Relations between combinatorial structures and Lie algebras: centers and derived Lie algebras

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Abstract

In this paper, we study how two important ideals of a given Lie algebra $\mathfrak{g}$ (namely, the center $Z(\mathfrak{g})$ and the derived Lie algebra $\mathcal{D}(\mathfrak{g})$) can be translated into the language of Graph Theory. In this way, we obtain some criteria and characterizations of these ideals using Graph Theory.

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1 Introduction

Finding relations between different fields of Mathematics is always an important goal in mathematical research. Both Lie Theory and Graph Theory are running in a high level due to their several applications in Engineering, Physics and Applied Mathematics, in addition to their theoretical study. There exists a close relation between both theories. For example, graphs have been essential for studying semisimple Lie algebras, because trees perform an important role to determine the Dynkin diagrams associated with such algebras [8]. Additionally, Graph Theory is also applied nowadays to study the representation of finite-dimensional algebras [7].

Our main goal consists in making new progress related to the link between Lie algebras and combinatorial structures (including graphs and other simplicial complexes). Hence, we are proceeding with previous work [1, 2, 3, 4, 6] in the literature opening this research line. Indeed, they all are based on the definition of a mapping between Lie algebras and certain types of combinatorial structures. This time, we study the translation of several operations on graphs and combinatorial structures into the language of Lie algebras.

This article is organized as follows: after reviewing some well-known results on Lie and Graph Theories in Section 2, Section 3 recalls the mapping introduced in [1], which associates combinatorial structures with Lie algebras. In that same section, we recall some properties proved in [1] for the above-mentioned mapping and apply them later in this paper. Finally, Sections 4 and 5 analyze the translation of the derived Lie algebra and the center of a given Lie algebra into the language of combinatorial structures. This translation provides some results and characterizations for these two important ideals of $\mathfrak{g}$ starting from its associated combinatorial structure.

In our opinion, the techniques and results introduced in this article are very useful and helpful to obtain a better knowledge and understanding of the Lie algebras and these two ideals, as well as of the relation between Lie algebras and simplicial complexes (including graphs). For instance, the classification of combinatorial structures may involve an easier method to advance in the classification problem of Lie algebras without using them.

2 Preliminaries

First, we show some preliminary concepts about Lie algebras, bearing in mind that the reader can consult [9] for a general overview. From here on, we only consider finite-dimensional Lie algebras over the complex number field $\mathbb{C}$.

Definition 1. *A Lie algebra $\mathfrak{g}$ is a vector space with a second bilinear inner composition law ($[\cdot,\cdot]$) called the* bracket product *or Lie bracket, which satisfies*

$$[X,X]=0,\ \forall X\in\mathfrak{g}\quad\text{and}\quad [[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0,\ \forall X,Y,Z\in\mathfrak{g}.$$

Given a basis $\{e_i\}_{i=1}^n$ of $\mathfrak{g}$, its structure (or Maurer-Cartan) constants *are defined by $[e_i,e_j]=\sum c_{i,j}^h e_h$, for $1\le i<j\le n$.*

Definition 2. *Given a Lie algebra $\mathfrak{g}$,*

a) its derived Lie algebra *is the ideal given by $\mathcal{D}(\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]=\operatorname{span}(\{[X,Y]\mid X,Y\in\mathfrak{g}\})$; and*

b) its center is defined as $Z(\mathfrak{g})=\{X\in\mathfrak{g}\mid [X,Y]=0,\ \forall Y\in\mathfrak{g}\}$.

Although the reader can consult [5] as an introductory reference on Graph Theory, some notions are recalled next in this section.

Definition 3. *A* graph *consists of an ordered pair $G=(V,E)$, where V is a non-empty set called the vertex set and E is a set of unordered pairs (edges) of two vertices, called the edge set. If the edges are ordered pairs of vertices, then the graph is named* digraph.

Definition 4. *Let $G=(V,E)$ be a graph. For a vertex $v\in V$, the* (open) neighbourhood *of v in G is the vertex subset $N(v)=\{w\in V\mid (v,w)\in E\}$. Thus, two vertices $u,v\in V$ are* twin *if they have the same neighbourhoods; i.e. $N(u)=N(v)$.*

Definition 5. *Given a (di)graph $G=(V,E)$, a* (oriented) cycle *in G is a sequence $v_1v_2\cdot v_r$ of vertices in V such that: a) $(v_i,v_{i+1})\in E$ for $i=1,\dots,r$; and b) $v_i\neq v_j$ for $i\neq j$ except for $v_1=v_r$. An n-cycle is a cycle of length n (with n vertices).*

For example, an oriented 2-cycle would be a double edge in a digraph (see Figure 1).

