Effective results on compositions of nonexpansive mappings

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Abstract

This paper provides uniform bounds on the asymptotic regularity for iterations associated to a finite family of nonexpansive mappings. We obtain our quantitative results in the setting of (r,δ)-convex spaces, a class of geodesic spaces which generalizes metric spaces with a convex geodesic bicombing.

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1 Introduction

Let X be a Hilbert space, $C \subseteq X$ a closed convex subset, $T_1, \ldots, T_N : C \to C$ (where $N \in \mathbb{Z}_+$) a finite family of nonexpansive mappings and (λ_n) a sequence in $[0,1]$. Given $u \in C$, one can define an iteration starting from u by

$$x_0 = u, \quad x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})T_{n+1}x_n, \tag{1}$$

where $T_n = T_{n \bmod N}$ and the mod N function takes values in $1, \ldots, N$. For the special case $N = 1$, this iteration coincides with the well-known Halpern iteration [8], whose strong convergence was proved by Wittmann [22] under suitable conditions on (λ_n), which are satisfied by the natural candidate $\lambda_n = \frac{1}{n+1}$.

The general iteration defined by (1) was first studied in Hilbert spaces by Lions [18], who assumed different hypotheses on (λ_n), with the drawback that $\lambda_n = \frac{1}{n+1}$ does not satisfy them. Bauschke [2] proved that the iteration (x_n) given by (1) converges strongly to the common fixed point of the mappings $T_1, \ldots, T_N$ which is nearest to u, under the assumptions that $F := \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ is nonempty, $F = \mathrm{Fix}(T_N T_{N-1} \cdots T_1) = \ldots = \mathrm{Fix}(T_1 T_N \cdots T_2) = \mathrm{Fix}(T_{N-1} \cdots T_1 T_N)$ and that (λ_n) satisfies

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} |\lambda_{n+N} - \lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty. \tag{2}$$

Bauschke's result is a generalization of Wittmanns's theorem to a finite family of mappings, since for $N = 1$ conditions (2) coincide with the ones used by Wittmann.

Another iteration that will be considered in this paper is the following one associated to any nonexpansive mapping $T : C \to C$,

$$x_{n+1} = T(\lambda_{n+1} u + (1 - \lambda_{n+1}) x_n). \tag{3}$$

This iteration, studied by Xu [23], is a discrete version of the approximating curve $z_t = T(tu + (1-t)z_t)$, $t \in (0, 1)$, analyzed by Combettes and Hirstoaga [7]. Strong convergence of the iteration (3) was established by Xu in the setting of uniformly smooth Banach spaces under appropriate assumptions on (λ_n), including those given by (2). The strong convergence results of Xu and Combettes and Hirstoaga were extended in [6] to more general approximating curves and iterations. As above, one can define for the iteration (3) a cyclic algorithm associated to the finite family of nonexpansive mappings $T_1, \ldots, T_N : C \to C$,

$$x_0 = u, \quad x_{n+1} = T_{n+1}(\lambda_{n+1} u + (1 - \lambda_{n+1}) x_n). \tag{4}$$

A very important concept in the study of the asymptotic behavior of nonlinear iterations is the so-called asymptotic regularity, introduced by Browder and Petryshyn [4] in their study of solutions of nonlinear functional equations using Picard iterations: $T : C \to C$ is asymptotically regular if $\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0$ for all $x \in C$. More generally, an iteration (x_n) associated to a mapping T is said to be *asymptotically regular* if $\lim_{n\to\infty} \|x_n - T x_n\| = 0$ for all starting points in C.

A natural question is to compute rates of asymptotic regularity for the iteration (x_n), i.e. rates of convergence of $(\|x_n - T x_n\|)$ towards 0. For the Halpern iteration this was done in a series of papers [15, 16, 13, 14, 17], corresponding to different classes of spaces. For the iteration given by (3), such rates were obtained in [6].

The notion of asymptotic regularity can be extended to sequences (x_n) associated to a family of mappings $T_1, \ldots, T_N : C \to C$, as it is the case in this paper. Thus, we say that (x_n) is *asymptotically regular* if $\lim\limits_{n\to\infty} \|x_n - T_{n+N} \cdots T_{n+1} x_n\| = 0$ for all starting points in C. The following asymptotic regularity result is contained in Bauschke's strong convergence proof for the iteration (1).

