Fixed Domain Transformations and Split–Step Finite Difference Schemes for Nonlinear Black–Scholes Equations for American Options

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Abstract. Due to transaction costs, illiquid markets, large investors or risks from an unprotected portfolio the assumptions in the classical Black–Scholes model become unrealistic and the model results in strongly or fully nonlinear, possibly degenerate, parabolic diffusion–convection equations, where the stock price, volatility, trend and option price may depend on the time, the stock price or the option price itself.

In this chapter we will be concerned with several models from the most relevant class of nonlinear Black–Scholes equations for American options with a volatility depending on different factors, such as the stock price, the time, the option price and its derivatives.

We will analytically approach the option price by following the ideas proposed by Ševčovič and transforming the free boundary problem into a fully nonlinear nonlocal parabolic equation defined on a fixed, but unbounded domain. Finally, we will present the results of a split–step finite difference scheme for various volatility models including the Leland model, the Barles and Soner model and the Risk adjusted pricing methodology model.

1 Introduction

The strong interest in pricing *financial derivatives* – among them in pricing options – arises from the fact that financial derivatives, also called contingent claims, can be used to minimize losses caused by price fluctuations of the underlying assets. This process of protection is called *hedging*. There is a variety of financial products on the market, such as futures, forwards, swaps and options. In this chapter we will focus on American Call options.

We recall that an *American Call* option is a contract where at any time before a prescribed time in the future, known as the *expiry date* T, the owner of the option, known as the *holder*, may purchase a prescribed asset, known as the *underlying asset* $S(t)$, for a prescribed amount, known as the exercise or *strike price* K. The opposite party, or the *writer*, has the obligation to sell the asset if the holder chooses to excercise his right. The value of the American Call option at the time of execution, known as the *pay-off function*, is

$$V(S,t) = (S-K)^+.$$

Option pricing theory has made a great leap forward since the development of the Black–Scholes option pricing model by Black and Scholes [6] in 1973 and previously by

Merton [44]. The solution of the famous *(linear) Black–Scholes equation*

$$0 = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV, \quad 0 < S < S_f(t),\, t \in (0,T), \tag{1}$$

where V denotes the value of the option and r the riskless interest rate, provides both an option pricing formula for an American Call option and a hedging portfolio that replicates the contingent claim. This is true under the assumption that the market is *complete*, which means that any derivative and any asset can be replicated or hedged with a portfolio of other assets in the market, cf. [61].

However, this assumption of a complete market is never fulfilled in reality. Due to transaction costs [4, 8, 41], large investor preferences [23, 24, 50] and incomplete markets [55] the classical model results in strongly or fully nonlinear, possibly degenerate, parabolic convection–diffusion equations, where both the volatility σ and the drift μ can depend on the time t, the stock price S or the derivatives of the option price V itself. Here, we will be concerned with several transaction cost models from the most relevant class of nonlinear Black–Scholes equations for American options with a constant drift μ and a nonconstant *modified volatility function*

$$\widetilde{\sigma}^2 := \widetilde{\sigma}^2(t, S, V_S, V_{SS}).$$

Under these circumstances (1) becomes the following *nonlinear Black–Scholes equation*:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS}) S^2 V_{SS} + rSV_S - rV, \quad S > 0,\, t \in (0,T) \tag{2}$$

Studying (1) for an American Call option would be redundant, since the value of an American Call option equals the value of a European Call option if no dividends are paid and the volatility is constant. In order to make the model more realistic, we will consider a modification of (2) for American options, where S pays out a *continuous dividend* $qSdt$ in a time step dt:

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS}) S^2 V_{SS} + (r-q)SV_S - rV, \quad S > 0,\, t \in (0,T), \tag{3}$$

where the dividend yield q is constant.

Remark 1.1 *Most dividend payments on an index — such as the Dow Jones Industrial Average (DJIA) or the Standard and Poor's 500 (S&P500) — are so frequent that they can be modeled as a continuous payment, which is the case in* (3)*. However, if companies only make two or four dividend payments per year, then one has to treat the dividend payments discretely and the question of how to incorporate* discrete dividend payments *into the Black–Scholes equation arises.*

Even though in this work we will focus on the case of continuous dividend payments, we briefly review the results for discrete dividend payments from [63] in the sequel.

We assume that there is only one dividend payment of the dividend yield q during the lifetime of the option at the dividend date t_q. *Neglecting other factors, such as taxes, the* *asset price S must decrease exactly by the amount of the dividend payment q at time t_q. Thus we have the* jump condition

$$S(t_q^+) = (1-q)S(t_q^-),$$

where t_q^-, t_q^+ denote the moments just before and after the dividend date t_q. This leads to the following effect on the option price:

$$V(S, t_q^-) = V((1-q)S, t_q^+), \tag{4}$$

i.e. the value of the option at S and time t_q^- is the same as the value immediately after the dividend date t_q but at the asset value $(1-q)S$. In order to calculate the value of a Call option with one dividend payment we solve the Black–Scholes equation from expiry $t = T$ until $t = t_q^+$ and use the relation (4) *to compute the values at $t = t_q^-$. Finally, we continue to solve the Black–Scholes equation backwards starting at $t = t_q^-$ using these values as the initial data. The boundary conditions, that are discussed in the next section, do not need to be modified for this case.*

In the mathematical sense equations (2) and (3) are called convection–diffusion equations. The second-order term $\frac{1}{2}\widetilde{\sigma}^2(t, S, V_S, V_{SS})S^2V_{SS}$ is responsible for the *diffusion*, the first-order term rSV_S or $(r-q)SV_S$ is the *convection* term and $-rV$ can be interpreted as the *reaction* term (cf. [53, 62]).

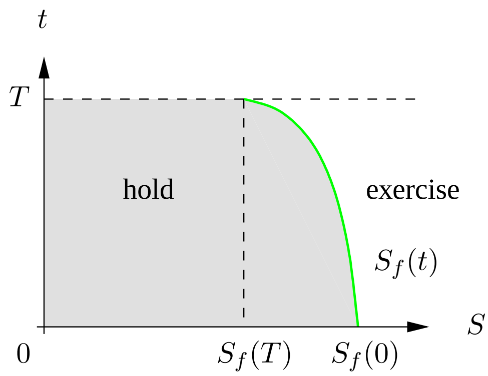
In the financial sense, the partial derivatives indicate the sensitivity of the option price V to the corresponding parameter and are called *Greeks*. The option delta is denoted by $\Delta = V_S$, the option gamma by $\Gamma = V_{SS}$ and the option theta by $\theta = V_t$ [33].

Since American options can be exercised at any time before expiry, we need to find the optimal time t of exercise, known as the *optimal exercise time*. At this time, which mathematically is a *stopping time*, the asset price reaches the *optimal exercise price* or *optimal exercise boundary* $S_f(t)$.

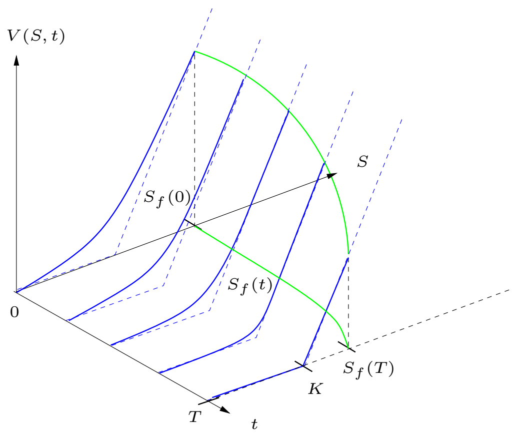
This leads to the formulation of the problem for American options by dividing the domain $[0,\infty[\times[0,T]$ of (3) into two parts along the curve $S_f(t)$ and analyzing each of them (see Fig. 1(a)). Since $S_f(t)$ is not known in advance but has to be determined in the process of the solution, the problem is called *free boundary value problem* (see e.g. [67]).

For different numerical approaches, the free boundary problem for American options can be reformulated into a *linear complementary problem*, a *variational inequality* and a *minimization problem* [26]. Here, we will only consider the formulation as a free boundary problem.

For the American Call option the *spatial* domain is divided into two regions by the *free boundary* $S_f(t)$, the *stopping region* $S_f(t) < S < \infty$, $0 \le t \le T$, where the option is exercised or dead with $V(S,t) = S - K$ and the *continuation region* $0 \le S \le S_f(t)$, $0 \le t \le T$, where the option is held or stays alive and equation (3) is valid under the



(a) Exercising and holding regions.



(b) Schematical value $V(S,t)$.

Figure 1: American Call.

following terminal and boundary conditions:

$$
\begin{aligned}
V(S,T) &= (S-K)^+ && \text{for } 0 \le S \le S_f(T)\\
V(0,t) &= 0 && \text{for } 0 \le t \le T\\
V(S_f(t),t) &= S_f(t) - K && \text{for } 0 \le t \le T\\
V_S(S_f(t),t) &= 1 && \text{for } 0 \le t \le T\\
S_f(T) &= \max(K, rK/q). &&
\end{aligned}
\tag{5}
$$

For the sake of simplicity we will assume $r > q$ in this chapter, and therefore we have $S_f(T) = rK/q$ for the American Call.

The structure of the value of an American Call can be seen Fig. 1(b), where we notice that the free boundary $S_f(t)$ determines the position of the exercise.

For American options, in general, analytic valuation formulae are not available, except for a few special types, which we are not going to address in this chapter. Those types are Calls on an asset that pays discrete dividends and *perpetual* Calls — meaning Calls with an infinite time to expiry [40]. For the other types, there are various kinds of analytical and numerical approximations that will be discussed in this chapter.

