. Our general conclusion is that the standard forward–Euler method gives unstable numerical solutions, while the three nonstandard schemes provide suitable integration procedures.

1 Introduction

The *cube–root differential equation* (CRDE) is [1]

$$\ddot{x} + x^{\frac{1}{3}} = 0, \tag{1.1}$$

where the 'dot' notation is used to indicate the derivative, i.e., $\dot{x} \equiv dx/dt$, etc. The mathematical properties of solutions to this equation has been investigated by Mickens [1, 2] and a summary of these results will be presented in Section 2. Our purpose here is to construct finite difference schemes for the CRDE and check, using numerical simulations, if they reproduce the periodic solutions known to exist for Eq. (1.1) [1, 2]. The discretizations formulated consist of a standard forward–Euler scheme and three nonstandard schemes based on the methodology of Mickens [3, 4, 5]. Thus, our interest is not in determining the overall accuracy of these schemes, but rather the evaluation of which can provide numerical solutions having the *same qualitative properties* of the solutions to the CRDE.

In Section 2, we provide a brief summary of the general properties of the solutions to Eq. (1.1). The four finite difference schemes are given in Section 3 with short discussions on their derivations. Section 4 provides a summary of our numerical experiments on the four finite difference schemes. Finally, in Section 5, we discuss our results and come to several conclusions regarding the construction of discrete models such as given by Eq. (1.1).

It should be noted that, for purposes of the implementation of the various finite difference schemes, care must be taken with regard to the evaluation of $x^{1/3}$. Certain computer software give ambiguous or no numerical evaluations for negative x–values, e.g. MATLAB returns using $x^{1/3}$ a complex valued root; for the needed real valued cubic root one has to use the MATLAB function `nthroot(x,3)`. However, we have overcome this weakness by replacing $x^{1/3}$ by the following equivalent expression

$$x^{1/3} = [\text{sign}(x)]\,(|x|)^{\frac{1}{3}}, \tag{1.2}$$

where

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

In the remainder of the paper, it should be assumed that whenever $x_k^{1/3}$ appears, the corresponding form given in Eq. (1.2) was used for the actual numerical simulations.

To avoid the repetition of writing certain expressions over several times, the following abbreviations are used in this paper:

CRDE	:	cube–root differential equation
NSFD	:	nonstandard finite difference
NSFE	:	nonstandard forward–Euler
ODE	:	ordinary differential equation
SFE	:	standard forward–Euler

2 The Cubic–Root Differential Equation

The initial value problem for the CRDE is [6]

$$\ddot{x} + x^{\frac{1}{3}} = 0, \qquad x(0) = A, \quad \dot{x}(0) = 0. \tag{2.1}$$

The more general set of initial conditions, $x(0) = A$, and $\dot{x}(0) = B$, is not required since it can be easily shown that the analysis of the special case, in Eq. (2.1), gives the same relevant results as the general situation for $A \neq 0$ and $B \neq 0$ [7].

The CRDE can be written as a system of two coupled first–order ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x^{\frac{1}{3}}, \qquad x(0) = A, \quad y(0) = 0. \tag{2.2}$$

The variables, x and y, constituent a two dimensional phase–space [7]. Note that the fixed–point or constant solution for this system is $(\bar{x}, \bar{y}) = (0, 0)$. Further, in the (x, y) phase–space, the trajectories, $y = y(x)$, are solutions to the first–order ODE [7]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^{\frac{1}{3}}}{y}. \tag{2.3}$$

Since this equation is separable, the following *first integral* results from integrating Eq. (2.3)

$$\frac{y^2}{2} + \left(\frac{3}{4}\right)x^{\frac{4}{3}} = \left(\frac{3}{4}\right)A^{\frac{4}{3}}. \tag{2.4}$$

The integration constant on the right–side was evaluated by use of the initial conditions in Eq. (2.2). Since all of the terms in the first integral are non–negative for all values of x and y, Eq. (2.4) represents a closed curve in the phase–space. This results implies that all of the solutions of the CRDE are periodic except for the fixed–point located at the origin [1, 6, 7].

The ODE for the trajectories in phase–space, Eq. (2.3), is invariant under the following three transformations

$$T_1 : x \to -x, \quad y \to y;$$
$$T_2 : x \to x, \quad y \to -y;$$
$$T_3 : x \to -x, \quad y \to -y.$$

They correspond, respectively, to reflection through the y–axis, reflection through the x–axis and inversion through the origin. These symmetries can also be used to show directly that all the trajectories in phase–space are closed curves and hence all solutions to the CRDE are periodic [1, 7].