Definition 6. *Given a digraph $G = (V, E)$, a vertex $v \in V$ is a* sink *(resp. a* source*) if all the edges incident with v are oriented towards v (resp. oriented from v). This definition is illustrated in Figure* 2.

Definition 7. *Given $n \in \mathbb{N}$, P_n is a weighted digraph of n vertices alternating sources with sinks.*

Examples of this definition are illustrated in Figures 3-5. This type of digraph was used in [1, Theorem 3.2].

3 Associating combinatorial structures with Lie algebras

Let $\mathfrak{g}$ be an n-dimensional Lie algebra with basis $\mathcal{B} = \{e_i\}_{i=1}^n$. The structure constants are given by $[e_i, e_j] = \sum_{k=1}^n c_{i,j}^k e_k$ and, hence, the pair $(\mathfrak{g}, \mathcal{B})$ is associated with a combinatorial structure built according to the following steps as introduced in [1]

a) Draw vertex i for each $e_i \in \mathcal{B}$.

b) Given three vertices $i < j < k$, draw the full triangle ijk if and only if $(c_{i,j}^k, c_{j,k}^i, c_{i,k}^j) \neq (0, 0, 0)$. Then, the edges ij, jk and ik have weights $c_{i,j}^k$, $c_{j,k}^i$ and $c_{i,k}^j$, respectively.

   b1) Use a discontinuous line (named *ghost edge*) for edges with weight zero.

   b2) If two triangles ijk and ijl with $1 \leq i < j < k < l \leq n$ satisfy $c_{i,j}^k = c_{i,j}^l$, draw only one edge between the vertices i and j shared by both triangles (see Figure 6).

c) Given two vertices i and j with $1 \leq i < j \leq n$ and such that $c_{i,j}^i \neq 0$ (resp. $c_{i,j}^j \neq 0$), draw a directed edge from j to i (resp. from i to j), with weight $c_{i,j}^i$ (resp. $c_{i,j}^j$). This can be seen in Figure 7.

Consequently, every Lie algebra with a given basis is associated with a combinatorial structure of this type, which turns out to be simplicial complexes of dimension less than 3.

Throughout the paper, we will refer to the following results from [1]:

Lemma 1. *[1, Lemma 3.1] Let $\mathfrak{g}$ be a Lie algebra associated with a digraph G. Then, the configurations shown in Figure* 8 *are forbidden in G, for any three different vertices i, j, k (independently of the weights of the edges).*

Theorem 1. *[1, Theorem 3.2] Let G be a digraph without (oriented) 3-cycles, associated with a Lie algebra. Then, G is*

(i) a unique double edge, or

(ii) a digraph of type P_n

Conversely, any digraph satisfying (i) or (ii) is associated with a Lie algebra.

Theorem 2. *[1, Theorem 3.6] Let G be a digraph containing (oriented) 3-cycles and associated with a Lie algebra. Then, G satisfies the following conditions*

(i) The double edges of G lie on the 3-cycles and there are no 3-cycles without double edges.

(ii) The adjacent vertices with the extreme vertices of the double edges are not mutually adjacent. Moreover, they appear in one of the configurations of Figure 9.

(iii) The subdigraph obtained from G by removing its double edges satisfies condition (ii) of Theorem 1.

Remark 1. *Note that $\{p_1, \ldots, p_r\}$ and $\{q_1, \ldots, q_s\}$ in Figure 9 are both sets of twin vertices.*

4 Derived Lie algebras

Next, if the combinatorial structure associated with a Lie algebra $\mathfrak{g}$ satisfies certain properties, we describe the derived Lie algebra $\mathcal{D}(\mathfrak{g})$ of $\mathfrak{g}$ in terms of the associated combinatorial structure.