Theorem 1.1. *Let X be a Hilbert space, $C \subseteq X$ convex, $T_1, \ldots, T_N : C \to C$ nonexpansive mappings and (λ_n) a sequence in $[0,1]$ satisfying (2). Let (x_n) be given by (1) and assume that (x_n) is bounded. Then,*

$$\lim_{n\to\infty} \|x_n - T_{n+N} \cdots T_{n+1} x_n\| = 0.$$

The main result of this paper is a quantitative version of Theorem 1.1 for both iterations (1) and (4). In order to get this result we apply methods of proof mining developed by Kohlenbach [12] with the aim of obtaining effective and uniform bounds from proofs where such information is not readily available. As a consequence, we provide for the first time effective and uniform rates of asymptotic regularity for the iterations (1) and (4).

Actually, we obtain our quantitative results in a setting more general than the one of normed space. More precisely, we introduce (r,δ)-*convex spaces*, a class of metric spaces which also includes Busemann spaces (and, hence, CAT(0) spaces), hyperconvex spaces, CAT(κ) spaces with $\kappa > 0$, as well as the so-called W-hyperbolic spaces (see [11]). Consequently, even when $N = 1$ and so (1) reduces in fact to the Halpern iteration, our results generalize ones obtained previously by the authors for CAT(κ) spaces with $\kappa > 0$ [17] and by the first author for normed [15] or W-hyperbolic spaces [16].

2 (r,δ)-convex spaces

Let (X,d) be a metric space. We recall first basic facts in geodesic geometry. Given $x, y \in X$, a *constant speed geodesic* from x to y is a mapping $\gamma : [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d(\gamma(s), \gamma(t)) = |s-t| d(x,y)$ for all $s, t \in [0,1]$. The image $\gamma([0,1])$ of γ is a *geodesic segment* which joins x and y. Note that a geodesic segment from x to y is not necessarily unique. Given $r \in (0,\infty]$, we say that (X,d) is a *(uniquely) r-geodesic space* if every two points $x, y \in X$ with $d(x,y) \le r$ can be joined by a (unique) geodesic segment. For $r = \infty$, we say simply that X is a (uniquely) geodesic space.

Let $r \in (0,\infty]$ and X be an r-geodesic space. We consider an r-geodesic bicombing Γ on X, that is, a choice of a constant speed geodesic $\gamma_{x,y}$ joining x and y for each pair of points $x, y \in X$ with $d(x,y) \le r$. When $r = \infty$, Γ is called a geodesic bicombing on X. If X is uniquely r-geodesic, then clearly one can define an r-geodesic bicombing in a unique way. For $\gamma_{x,y} \in \Gamma$, we denote by $[x,y]$ the geodesic segment $\gamma_{x,y}([0,1])$. A subset C of X is *r-convex* if $[x,y] \in C$ for all $x, y \in C$ with $d(x,y) \le r$. Given $t \in [0,1]$, we use the notation $(1-t)x + ty$ for

$\gamma_{x,y}(t)$. Then, $d(x,(1-t)x+ty) = td(x,y)$ and $d(y,(1-t)x+ty) = (1-t)d(x,y)$. The r-geodesic bicombing is convex if it satisfies

$$d((1-t)x + ty, (1-t)x + tz) \leq td(y,z) \tag{5}$$

for all $x, y, z \in X$ with $d(x,y), d(x,z), d(z,y) \leq r$ and all $t \in [0,1]$.

Normed spaces are obviously geodesic spaces with a convex geodesic bicombing. Another natural example are Busemann spaces, which were used for the first time by Busemann [5] to give a definition of nonpositive curvature in geodesic spaces. Thus, geodesic spaces with the property that each point has a convex neighborhood which is a Busemann space are 'nonpositively curved' spaces in the sense of Busemann, who called them G-spaces. We refer to [20] for a nice exposition of this very important class of geodesic spaces. It turns out that Busemann spaces are uniquely geodesic spaces with a (unique) convex geodesic bicombing.

A related example of metric spaces with a convex geodesic bicombing are the so-called W-hyperbolic spaces, defined in [11] as metric spaces together with a convexity mapping $W : X \times X \times [0,1] \to X$ satisfying suitable properties. As it was remarked in [1], Busemann spaces are exactly the uniquely geodesic W-hyperbolic spaces.