The structure of this chapter is as follows. In Section 2 several nonconstant volatility models that lead to the nonlinearity of the Black–Scholes equation will be introduced. The focus of this chapter is the solution of the resulting nonlinear problems for American Call options. Since in general, a closed–form solution to the nonlinear Black–Scholes equation for American options does not exist (even in the linear case), we have to solve the problems numerically. The numerical solution and the comparison study for American options will be achieved by the transformation of the free boundary problem (3) subject to (5) into a forward-in-time parabolic equation defined on a fixed (but unbounded) spatial domain (Section 3). This new problem will be numerically solved by the method of finite differences using an operator splitting technique (Section 4). It will then be evaluated and concisely discussed in Section 5 thereafter.

2 Volatility Models

The essential parameter of the standard Black–Scholes model, that is not directly observable and is assumed to be constant, is the volatility σ. There have been many approaches to improve the model by treating the volatility in different ways and using a *modified volatility function* $\widetilde{\sigma}(\cdot)$ to model the effects of transaction costs, illiquid markets and large traders, which is the reason for the nonlinearity of (2) and (3). In this section we will first give a brief overview of several volatility models and then focus on the volatility models of transaction costs.

- The constant volatility σ in the standard Black–Scholes model can be replaced by the estimated volatility from the former values of the underlying. This volatility is known as the *historical volatility* [26].

- If the price of the option and the other parameters are known, which is e.g. the case for the European Call and Put options, then the *implied volatility* can be calculated from those Black–Scholes formulae. The implied volatility is the value σ, for which the Black–Scholes equation. is true compared to the real market data. It can be calculated implicitly via the difference between the observed option price V (from the market data) and the Black–Scholes formulae, where all the parameters - except for the implied volatility σ - are taken from the market data (the stock price S, the time t, the expiration date T, the strike price K, the interest rate r the dividend rate q).

  Considering options with different strike prices K but otherwise identical parameters, we see that the implicit volatility changes depending on the strike price. If the implicit volatility for a certain strike price K is less than the implicit volatility for both the strike price greater and less than K, this effect is called *volatility smile* (see e.g. [39]).

- Replacing the constant volatility with the observed implicit volatilities at each stock price and time leads to the term of the *local volatility* $\widetilde{\sigma} := \widetilde{\sigma}(S,t)$. Dupire [16] examines the dependencies and expresses the local volatility as a function of implicit volatilities.

- Hull and White [32] and Heston [28] develop a model, in which the volatility follows the dynamics of a stochastic process. This is known as the *stochastic volatility*.

- The assumption, that each security is available at any time and any size, or that individual trading will not influence the price, is not always true. Therefore, illiquid markets and large trader effects have been modeled by several authors. In [23] Frey and Stremme and later Frey and Patie [24] consider these effects on the price and come up with the result

$$\widetilde{\sigma} = \frac{\sigma}{1 - \rho\lambda(S)SV_{SS}}, \tag{6}$$

where σ is the historical volatility, ρ is constant, $\lambda(S)$ is a strictly convex function and $\lambda(S) \geq 1$. The function $\lambda(S)$ depends on the pay-off function of the financial derivative. For the European Call option, Frey and Patie show that $\lambda(S)$ is a smooth, slightly increasing function for $S \geq K$. Bordag and Chmakova [7] assume that $\lambda(S)$ is constant and solve the problem (2) with the modified volatility (6) explicitly using Lie-group theory (see also [12]).

The main scope of this chapter is the numerical solution of the nonlinear Black–Scholes equation for the American Call option, where the nonlinearity results from transaction costs. Therefore, after this general overview, we devote our attention to a more detailed description of several transaction cost models.

2.1 Transaction Costs

The Black–Scholes model requires a continuous portfolio adjustment in order to hedge the position without any risk. In the presence of transaction costs it is likely that this adjustment easily becomes expensive, since an infinite number of transactions is needed [40]. Thus, the hedger needs to find the balance between the transaction costs that are required to rebalance the portfolio and the implied costs of hedging errors. As a result to this "imperfect" hedging, the option might be over- or underpriced up to the extent where the riskless profit obtained by the arbitrageur is offset by the transaction costs, so that there is no single equilibrium price but a range of feasible prices. It has been shown that in a market with transaction costs there is no replicating portfolio for the European Call option and the portfolio is required to dominate rather than replicate the value of the option (see [4]). Soner, Shreve and Cvitanič prove in [54] that the minimal hedging portfolio that dominates a European Call is the trivial one (hence holding one share of the stock that the Call is written on), so that efforts have been made to find an alternate relaxation of the hedging conditions to better replicate the pay-offs of derivative securities.

2.1.1 The model of Leland

Leland's idea of relaxing the hedging conditions is to trade at discrete times [41], which promises to reduce the expenses of the portfolio adjustment. He assumes that the transaction cost $\kappa|\Delta|S/2$, where κ denotes the round trip transaction cost per unit dollar of the transaction and Δ the number of assets bought ($\Delta > 0$) or sold ($\Delta < 0$) at price S, is proportional to the monetary value of the assets bought or sold. Leland derives the relation

$$rB\delta t - \frac{\kappa}{2}|\delta\Delta|S = (V_t + \frac{\sigma^2}{2}S^2V_{SS})\delta t, \tag{7}$$

where B is the bond and r the riskless interest rate, and shows that

$$\frac{\kappa}{2}|\delta\Delta|S = \frac{\sigma^2}{2}\,\mathrm{Le}\,S^2|V_{SS}|\delta t. \tag{8}$$

Here, Le denotes the *Leland number*, which is given by

$$\mathrm{Le} = \sqrt{\frac{2}{\pi}}\left(\frac{\kappa}{\sigma\sqrt{\delta t}}\right), \tag{9}$$

with δt being the transaction frequency (interval between successive revisions of the portfolio) and κ the round trip transaction cost per unit dollar of the transaction. Plugging (8) and $B = \Pi - \Delta S = V - SV_S$ into the equation (7) becomes

$$rV - rSV_S - \frac{\sigma^2}{2}LeS^2|V_{SS}| = V_t + \frac{\sigma^2}{2}S^2V_{SS}. \tag{10}$$

Therefore, Leland deduces that the option price is the solution of the nonlinear Black–Scholes equation

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2S^2V_{SS} + rSV_S - rV,$$

with the *modified volatility*

$$\widetilde{\sigma}^2 = \sigma^2\Big(1 + \mathrm{Le}\,\mathrm{sign}(V_{SS})\Big), \tag{11}$$

where σ represents the historical volatility and Le the Leland number. It follows from the definition of the Leland number (9) that the more frequent the rebalancing (δt smaller), the higher the transaction cost and the greater the value of V.
Leland's model has played a significant role in financial mathematics, even though it has been partly criticized by e.g. Kabanov and Safarian in [37], who prove that Leland's result has a hedging error. The restriction of his model is the convexity of the resulting option price V (hence $V_{SS} > 0$) and the possibility to only consider one option in the portfolio. Hoggard, Whalley and Wilmott study equation (2) with the modified volatility (11) for several underlyings in [30]. An extension to this approach to general pay–offs is obtained by Avellaneda and Parás in [3].

2.1.2 The model of Barles and Soner

In [4] Barles and Soner derive a more complicated model by following the *utility function approach* of Hodges and Neuberger [29].

Supposing that the proportional transaction cost κ is equal to $a\sqrt{\varepsilon}$ for some constant $a > 0$, they prove that as ε and κ go to 0, V is the unique (viscosity) solution of the nonlinear Black–Scholes equation

$$0 = V_t + \frac{1}{2}\widetilde{\sigma}^2S^2V_{SS}^2 + rSV_S - rV,$$

where

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + \Psi(e^{r(T-t)}a^2S^2V_{SS})\right). \tag{12}$$

Here σ denotes the historical volatility, $a = \kappa/\sqrt{\varepsilon}$ and $\Psi(x)$ is the solution to the following nonlinear ordinary differential equation (ODE)

$$\Psi'(x) = \frac{\Psi(x)+1}{2\sqrt{x\Psi(x)} - x},\ x \neq 0, \tag{13a}$$

with the initial condition

$$\Psi(0) = 0. \tag{13b}$$

The analysis of this ODE (13) by Barles and Soner in [4] implies that

$$\lim_{x\to\infty}\frac{\Psi(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to-\infty}\Psi(x) = -1. \tag{14}$$

The property (14) encourages to treat the function $\Psi(\cdot)$ as the identity for large arguments and therefore to simplify the calculations. In this case the volatility becomes

$$\widetilde{\sigma}^2 = \sigma^2(1 + e^{r(T-t)}a^2S^2V_{SS}). \tag{15}$$

The existence of a viscosity solution to (2) for European options with the volatility given by (12) is proved by Barles and Soner in [4] and their numerical results indicate an economically significant price difference between the standard Black–Scholes model and the nonlinear model with transaction costs.

2.1.3 Risk Adjusted Pricing Methodology

In this model, proposed by Kratka in [39] and improved by Jandačka and Ševčovič in [35], the optimal time-lag δt between the transactions is found to minimize the sum of the rate of the transaction costs and the rate of the risk from an unprotected portfolio. That way the portfolio is still well protected with the Risk Adjusted Pricing Methodology (RAPM) and the *modified volatility* is now of the form

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + 3\big(\frac{C^2M}{2\pi}SV_{SS}\big)^{\frac{1}{3}}\right), \tag{16}$$

where $M \geq 0$ is the transaction cost measure and $C \geq 0$ the risk premium measure.

It is worth mentioning that these nonlinear transaction cost models that are described above are all consistent with the linear model if the additional parameters for transaction costs are equal to zero and vanish (Le, $\Psi(\cdot)$, M). We will study these models – more precisely equations (2) and (3) where the volatility is given by the equations (11), (12), (15) and (16) – for American Call options.

