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Determining Asymptotic Behaviour from the Dynamics on Attracting Sets

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Proposed running head: Determining asymptotic behaviour

Two tracking properties for trajectories on attracting sets are studied. We prove that trajectories on the full phase space can be followed arbitrarily closely by skipping from one solution on the global attractor to another. A sufficient condition for asymptotic completeness of invariant exponential attractors is found, obtaining similar results as in the theory of inertial manifolds. Furthermore, such sets are shown to be retracts of the phase space, which implies that they are simply connected.

KEY WORDS: Global attractors, inertial manifolds, exponential attractors, asymptotic completeness, connectedness.

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1. INTRODUCTION

It is known that the theory of inertial manifolds (finite-dimensional, positively invariant, exponentially attracting Lipschitz manifolds) for dissipative dynamical systems (see Foias *et al.*, 1988; Temam, 1990) enables the dynamics of many partial differential equations to be described by a finite dimensional system of ODEs. This is strongest when we can show that any trajectory of the flow in the infinite dimensional Hilbert space H tends to a trajectory on the manifold, so that we can track, exponentially fast, every trajectory in H by a trajectory moving on the finite-dimensional inertial manifold. This property is called *asymptotic completeness* or *exponential tracking* (see Constantin *et al.*, 1998; Chow *et al.*, 1992; Foias *et al.*, 1989; Robinson, 1996a). In Robinson (1996a) it is shown that the property of *flow-normal hyperbolicity* is a sufficient condition for an inertial manifold to be asymptotically complete.

On the other hand, the concept of *exponential attractors* (also called *inertial sets*) was introduced in 1990 by Eden *et al.* (see also their 1994 monograph). An exponential attractor is a compact set, with finite fractal dimension, which is invariant for the forward flow, and attracts all the orbits at an exponential rate.

A natural question for any kind of attracting set is how much information can be obtained from the dynamics on the attracting set that could be useful for the understanding of the full dynamics on H.

In this paper, we first prove (section 3) that we can track arbitrarily closely any trajectory in H for arbitrary large time-lengths by trajectories on the global attractor. Furthermore, it is shown that every trajectory on the phase space H can be followed for all time by skipping from one trajectory on the global attractor to another. Despite being mathematically rather obvious, the results of this section indicate exactly how the dynamics on the global attractor, in general, “determine” the asymptotic dynamics of the full equation.

Our second result (section 4) shows that, for *invariant* exponential attractors, and under the hypothesis of flow normal hyperbolicity, we have asymptotic completeness.

Finally, we show (section 5) that a flow normally hyperbolic set X must be a retract of the phase space, i.e. that is there is a continuous map $\pi : H \to X$ which is the identity on X. This implies that X is both arcwise and simply connected, and may prove useful in deducing new properties of global attractors.

2. BRIEF FORMULATION OF THE PROBLEM

Suppose we have a dissipative evolution equation of the form

$$du/dt + Au = f(u) \qquad u(0) = u_0 \tag{2.1}$$

in a separable Hilbert space H. The linear operator A is assumed to be positive, self-adjoint, unbounded and with A^{-1} compact. We further assume that the nonlinear term f is locally Lipschitz from $D(A^\alpha)$ into $D(A^\beta)$ $(0 \leq \alpha - \beta < 1)$; under these conditions the initial value problem (2.1) is solved by a semigroup of nonlinear operators $\{S(t)\}_{t\geq 0}$, that is continuous from $D(A^\alpha)$ into itself, for $t \geq 0$ (Henry, 1981). Finally, we suppose that there exists a compact, absorbing set B, invariant for the forward flow of (2.1), which implies (Temam, 1988) the existence of a finite dimensional global attractor $\mathcal{A}$, i.e., a compact invariant set that attracts all the solutions of (2.1). In what follows, $|\,.\,|_\alpha$ denotes the norm in $D(A^\alpha)$ (i.e. $|u|_\alpha = |A^\alpha u|_H$).

Under these conditions, we have the following simple lemma on continuity of solutions with respect to initial conditions, which will be one of the main tools in what follows (cf. Henry, 1981).

Lemma 2.1. *Let $u_1(t), u_2(t)$ be two solutions of (2.1) corresponding to initial data $u_1(0)$,$u_2(0) \in B$ respectively. Then, there exists $k > 0$ (that depends on α, β and the*

Lipschitz constant of f on the absorbing set B) such that

$$|u_1(t) - u_2(t)|_\alpha \le |u_1(0) - u_2(0)|_\alpha e^{kt^\theta} \tag{2.2}$$

for all $t \ge 0$, with $\theta = 1 - (\alpha - \beta)$.

