# Continuity for s-convex fuzzy processes

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In a previous paper we introduced the concept of s-convex fuzzy mapping and established some properties. In this work we study the continuity for s-convex fuzzy processes.

## 1 Introduction

The notion of convex process was introduced by Rockafellar [19] (see also [20]). These processes are set-valued maps whose graphs are closed convex cones. They can be seen as the set-valued version of a continuous linear operator. Derivatives of some set-valued maps are closed convex processes, which is a desirable property for a derivative (see [1]). An important property of convex processes is that it is possible to transpose closed convex processes and to use the benefits of duality theory. As it is well known, these facts are very useful in optimization theory (see for example [2], [16], [17], [18], [3]). The extension of this notion to the fuzzy framework was done by Matłoka [15]. Recently, Syau, Low and Wu [26] observed that Matłoka's definition is very strict. They gave another definition that extends Matłoka's one. The concept of $m$-convex fuzzy mapping was introduced in [7]. When $m = 1$ this concept and the definition of convex fuzzy process given in [26] coincide (see Theorem 3.4, p. 195 in [26]). As a generalization of convex functions, Breckner [4] introduced s-convex functions and in [5] he studied the set-valued version of these functions. Convex processes are a particular case of s-convex set-valued maps. Breckner also proved the important fact that a set-valued map is s-convex if and only if its support function is a s-convex function. Other related works are [6], [24], [25]. The fuzzy version of Breckner's definition was introduced in [8], that was called s-convex fuzzy process. In this work it was proved the equivalence with the s-convexity of the support fuzzy function and other properties were also studied. Generalizations of the Hadamard and Jensen inequalities for s-convex fuzzy process were established in [22]. In this work we will continue the study of these processes.

In convex analysis it is well known the relation between continuity and boundedness for convex functions (see for instance [13], [9], [10]). These results have been generalized for multifunctions by several authors. An important work in this line is the one in [5], where the author define the generalized convex and generalized concave set-valued mappings. Later, [23] introduced the concept of Hölder continuity of a set valued mapping between topological linear spaces, using the mean of Minkowski function. He studied the relationship between this concept and the continuity of a s-convex set-valued mapping.

In this work, we extend the above cited continuity results to the fuzzy context. These questions have not been considered in the works [15] and [26]. Here we give results for the finite dimensional case. Actually, we are studying the infinite dimensional case and the obtained results will be published in a future paper.

The structure of this paper is as follows. In section 2 we introduce some notation and give the basic concepts. In Section 3 we show the main results.

## 2 Preliminaries

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space and let $C \subseteq \mathbb{R}^n$ be a convex set. Let $s \in ]0,1]$ and let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function such that for all $a \in [0,1]$ and for all $x, y \in C$, the following inequality holds

$$f(ax + (1-a)y) \leq a^s f(x) + (1-a)^s f(y). \tag{1}$$

These functions are called s-convex and they have been introduced by Breckner [4], where it is also possible to find examples of such functions (see also [11]).

Let $P(\mathbb{R}^n)$ denote the set of all nonempty subsets of $\mathbb{R}^n$. In [5], Breckner generalized the notion of s-convexity for a set-valued mapping $F : C \subseteq \mathbb{R}^m \to P(\mathbb{R}^n)$. $F$ is said to be a s-convex set-valued mapping on $C$ if the following relation is verified

$$(1-a)^s F(x) + a^s F(y) \subseteq F\left\{(1-a)x + ay\right\} \tag{2}$$

for all $a \in [0,1]$ and all $x, y \in \mathbb{R}^m$. Also, if $a \in \mathbb{Q} \cap [0,1]$, that is, $a$ is a rational number, we say that $F$ is a rational s-convex set-valued mapping.