It is well-known that a hyperconvex space X also admits a convex geodesic bicombing obtained by embedding X isometrically into $\ell^\infty(X)$ and using the existence of a nonexpansive retraction from $\ell^\infty(X)$ into X to define the geodesic bicombing through the convex linear geodesic bicombing on $\ell^\infty(X)$ (for more details see, for instance, Chapter 13 in [10]).

In the following we define a natural generalization of metric spaces with a convex r-geodesic bicombing.

Definition 2.1. *Let $r \in (0,\infty]$ and $\delta \in [0,1]$. A metric space (X,d) with an r-geodesic bicombing is said to be (r,δ)-convex if for all $x,y,z \in X$ with $d(x,y), d(x,z), d(z,y) \leq r$ and all $t \in [0,1]$,*

$$d((1-t)x + ty, (1-t)x + tz) \leq (t + \delta(1-t))d(y,z).$$

If $r = \infty$, we say that X is δ-convex.

An example of such spaces are CAT(κ) spaces with $\kappa > 0$. CAT(κ) spaces are defined in terms of comparisons with the model spaces M_κ^2 (see [3] for more details). Denote $D_\kappa = \pi/\sqrt{\kappa}$.

Proposition 2.2. *A CAT(κ) space X is $\left(\frac{\mu D_\kappa}{2}, 1 - \cos\frac{\mu\pi}{2}\right)$-convex for any $\mu \in (0,1]$.*

Proof. Let $x,y,z \in X$ with $d(x,y), d(x,z), d(z,y) \leq \frac{\mu D_\kappa}{2}$ and $t \in [0,1]$. By [21, Lemma 3.3] (see also [17, Lemma 4.1]) we have that $d((1-t)x+ty, (1-t)x+tz) \leq$

$\dfrac{\sin\frac{t\mu\pi}{2}}{\sin\frac{\mu\pi}{2}}d(y,z)$. Note that

$$\begin{aligned} 1-\frac{\sin\frac{t\mu\pi}{2}}{\sin\frac{\mu\pi}{2}} &= \frac{2\cos\frac{(1+t)\mu\pi}{4}\sin\frac{(1-t)\mu\pi}{4}}{\sin\frac{\mu\pi}{2}} \geq \frac{2\cos\frac{\mu\pi}{2}\sin\frac{(1-t)\mu\pi}{4}}{\sin\frac{\mu\pi}{2}} \\ &\geq (1-t)\cos\frac{\mu\pi}{2}, \quad \text{since } \sin\frac{(1-t)\mu\pi}{4} \geq \frac{1-t}{2}\sin\frac{\mu\pi}{2}. \end{aligned}$$

Hence,

$$\frac{\sin\frac{t\mu\pi}{2}}{\sin\frac{\mu\pi}{2}} \leq 1-(1-t)\cos\frac{\mu\pi}{2} = t+\left(1-\cos\frac{\mu\pi}{2}\right)(1-t).$$

□

We point out that CAT(κ) spaces with $\kappa > 0$ do not have in general a convex r-geodesic bicombing for $r < D_\kappa$ (to see this it suffices to consider the spherical space $\mathbb{S}^2$).

Let us recall another notion of convexity for metric spaces, introduced by Ohta [19]. Given $L_1, L_2 \in [0,\infty)$, a geodesic space X is said to be *L-convex* for (L_1, L_2) if for any $x, y, z \in X$, any constant speed geodesics $\gamma, \xi : [0,1] \to X$ with $\gamma(0) = \xi(0) = x$, $\gamma(1) = y$, $\xi(1) = z$ and for every $t \in [0,1]$,

$$d(\gamma(t),\xi(t)) \leq \left(1+L_1\frac{\min\{d(x,y)+d(x,z),2L_2\}}{2}\right)td(y,z).$$

An additional related notion says that an r-geodesic bicombing on a metric space X is *weakly convex* if there exists a constant $C \geq 1$ such that

$$d((1-t)x+ty,(1-t)x+tz) \leq Ctd(y,z),$$

for all $t \in [0,1]$ and all $x, y, z \in X$ as in Definition 2.1. One can easily see that an L-convex space for (L_1, L_2) has a weakly convex r-geodesic bicombing, where $r \in (0,\infty]$ and $C := 1 + L_1\min\{r, L_2\}$.