In general, an exact analytical solution leading to a closed expression is not known for American options in a market with transaction costs. In the next section we will analytically approach the solution of (3) by a transformation, that facilitates the numerical solution in the section thereafter. We will compare and evaluate the results in the section thereafter.

3 The fixed Domain Transformation

The equation (3) subject to (5) is a backward-in-time free boundary problem. In order to ease the numerical solution of (3) (5) for American Call options, we transform the problem into a problem posed on a a fixed (unbounded) domain additionally to the forward transformation in time. Hence, the domain does not depend on the free boundary $S_f(t)$ anymore and we simply calculate an algebraic constraint equation for the position of the free boundary. Following the idea of Ševčovič [52] we change the variables to:

$$\tau = T - t, \quad x = \ln\left(\frac{\varrho(\tau)}{S}\right) \Leftrightarrow S = e^{-x}\varrho(\tau), \quad \varrho(\tau) = S_f(T-\tau),$$

such that $x \in \mathbb{R}^+$ and $\tau \in [0, T]$.
Then, we construct a portfolio

$$\Pi(x,\tau) = V(S,t) - SV_S(S,t)$$

by buying $\Delta = V_S$ shares S and selling an option V. Differentiating Π with respect to x and τ gives us

$$\Pi_x = V_S S_x - S_x V_S - SV_{SS}S_x = S^2 V_{SS}$$

and

$$\begin{aligned}\Pi_\tau &= V_S S_\tau + V_t t_\tau - S_\tau V_S - S(V_{SS}S_\tau + V_{St}t_\tau)\\ &= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)}S^2 V_{SS} + SV_{St}\\ &= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)}\Pi_x - S\partial_S(-V_t).\end{aligned} \tag{17}$$

Substituting

$$-V_t = \frac{\widetilde{\sigma}^2}{2}S^2 V_{SS} - r(V - SV_S) - qSV_S = \frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi - qSV_S$$

from (3) into (17) and using the fact that $-S\partial_S = \partial_x$, we get

$$\begin{aligned}\Pi_\tau &= \frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi - qSV_S - \frac{\varrho'(\tau)}{\varrho(\tau)}\Pi_x + \partial_x\left(\frac{\widetilde{\sigma}^2}{2}\Pi_x - r\Pi\right) + S\partial_S(qSV_S)\\ &= \left(\frac{\widetilde{\sigma}^2}{2} - \frac{\varrho'(\tau)}{\varrho(\tau)} - r + q\right)\Pi_x - r\Pi + \frac{1}{2}\partial_x(\widetilde{\sigma}^2\Pi_x).\end{aligned}$$

We therefore obtain

$$0 = \Pi_\tau + \big(b(\tau) - \frac{\widetilde{\sigma}^2}{2}\big)\Pi_x - \frac{1}{2}\partial_x(\widetilde{\sigma}^2\Pi_x) + r\Pi, \tag{18}$$

defined on $x \in \mathbb{R}^+$, $0 \le \tau \le T$, where the coefficient $b(\tau)$ is

$$b(\tau) = \frac{\varrho'}{\varrho}(\tau) + r - q.$$

The terminal condition from (5) in the original variables (S,T) becomes the initial condition in the new variables $(x,0)$:

$$\begin{aligned}\Pi(x,0) &= V(S,T) - SV_S(S,T)\\ &= \begin{cases} -K & \text{for } S > K \Leftrightarrow x < \ln\frac{\varrho(0)}{K} \\ 0 & \text{otherwise}\end{cases}.\end{aligned} \tag{19a}$$

and the boundary conditions from (5) transform to

$$\Pi(x,\tau) = 0 \qquad \text{as } x \to \infty,\ 0 \le \tau \le T, \tag{19b}$$

$$\Pi(0,\tau) = -K \qquad \text{for } 0 \le \tau \le T. \tag{19c}$$

To complete the system of equations that enables the computation of the portfolio Π we need to use the last two conditions of (5) to obtain an expression at the free boundary position $\varrho(\tau)$. Differentiating and evaluating $V(S_f(t),t) = S_f(t) - K$ at the free boundary gives us

$$V_S(S_f(t),t)S_f'(t) + V_t(S_f(t),t) = S_f'(t).$$

Using (5), we conclude that

$$V_t(S_f(t),t) = 0 \text{ for } 0 \le \tau \le T.$$

Computing (3) at the point $(S_f(t),t)$ or at $(0,\tau)$ in the transformed variables yields:

$$\begin{aligned}0 &= V_t(S_f(t),t) + \frac{1}{2}\widetilde{\sigma}^2\Pi_x(0,\tau) + (r-q)S_f(t)V_S(S_f(t),t) - rV(S_f(t),t)\\ &= \frac{1}{2}\widetilde{\sigma}^2\Pi_x(0,\tau) + rK - q\varrho(\tau).\end{aligned}$$

We remind the reader that we have assumed $r \ge q$ and therefore we obtain the last condition:

$$\varrho(\tau) = \frac{1}{2q}\widetilde{\sigma}^2\Pi_x(0,\tau) + \frac{rK}{q} \quad \text{with} \quad \varrho(0) = \frac{rK}{q}, \tag{19d}$$

where $0 \le \tau \le T$ and $\widetilde{\sigma}^2$ depends on the volatility model we choose. The volatility (11) from Leland's model becomes

$$\widetilde{\sigma}^2 = \sigma^2\Big(1 + Le\,\mathrm{sign}(\Pi_x)\Big), \tag{20a}$$

for Barles' and Soner's model (12) we get

$$\widetilde{\sigma}^2 = \sigma^2(1 + \Psi(e^{r\tau}a^2\Pi_x)), \tag{20b}$$

for the identity model (15) we obtain

$$\widetilde{\sigma}^2 = \sigma^2(1 + e^{r\tau}a^2\Pi_x), \tag{20c}$$

and for the Risk Adjusted Princing Methodology (16) we derive

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + 3\big(\frac{C^2 M}{2\pi}\Pi_x \varrho(\tau) e^{-x}\big)^{\frac{1}{3}}\right). \tag{20d}$$

This transformed problem (18) subject to (19) with the corresponding volatilities (20) is solved by the split-step finite-difference method proposed by Ševčovič [52]

Once we have numerically solved the transformed problem by calculating the solution to our portfolio $\Pi(x,\tau)$ and the free boundary $\varrho(\tau)$, we calculate the value of the American Call $V(S,t)$ option by transforming

$$\Pi(x,\tau) = V(S,t) - SV_S(S,t)$$

back to the original variables. Since we know that

$$\frac{\Pi(x,\tau)}{S^2} = \frac{V(S,t)}{S^2} - \frac{V_S(S,t)}{S} = \partial_S\left(-\frac{V(S,t)}{S}\right),$$

we integrate the above equation from S to $S_f(t)$, take into account the boundary condition $V(S_f(t),t) = S_f(t) - K$ and obtain:

$$\begin{aligned}
\int_S^{S_f(t)} \frac{\Pi(\ln(\varrho(\tau)/S),\tau)}{S^2} dS &= \int_S^{S_f(t)} \partial_S\left(-\frac{V(S,t)}{S}\right) dS \\
\int_{\ln\frac{\varrho(\tau)}{S}}^{\ln\frac{\varrho(\tau)}{S_f(t)}} \frac{\Pi(x,\tau)}{S^2}(-S)dx &= -\frac{V(S_f(t),t)}{S_f(t)} + \frac{V(S,t)}{S} \\
S\int_0^{\ln\frac{\varrho(\tau)}{S}} \frac{\Pi(x,\tau)}{e^{-x}\varrho(\tau)} dx &= -S\frac{\varrho(\tau)-K}{\varrho(\tau)} + V(S,t)
\end{aligned}$$

$$V(S,T-\tau) = \frac{S}{\varrho(\tau)}\left(\varrho(\tau) - K + \int_0^{\ln\frac{\varrho(\tau)}{S}} e^x \Pi(x,\tau)dx\right). \tag{21}$$

Therefore, (21) yields the price of the American Call option $V(S,t)$ in the presence (and absence) of transaction costs.

4 Numerical Solution

Due to the lack of general closed–form solutions to the Black–Scholes equations, there are various numerical methods for solving Black–Scholes equations for American options.

For European Call and Put options, the Black–Scholes formulae provide the correct answer, but for more complicated contracts in more general settings analytical formulae are seldom available and numerical methods have to be used to solve the problem. These vary from lattice methods (including binomial and trinomial approximations [14]), Monte-Carlo methods using the least-square techniques [34], analytical approximations [5, 11, 42], finite-element discretizations [26] to finite-difference methods [2, 9, 13].

There are numerous other methods for pricing American options including the method of lines [45], front-tracking algorithms [64], penalty methods [68] and many others. One of the standard approaches for solving the Black–Scholes equation for American options consists of the transformation of the original equation into the heat equation posed on a semi–unbounded domain with a free boundary $S_f(t)$ [53, 63]. For a new alternative direct method using the Mellin transformation we refer to [36, 47].

Up to now, an exact analytical formula for the free boundary profile $S_f(t)$ in (3) subject to (5) is not known, but several authors derived approximate expressions to evaluate American Call and Put options in the linear case [25]. Recently, in a promising approach [51], Ševčovič obtained a semi–explicit formula for an American Call in the case of $r > q$. By transforming the linear Black–Scholes equation for the American Call option into a nonlinear parabolic equation on a fixed domain and applying Fourier sine and cosine transformations, he derives a nonlinear singular integral equation determining the shape of the free boundary. This integral equation can be solved effectively by the means of successive iterations.

Another standard method consists of the reformulation of the free boundary problem into a linear complementary problem (LCP) and the solution by the *Projected Successive Over Relaxation (PSOR)* method of Cryer [15]. Alternatively, penalty and front–fixing methods are developed (e.g. in [22, 46]). A disadvantage of these methods is the change of the underlying model.