We denote by $\mathcal{K}(\mathbb{R}^m)$ the subset of $P(\mathbb{R}^m)$ whose elements are compact and nonempty and by $\mathcal{K}_c(\mathbb{R}^m)$ the subset of $\mathcal{K}(\mathbb{R}^m)$ whose elements are convex. If $A \in \mathcal{K}(\mathbb{R}^m)$, then the support function $\sigma(A, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is defined as

$$\sigma(A,\psi)=\sup_{a\in A}<\psi,a>,\ \forall\psi\in\mathbb{R}^m.$$

It is important to remark that if $A,B\in\mathcal{K}_c(\mathbb{R}^m)$, then, as a direct consequence of the separation Hahn-Banach theorem, we obtain that $\sigma(A,\cdot)=\sigma(B,\cdot)\Leftrightarrow A=B$.

A fuzzy subset of $\mathbb{R}^n$ is a function $u:\mathbb{R}^n\to[0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ denote the set of all fuzzy sets in $\mathbb{R}^n$. We define the addition and the scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ by the usual extension principle as follows:

$$(u+v)(y)=\sup_{y_1,y_2:\ y_1+y_2=y}\min\{u(y_1),v(y_2)\}$$

and

$$(\lambda u)(y)=\begin{cases}u(\frac{y}{\lambda}) & if\ \lambda\neq 0,\\ \chi_{\{0\}}(y) & if\ \lambda=0,\end{cases}$$

where for any subset $A\subset\mathbb{R}^n$, $\chi_A$ denotes the characteristic function of $A$.

We can define a partial order $\subseteq$ on $\mathcal{F}(\mathbb{R}^n)$ by setting

$$u\subseteq v\Leftrightarrow u(y)\leq v(y),\ \ \forall y\in\mathbb{R}^n.$$

Let $u\in\mathcal{F}(\mathbb{R}^n)$. For $0<\alpha\leq 1$, we denote by $[u]^\alpha=\{y\in\mathbb{R}^n\ \ /\ \ u(y)\geq\alpha\}$ the $\alpha$-level set of $u$. $[u]^0=\text{supp}(u)=\overline{\{y\in\mathbb{R}^n\ \ /\ \ u(y)>0\}}$ it is called the support of $u$.

A fuzzy set $u$ is called convex if (see [14])

$$u\{\lambda y_1+(1-\lambda)y_2\}\geq\min\{u(y_1),u(y_2)\},$$

for all $y_1,y_2\in\text{supp}(u)$ and $\lambda\in]0,1[$.

A fuzzy set $u:\mathbb{R}^n\to[0,1]$ is said to be a fuzzy compact set if $[u]^\alpha$ is compact for all $\alpha\in[0,1]$. If $u\in\mathcal{F}(\mathbb{R}^n)$ is convex, then $[u]^\alpha$ is convex for all $\alpha\in[0,1]$.

We denote by $\mathcal{F}_K(\mathbb{R}^n)$ ($\mathcal{F}_C(\mathbb{R}^n)$) the space of all fuzzy compact (compact convex) sets. Given $u,v\in\mathcal{F}_K(\mathbb{R}^n)$, it is verified that

$(a)$ $u\subseteq v\Leftrightarrow[u]^\alpha\subseteq[v]^\alpha,\ \ \forall\alpha\in[0,1]$,
$(b)$ $[\lambda u]^\alpha=\lambda[u]^\alpha,\ \ \forall\lambda\in\mathbb{R},\ \forall\alpha\in[0,1]$,
$(c)$ $[u+v]^\alpha=[u]^\alpha+[v]^\alpha,\ \ \forall\alpha\in[0,1]$.

Any application $F:\mathbb{R}^m\to\mathcal{F}(\mathbb{R}^n)$ is called a fuzzy process. For each $\alpha\in[0,1]$ we define the set-valued mapping $F_\alpha:\mathbb{R}^m\to P(\mathbb{R}^n)$ by

$$F_\alpha(x)=[F(x)]^\alpha.$$

For any $u\in\mathcal{F}_C(\mathbb{R}^n)$ the support function of $u$, $S(u,(\cdot,\cdot)):[0,1]\times\mathbb{S}^m\to\mathbb{R}$, where $\mathbb{S}^m=\{\psi\in\mathbb{R}^m/\,\|\psi\|\leq 1\}$ and $\|.\|$ denotes the Euclidean norm, is defined as

$$S(u,(\alpha,\psi))=\sigma([u]^\alpha,\psi).$$

For details about support functions see for example [21].