Remark 2.3. *We remark that we could have defined an even more general notion: given $r > 0$ and $\eta : [0,1] \to [0,\infty)$, a metric space with an r-geodesic bicombing is* (r,η)-convex *if*

$$d((1-t)x+ty,(1-t)x+tz) \leq (t+\eta(t))d(y,z),$$

for all $t \in [0,1]$ and all $x, y, z \in X$ as in Definition 2.1.

This very general definition has the advantage that it covers the case of metric spaces with a weakly convex r-geodesic bicombing and, thus, of L-convex spaces.

However, we use in this paper Definition 2.1, as this is the notion which allows us to get the effective results from the next section.

3 Effective rates of asymptotic regularity

Let $r \in (0,\infty], \delta \in [0,1)$, X be an (r,δ)-convex space, $C \subseteq X$ a convex subset and $T_1,\ldots,T_N : C \to C$ be nonexpansive mappings, where $N \in \mathbb{Z}_+$. If (λ_n) is a sequence in $[0,1]$ and $u \in C$, one can, obviously, define the iterations (1) and (4) starting with u in this setting, too:

$$x_0 = u, \quad x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})T_{n+1}x_n, \tag{6}$$

$$x_0 = u, \quad x_{n+1} = T_{n+1}(\lambda_{n+1}u + (1-\lambda_{n+1})x_n), \tag{7}$$

where $T_n = T_{n \bmod N}$ and the mod N function takes values in $1,\ldots,N$. We use the following notation $T_{n,N} := T_{n+N}\cdots T_{n+1}$.

The main theorem of the paper is a quantitative result on the asymptotic regularity of the above iterations.

Theorem 3.1. *Let $\varepsilon > 0$, $M > 0$ be such that $M \le \frac{r}{2}$, $\alpha,\gamma : (0,\infty) \to \mathbb{Z}_+$ and $\theta : \mathbb{Z}_+ \to \mathbb{Z}_+$. Suppose that*

(i) $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$ *with rate of divergence θ;*

(ii) $\sum\limits_{n=1}^{\infty} |\lambda_{n+N} - \lambda_n|$ *converges with Cauchy modulus γ.*

Let

$$\begin{aligned}\tilde{\Phi}(\varepsilon, M, \gamma, \theta, \delta) &= \theta\left(\left\lceil \frac{1}{1-\delta} \right\rceil \left(\gamma\left(\frac{\varepsilon}{4M}\right) + \max\left\{ \left\lceil \ln\left(\frac{4M}{\varepsilon}\right)\right\rceil, 1 \right\} \right) \right),\\ \Phi(\varepsilon, M, \gamma, \theta, \delta, N, \alpha) &= \max\left\{ \tilde{\Phi}\left(\frac{\varepsilon}{2}, M, \gamma, \theta, \delta\right), \alpha\left(\frac{\varepsilon}{4MN}\right) \right\}.\end{aligned}$$

Assume either

(i) (x_n) is given by (6) with $d(x_n,u) \le M$ for all $n \ge 1$ and $d(u,T_iu) \le M$ for each $i = 1,\ldots,N$, or

(ii) (x_n) is given by (7) with $d(x_n,u) \le M$ for all $n \ge 1$.

Then $\lim\limits_{n\to\infty} d(x_n, x_{n+N}) = 0$ with rate of convergence $\tilde{\Phi}$. Furthermore, if $\lim\limits_{n\to\infty} \lambda_{n+1} = 0$ with rate of convergence α, then $\lim\limits_{n\to\infty} d(x_n, T_{n,N}(x_n)) = 0$ with rate of convergence Φ.

We give the proof of the theorem in the next section. Let us state now some immediate consequences.

Corollary 3.2. *Let $\varepsilon, M, (\lambda_n), \alpha, \gamma, \theta, \tilde{\Phi}, \Phi$ be as above. Assume moreover that C is bounded and M is an upper bound on its diameter.*

If (x_n) is given by either (6) or (7), then $\lim\limits_{n\to\infty} d(x_n, x_{n+N}) = 0$ with rate of convergence $\tilde{\Phi}$ and $\lim\limits_{n\to\infty} d(x_n, T_{n,N}(x_n)) = 0$ with rate of convergence Φ.

Thus, for C bounded we obtain a highly uniform rate of asymptotic regularity Φ which does not depend at all on the starting point u and the nonexpansive mappings $T_1, \ldots, T_N$. Moreover, the dependence on the set C and the space X is very weak: via δ and a bound $M \leq \frac{r}{2}$ on the diameter of C.