A different approach [31] is based on a recursive calculation of the early exercise boundary, estimating the boundary only at some points and then approximating the whole boundary by Richardson extrapolation. Explicit boundary tracking algorithms are e.g. a *finite-difference bisection scheme* [38] or the *front–tracking strategy* of Han and Wu [27].

This emphasis of this chapter is on finite-difference schemes, thus other methods will not be further elaborated on here. For more information on numerical methods we refer the reader to [48, 49, 66] and the references therein.

4.1 American Call option

Now we want to solve the transformed problem from the previous section.

$$0 = \Pi_\tau + \big(b(\tau) - \frac{\widetilde{\sigma}^2}{2}\big)\Pi_x - \frac{1}{2}\partial_x(\widetilde{\sigma}^2\Pi_x) + r\Pi, \quad x \in \mathbb{R}^+, \quad 0 \le \tau \le T \qquad (22)$$

with the corresponding volatilities (20) subject to the conditions

$$
\begin{aligned}
&\Pi(x,0) = \begin{cases} -K & \text{for } x < \ln \frac{\varrho(0)}{K} \\ 0 & \text{otherwise} \end{cases} \\
&\Pi(x,\tau) = 0 \quad \text{as } x \to \infty,\ 0 \le \tau \le T, \\
&\Pi(0,\tau) = -K \quad \text{for } 0 \le \tau \le T,
\end{aligned} \tag{23}
$$

and the constraint

$$
\varrho(\tau) = \frac{1}{2q}\widetilde{\sigma}^2 \Pi_x(0,\tau) + \frac{rK}{q} \quad \text{with} \quad \varrho(0) = \frac{rK}{q}. \tag{24}
$$

We therefore first describe the solution of (22) subject to (23) and (24) with the corresponding volatilities (20) by finite-difference schemes and then present the numerical results.

4.2 Finite-Difference Schemes

There have been many approaches to calculate the value of an American option numerically by compact finite-difference schemes in the absence of transaction costs. Recently, Tangman et al. [59, 60] introduced a compact scheme of order $(2, 4)$. Two other compact schemes, known as the *Numerov-type* (see [58,66]) and the *Crandall-Douglas scheme* (see [43]), are analyzed for linear Black–Scholes equations. However, these schemes are not directly transferable to the model in the presence transaction costs.

In order to find a solution for the nonlinear Black–Scholes equation (22) subject to (23) with the corresponding volatilities (20) and the constraint (24), Ševčovič suggests to combine two approaches that solve the problem for the American Call with a constant volatility numerically [52]. One of them is the transformation of the problem into a variational inequality and its solution by the PSOR algorithm [26, 53]. The other one is the derivation of a nonlinear integral equation for the position of the free boundary without the knowledge of the price itself [40, 64].

Even though these methods are not directly applicable, since they require a constant volatility σ, this approach is successful when it is combined with an operator splitting technique. The idea is to discretize (22) in time, to split the equation into a convective and a diffusive part and to find an approximation for the solution pair (Π, ϱ) at each time level. The detailed derivation is given in the sequel.

4.2.1 Grid

We discretize the problem (22) subject to the conditions (23) with the corresponding volatilities (20) by confining the unbounded domain $x \in \mathbb{R}^+$ and $\tau \in [0, T]$ to $x \in (0, R)$ with $R > 0$ sufficiently large (see [52]). For the calculation Ševčovič chooses to take $R = 3$, since this is equivalent to $S \in (S_f(t)e^{-R}, S_f(t))$ and yields a good approximation for $S \in (0, S_f(t))$ (as the transformation was $S = S_f(t)e^{-x}$). In the sequel we refer to $h > 0$

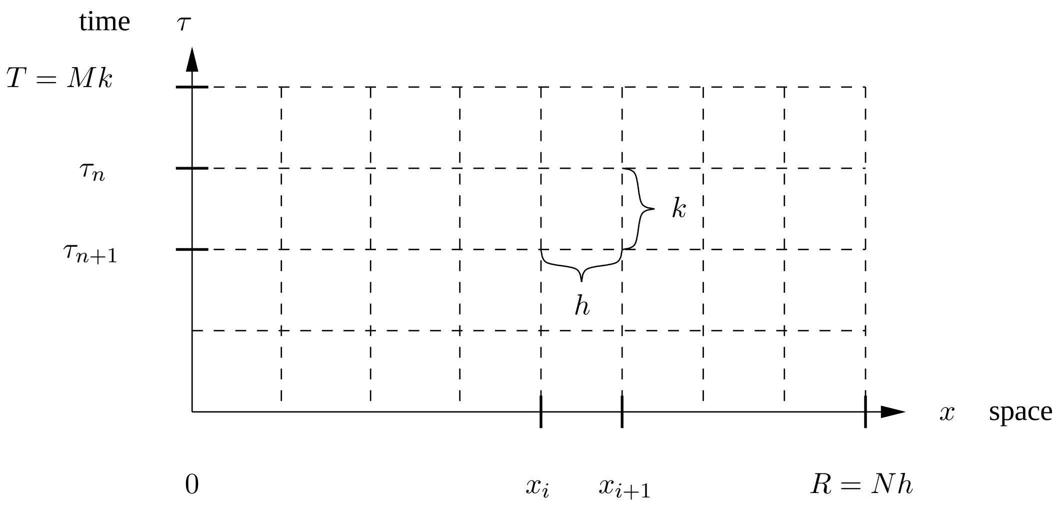


Figure 2: Uniform grid for an American Call option.

as the spatial step and to $k > 0$ as the time step, $x_i = ih$, $i \in [0, N]$, $R = Nh$ and $\tau_n = nk$, $n \in [0, M]$, $T = Mk$ (see Fig. 2).

The approximate solution of (22) in x_i at time τ_n is denoted by $\Pi_i^n := \Pi(x_i, \tau_n)$, the value of the free boundary at time τ_n by $\varrho^n := \varrho(\tau_n)$ and the value of the coefficient $b(\tau)$ at τ_n by $b^n := b(\tau_n)$.

We treat the initial and boundary conditions (23) in the following way:

$$\begin{aligned} \Pi_i^0 &= \Pi(x_i, 0) = \begin{cases} -K & \text{for } x_i < \ln \frac{\varrho(0)}{K} = \ln \frac{r}{q} \\ 0 & \text{otherwise} \end{cases}, \\ \Pi_0^n &= -K, \\ \Pi_N^n &= 0. \end{aligned} \tag{25}$$

4.2.2 Difference Quotients

We denote the forward difference quotient with respect to the spatial variable in x_i at time τ_n with the spatial step size h by:

$$D_h^+ \Pi_i^n := \frac{\Pi_{i+1}^n - \Pi_i^n}{h} \approx \Pi_x(x_i, \tau_n),$$

the backward difference quotient by:

$$D_h^- \Pi_i^n := \frac{\Pi_i^n - \Pi_{i-1}^n}{h} \approx \Pi_x(x_i, \tau_n)$$

and the central difference quotient by

$$D_h^0 \Pi_i^n := \frac{\Pi_{i+1}^n - \Pi_{i-1}^n}{2h} \approx \Pi_x(x_i, \tau_n),$$

omitting the truncation error $\mathrm{O}(h)$, $\mathrm{O}(h)$ and $\mathrm{O}(h^2)$, respectively. For the second spatial derivative we introduce the standard difference quotient

$$D_h^2 \Pi_i^n := \frac{\Pi_{i+1}^n - 2\Pi_i^n + \Pi_{i-1}^n}{h^2} \approx \Pi_{xx}(x_i, \tau_n),$$

with the error term $\mathrm{O}(h^2)$.

4.3 Volatility Functions

The volatilities (20) can all be written in the form

$$\left(\widetilde{\sigma}_i^n\right)^2 = \sigma^2(1 + s_i^n),$$

where s_i^n denotes the volatility correction in x_i at time τ_n. We choose forward differences to approximate Π_x in the volatility formulae, so that for Leland's model with the volatility (20a) our volatility correction becomes

$$s_i^n = Le\, \mathrm{sign}(D_h^+ \Pi_i^n), \tag{26a}$$

for the volatility correction in Barles' and Soner's model with the volatility (20b) we get

$$s_i^n = \Psi\big(e^{r\tau_n} a^2 D_h^+ \Pi_i^n\big), \tag{26b}$$

for the volatility correction in case of treating $\Psi(\cdot)$ as the identity with the original volatility (20c) we obtain

$$s_i^n = e^{r\tau_n} a^2 D_h^+ \Pi_i^n, \tag{26c}$$

and for the volatility (20d) in the Risk Adjusted Pricing Methodology (RAPM) the volatility correction is

$$s_i^n = 3\left(\frac{C^2 M}{2\pi} D_h^+ \Pi_i^n \varrho^n e^{-x_i}\right)^{\frac{1}{3}}. \tag{26d}$$

4.4 The Treatment of the Free Boundary

We discretize the free boundary (24) by approximating the spatial derivative at the origin $x = 0$ by forward differences and obtain:

$$\varrho^n = \frac{1}{2q}\sigma^2(1 + s_0^n) D_h^+ \Pi_0^n + \frac{rK}{q} \quad \text{with} \quad \varrho^0 = \frac{rK}{q}, \tag{27}$$

where s_0^n denotes (26) at $x = 0$ depending on the volatility model.

Note, that in case of the RAPM, where the volatility correction is given by equation (26d), s_0^n depends on ϱ^n and therefore ϱ^n in (27) is expressed by a fixed point equation.