A fuzzy process $F : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is called convex if it satisfies the following relation

$$F\left\{(1-a)x_1 + ax_2\right\}(y) \geq \sup_{y_1,y_2:(1-a)y_1+ay_2=y} \min\{F(x_1)(y_1), F(x_2)(y_2)\},$$

for all $x_1, x_2 \in \mathbb{R}^m$, $a \in ]0,1[$ and $y \in \mathbb{R}^n$. This notion of convex fuzzy process was recently introduced in [26] and it extends Matłoka's definition given in [15].

**Definition 1.** *Let $M \subset \mathbb{R}^m$ and $x_0 \in M$. Let $F : M \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy valued mapping and let $\{F_\alpha\}_{\alpha\in[0,1]}$ be the family of the $\alpha$-level set-valued mappings associated to $F$. Then $F$ is said to be*

1) ***lower semicontinuous (lsc)*** *at $x_0$ if for all $\rho > 0$ there exists $\delta > 0$ such that*
$$F_\alpha(x_0) \subseteq F_\alpha(x) + B_n(0;\rho),$$
*for all $\alpha \in [0,1]$ and $x \in M \cap B_m(x_0;\delta)$, where $B_n(x;\rho) = \{\psi \in \mathbb{R}^n / \|\psi - x\| < \rho\}$.*
2) ***upper semicontinuous (usc)*** *at $x_0$ if all for all $\rho > 0$ there exists $\delta > 0$ such that*
$$F_\alpha(x) \subseteq F_\alpha(x_0) + B_n(0;\rho),$$
*for all $\alpha \in [0,1]$ and $x \in M \cap B_m(x_0;\delta)$.*
3) ***continuous*** *at $x_0$ if it is usc and lsc at $x_0$.*
4) ***locally bounded*** *at $x_0$ if for each $\rho > 0$ there exist $\delta > 0$ and $a > 0$ such that*
$$\bigcup_{\alpha\in[0,1]} F_\alpha(B_m(x_0;\delta) \cap M) \subseteq aB_n(0;\rho).$$
5) ***bounded*** *at $x_0$ if the set $\cup_{\alpha\in[0,1]} F_\alpha(x_0)$ is bounded, that is, if for each $\gamma > 0$ there exists $a > 0$ such that*
$$\bigcup_{\alpha\in[0,1]} F_\alpha(x_0) \subseteq aB_n(0;\gamma).$$
6) ***locally s-Hölder continuous*** *at $x_0$ if for all $\epsilon > 0$ there exists $a > 0$ such that*
$$H(F_\alpha(x), F_\alpha(y)) \leq a \parallel x - y \parallel^s,$$
*for all $\alpha \in [0,1]$ and all $x, y \in B_m(x_0;\epsilon) \cap M$, where $H(\cdot,\cdot)$ is the Hausdorff metric defined by*
$$H(A,B) = \sup\{\sup_{b\in B} \inf_{a\in A} \|a-b\|, \sup_{a\in A} \inf_{b\in B} \|a-b\|\}.$$

*The fuzzy valued mapping $F$ is* ***continuous*** *(respectively locally bounded, bounded, locally s-Hölder continuous) on $M$ if it is* ***continuous*** *(respectively locally bounded, bounded, locally s-Hölder continuous) at each point of $M$.*

## 3 Main results

In this Section, we prove some continuity results for s-convex processes.