4.1 Derived Lie algebras and digraphs

Proposition 1. *If $G = (V, E)$ is a digraph associated with a Lie algebra $\mathfrak{g}$, then* $\mathcal{D}(\mathfrak{g}) = \text{span}(\{e_i \mid i \text{ is a sink}\} \cup \{c^j_{j,k} e_j + c^k_{j,k} e_k \mid \{j,\ k\} \text{ is an oriented } 2-\text{cycle}\})$.

Proof. If G does not contain oriented 2-cycles, then G is a weighted digraph P_n, alternating sources and sinks. In these graphs, sinks correspond to vectors in the generator system of $\mathcal{D}(\mathfrak{g})$. Obviously, the second set in the expression to be proved for $\mathcal{D}(\mathfrak{g})$ is empty in these graphs.

If G contains 2-cycles, but not 3-cycles, then G must be a unique double edge, according to Theorem 1 and $\mathcal{D}(\mathfrak{g})$ is spanned by the only vector resulting from the bracket product, which belongs to the second set. Obviously, the first set in the expression to be proved for $\mathcal{D}(\mathfrak{g})$ is empty in these graphs.

Finally, if G contains 3-cycles, Theorem 2 implies that the vertices of G must be only either sinks or those from an oriented 2-cycle (as can be seen in Figure 9). Therefore, $\mathcal{D}(\mathfrak{g})$ is spanned by those vectors associated with sinks and the sum of the vectors corresponding to each oriented 2-cycle. □

Next, we show the implementation of an algorithmic procedure to compute the derived Lie algebra from a weighted digraph associated with a Lie algebra. This algorithm consists of the following two steps:

a) Check if the digraph inserted is associated with a Lie algebra.

b) Compute the derived Lie algebra according to Proposition 1.

In order to implement the algorithm, we have used the symbolic computation package MAPLE 12. The libraries `DifferentialGeometry`, `LieAlgebras`, `GraphTheory` and `ListTools` must be loaded to use commands related to lists, graph theory and Lie algebras.

The first step of this algorithm is executed by the routine `associated`. This routine checks if a given digraph is associated with a Lie algebra. It receives the following two inputs: the list `V` with the vertices of the digraph and the set `E` with its directed, weighted edges. As output, we obtain the vector space with basis $\{e_i\}_{i=1}^{n}$, where e_i corresponds to vertex i in the list `V`, and the brackets associated with the edges in the set `E`.

```
> associated:=proc(V,E)
> local L;
> L:=[];
> for i from 1 to nops(E) do
> if E[i][1][1] < E[i][1][2] then
> L:=[op(L),[[E[i][1][1],E[i][1][2],E[i][1][2]],E[i][2]]];
> else L:=[op(L),[[E[i][1][2],E[i][1][1],E[i][1][2]],E[i][2]]];
> end if;
> end do;
> return _DG([["LieAlgebra",Alg1,[nops(V)]],L]);
> end proc:
```

Once the vector space and the structure constants (i.e. the bracket product) are generated by the routine `associated`, we must define the law corresponding to these, which is done by evaluating the sentence

```
> DGsetup(associated(V,E));
```

After defining this vector space, saved as `Alg1`, we can operate over it. Now, we can execute a sentence in order to test if the Jacobi identities hold for `Alg1`. The output is saved in the variable `Jacobi`.

```
Alg1 > assign(Jacobi,Query(Alg1,"Jacobi"));
```

The vector space `Alg1`, defined by the output of `associated`, is a Lie algebra if and only if the variable `Jacobi` equals `true`.

Now, we show the implementation of the second step of our algorithm, where we compute the derived Lie algebra by applying Proposition 1. The inputs in this routine are again the sets V and E. Firstly, we write a conditional sentence so that if the Jacobi identities are not satisfied, then the program returns a message saying that the given digraph is not associated with any Lie algebra. In the implementation, we define two local variables: S, which saves the basis of the derived algebra; and F, which starts as the set E of edges and finishes as the complement of the subset of oriented 2-cycles with respect to E. The variable F is modified by deletion of oriented 2-cycles after their use and, finally, F consists of the edges incident with sink vertices.

```
> derived:=proc(V,E)
> local S, F;
> S:={}; F:=E;
> for i from 1 to nops(E)-1 do
> for j from i+1 to nops(E) do
> if E[i][1]=Reverse(E[j][1]) then
> S:={op(S),E[i][2]*e[E[i][1][2]]+E[j][2]*e[E[j][1][2]]};
> F:=F minus {E[i],E[j]};
> end if;end do;end do;
> for k from 1 to nops(F) do
> S:={op(S),F[k][2]*e[F[k][1][2]]};
> end do;
> return S;
> end proc:
```