Corollary 3.3. *Suppose that* $\lambda_n = \dfrac{1}{n+1}$, $n \geq 1$. *Then* $\lim_{n\to\infty} d(x_n, x_{n+N}) = \lim_{n\to\infty} d(x_n, T_{n,N}(x_n)) = 0$ *with a common rate of convergence*

$$\Psi(\varepsilon, M, N, \delta) = \exp\left(\left\lceil \frac{1}{1-\delta} \right\rceil \left(\left\lceil \frac{8M(N+1)}{\varepsilon} \right\rceil + 2\right) \ln 4\right).$$

Proof. We can take $\theta(n) = \exp(n \ln 4)$, $\gamma(\varepsilon) = \left\lceil \dfrac{N}{\varepsilon} \right\rceil$ and $\alpha(\varepsilon) = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$. □

As mentioned before, in the case $N = 1$, the iterative scheme defined by (6) yields the usual Halpern iteration, for which rates of asymptotic regularity have already been computed in the setting of CAT(κ) spaces [17]. Our main theorem recovers (with a slightly modified rate) [17, Proposition 3.2] since, for $M < \dfrac{D_\kappa}{2}$, one takes $\mu = \dfrac{2M}{D_k}$ in Proposition 2.2 to get that any CAT(κ) space is $(M, 1 - \cos(M\sqrt{\kappa}))$-convex and then apply Corollary 3.2. Furthermore, we generalize with basically the same bounds the results obtained for the Halpern iteration in normed spaces [15] and, more general, W-hyperbolic spaces [16].

As we have already pointed out, in this paper we obtain for the first time, even for Banach spaces, effective bounds on the asymptotic regularity of the iterations (6) and (7). Recently, using proof mining methods as well, Khan and Kohlenbach [9] obtained in the setting of uniformly convex Busemann spaces effective results on the asymptotic behavior of a different iteration associated to a finite family of nonexpansive mappings which extends the Krasnoselski-Mann iteration of a single nonexpansive mapping.

4 Proof of the main result

Assume the hypothesis of Theorem 3.1. As in the case of the Halpern iteration associated to a single mapping [15, 16, 14, 17], we shall apply the following quantitative lemma.

Lemma 4.1. *[17, 14] Let* $(\alpha_n)_{n\geq 1}$ *be a sequence in* $[0,1]$ *and* $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ *be sequences in* $\mathbb{R}_+$ *such that*

$$a_{n+1} \leq (1 - \alpha_{n+1})a_n + b_n \quad \text{for all } n \in \mathbb{Z}_+. \tag{8}$$

Assume that $\sum_{n=1}^{\infty} b_n$ *is convergent with Cauchy modulus* γ *and* $\sum_{n=1}^{\infty} \alpha_{n+1}$ *diverges with rate of divergence* θ.

Then, $\lim_{n\to\infty} a_n = 0$ *with rate of convergence* Σ *given by*

$$\Sigma(\varepsilon, P, \gamma, \theta) = \theta\left(\gamma\left(\frac{\varepsilon}{2}\right) + \max\left\{\left\lceil \ln\left(\frac{2P}{\varepsilon}\right)\right\rceil, 1\right\}\right) + 1 \tag{9}$$

where $P > 0$ *is an upper bound on* (a_n)*.*

The next lemma is the second main tool for the proof of our main result.

Lemma 4.2. *Let* $M > 0$ *satisfy* $2M \leq r$*.*

(i) Assume that (x_n) *is given by* (6)*,* $d(u, T_i u) \leq M$ *for each* $i = 1, \ldots, N$ *and* $d(x_n, u) \leq M$ *for all* $n \in \mathbb{N}$*. Then, for all* $n \geq 1$*,*

$$d(x_{n+1}, T_{n+1}x_n) \leq 2M\lambda_{n+1},$$
$$d(x_n, x_{n+N}) \leq (1-(1-\delta)\lambda_n)d(x_{n-1}, x_{n+N-1}) + 2M|\lambda_{n+N} - \lambda_n|.$$

(ii) Assume that (x_n) *is given by* (7) *and* $d(x_n, u) \leq M$ *for all* $n \in \mathbb{N}$*. Then, for all* $n \geq 1$*,*

$$d(x_{n+1}, T_{n+1}x_n) \leq M\lambda_{n+1},$$
$$d(x_n, x_{n+N}) \leq (1-(1-\delta)\lambda_n)d(x_{n-1}, x_{n+N-1}) + M|\lambda_{n+N} - \lambda_n|.$$