Proof. If we write the solutions $u_1(t), u_2(t)$ using the variation of constants formula, we get

$$u_i = u_i(0)e^{-At} + \int_0^t e^{-A(t-s)} f(u_i(s))\, ds,\ \ i = 1, 2.$$

If we call $\delta(t) = u_1(t) - u_2(t)$, $\delta(t)$ satisfies

$$\delta(t) = \delta(0)e^{-At} + \int_0^t e^{-A(t-s)}(f(u_1(s)) - f(u_2(s)))\, ds.$$

We want to evaluate how the trajectories $u_1(t), u_2(t)$ separate in $D(A^\alpha)$, so, taking the norm in $D(A^\alpha)$,

$$|A^\alpha \delta(t)| \le |A^\alpha e^{-At}\delta(0)| + \int_0^t |A^\alpha e^{-A(t-s)}(f(u_1) - f(u_2))|\, ds$$

$$\le \|e^{-At}\|_{op}|A^\alpha \delta(0)| + \int_0^t \|A^{\alpha-\beta}e^{-A(t-s)}\|_{op}|A^\beta(f(u_1) - f(u_2))|\, ds$$

$$\le |\delta(0)|_\alpha + cc_{\alpha-\beta}\int_0^t (t-s)^{-(\alpha-\beta)}|u_1(s) - u_2(s)|_\alpha\, ds,$$

(bounding the operator norms by standard expressions, see Henry (1981) or Temam (1988)) and so, by Gronwall's lemma,

$$|\delta(t)|_\alpha \le |\delta(0)|_\alpha e^{kt^\theta}.$$

with $k = cc_{\alpha-\beta}/\theta$, and $\theta = 1 - (\alpha - \beta)$. ∎

3. TRACKING TRAJECTORIES FOR ARBITRARILY LONG TIMES

Using the previous result it is now straightforward to prove the following one, which gives us some information (for finite time intervals) about the relationship between trajectories of (2.1) on H and trajectories on the global attractor $\mathcal{A}$.

Proposition 3.1. *Given a trajectory $u(t)$ of (2.1), $\epsilon > 0$ and $T > 0$, there exists a time $\tau = \tau(\epsilon, T) > 0$ and a point $v_0 \in \mathcal{A}$ such that*

$$|u(\tau + t) - S(t)v_0|_\alpha \leq \epsilon \qquad \text{for all} \qquad 0 \leq t \leq T. \tag{3.1}$$

Proof. Since $\mathcal{A}$ is a global attractor, the trajectory $u(t)$ tends towards $\mathcal{A}$. Thus, given $\epsilon > 0$ and $T > 0$, and using the compactness of $\mathcal{A}$, there exists a time τ and a point $v_0 \in \mathcal{A}$ such that

$$\operatorname{dist}(u(\tau), \mathcal{A}) = |u(\tau) - v_0|_\alpha \leq \epsilon e^{-kT^\theta}. \tag{3.2}$$

We now consider the trajectory $v(t)$ on $\mathcal{A}$ with $v(0) = v_0$. Then, the two trajectories $u(t)$ (seen as a trajectory starting at the point $u(\tau)$) and $v(t) = S(t)v_0$ satisfy, by (2.2),

$$\begin{aligned} |u(\tau + t) - S(t)v_0|_\alpha &\leq |u(\tau) - v_0|_\alpha e^{kt^\theta} \qquad \text{for all } t \geq 0 \\ &\leq |u(\tau) - v_0|_\alpha e^{kT^\theta} \qquad \text{for } 0 \leq t \leq T \\ &\leq \epsilon \text{ by (3.2).} \end{aligned}$$

∎

In fact, this proposition gives us a little more information, since it says that, given ϵ_1 and $T > 0$, there exists a time τ_1 such that, for all $t \geq \tau_1$,

$$\operatorname{dist}(u(t), \mathcal{A}) \leq \epsilon_1 e^{-kT^\theta}.$$

So, we can track the trajectory $u(t)$ within a distance ϵ_1 for a time T starting at any time $t \geq \tau_1$.

We can replace T by $2T$ and apply the same argument for $\epsilon_2 < \epsilon_1$, that is, there exists a time τ_2 such that, for all $t \geq \tau_2$

$$\operatorname{dist}(u(t), \mathcal{A}) \leq \epsilon_2 e^{-k(2T)^\theta},$$

and then, the trajectory $u(t)$ can be tracked for a time $2T$ starting at any time $t \geq \tau_2$.