Remark 4.1 *For the American Call option (in contrast to the American Put option) it is possible to derive a series for the location of the optimal exercise boundary close to expiry using standard asymptotic analysis [1, 63]. This local analysis of the free boundary* $S_f(t)$ *yields*

$$S_f(t) \sim S_f(T)\left(1+\xi_0\sqrt{\frac{1}{2}\sigma^2(T-t)}+\dots\right), \qquad \textit{as } t \to T, \tag{28}$$

where $\xi_0 = 0.9034\dots$ *is a* universal constant *of Call option pricing. Equation* (28) *can be rewritten as*

$$\varrho(\tau) \sim \varrho(0)\left(1+\xi_0\sqrt{\frac{1}{2}\sigma^2(\tau)}+\dots\right), \qquad \textit{as } \tau \to 0. \tag{29}$$

With only very few terms we get a fairly accurate result for the free boundary and thus equation (29) *will serve us as a check for the case of a constant volatility* $\widetilde{\sigma}^2 = \sigma^2$ *(see Fig. 3). Note that this result is especially useful in the first time levels of a numerical calculation where rapid changes in* $\varrho(\tau)$ *influence the whole solution region.*

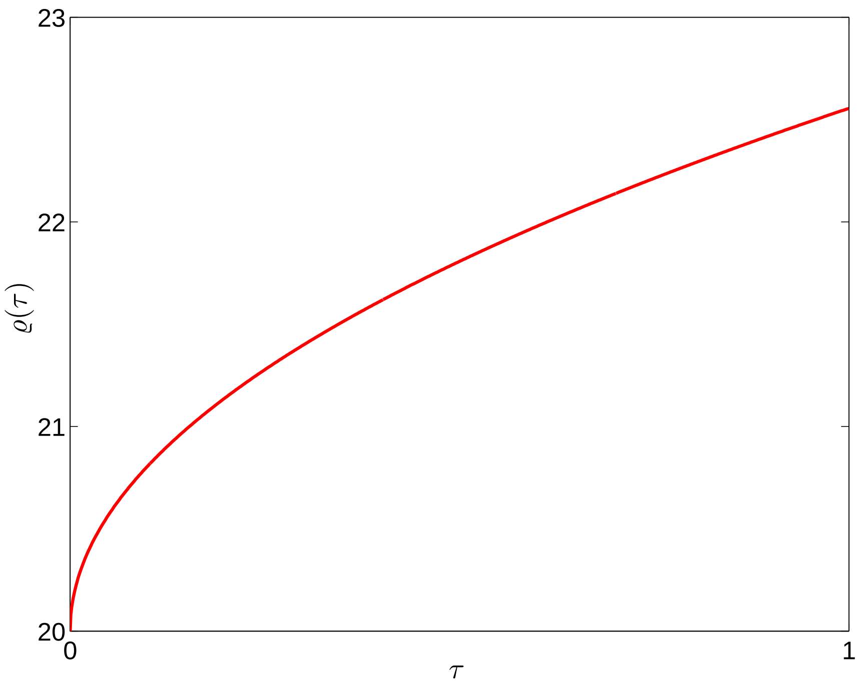


Figure 3: Asymptotic solution for the free boundary $\varrho(\tau)$ with $T = 1$, $K = 10$, $\sigma = 0.2$, $r = 0.1$, $q = 0.05$.

4.5 The Splitting in Time Method

We approximate the time derivative of (22) by backward differences $D_k^- \Pi_i^n$, the first and second spatial derivatives by central differences $D_h^0 \Pi_i^n$ and $D_h^2 \Pi_i^n$. Then, (22) becomes:

$$0 = D_k^- \Pi_i^n + \big(b^n - \frac{\sigma^2}{2}(1+s_i^n)\big) D_h^0 \Pi_i^n - \frac{1}{2}\partial_x \big(\sigma^2 (1+s_i^n) D_h^0 \Pi_i^n\big) + r\Pi_i^n \tag{30}$$

subject to the Dirichlet conditions (25). We introduce an intermediate step at time $\tau_{n-\frac{1}{2}}$, such that

$$D_k^- \Pi_i^n = \frac{\Pi_i^n - \Pi_i^{n-1}}{k} = \frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}} + \Pi_i^{n-\frac{1}{2}} - \Pi_i^{n-1}}{k},$$

and then split the problem (30) into a *convective part* with the linear first-order term $b^n D_h^0 \Pi_i^n$:

$$0 = \frac{\Pi_i^{n-\frac{1}{2}} - \Pi_i^{n-1}}{k} + b^n D_h^0 \Pi_i^n \tag{31}$$

and a *diffusive part* with the nonlinear first- and second-order terms $\sigma^2/2(1+s_i^n) D_h^0 \Pi_i^n$ and $-\partial_x\big(\sigma^2/2(1+s_i^n) D_h^0 \Pi_i^n\big)$:

$$0 = \frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}}}{k} - \frac{\sigma^2}{2}(1+s_i^n) D_h^0 \Pi_i^n - \frac{1}{2}\partial_x\big(\sigma^2(1+s_i^n) D_h^0 \Pi_i^n\big) + r\Pi_i^n. \tag{32}$$

Assuming that $D_h^0 \Pi_i^n \approx D_h^0 \Pi_i^{n-\frac{1}{2}}$, which is reasonable for small time steps k, we can approximate the convective part (31) as

$$0 = \frac{\Pi_i^{n-\frac{1}{2}} - \Pi_i^{n-1}}{k} + b^n D_h^0 \Pi_i^{n-\frac{1}{2}}. \tag{33}$$

Now the solution to (32)-(33) gives a good approximation to the solution of (30) (see [52]). This decomposition of the problem is called *Lie-Splitting* and is a spitting of order 1 in time.

4.5.1 Convective part

First, we solve the convective part (33), which can be approximated by an explicit solution to the *transport equation*

$$\Pi_\tau + b(\tau)\Pi_x = 0, \tag{34}$$

for $(x,\tau) \in \mathbb{R} \times [0,T]$, subject to the boundary and initial conditions

$$\begin{aligned} \Pi(0,\tau) &= -K, \\ \Pi(x,0) &= \begin{cases} -K & \text{for } x < \ln\frac{r}{q} \\ 0 & \text{otherwise} \end{cases} = \Pi^0(x). \end{aligned} \tag{35}$$

We then know by the theory of partial differential equations (see e.g. [20]) that the solution for this problem (34)–(35) is

$$\Pi(x,\tau) = \Pi(x - \int_0^\tau b(s)\,ds, 0) = \Pi^0(x - \int_0^\tau b(s)\,ds) \tag{36}$$

with the primitive function $\int b(s)\,ds = B(\tau) + c = \ln \varrho(\tau) + (r-q)\tau + c$. Hence, considering the problem (34) for $(x_i, \tau_j) \in \mathbb{R} \times [\tau_{n-1}, \tau_n]$ subject to the boundary and initial conditions

$$\begin{aligned} \Pi(0,\tau_j) &= -K, \\ \Pi(x_i, \tau_{n-1}) &= \Pi^{n-1}(x_i), \end{aligned} \tag{37}$$

we know that the solution is given by

$$\begin{aligned} \Pi(x_i,\tau_j) &= \Pi\Big(x_i - \int_{\tau_{n-1}}^{\tau_j} b(s)\,ds, \tau_{n-1}\Big) \\ &= \begin{cases} \Pi(\xi_i^j, \tau_{n-1}) & \text{for } \xi_i^j > 0 \\ -K & \text{otherwise}, \end{cases}. \end{aligned} \tag{38}$$

where we set $\xi_i^j = x_i - B(\tau_j) + B(\tau_{n-1}) = x_i - \ln \frac{\varrho^j}{\varrho^{n-1}} - (\tau_j - \tau_{n-1})(r-q)$. Then we can write

$$\Pi_i^{n-\frac{1}{2}} = \begin{cases} \Pi(\xi_i^n, \tau_{n-1}) & \xi_i^n = x_i - \ln \frac{\varrho^n}{\varrho^{n-1}} - k(r-q) > 0 \\ -K & \text{otherwise.} \end{cases} \tag{39}$$

Here, we use a linear approximation between the discrete values $\Pi(x_i, \tau_{n-1})$, $i \in \mathbb{N}$ in order to compute the value of $\Pi(\xi_i^n, \tau_{n-1})$.
Hence, (39) is the solution to the convective part (33) of the problem (30).

4.5.2 Diffusive part

We solve the diffusive part (32) of the problem (30) by the finite-difference method. We approximate the second spatial derivative by central differences $D_h^2 \Pi_i^n$ and the first spatial derivative by both central $D_h^0 \Pi_i^n$ and backward differences $D_h^- \Pi_i^n$. Then, (32) becomes:

$$\begin{aligned} 0 &= \frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}}}{k} - \frac{\sigma^2}{2}(1+s_i^n)\frac{\Pi_{i+1}^n - \Pi_{i-1}^n}{2h} + r\Pi_i^n \\ &\quad - \frac{\sigma^2}{2}\Big((1+s_i^n)\frac{\Pi_{i+1}^n - 2\Pi_i^n + \Pi_{i-1}^n}{h^2} + \frac{(1+s_i^n) - (1+s_{i-1}^n)}{h}\frac{\Pi_i^n - \Pi_{i-1}^n}{h}\Big) \\ &= \frac{\Pi_i^n - \Pi_i^{n-\frac{1}{2}}}{k} - \frac{\sigma^2}{2}(1+s_i^n)\frac{\Pi_{i+1}^n - \Pi_{i-1}^n}{2h} + r\Pi_i^n \\ &\quad - \frac{\sigma^2}{2}\Big((1+s_i^n)\frac{\Pi_{i+1}^n - \Pi_i^n}{h^2} - (1+s_{i-1}^n)\frac{\Pi_i^n - \Pi_{i-1}^n}{h^2}\Big). \end{aligned}$$