Since Eq. (2.1) has (all) periodic solutions, it is of interest to determine the period T of these solutions, i.e.,

$$x(t+T) = x(t). \tag{2.5}$$

The following expression gives the period [2]

$$T(A) = \sqrt{\frac{32}{2}} \int_0^A \frac{dx}{\sqrt{A^{\frac{4}{3}} - x^{\frac{4}{3}}}}. \tag{2.6}$$

After a rather involved set of calculations, we obtain

$$T(A) = (2\sqrt{6})\phi A^{\frac{1}{3}}, \tag{2.7}$$

where

$$\phi = \int_0^1 \sqrt{\frac{w}{(1+w)(1-w)}}\, dw \tag{2.8}$$

has a known numerical value represented in terms of complete elliptic integrals of the first and second kinds [8]. Putting all this together gives

$$T(A) = (5.86966)A^{\frac{1}{3}}. \tag{2.9}$$

From a physical point of view, the CRDE represents a nonlinear conservative oscillator for which the elastic force term is $x^{1/3}$. This fact will be useful for the construction of one of our NSFD schemes.

In summary, the CRDE corresponds to a nonlinear conservative oscillator. Except for the fixed–point or equilibrium solution, all other solutions are periodic with constant amplitudes.

3 Discretizations of the CRDE

The first finite difference scheme for the CRDE is the standard forward–Euler representation [9, 10] for the system equations; see Eq. (2.2). For this case, we have

$$\frac{x_{k+1}-x_k}{h} = y_k, \quad \frac{y_{k+1}-y_k}{h} = -x_k^{\frac{1}{3}}, \tag{3.1}$$

where

$$t \to t_k = hk, \quad h = \Delta t; \quad k = 0, 1, 2, \dots,$$
$$x(t) \to x(t_k) = x_k.$$

Solving for x_{k+1} and y_{k+1} gives

$$x_{k+1} = x_k + hy_k, \quad y_{k+1} = y_k - hx_k^{\frac{1}{3}}. \tag{3.2}$$

We call this the *SFE scheme* for the CRDE.

The second scheme is

$$\frac{x_{k+1}-x_k}{h} = y_k, \quad \frac{y_{k+1}-y_k}{h} = -x_{k+1}^{\frac{1}{3}}. \tag{3.3}$$

Solving these expressions for x_{k+1} and y_{k+1} gives the following expressions

$$x_{k+1} = x_k + hy_k, \quad y_{k+1} = y_k - h(x_k + hy_k)^{\frac{1}{3}}. \tag{3.4}$$

Note that for this scheme, the $x^{1/3}$ term is evaluated at t_{k+1} rather than t_k, as was the case for the SFE discretization. We denote the scheme, Eq. (3.4), by the notation *NSFE–1*.

For the third scheme we use the discrete construction

$$\frac{x_{k+1}-x_k}{h} = y_{k+1}, \quad \frac{y_{k+1}-y_k}{h} = -x_k^{\frac{1}{3}}, \tag{3.5}$$

and these equations can be rewritten to the forms

$$x_{k+1} = x_k + hy_k - h^2 x_k^{\frac{1}{3}}, \quad y_{k+1} = y_k - hy_k^{\frac{1}{3}}. \tag{3.6}$$

This is scheme *NSFE–2*.

The fourth scheme is derived from formulations derived by Mickens [11], and Anguelov and Lubuma [12]. While the derivations appear to be completely different, for the CRDE the same NSFD scheme is obtained. The basic idea is to construct a discrete form for the first integral of the CRDE; see Eq. (2.4). Denoting this by $H_k = H(x_k, x_{k-1})$, we have [11]

$$H_k = \frac{1}{2}\Big(\frac{x_k - x_{k-1}}{h}\Big)^2 + \frac{3}{4}\, x_k^{\frac{2}{3}}\, x_{k-1}^{\frac{2}{3}} = \text{const.} \tag{3.7}$$

Since H_k is constant, then

$$H_{k+1} = H_k. \tag{3.8}$$

After some algebraic work, Eq. (3.8) can be rewritten to the form

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + \left[\frac{3x_k^{\frac{2}{3}}}{x_{k+1}^{\frac{2}{3}} + x_{k+1}^{\frac{1}{3}} x_{k-1}^{\frac{1}{3}} + x_{k-1}^{\frac{2}{3}}}\right]\left(\frac{x_{k+1}^{\frac{1}{3}} + x_{k-1}^{\frac{1}{3}}}{2}\right) = 0 \tag{3.9}$$

Observe that the second–order derivative d^2x/dt^2 is represented by a (standard) central difference scheme, while the $x^{1/3}$ term has a very complex algebraic structure. It is very clear that the scheme of Eq. (3.9) would not be formulated within the context of standard numerical methods for ODEs [11, 12]. We denote the scheme of Eq. (3.9) by *NSFE–3*.