Proof. (i) First, let us note that $d(u, T_n x_m) \leq 2M$ for all $m, n \geq 1$. It follows that for all $n \geq 1$,

$$d(x_{n+1}, T_{n+1}x_n) = \lambda_{n+1} d(u, T_{n+1}x_n) \leq 2M\lambda_{n+1}$$

and

$$\begin{aligned} d(x_n, x_{n+N}) &= d(\lambda_n u + (1-\lambda_n)T_n x_{n-1}, \lambda_{n+N}u + (1-\lambda_{n+N})T_n x_{n+N-1}) \\ &\leq d(\lambda_n u + (1-\lambda_n)T_n x_{n-1}, \lambda_n u + (1-\lambda_n)T_n x_{n+N-1}) \\ &\quad + d(\lambda_n u + (1-\lambda_n)T_n x_{n+N-1}, \lambda_{n+N}u + (1-\lambda_{n+N})T_n x_{n+N-1}) \\ &\leq (1-(1-\delta)\lambda_n)d(x_{n-1}, x_{n+N-1}) + 2M|\lambda_{n+N} - \lambda_n|. \end{aligned}$$

(ii) The proof is similar, using that

$$d(x_{n+1}, T_{n+1}x_n) \leq d(\lambda_{n+1}u + (1-\lambda_{n+1})x_n, x_n) \leq M\lambda_{n+1}$$

and

$$\begin{aligned} d(x_n, x_{n+N}) &= d(T_n(\lambda_n u + (1-\lambda_n)x_{n-1}), T_n(\lambda_{n+N}u + (1-\lambda_{n+N})x_{n+N-1})) \\ &\leq d(\lambda_n u + (1-\lambda_n)x_{n-1}, \lambda_{n+N}u + (1-\lambda_{n+N})x_{n+N-1}). \end{aligned}$$

for all $n \geq 1$. ∎

4.1 Proof of Theorem 3.1

Let (x_n) be given by either (6) or (7). Denote $t_{n+1} = (1-\delta)\lambda_n \in [0,1]$. As an immediate consequence of Lemma 4.2, we get that

$$d(x_n, x_{n+N}) \leq (1-t_{n+1})d(x_{n-1}, x_{n+N-1}) + 2M|\lambda_{n+N} - \lambda_n|.$$

Note that $\sum_{n=1}^{\infty} 2M|\lambda_{n+N} - \lambda_n|$ converges with Cauchy modulus $\tilde{\gamma}(\varepsilon) = \gamma\left(\frac{\varepsilon}{2M}\right)$ and $\sum_{n=1}^{\infty} t_{n+1} = \infty$ with rate of divergence $\tilde{\theta}(n) = \theta\left(\left\lceil \frac{1}{1-\delta}\right\rceil n\right)$.

We apply Lemma 4.1 with $\alpha_n := t_n$, $P := 2M$, $a_n := d(x_{n-1}, x_{n+N-1})$ and $b_n := 2M|\lambda_{n+N} - \lambda_n|$ to obtain that $\lim\limits_{n\to\infty} d(x_{n-1}, x_{n+N-1}) = 0$ with rate of convergence $\tilde{\Phi}+1$. Hence, $\lim\limits_{n\to\infty} d(x_n, x_{n+N}) = 0$ with rate of convergence $\tilde{\Phi}$.

Assume now that $\lim\limits_{n\to\infty} \lambda_{n+1} = 0$ with rate of convergence α. By Lemma 4.2, it follows that $d(x_{n+1}, T_{n+1}(x_n)) \leq 2M\lambda_{n+1}$, hence

$$d(x_{n+1}, T_{n+1}(x_n)) \leq \frac{\varepsilon}{2N} \text{ for all } n \geq \alpha\left(\frac{\varepsilon}{4MN}\right).$$

One can easily see that

$$d(x_n, T_{n,N}(x_n)) \leq d(x_n, x_{n+N}) + \sum_{i=1}^{N} d(x_{n+i}, T_{n+i}(x_{n+i-1})).$$

Therefore, $d(x_n, T_{n,N}(x_n)) \leq \varepsilon$ for all $n \geq \Phi$. □

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