Example 1. *Consider the digraph determined for the following sets of vertices and edges.*

```
> V:=[1,2,3,4];
> E:={[[1,2],-1],[[2,1],1],[[1,3],1],[[2,3],1],[[4,3],1]};
```

To obtain a representation of this digraph (see Figure 10*), we execute the sentences*

```
> G:=Digraph(V,E);
> DrawGraph(G);
```

Now, we use the routine associated *in order to check if this digraph is associated with a Lie algebra.*

```
> associated(V,E);
```

Finally, we check the Jacobi identities and we apply the routine derived*.*

```
> DGsetup(V,E);
Alg1 > Query(Alg1,"Jacobi");
>             true
```

The output obtained is the derived Lie algebra, so the variable Jacobi *equals* true *and the digraph in Figure* 10 *is associated with a* 4*-dimensional Lie algebra.*

```
> derived(A,B);
>          {e[3],-e[2]+e[1]}
```

4.2 Derived Lie algebras and full triangles

Proposition 2. *If T is a full triangle with vertices i, j and k associated with a Lie algebra, then the following holds:*

- *If all the edges in T are full, then $\mathfrak{g}$ is a perfect Lie algebra (i.e. $\mathcal{D}(\mathfrak{g}) = \mathfrak{g}$).*
- *If ij is the unique ghost edge in T, then $\mathcal{D}(\mathfrak{g}) = \mathrm{span}(\{e_i, e_j\})$.*
- *If ij and ik are ghost edges in T, then $\mathcal{D}(\mathfrak{g}) = \mathrm{span}(\{e_i\})$.*

Proof. It follows straight from the law of the Lie algebra associated with T. □

4.3 Derived Lie algebras and combinatorial structures

Assume that G is a combinatorial structure associated with a Lie algebra $\mathfrak{g}$. Then, the following algorithm can be applied to obtain a generator system for $\mathcal{D}(\mathfrak{g})$. For each pair of vertices (i, j), we follow this procedure

1. If i and j are vertices of an edge in a digraph not belonging to a full triangle, then we add to the generator system for $\mathcal{D}(\mathfrak{g})$
   (a) the vector e_i, if i is a sink (analogously for j) or;
   (b) the vector $c_{i,j}^i e_i + c_{i,j}^j e_j$, if i and j determine an oriented 2-cycle.
2. If i and j are vertices of an edge in a full triangle, not determining a directed edge, then we consider all the full triangles containing the vertices i and j and we follow the following steps
   (a) Consider a vector $v = 0$.
   (b) Label the vertices determining a full triangle with i and j. These vertices form the set $\{k_\alpha\}_{\alpha \in I}$.
   (c) For each vertex k_α such that the edge ij is full in the triangle ijk_α, write $v = v + c_{ij}^{k_\alpha} e_{k_\alpha}$.
   (d) Add vector v to the generator system for the derived Lie algebra $\mathcal{D}(\mathfrak{g})$.
3. If i and j are vertices of an edge in a digraph (resp. a full triangle) and only one of them is a common vertex to a full triangle (resp. a digraph), then we follow the procedure indicated in Step 1 (resp. Step 2).
4. If both i and j are common vertices of a directed edge and a full triangle, then we follow this procedure
   (a) Since ij is an edge of a full triangle, consider the vector v as we did in Step 2.

(b) Since ij is an edge in a digraph, add to v the corresponding vector associated with the edge ij according to Step 1.

(c) Add the vector v to the generator system for the derived Lie algebra $\mathcal{D}(\mathfrak{g})$.

5 Centers

The present section characterizes the structure of the center $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ associated with a combinatorial structure.