Thus, $u(t)$ can be followed from τ_1 to τ_2 by a distance ϵ_1 with a finite number of trajectories on $\mathcal{A}$ of time-length T, and when we reach τ_2, we can start to track $u(t)$ within a distance ϵ_2 with trajectories on $\mathcal{A}$ of time-length $2T$, until we reach the corresponding τ_3, etc. Furthermore, note that, for two of these consecutive trajectories on $\mathcal{A}$, the "jumps" are bounded by $\epsilon_k + \epsilon_{k+1}$, since

$$\begin{aligned} &|v_{k+1} - S(t_{k+1} - t_k)v_k|_\alpha \\ &\qquad \leq |v_{k+1} - u(t_{k+1})|_\alpha + |u(t_k + (t_{k+1} - t_k)) - S(t_{k+1} - t_k)v_k|_\alpha \\ &\qquad \leq \epsilon_{k+1} + \epsilon_k. \end{aligned}$$

If we apply this process inductively, we obtain the following corollary (reminiscent of a results of Vishik (1992) which use "finite-dimensional combined trajectories" to approximate full trajectories of evolution equations under certain conditions):

Corollary 3.2. *Given a solution $u(t)$ of (2.1), there exists $\{\epsilon_m\}_{m=1}^\infty$, $\epsilon_m > 0$, $\epsilon_m \to 0$, a sequence of times $\{t_m\}_{m=1}^\infty$ and a sequence of points $\{v_m\}_{m=1}^\infty$, with $v_m \in \mathcal{A}$, such that*

$$t_{m+1} > t_m, \ \forall m \in N, \quad t_{m+1} - t_m \to \infty \ \textit{as} \ m \to \infty,$$

and

$$|u(t) - S(t - t_m)v_m|_\alpha \leq \epsilon_m \qquad \textit{for all} \ \ t_m \leq t \leq t_{m+1}.$$

Furthermore, the jumps $|v_{m+1} - S(t_{m+1} - t_m)v_m|_\alpha$ decrease to zero.

Note that our sequence of times $\{t_m\}_{m=1}^\infty$ verifies $t_{m+1} - t_m \to \infty$, and so the trajectory $u(t)$ is followed more and more closely for longer and longer lengths of time as $m \to \infty$.

Although elementary, this result is instructive. Indeed, it shows exactly how the dynamics on $\mathcal{A}$ can be said to determine the asymptotic behaviour of trajectories on H.

A trivial example (cf. Robinson, 1996a), the three dimensional system

$$dx/dt = z(x-y)$$

$$dy/dt = z(y-x)$$

$$dz/dt = -\lambda z|z|,$$

has $z \equiv 0$ as an attractor, on which the dynamics are trivial. However, trajectories that start off $z \equiv 0$, approach algebraically slowly according to

$$z(t) = \frac{z_0}{1+\mu|z_0|t}.$$

Thus for an initial condition (r, θ_0, z_0) (where the x and y have been turned into polar co-ordinates), $\theta(t)$ is given by

$$\theta(t) = \theta_0 + \frac{1}{\mu}\ln(1+\mu|z_0|t).$$

The trajectory is "determined" by that on $z \equiv 0$ inasmuch as it remains constant (to within an ϵ error) for longer and longer time intervals. Without a result like corollary 3.2, this interpretation is not necessarily an obvious one.

Since the dynamics are determined by those on the global attractor, constructing a finite dimensional system which reproduces the dynamics on (the finite dimensional set) $\mathcal{A}$ is a good way to study the asymptotic behaviour of (2.1).

Eden *et al.* (1994, chapter 10) make a start on this in their monograph on exponential attractors. The existence of a projection $P : H \to \mathbb{R}^D$, injective on $\mathcal{A}$, is ensured by a result of Mañé (1981), provided $D > 2d_F(\mathcal{A})+1$, where $d_F(\mathcal{A})$ is the fractal (box counting) dimension of $\mathcal{A}$. Eden *et al.* project the dynamics on $\mathcal{A}$ into $\mathbb{R}^D$ by such an orthogonal projection P, and construct the following finite dimensional non-smooth system of ODEs