4 Numerical Results

Numerical simulations were carried out for the four finite difference schemes constructed for the CRDE using the step size $h = 0.01$ until the final time $T = 100$ (i.e. 10,000 steps). For all of the simulations, the initial values were selected to be

$$x(0) = 1, \qquad \dot{x}(0) = y(0) = 0; \tag{4.1}$$

consequently, $x_0 = 1$ and $y_0 = 0$. The NSFE–3 scheme is a second order difference equation and therefore both x_0 and x_1 are needed. The value for x_1 was calculated by the following evaluation:

$$\begin{aligned} x_1 = x(h) &= x(0) + h\,\dot{x}(0) + \frac{h^2}{2}\,\ddot{x}(0) + \mathrm{O}(h^3) \\ &= x_0 + hy(0) + \frac{h^2}{2}\Big[-(x_0)^{\frac{1}{3}}\Big] + \mathrm{O}(h^3) \\ &= 1 - \frac{h^2}{2} + \mathrm{O}(h^3). \end{aligned} \tag{4.2}$$

The first two terms on the right–side of Eq. (4.2) were then used to calculate x_1.

Figure 1 presents the results for the standard forward–Euler scheme; see Eq. (3.2). Observe that this scheme produces numerical solutions that oscillate with increasing amplitude. This behaviour is clearly inconsistent with the known properties of the CRDE: all its solutions oscillate with constant amplitude, i.e., the general solution (except for the fixed–point) is periodic. Therefore, we conclude that the SFE scheme is unstable and should not be used to calculate numerical solutions for the CRDE.

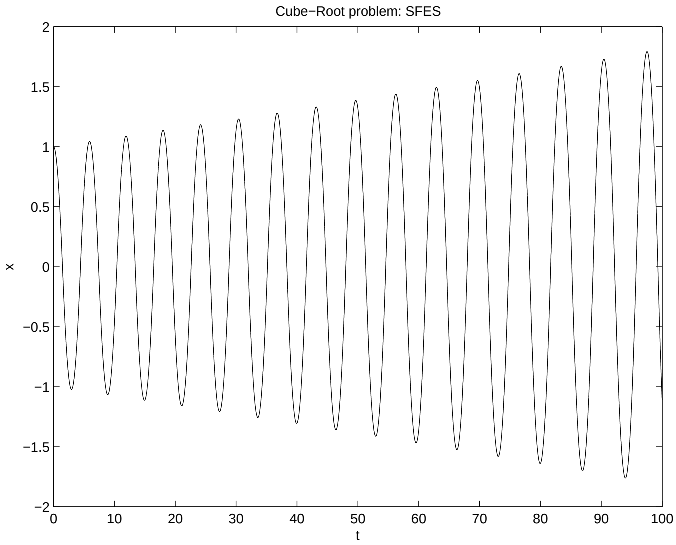

Figure 1: Numerical solution to the CRDE using the SFE scheme.

It should be indicated, for dynamic systems having either a linear or nonlinear center fixed–point, the use of forward–Euler discretizations, in general, transforms the center into an unstable node for the corresponding difference scheme [13]. In addition to the paper of Wang et al. [13], further work on this issue has been carried out by Mickens [14], Sanz–Serna [15], Mickens [16], and Roeger [17].

Of both interest and importance, the three nonstandard finite difference schemes gave essentially identical numerical results for the integration of the CRDE; the corresponding three schemes are given in Eqs. (3.4), (3.6), and (3.9). Observe, from Figure 2, that for all three nonstandard schemes the amplitudes of the oscillations are constant and that computationally

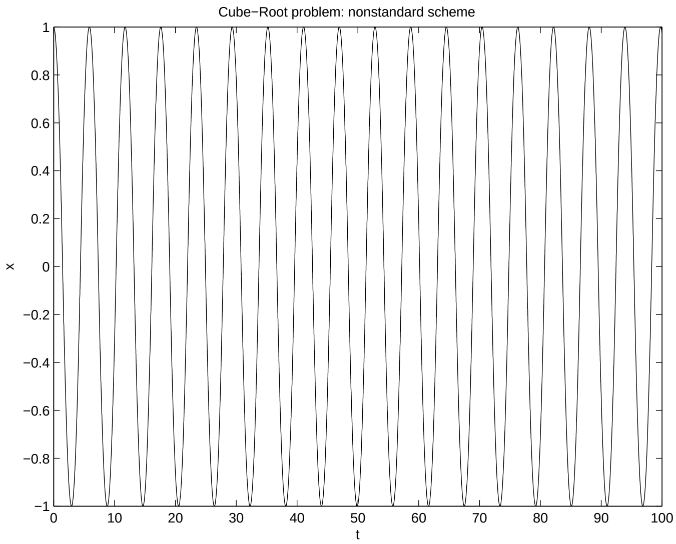

Figure 2: Numerical solution to the CRDE using one of the nonstandard schemes.

the solutions are periodic. Clearly the NSFE schemes produce numerical solutions having all of the essential mathematical features of the solutions of the CRDE.