Rearranging leads to a tridiagonal system of equations

$$\Pi_i^{n-\frac{1}{2}} = a_i^n \Pi_{i-1}^n - b_i^n \Pi_i^n + c_i^n \Pi_{i+1}^n, \tag{40}$$

with the coefficients

$$\begin{aligned}
a_i^n &= \frac{\sigma^2}{2}(1+s_i^n)\frac{k}{2h} - \frac{\sigma^2}{2}(1+s_{i-1}^n)\frac{k}{h^2},\\
b_i^n &= 1 + kr + \frac{\sigma^2}{2}(1+s_i^n)\frac{k}{h^2} + \frac{\sigma^2}{2}(1+s_{i-1}^n)\frac{k}{h^2},\\
c_i^n &= -\frac{\sigma^2}{2}(1+s_i^n)\frac{k}{2h} - \frac{\sigma^2}{2}(1+s_i^n)\frac{k}{h^2}.
\end{aligned}$$

Equation (40) can be written in the form of matrices:

$$\Pi^{n-\frac{1}{2}} = A^n \Pi^n + d^n, \tag{41}$$

where

$$\Pi^n = \left(\Pi_1^n, \cdots, \Pi_{N-1}^n\right)^\top \in \mathbb{R}^{N-1},$$

$$A^n = \begin{pmatrix}
b_1^n & c_1^n & 0 & \cdots & 0\\
a_2^n & b_2^n & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & b_{N-2}^n & c_{N-2}^n\\
0 & \cdots & 0 & a_{N-1}^n & b_{N-1}^n
\end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)},$$

and

$$d^n = \left(a_1^n \Pi_0^n, \quad 0, \quad \cdots, \quad 0, \quad c_{N-1}^n \Pi_N^n\right)^\top \in \mathbb{R}^{N-1}.$$

Therefore, (41) solves the diffusive part (32) of the problem (30).

Now, we have a set of nonlinear equations (26), (27), (39) and (41) that delivers the solution to our portfolio $\Pi(x,\tau)$ and to the free boundary $\varrho(\tau)$, from which we can calculate the value of the American Call option $V(S,t)$ with equation (21).

In order to see the dependencies of the equations, we rewrite them in the following abstract form:

$$\begin{aligned}
s^n &= \mathcal{D}(\Pi^n, \varrho^n),\\
\varrho^n &= \mathcal{F}\big(\Pi^n, s^n\big) = \mathcal{F}\big(\Pi^n, \varrho^n\big),\\
\Pi^{n-\frac{1}{2}} &= \mathcal{G}\big(\Pi^{n-1}, \varrho^n, \varrho^{n-1}\big) = \mathcal{G}\big(\Pi^{n-1}, \varrho^n\big),\\
A\big(s^n\big)\Pi^n = A\big(\Pi^n, \varrho^n\big)\Pi^n &= \Pi^{n-\frac{1}{2}} - d(s^n),
\end{aligned} \tag{42}$$

where

$$s^n = \left(\; s_0^n, \quad \cdots, \quad s_N^n \; \right)^\top \in \mathbb{R}^{N+1},$$

$\mathcal{D}(\cdot)$ denotes the right-hand side of (26), $\mathcal{F}(\cdot)$ is the right-hand side of (27), $\mathcal{G}(\cdot)$ is the right-hand side of the transport equation (39), $A(\cdot)$ is the tridiagonal matrix and $d(s^n)$ the vector as defined in (41).

As we can see by this notation (42), both ϱ^n and Π^n are given in terms of themselves, hence each is given in terms of ϱ^n and Π^n. This problem can be approximately solved by a successive fixed point iteration over $p \in \mathbb{N}$ at each time level n.

Following Ševčovič [52] we define for $n \geq 1$: $\Pi^{n,0} = \Pi^{n-1}$, $\varrho^{n,0} = \varrho^{n-1}$ and $s^{n,0} = s^{n-1}$. Then the (p+1)-th approximation of Π^n, ϱ^n and s^n is obtained as the solution of the system:

$$\begin{aligned} s^{n,p+1} &= \mathcal{D}(\Pi^{n,p}, \varrho^{n,p}), \\ \varrho^{n,p+1} &= \mathcal{F}\big(\Pi^{n,p}, s^{n,p+1}\big), \\ \Pi^{n-\frac{1}{2},p+1} &= \mathcal{G}\big(\Pi^{n-1,p}, \varrho^{n,p+1}\big), \\ A\big(s^{n,p+1}\big)\Pi^{n,p+1} &= \Pi^{n-\frac{1}{2},p+1} - d(s^{n,p+1}). \end{aligned} \tag{43}$$

Both the volatility correction $s_i^{n,p+1}$, the free boundary $\varrho^{n,p+1}$ and the solution $\Pi^{n-\frac{1}{2},p+1}$ to the convective part (31) can be directly computed from (26), (27) and (39) respectively. The solution $\Pi^{n,p+1}$ to the diffusive part (32) has to be calculated from the system of equations (41).

Assuming that the system (43) converges to some limiting values $s^{n,p_{max}}$, $\varrho^{n,p_{max}}$, $\Pi^{n-\frac{1}{2},p_{max}}$ and $\Pi^{n,p_{max}}$ at each time level n [52], we can calculate $V(S_i, t_n) = V(e^{-x_i}\varrho^n, T-\tau_n)$ with these values and proceed to the next time level $n+1$.

From (21) we then know that:

$$V(S_i, t_n) = e^{-x_i}\big(\varrho^n - K + \mathcal{I}_i\big), \tag{44}$$

where

$$\begin{aligned} \mathcal{I}_i &= \sum_{j=0}^{i-1} \mathcal{I}_k + \int_{x_{i-1}}^{x_i} e^x \Pi(x,\tau)dx \\ &= \sum_{j=0}^{i-1} \mathcal{I}_k + \frac{x_i - x_{i-1}}{2}\bigg(e^{x_{i-1}}\Pi_{i-1}^n + e^{x_i}\Pi_i^n\bigg). \end{aligned}$$

Here, we use the *trapezoidal rule* in order to approximate the integral in equation (21).

We summarize the calculation of the price $V(S,t)$ for the American Call option in the presence or absence of transaction costs by the Algorithm 1 given in the appendix.

5 Comparison Study

Based on the iterative algorithm described in the previous section (Algorithm 1), we solve the transformed Black–Scholes equation (22) with the corresponding volatilities (20) for the American Call option and finally transform $\Pi(x,\tau)$ back to the original option price $V(S,t)$.

The main purpose of this section is to compare the resulting option value $V(S,t)$ and the free boundary $S_f(T-t) = \varrho(\tau)$ for the four different transaction cost models (26) to the linear model (σ constant) and to each other.

We choose $p_{max} = 5$ for the successive iteration over p in our algorithm in order to solve the system (42) with the precision of 10^{-7} [52]. We use the following parameters to calculate $\Pi(x,\tau)$ and $\varrho(\tau)$:

$$r = 0.1, \quad \sigma = 0.2, \quad K = 10, \quad T = 1 \text{ (one year)}, \quad R = 3.$$

We start by comparing the free boundary $\varrho(\tau)$ computed with Algorithm 1 to the asymptotic solution (29) from Remark 4.1 for the linear case ($s_i^n = 0$). In Fig. 4 we observe that for smaller spatial steps $h \to 0$ the free boundary computed by the iterative algorithm converges monotonically towards the asymptotic solution (29) from below.

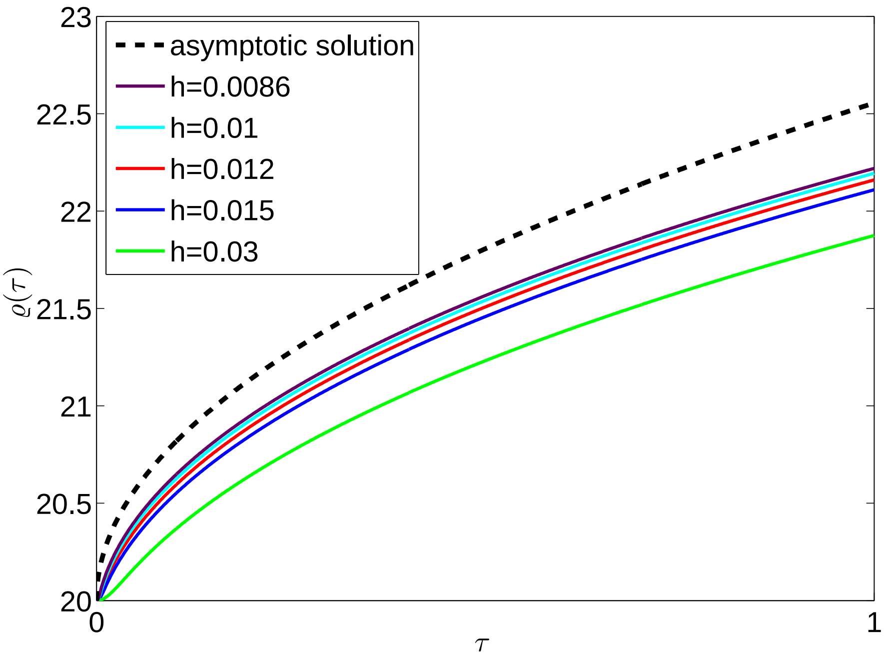


Figure 4: Free boundary for various spatial steps h with a constant time step $k = 0.0008$ and a constant volatility σ^2 computed by Algorithm 1 vs. the asymptotic solution of (29).