$$dx/dt = \alpha(\nu(x)-x) + T(\nu(x)), \tag{3.3}$$

where $\alpha > 0$, $\nu(x)$ is a point in $P\mathcal{A}$ such that $\text{dist}(x, P\mathcal{A}) = |x-\nu(x)|$, and $T(v) = PF(P^{-1}v)$, with $F(u) = -Au + f(u)$ (note that $T(v)$ is well-defined for $v \in P\mathcal{A}$ since P is injective on $\mathcal{A}$). Furthermore, the projected set $P\mathcal{A}$ is exponentially attracting for (3.3). It would be very interesting to apply the corollary 3.2 to this system of ODEs, since it would mean that the dynamics of (3.3) is also limited by the dynamics on $\mathcal{A}$. However, the non-smooth terms in (3.3) mean that trajectories of this system do not depend continuously on the initial conditions (as in lemma 2.1) and so, it is not possible to apply the above argument to this system. Therefore, it would be very interesting to improve (3.3) by obtaining a smooth set of ODEs.

Note that in the results of this section we can replace the global attractor $\mathcal{A}$ by any *positively* invariant attracting set.

4. ASYMPTOTIC COMPLETENESS FOR INVARIANT EXPONENTIAL ATTRACTORS

We now suppose that we have an exponential attractor $\mathcal{E}$ for $(\{S(t)\}_{t\geq 0}, B)$, that is, $\mathcal{E}$ is a compact set, positively invariant, contains the global attractor $\mathcal{A}$ and attracts exponentially fast every solution with initial data in B. We can consider an exponential attractor as an intermediate step between global attractors and inertial manifolds. Thus, we try to find a result about the asymptotic completeness property similar to some of the results in the theory of inertial manifolds. However, without the hypothesis of the invariance of the exponential attractor (that is, $S(t)\mathcal{E} = \mathcal{E},\ \forall t \geq 0$) it seems difficult. So, we have restricted to the case of invariant exponential attractors, obtaining a sufficient condition for this attractor to be asymptotically complete. Nevertheless, even with this restriction, the proof improves on that of Robinson (1996a), based on inertial manifolds given as graphs, since it makes no assumptions on the form of the dynamics on the set.

We start with the following definition:

Definition 4.1. *An invariant exponential attractor $\mathcal{M}$ is flow-normally hyperbolic* *if*

$$\text{dist}(u(t), \mathcal{M}) \leq ce^{-\nu t} \textit{ for all } t \geq 0,$$

for any solution $u(t)$ *of (2.1), and*

$$|v_1(t) - v_2(t)|_\alpha \leq De^{-\gamma t}|v_1(0) - v_2(0)|_\alpha \textit{ for all } t \leq 0,$$

for two solutions $v_1(t)$ *and* $v_2(t)$ *lying on* $\mathcal{M}$*, where* $\nu > \gamma$*.*

This concept is a generalisation of the classical definition of *linearised normal hyperbolicity*, which is based on a linearised version of the flow on the manifold, and is used in analyses that show the persistence of invariant manifolds (e.g. Fenichel, 1971).

Theorem 4.2. *If an invariant exponential attractor is flow-normally hyperbolic, then it is asymptotically complete. Furthermore, the rate of tracking is the same as the rate of attraction towards the exponential attractor.*

Proof. Since $\mathcal{M}$ is an exponential attractor, we have that for all $u_0 \in D(A^\alpha)$

$$\text{dist}(u(t; u_0), \mathcal{M}) \leq c(u_0)e^{-\nu t}, \; t \geq 0,$$

and so, for all $t \geq 0$, there exists $v_t \in \mathcal{M}$ such that

$$|u(t) - v_t|_\alpha \leq ce^{-\nu t}. \tag{4.1}$$

Define

$$v_\infty(t) = \lim_{T\to\infty} v(t; T, v_T), \tag{4.2}$$

where $v(t; T, v_T)$ denotes the solution of (2.1) with $v(T; T, v_T) = v_T \in \mathcal{M}$, and hence $|u(T) - v_T|_\alpha \leq ce^{-\nu T}$.

Since $\mathcal{M}$ is invariant, the trajectory $v(t; T, v_T)$ is on $\mathcal{M}$. Defining $v_\infty(t)$ in the form we have done is a logical step because, roughly speaking, we want a trajectory that is ‘equal’ to $u(t)$ when '$T = \infty$' (see Marlin & Struble (1969) and Robinson (1996a) for a similar construction). We are going to show that $v_\infty(t)$ is well defined, is a solution of (2.1), and is a tracking trajectory for $u(t)$.