**Proposition 1.** *Let $M$ be a nonempty subset of $\mathbb{R}^m$, let $x_0$ be a point in $M$ and let $F : M \to \mathcal{F}(\mathbb{R}^n)$ be a fuzzy valued mapping. Then*

*a) If $F$ is usc and bounded at $x_0$, then it is locally bounded at $x_0$.*
*b) If $F$ is locally s-Hölder continuous at $x_0$, then it is continuous at $x_0$.*

*Proof.* (a) Since $F$ is usc at $x_0$, for all $\gamma > 0$ there exists $\delta > 0$ such that

$$F_\alpha(x) \subseteq F_\alpha(x_0) + B_n(0; \gamma/2),$$

for all $\alpha \in [0, 1]$ and $x \in M \cap B_m(x_0; \delta)$. Consequently,

$$\bigcup_{\alpha\in[0,1]} F_\alpha\left(M \cap B_m(x_0; \delta)\right) \subseteq \bigcup_{\alpha\in[0,1]} F_\alpha(x_0) + B_n(0; \gamma/2).$$

Also, by using the fact that $F$ is bounded at $x_0$, we have that for all $\gamma > 0$ there exists $a \geq 1$ such that

$$\bigcup_{\alpha\in[0,1]} F_\alpha(x_0) \subseteq aB_n(0; \gamma/2).$$

Therefore,

$$\bigcup_{\alpha\in[0,1]} F_\alpha(M \cap B_m(x_0; \delta)) \subseteq aB_n(0; \gamma/2) + B_n(0; \gamma/2) \subseteq aB_n(0; \gamma),$$

which completes the proof of a).

(b) The assertion in b) is an immediate consequence of the definition of the Hausdorff metric. □

**Proposition 2.** *Let $M$ be a nonempty convex subset of $\mathbb{R}^m$, let $x_0$ be an interior point in $M$ and let $F : M \to \mathcal{F}(\mathbb{R}^n)$ be a rationally s-convex process, which is locally bounded at $x_0$. Then $F$ is continuous at $x_0$.*

*Proof.* Let $\rho > 0$. Since $F$ is locally bounded at $x_0$, there exist $a > 0$ and $\delta > 0$ such that $B_m(x_0; \delta) \subset M$ and

$$\bigcup_{\alpha\in[0,1]} F_\alpha\left(B_m(x_0; \delta)\right) \subseteq aB_n(0; \rho/2).$$

Choose a rational number $r \in (0, 1)$ such that

$$r^s a < 1, \quad \left[1 - (1 - r)^s\right] a < 1 \quad \text{and} \quad \left[1 - \left(\frac{1}{r + 1}\right)^s\right] a < 1.$$

We claim that
$$F_\alpha(x_0) \subseteq F_\alpha(x) + B_n(0;\rho)$$
and
$$F_\alpha(x) \subseteq F_\alpha(x_0) + B_n(0;\rho)$$
for all $\alpha \in [0,1]$ and all $x \in B_m(x_0;\delta r)$.

To prove this, we fix $\alpha \in [0,1]$ and $x \in B_m(x_0;\delta r)$. Then, the points $y = x_0 + \frac{1}{r}(x - x_0)$ and $z = x_0 + \frac{1}{r}(x_0 - x)$ lie both in $B_m(x_0;\delta)$. As $F$ is s-convex, we have
$$(1-r)^s F_\alpha(x_0) + r^s F_\alpha(y) \subseteq F_\alpha(x).$$
Thus,
$$(1-r)^s F_\alpha(x_0) \subseteq F_\alpha(x) - r^s F_\alpha(y).$$
Let $v \in F_\alpha(x_0)$ be arbitrary, then
$$\begin{aligned} v &= [1-(1-r)^s]\, v + (1-r)^s v \\ &\in [1-(1-r)^s]\, F_\alpha(x_0) + F_\alpha(x) - r^s F_\alpha(y) \\ &\subseteq [1-(1-r)^s] B_n(0;\rho/2) + F_\alpha(x) - r^s B_n(0;\rho/2) \\ &\subseteq F_\alpha(x) + B_n(0;\rho), \end{aligned}$$
which proves that $F$ is lsc. Analogously, one can see that $F$ is usc. □