We keep the time step $k = 0.0008$ constant and see that for $h = 0.0086$ (purple line) the free boundary at T is computed by our algorithm as $\varrho(T) \approx 22.2201$. The asymptotic solution at T is $\varrho(T) \approx 22.5552$, which means a relative error of 1.49%. The free boundary values for the other spatial steps can be seen in Table 1.

h	0.03	0.015	0.012	0.01	0.0086
$\varrho(T)$	21.8764	22.1111	22.1619	22.1955	22.2201

Table 1: Values of the free boundary position for various spatial steps h with a constant time step $k = 0.0008$ and a constant volatility σ^2.

Since the asymptotic solution of (29) is only an approximation, we are satisfied by our results and take the free boundary $\varrho(T) \approx 22.1111$ for $k = 0.0008$, $h = 0.015$ (blue line in Fig. 4) as our reference solution in the absence of transaction costs for the sake of the computational time.

Fig. 5 shows the structure of the price for the American Call option $V(S,t)$ without transaction costs with $k = 0.0008$ and $h = 0.015$. It is computed with the iterative algorithm described in the previous sections and the parameters above.

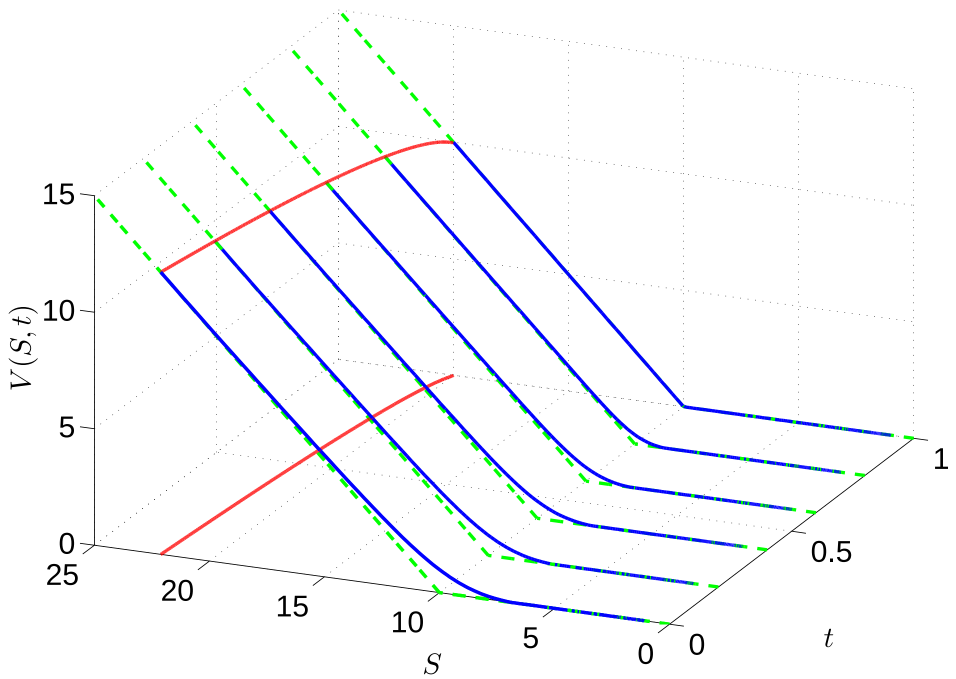
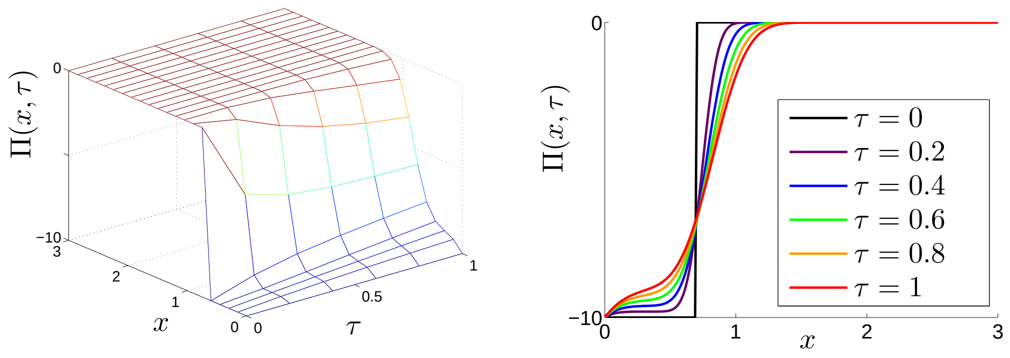


Figure 5: Value of an American Call option $V(S,t)$ in the absence transaction costs computed with Algorithm 1 determined by the free boundary (red line).

The corresponding synthetic portfolio $\Pi(x,\tau)$ in the absence of transaction costs is illustrated in Fig. 6. Note, that we include rounding and discretization errors when transforming $\Pi(x,\tau)$ back into $V(S,t)$, since equation (44) involves an integral approximation. However, the analysis of $V(S,t)$ is more interesting for us and we therefore assume that these errors are sufficiently small due to the chosen mesh.

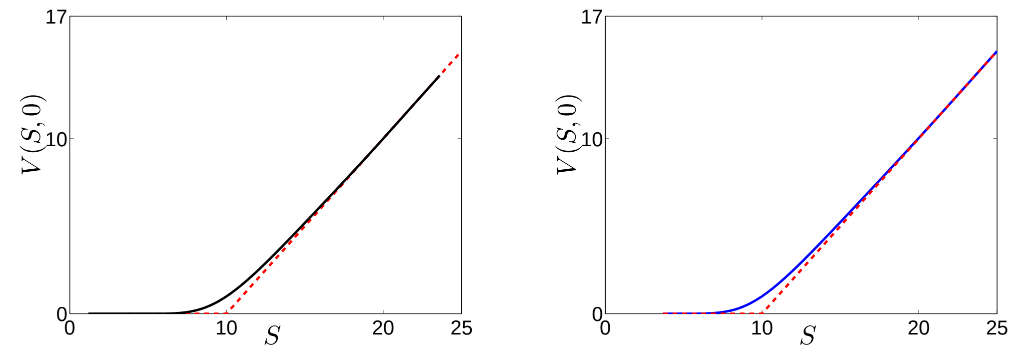


(a) 3-D profile.

(b) Profile at different time points.

Figure 6: Value of the synthetic portfolio $\Pi(x,\tau)$ in the absence of transaction costs computed with Algorithm 1.

We now compare the price $V(S,0)$ computed by Algorithm 1 to the price $V_{PSOR}(S,0)$ computed by the PSOR algorithm in the linear case $s_i^n = 0$. Fig. 7 shows that with the given mesh size $k = 0.0008$ and $h = 0.015$ the price computed by our algorithm (Fig. 7(a)) only slightly differs from the price computed by the PSOR algorithm (Fig. 7(b)).



(a) Computed with Algorithm 1.

(b) Computed with PSOR.

Figure 7: Price of an American Call option $V(S,0)$ in the absence of transaction costs and the pay-off $V(S,T)$ (red dotted line).

We calculate the error of accuracy of our computation one year to expiry at $t = 0$, denoted by the ℓ^2-error

$$\mathrm{err}_2(0) = \left(h \sum_{i=0}^{N} |V_{\mathrm{PSOR}}(S_i,0) - V_i^0|^2 \right)^{\frac{1}{2}},$$

where $V_{\mathrm{PSOR}}(S_i,0)$ denotes the solution computed by the PSOR algorithm at $S_i = e^{-ih}\varrho(T)$ and $\varrho(T)$ depends on the step size h. For this purpose, we interpolate the solution computed by the PSOR algorithm by the `MATLAB` routines `spline` and `ppval`. For V_i^0 we use our corresponding solution, where $k = 0.0008$. The error can be seen in Table 2, which reveals that it is reasonable to assume the accuracy $\mathrm{O}(h)$.

h	0.03	0.015	0.012	0.01	0.0086
ℓ^2-error	0.0365	0.0162	0.0257	0.0084	0.0167

Table 2: ℓ^2-error of accuracy of Algorithm 1 compared to the PSOR algorithm in the absence of transaction costs.

We further compute the free boundary profiles for the four different transaction cost models (26) by Algorithm 1 and compare them to the profile of the free boundary in the absence transaction costs. For our computations we take $k = 0.0008$ and $h = 0.015$.

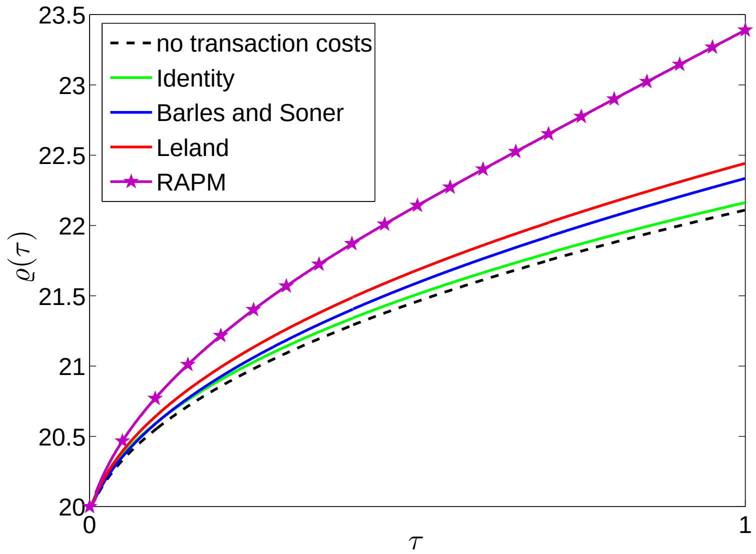


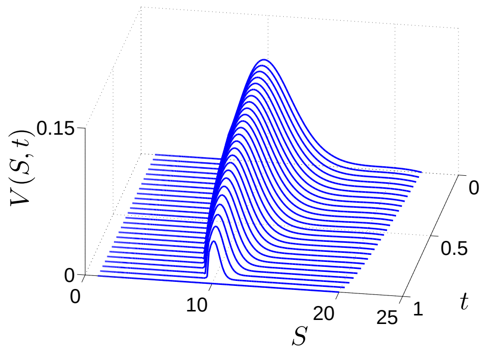
Figure 8: Free boundary positions for various transaction cost models vs. the free boundary profile in the absence of transaction costs.