5.1 Centers and digraphs

Lemma 2. *Let G be a digraph associated with a Lie algebra $\mathfrak{g}$ whose decomposition into connected components is given by $G = \bigcup_{\alpha \in I} C_\alpha$. Then, the Lie algebra $\mathfrak{g}$ can be decomposed into the direct sum $\mathfrak{g} = \bigoplus_{\alpha \in I} \mathfrak{c}_i$, where $\mathfrak{c}_i$ is the Lie algebra associated with C_i, as well as being a Lie ideal of $\mathfrak{g}$.*

Proof. The bracket between a vector of $\mathfrak{c}_i$ and other of $\mathfrak{g} \setminus \mathfrak{c}_i$ is zero since these two vectors are associated with vertices from different connected components and, hence, there does not exist any edge between these vertices. □

Proposition 3. *Let $k \in \mathbb{N}$ and the Lie algebra $\mathfrak{g}$ is associated with the digraph P_{2k+1} such that both end vertices are sources. The center $Z(\mathfrak{g})$ is of dimension one and spanned by the vector $u = \lambda_0 e_1 + \lambda_1 e_3 + \lambda_2 e_5 + \ldots + \lambda_{k-2} e_{2k-3} + \lambda_{k-1} e_{2k-1} + \lambda_k e_{2k+1}$, where the coefficients satisfy the following conditions*

$$\lambda_0 = c_{2,3}^2, \lambda_1 = \lambda_0 \cdot \frac{c_{1,2}^2}{c_{2,3}^2}, \lambda_2 = \lambda_1 \cdot \frac{c_{3,4}^4}{c_{4,5}^4}, \lambda_3 = \lambda_2 \cdot \frac{c_{5,6}^6}{c_{6,7}^6}, \ldots, \lambda_{\lfloor \frac{k}{2} \rfloor} = \lambda_{\lfloor \frac{k}{2} \rfloor - 1} \cdot \frac{c_{k-2,k-1}^{k-1}}{c_{k-1,k}^{k-1}},$$

$$\lambda_{\lfloor \frac{k}{2} \rfloor + 1} = \lambda_{\lfloor \frac{k}{2} \rfloor + 2} \cdot \frac{c_{k+3,k+4}^{k+3}}{c_{k+2,k+3}^{k+3}}, \ldots, \lambda_{k-3} = \lambda_{k-2} \cdot \frac{c_{2k-4,2k-3}^{2k-4}}{c_{2k-5,2k-4}^{2k-4}}, \lambda_{k-2} = \lambda_{k-1} \cdot \frac{c_{2k-2,2k-1}^{2k-2}}{c_{2k-3,2k-2}^{2k-2}}$$

$$\lambda_{k-1} = \lambda_k \cdot \frac{c_{2k,2k+1}^2 k}{c_{2k-1,2k}^{2k}}, \lambda_k = c_{2k-1,2k}^{2k},$$

$$\lambda_{\lfloor \frac{k}{2} \rfloor} \cdot c_{k,k+1}^{k+1} = c_{k+1,k+2}^{k+1} \cdot \lambda_{\lfloor \frac{k}{2} \rfloor + 1}.$$

Proof. It is sufficient to consider the digraph P_{2k+1} shown in Figure 3 and impose $[u, e_i] = 0$, for $i = 1, \ldots, 2k+1$, where $u = \sum_{i=1}^{2k+1} \lambda_i e_i$. □

Proposition 4. *Let G be a digraph of type P_n different from the one considered in Proposition 3. Then, its associated Lie algebra $\mathfrak{g}$ has a trivial center $Z(\mathfrak{g}) = \{0\}$.*

Proof. We analyze the two unique possible configurations

1. Consider the graph P_{2k+1} whose end vertices are sinks (see Figure 4). If $u = \sum_{i=1}^{2k+1} \lambda_i e_i \in Z(\mathfrak{g})$, we must impose that u commutes with each basis vector e_i. Since this bracket is

$$[u, e_i] = \begin{cases} -\lambda_2 c_{1,2}^1 e_1, & \text{if } i = 1; \\ \lambda_{i-1} c_{i-1,i}^{i-1} e_{i-1} - \lambda_{i+1} c_{i,i+1}^{i+1} e_{i+1}, & \text{if } 2 \leq i \leq 2k \text{ even}; \\ \lambda_{i-1} c_{i-1,i}^{i} e_i - \lambda_{i+1} c_{i,i+1}^{i+1} e_i, & \text{if } 3 \leq i \geq 2k \text{ odd}; \end{cases}$$

the solution of the system $\{[u, e_i] = 0\}_{1 \leq i \leq 2k}$ is $\{\lambda_i = 0\}_{1 \leq i \leq 2k+1}$ and, therefore, $u = 0$.