**Proposition 3.** *Let $M$ be a nonempty convex subset of $\mathbb{R}^m$, let $x_0$ be an interior point in $M$ and let $F : M \to \mathcal{F}(\mathbb{R}^n)$ be a rationally s-convex process, which is continuous at $x_0$. If, in addition, either $s = 1$ or $s \in (0,1)$ and $F$ is bounded at $x_0$, then $F$ is locally s-Hölder continuous at $x_0$.*

*Proof.* Let $\rho > 0$. If $s \in (0,1)$, using the fact that $F$ is bounded at $x_0$, we have that there exists a number $\overline{a} > 0$ such that
$$\bigcup_{\alpha\in[0,1]} F_\alpha(x_0) \subseteq \frac{\overline{a}}{4} B_n(0;\rho).$$
By other hand, the continuity of $F$ at $x_0$ implies that there exists $\delta > 0$ such that
$$F_\alpha(x_0) \subseteq F_\alpha(x) + B_n(0;\rho/4) \tag{3}$$
and
$$F_\alpha(x) \subseteq F_\alpha(x_0) + B_n(0;\rho/4) \tag{4}$$
for all $\alpha \in [0,1]$ and all $x \in B_m(x_0;\delta) \subseteq M$.

We claim that
$$\sup_{v\in F_\alpha(y)} \inf_{w\in F_\alpha(x)} \|u - v\| \le a\|x-y\|^s, \tag{5}$$
for all $\alpha \in [0,1]$ and for all $x, y \in B_m(x_0;\delta/2)$, where

$$a = \begin{cases} 1 & \text{if } s = 1, \\ \max\{\overline{a}, 1\} & \text{if } s \in (0,1). \end{cases}$$

To prove (5) we fix $\alpha \in [0,1]$ and $x, y \in B_m(x_0; \delta/2)$. Let $r$ be any rational number such that $r > \|x - y\|$. Let $v$ be any point in $F_\alpha(y)$. Consider the point $z = x + (x - y)/r$. Select a number $t \in (0, r)$ such that $x - y \in tB_m(0; \delta/2)$. We have

$$z = x + \frac{t}{r}\frac{1}{t}(x - y) \in B_m(x_0; \delta/2) + \frac{t}{r}B_m(x_0; \delta/2) \subseteq B_m(x_0; \delta).$$

Now, from $x = \frac{r}{r+1}z + \frac{1}{r+1}y$, we get

$$\begin{aligned} F_\alpha(x) &\supseteq \left(\frac{r}{r+1}\right)^s F_\alpha(z) + \left(\frac{1}{r+1}\right)^s F_\alpha(y) \qquad (6) \\ &\supseteq \left(\frac{r}{r+1}\right)^s F_\alpha(z) + \left(\frac{1}{r+1}\right)^s v. \end{aligned}$$

Since $y \in B_m(x_0; \delta/2) \subseteq B_m(x_0; \delta)$, from (3) it follows that $v \in F_\alpha(x_0) + B_n(0; \rho/4)$, hence there exists a point $v_0 \in F_\alpha(x_0)$ such that $v - v_0 \in B_n(0; \rho/4)$. As $z \in B_m(x_0; \delta)$, we deduce by virtue of (4) that $v_0 \in F_\alpha(z) + B_n(0; \rho/4)$, hence there exists a point $w \in F_\alpha(z)$ such that $v_0 - w \in B_n(0; \rho/4)$. From (6) it follows that there exist a point $\overline{u} \in F_\alpha(x)$ such that

$$\overline{u} = \left(\frac{r}{r+1}\right)^s w + \left(\frac{1}{r+1}\right)^s v.$$