First of all, we want to evaluate

$$|v(t; s+h, v_{s+h}) - v(t; s, v_s)|_\alpha = |v(t-s; 0, v(s; s+h, v_{s+h})) - v(t-s; 0, v_s)|_\alpha.$$

Since $\mathcal{M}$ is flow-normally hyperbolic, we obtain that this expression is

$$\begin{aligned} &\leq De^{-\gamma(t-s)}|v(s; s+h, v_{s+h}) - v_s|_\alpha \\ &= De^{-\gamma(t-s)}|v(s; s+h, v_{s+h}) - v(s; s+h, S(h)v_s)|_\alpha, \end{aligned}$$

and again by the flow-normal hyperbolicity

$$\leq De^{-\gamma(t-s)}De^{\gamma h}|v_{s+h} - S(h)v_s|_\alpha. \tag{4.3}$$

Let us evaluate this last expression:

$$|v_{s+h} - S(h)v_s|_\alpha \leq |v_{s+h} - u(s+h)|_\alpha + |u(s+h) - S(h)v_s|_\alpha,$$

and by definition of v_{s+h} and (2.2) (the result of lemma 2.1)

$$\begin{aligned} &\leq ce^{-\nu(s+h)} + Ee^{kh^\theta}|u(s) - v_s|_\alpha \\ &\leq ce^{-\nu(s+h)} + Ee^{kh^\theta}ce^{-\nu s}. \end{aligned}$$

Thus, returning to (4.3)

$$\begin{aligned} |v(t; s+h, v_{s+h}) - v(t; s, v_s)|_\alpha &\leq De^{-\gamma(t-s)}De^{\gamma h}[ce^{-\nu(s+h)} + Ee^{kh^\theta}ce^{-\nu s}] \\ &\leq Ke^{-\gamma(t-s)}e^{-\nu s}, \end{aligned}$$

for $h < h^*$, choosing some $h^* > 0$.

Using this expression, it is now clear that (4.2) converges uniformly on bounded intervals of $[0, +\infty)$, since, for any $\tau > T$,

$$\begin{aligned} |v(t; T, v_T) - v(t; \tau, v_\tau)|_\alpha &\leq K\sum_{n=0}^{\infty} e^{-\gamma t}e^{-(\nu-\gamma)(T+nh)} \\ &\leq Ke^{-\gamma t}e^{-(\nu-\gamma)T}[1 - e^{-(\nu-\gamma)h}]^{-1} \\ &\leq \mathcal{K}e^{-(\nu-\gamma)T}, \end{aligned}$$

which tends to zero uniformly on $[0, t_0]$, for all $t_0 > 0$, as $T \to \infty$. Therefore, the limit in (4.2) exists and satisfies equation (2.1), since it is the uniform limit of solutions of (2.1).

To see that $v_\infty(t)$ has the tracking property for $u(t)$ is now straightforward, since

$$\begin{aligned} |v_\infty(t) - u(t)|_\alpha &\le |v_\infty(t) - v_t|_\alpha + |v_t - u(t)|_\alpha \\ &\le K \sum_{n=0}^{\infty} e^{-\gamma t} e^{-(\nu-\gamma)(t+nh)} + ce^{-\nu t} \\ &\le K e^{-\gamma t} e^{-(\nu-\gamma)t} [1 - e^{-(\nu-\gamma)h}]^{-1} + ce^{-\nu t} \\ &\le \mathcal{K}' e^{-\nu t}. \end{aligned}$$

Note that we also have ν as the exponential rate of tracking. ∎

In applications it may be difficult to check the normal hyperbolicity condition, particularly the backwards separation of trajectories on the attractor. In the infinite-dimensional case, one generally does not expect the backwards separation to obey an exponential inequality. Indeed, on the attractor one would only expect the expression

$$F(u) = -Au + f(u)$$

to be Hölder: if $\mathcal{E}$ is bounded in $D(A^{3/2})$, for example, then

$$|Au - Av| \le c|u - v|^{1/3} |A^{3/2}(u - v)|^{2/3} \le K|u - v|^{1/3},$$

for $u, v \in \mathcal{E}$, using a standard interpolation inequality (see Temam (1988), for example). Thus $F : D(A^\alpha) \to D(A^\alpha)$ will also be Hölder with exponent 1/3, and the backwards separation not exponentially bounded in general.