2. Consider the graph P_{2k} (see Figure 5). We reproduce the previous reasoning with the vector $u = \sum_{i=1}^{2k} \lambda_i e_i \in Z(\mathfrak{g})$, but taking into account that the bracket between the vector u and the basis vector e_i is now as follows

$$[u, e_i] = \begin{cases} -\lambda_2 c_{1,2}^2 e_2, & \text{if } i = 1; \\ \lambda_{i-1} c_{i-1,i}^{i} e_i - \lambda_{i+1} c_{i,i+1}^{i} e_i, & \text{if } 2 \leq i \leq 2k-1 \text{ even}; \\ \lambda_{i-1} c_{i-1,i}^{i-1} e_{i-1} - \lambda_{i+1} c_{i,i+1}^{i+1} e_{i+1}, & \text{if } 3 \leq i \geq 2k-1 \text{ odd}; \\ \lambda_{2k} c_{2k-1,2k}^{2k} e_{2k}, & \text{if } i = 2k; \end{cases}$$

The solution of the system $\{[u, e_i] = 0\}_{1 \leq i \leq 2k}$ is $\{\lambda_i = 0\}_{1 \leq i \leq 2k}$ again and, consequently, $u = 0$.

□

Proposition 5. *If G is an oriented 2-cycle or a digraph corresponding to the configurations in Theorem 2, then the center $Z(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ associated with G is trivial.*

Proof. In virtue of Lemma 2, the center of a direct sum of Lie algebras with trivial center is trivial. Therefore, we can assume that G is connected. If G is an oriented 2-cycle with vertices i and j, then its associated Lie algebra is given by the bracket $[e_i, e_j] = c_{i,j}^i e_i + c_{i,j}^j e_j$, where $c_{i,j}^i, c_{i,j}^j \neq 0$. Thus, its center is trivial.

Next, we consider the first class of digraphs in Theorem 2, but only considering a unique twin vertex (see Figure 11). If $u = \alpha e_i + \beta e_j + \gamma e_k \in Z(\mathfrak{g})$, then $[e_i, u] = \beta(c_{i,j}^i e_i + c_{i,j}^j e_j) + \gamma c_{i,k}^k e_k = 0$ and $[e_j, u] = -\alpha(c_{i,j}^i e_i + c_{i,j}^j e_j) + \gamma c_{j,k}^k e_k = 0$. These brackets lead to a system whose solution is $u = 0$. The same conclusion can be obtained when considering an arbitrary amount of twin vertices in the structure.

Finally, we study the second digraph in Theorem 2, but only considering a unique twin vertex again (see Figure 12). By using an analogous reasoning, we can also prove that $Z(\mathfrak{g}) = \{0\}$ and this can also be proved for an arbitrary number of twin vertices. □

Corollary 1. *Let G be a digraph whose connected components are digraphs P_n, oriented 2-cycles or configurations from Figure 9. Then, the center $Z(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ associated with G is spanned by the vectors associated with the isolated vertices and the vectors of type u shown in Proposition 3.* □

5.2 Centers and full triangles

Proposition 6. *If G consists of full triangles and is associated with a Lie algebra $\mathfrak{g}$, then the center $Z(\mathfrak{g})$ is spanned by the vertices being incident with ghost edges.*

Proof: First, we prove that vertices being incident with full edges are not associated with vertices in the center $Z(\mathfrak{g})$. Effectively, if v is incident with a full edge vw, then $[e_v, e_w] \neq 0$ and $v \notin Z(\mathfrak{g})$.

Next, we prove that the remaining vertices correspond to vectors belonging to the center $Z(\mathfrak{g})$. Consider a full triangle with a vertex incident with two ghost edges. Hence, its associated Lie algebra $\operatorname{span}(\{e_1, e_2, e_3\})$ has non-zero brackets $[e_i, e_j] = c_{i,j}^k e_k$, where $1 \leq i < j \leq 3$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$. We define $u = \sum_{h=1}^{3} \alpha_h e_h \in Z(\mathfrak{g})$. In a full triangle there is at least a full edge and if this full edge is ij, then $c_{i,j}^k \neq 0$. Conditions $[e_i, u] = [e_j, u] = 0$ imply $\alpha_i = \alpha_j = 0$. Therefore, the vertices incident with full edges do not appear as terms in u. □

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