If $s = 1$, we have

$$\begin{aligned} \overline{u} - v &= \left(\frac{r}{r+1}\right)(w - v) = \left(\frac{r}{r+1}\right)(w - v_0) - \left(\frac{r}{r+1}\right)(v - v_0) \\ &\in \left(\frac{r}{r+1}\right) B_n(0; \rho/4) + \left(\frac{r}{r+1}\right) B_n(0; \rho/4) \subseteq arB_n(0; \rho). \end{aligned}$$

If $s \in (0,1)$, we have

$$\begin{aligned} \overline{u} - v &= \left(\frac{r}{r+1}\right)^s w + \left[\left(\frac{1}{r+1}\right)^s - 1\right] v = \left(\frac{r}{r+1}\right)^s (w - v_0) - \\ &\quad - \left[1 - \left(\frac{1}{r+1}\right)^s\right](v - v_0) + \left[\left(\frac{r}{r+1}\right)^s + \left(\frac{1}{r+1}\right)^s - 1\right] v_0 \\ &\in \left(\frac{r}{r+1}\right)^s B_n(0; \rho/4) + \left[1 - \left(\frac{1}{r+1}\right)^s\right] B_n(0; \rho/4) \\ &\quad + \left[\left(\frac{r}{r+1}\right)^s + \left(\frac{1}{r+1}\right)^s - 1\right] F_\alpha(x_0) \\ &\subseteq \left(\frac{r}{r+1}\right)^s B_n(0; \rho/4) + \left[1 - \left(\frac{1}{r+1}\right)^s\right] B_n(0; \rho/4) \\ &\quad + \overline{a}\left(\frac{r}{r+1}\right)^s B_n(0; \rho/4) + \overline{a}\left[1 - \left(\frac{1}{r+1}\right)^s\right] B_n(0; \rho/4). \end{aligned}$$

Since $0 \leq 1 - (\frac{1}{r+1})^s \leq (\frac{r}{r+1})^s$, we deduce that $\overline{u} - v \in ar^s B_n(0;\rho)$.

Thus, in both cases we have $\overline{u} - v \in ar^s B_n(0;\rho)$. Therefore,

$$\inf_{u \in F_\alpha(x)} \|u - v\| \leq \|\overline{u} - v\| \leq ar^s.$$

As $v$ was arbitrarily chosen in $F_\alpha(y)$, it follows that

$$\sup_{v \in F_\alpha(y)} \inf_{w \in F_\alpha(x)} \|v - w\| \leq ar^s.$$

Since $r$ is any rational number satisfying the inequality $r > \|x - y\|$, the inequality (5) must be valid.

Analogously, it can be proved that

$$\sup_{v \in F_\alpha(x)} \inf_{w \in F_\alpha(y)} \|v - w\| \leq a\|x - y\|^s \tag{7}$$

for all $\alpha \in [0,1]$ and for all $x, y \in B_m(x_0;\rho/2)$. From (5) and (7) it follows that $F$ is locally s-Hölder continuous at $x_0$. □

The following results are consequences of the above propositions.

**Theorem 1.** *Let $M$ be a nonempty convex subset of $\mathbb{R}^m$, let $x_0$ be an interior point in $M$, and let $F : M \to \mathcal{F}(\mathbb{R}^n)$ be a rationally s-convex process. If $s = 1$ or $s \in (0,1)$ and $F$ is bounded at $x_0$, then the following assertions are equivalent:*

(1) *$F$ is continuous at $x_0$.*
(2) *$F$ is locally s-Hölder continuous at $x_0$.*

*If $F$ is bounded at $x_0$, the above assertions are equivalent to*

(3) *$F$ is locally bounded at $x_0$.*

**Theorem 2.** *Let $M$ be a nonempty convex subset of $\mathbb{R}^m$, let $x_0$ be an interior point in $M$ and let $F : M \to \mathcal{F}(\mathbb{R}^n)$ be a rationally s-convex process, which is locally bounded at a point of $M$. Then $F$ is locally s-Hölder continuous on $M$.*

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