As expected, we see that for all the transaction cost models the free boundary values are greater than in the case without transaction costs (Fig. 8). With the given parameters the free boundary in the absence of transaction costs is $\varrho(T) \approx 22.11$, followed by the identity model with $a = 0.02$ ($\varrho(T) \approx 22.16$), Barles' and Soner's model with $a = 0.02$ ($\varrho(T) \approx 22.34$), Leland's model with $\delta t = 0.1$, $\kappa = 0.02$ ($\varrho(T) \approx 22.44$) and finally the RAPM with $C = 0.01$, $R = 30$ ($\varrho(T) \approx 23.39$).

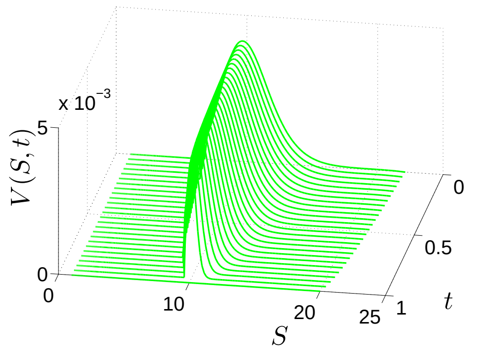
Furthermore, we compute the corresponding values $V(S,t)$ for the American Call option by Algorithm 1 and check the price difference between the American Call option with transaction costs and the American Call option without transaction costs

$$V_{\text{nonlinear}}(S,t) - V_{\text{linear}}(S,t).$$

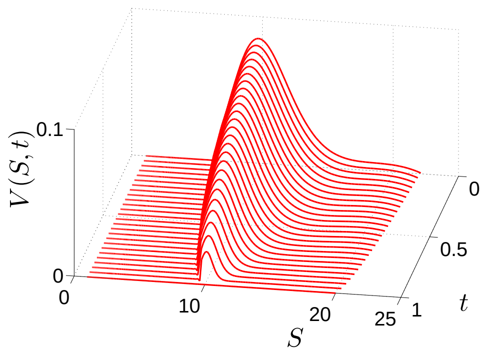
The influence of transaction costs for the four models can be seen in Fig. 9. We notice that the difference is maximal one year to expiry at $t = 0$ and $S \approx 9.5$. The difference is not symmetric, but decreases towards the expiry. This seems plausible, since towards expiry the portfolio can not be adjusted as often at it could be adjusted before. Hence, the transaction costs and the value of the American Call option with transaction costs decrease towards $t = 1$.



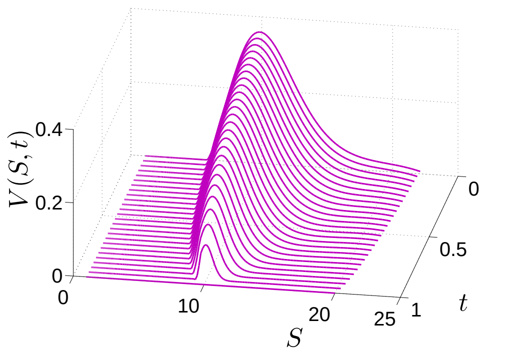
(a) Barles' and Soner's model ($a = 0.02$) vs. linear model.



(b) $\Psi(x) := x$ chosen as the identity ($a = 0.02$) vs. linear model.



(c) Leland's model ($\delta t = 0.1$, $\kappa = 0.02$) vs. linear model.



(d) RAPM model ($M = 0.01$, $C = 30$) vs. linear model.

Figure 9: The influence of transaction costs $V_{\text{nonlinear}}(S,t) - V_{\text{linear}}(S,t)$.

The corresponding prices $V(S,0)$ in the presence of transaction costs can be seen in Fig. 10. At $S \approx 9.5$ with the parameters as indicated above and $k = 0.0008$, $h = 0.015$ the price of the American Call option evaluated with the RAPM transaction cost model is the highest (≈ 1.06). It is followed by Barles' and Soner's model (≈ 0.82), Leland's model

(≈ 0.78), the identity model (≈ 0.74) and finally the model in the absence of transaction costs (≈ 0.71). As already shown in Table 2, the linear price computed by our algorithm (light blue solid line in Fig. 10) only slightly deviates from the price computed by the PSOR algorithm (black dotted line in Fig. 10).

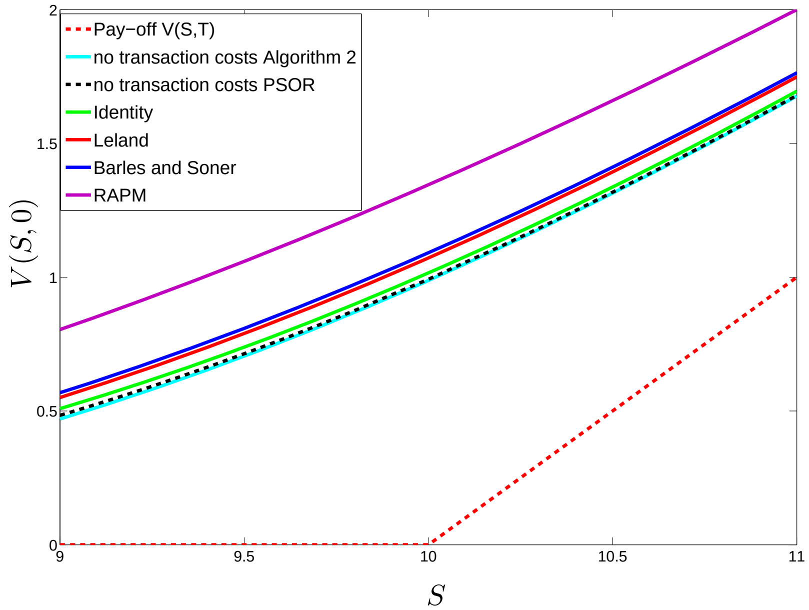


Figure 10: Price of an American Call option $V(S,0)$ for different transaction cost models vs. the price without transaction costs.

For other numerical experiments in the future is recommendable to use rather `C` or `C++` in order to reduce the computational time which is relatively high in `MATLAB`.

Conclusion

In this chapter we solved the nonlinear Black–Scholes equation for American options in the presence of transaction costs. Summing up, our numerical results showed a considerable price difference between linear and nonlinear prices for American Call options.

While we focused in this chapter on standard options (known as *plain–vanilla options*) of American type, our future work will deal with extensions: forward and future contracts, options on futures, more general pay–off functions (e.g. 'cash–or–nothing call') with transaction costs and instalment options.

Moreover, we will consider a higher-order splitting in time, e.g. the well–known *Strang–Splitting* [56] and combine this with modern compact finite difference of high spatial order, like the *Crandall–Douglas Scheme* [43] which is fourth-order accurate in 'space' (i.e. asset price) or the high–order compact methods proposed in [59], [60], [66]. Especially, the method of [60] is promising, since it is already an improvement of the Han and Wu method [27] with a higher order interior scheme and more accurate tracking of the free boundary.

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Algorithm 1 Computation of the price $V(S,t)$ for the American Call

Appendix

Require: R, T, h, k, M, N, r, K, D, σ, Le, a, C, M

solve the ODE (13) required for the volatility model of Barles and Soner and interpolate the solution
initialize Π^0
initialize the free boundary $\varrho^0 = rK/q$
transform Π^0 into V^0
set $\pi = \Pi^0$ and $v = V^0$
set $\Pi^{1,0} = \Pi^0$ and $\varrho^{1,0} = \varrho^0$
calculate Π^n, ϱ^n at each time level
for $n = 1 : M$ **do**
    calculate $s^{n,p}$, $\varrho^{n,p}$, $\Pi^{n-1/2,p}$ and $\Pi^{n,p}$ in the successive loop over p
    for $p = 1 : p_{max}$ **do**
        calculate the volatility correction $s^{n,p}$ depending on the volatility model using $\Pi^{n,p-1}$ and $\varrho^{n,p-1}$ (in the case of Barles' and Soner's model use the interpolated solution of (13), in the case without transaction costs $s^{n,p} = (0, \cdots, 0)^\top \in \mathbb{R}^{N+1}$)
        calculate $\varrho^{n,p}$ using $\Pi^{n,p-1}$ and $s^{n,p}$
        calculate $\Pi^{n-1/2,p}$ using Π^{n-1} and $\varrho^{n,p}$
        fill the matrix $A^{n,p}$ and the vector $d^{n,p}$ with the corresponding coefficients using $s^{n,p}$
        L-R-decompose $A^{n,p} = L^{n,p}R^{n,p}$
        solve $L^{n,p}y^{n,p} = \Pi^{n-1/2,p} - d^{n,p}$ for $y^{n,p}$
        solve $R^{n,p}\Pi^{n,p} = y^{n,p}$ for $\Pi^{n,p}$
        start over with the loop over p
    end for
    set $\Pi^n = \Pi^{n,p}$ and $\varrho^n = \varrho^{n,p}$
    transform Π^n into V^n
    save the solution in the transformed variables in the array
    $\pi = \begin{bmatrix}\pi & [-K; \Pi^n; 0]\end{bmatrix}$
    save the solution in the original variables in the array
    $v = \begin{bmatrix}v & [\varrho^n - K; V^n; 0]\end{bmatrix}$
    start over with the loop over n
end for
plot v at each time level and each stock price, plot ϱ at each time level

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