Finally, we tried for a comparison, the standard MATLAB one–step ODE solver `ode45` that is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair [18]. We prescribed a relative error tolerance of 10^{-4} and computed the solution until the extended final time $T = 300$. In Figure 3, one clearly observes that the amplitude of the oscillations is decaying in time. Thus, this solution behaviour is another motivation to use nonstandard schemes like Eqs. (3.4), (3.6), and (3.9) to solve the CRDE numerically.

5 Discussion

Our work has demonstrated that the use of nonstandard finite difference schemes allows the construction of dynamic consistent [19] discretizations for the CRDE. However, it would be of value to gain insight as to why the SFE scheme did not work, i.e., why it produced an unstable solution. To gain this insight into a possible mechanism, we will analyze a much simpler

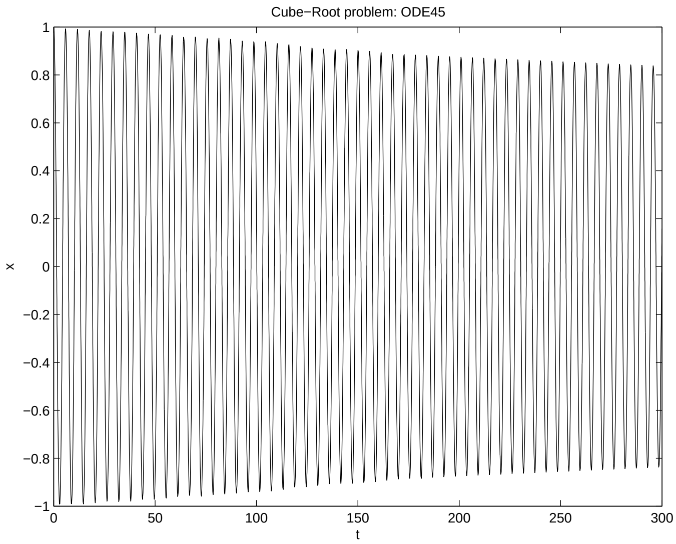

Figure 3: Numerical solution to the CRDE using the MATLAB solver `ode45`.

system, namely, the linear harmonic oscillator [4]

$$\ddot{x} + x = 0, \tag{5.1}$$

which can be written in the system form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x. \tag{5.2}$$

The SFE scheme for these equations are

$$\frac{x_{k+1} - x_k}{h} = y_k, \quad \frac{y_{k+1} - y_k}{h} = -x_k. \tag{5.3}$$

Eliminating y_k and shifting the index k down by one gives the following second–order difference equation

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + x_{k-1} = 0. \tag{5.4}$$

To find the *approximating differential equation* [4] make an expansion of x_{k-1},

$$\begin{aligned} x_{k-1} &= x(t_k - h) \\ &= x(t_k) - h\dot{x}(t_k) + \mathrm{O}(h^2), \end{aligned} \tag{5.5}$$

substitute this expression into Eq. (5.4), and keep terms to order h. Carrying out this procedure gives the ODE

$$\ddot{x} - h\dot{x} + x = 0. \tag{5.6}$$

Note that the solutions to this *modified harmonic oscillator equation* oscillate with increasing amplitude [4]. Thus, the conclusion is that the SFE scheme for the harmonic oscillator equation changes the center fixed–point, for the ODE, into an unstable node for the difference scheme.

We believe that the same mechanism is at work when the SFE scheme is applied to the CRDE. This clearly is indicated by the elimination of y_k in Eq. (3.1) to obtain

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + x_{k-1}^{\frac{1}{3}} = 0, \tag{5.7}$$

and this second–order nonlinear difference equation has a structure similar to Eq. (5.4).

In summary, we have constructed four finite difference schemes to numerically integrate the CRDE

$$\ddot{x} + x^{\frac{1}{3}} = 0. \tag{5.8}$$

All of the NSFD schemes produced numerical solutions that were dynamically consistent with the actual solutions of the original differential equation. However, the SFE scheme solutions were unstable and this fits in with previous experiences with this scheme [13]–[17]. The conclusion is that NSFD schemes can provide valid discretizations for the CRDE. A future research problem is to carry out a mathematical and discretization analysis of the related nonlinear ODE

$$\ddot{x} + \frac{1}{x^{\frac{1}{3}}} = 0. \tag{5.9}$$

Phase space analysis and the existence of a first–integral shows that all solutions to Eq. (5.9) are periodic.

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