Nonetheless, there do exist some interesting cases where we find invariant exponential attractors. For example, in Robinson (1996b), the limiting behaviour of a family of problems

$$du/dt + Au + f_N(u) = 0 \tag{4.4}$$

is studied, with $f_N \to f$ in the C^0 norm, where each of the equations of (4.4) has an inertial manifold $\mathcal{M}_N$; it is shown that the inertial manifolds $\mathcal{M}_N$ converge to a finite dimensional invariant exponential attractor $\mathcal{M}_\infty$ which, in general, is not given as a graph. It is clear that we could apply this result to the exponential attractor $\mathcal{M}_\infty$ provided we had the flow normal hyperbolicity property for $\mathcal{M}_\infty$, which will follow also from the limiting procedure provided all the manifolds $\mathcal{M}_N$ are flow-normally hyperbolic with the same constants.

The theory is also applicable to inertial manifolds which are not given in the standard form of graphs.

In finite dimensional systems of ordinary differential equations one would indeed expect such an exponential bound on the separation, and this will be of interest in the next section. Note, however, that although Eden *et al.* (1994) prove that the projected attractor $P\mathcal{A}$ is an invariant exponential attractor for the finite-dimensional system (3.3), we find again that the non-smooth terms in this system obstruct the application of our result, since we do not have continuous dependence on initial conditions. Once more, this problem would be alleviated by obtaining a smooth system.

5. FLOW NORMAL HYPERBOLICITY & RETRACTIONS

The topology of the global attractor in general is not well understood, and standard results only ensure that it is connected. In this section we show that if the attractor is flow normally hyperbolic then it is simply connected, and in fact integrally connected in every dimension (see definition 6.2) - this follows from showing that the set is a retract of the phase space. A retract is simply a set X onto which there exists a retraction, i.e. a continuous function $r : H \to X$ such that r is the identity on X.

Theorem 5.1. *If an invariant exponential attractor is flow normally hyperbolic, then it is a retract of the phase space.*

Proof. We show, following Rosa & Temam (1994), that the map $u_0 \mapsto \pi u_0$, where πu_0 is the point $v_\infty(0)$ given by (4.2) in theorem 4.2, is a retraction. Clearly $\pi u = u$ if $u \in \mathcal{M}$, so it only remains to show that π is continuous. Suppose not. Then there exists a sequence $u_k \to u_0$ such that

$$|\pi u_k - \pi u_0| \geq \epsilon, \tag{5.1}$$

for some $\epsilon > 0$. Now consider the sequence $\{\pi u_k\}$. Since $\mathcal{A}$ is compact, there is a subsequence u_{k_j} such that

$$\pi u_{k_j} \to v_0 \in \mathcal{A},$$

and by (5.1) $|v_0 - \pi u_0| \geq \epsilon$. Since

$$|S(t)u_{j_k} - S(t)\pi u_{j_k}| \leq Ce^{-kt},$$

then taking limits (by lemma 2.1) yields

$$|S(t)u_0 - S(t)v_0| \leq Ce^{-kt},$$

and combining this with the tracking of $S(t)u_0$ by $S(t)\pi u_0$ shows that

$$|S(t)\pi u_0 - S(t)v_0| \leq 2Ce^{-kt}.$$

However, by the flow-normal hyperbolicity assumption two trajectories on $\mathcal{M}$ cannot approach faster than $e^{-\gamma t}$, which is a contradiction. ∎

An immediate corollary is a general connectedness result for flow-normally hyperbolic sets. Indeed, it is "connected in dimension n" for any n.

Definition 5.2. ***(Kuratowski, 1968).*** *A set X is* connected in dimension n *if given a continuous function f from the n-sphere into X, $f : \mathcal{S}^n \to X$, there is a continuous extension F of f, from the n-ball into X, $F : \mathcal{Q}^n \to X$.*

Note that if $n = 0$ then X is arcwise connected, and if $n = 1$ then X is simply connected.

Proposition 5.3. *Let H be connected in dimension n, and X is a flow normally hyperbolic invariant subset of H. Then X is connected in dimension n.*

Proof. By theorem 5.1 there exists a retraction π from H onto X. Now take a function $f : \mathcal{S}^n \to X$. This can also be viewed as a function $f : \mathcal{S}^n \to H$, and so can be extended to a continuous map $\mathcal{F} : \mathcal{Q}^n \to H$ since H is n-connected. Then $F = \pi \circ \mathcal{F}$ is a continuous extension of f which maps $\mathcal{Q}^n$ onto X. ∎

Since in general the phase space H is connected in all dimensions, any flow-normally hyperbolic set will also be connected in all dimensions. This shows that a flow-normally hyperbolic set is simply connected: current results for global attractors (e.g. Hale, 1988; Temam, 1988) only guarantee standard connectedness rather than the stronger properties that arise from the retraction.

For any finite-dimensional global attractor X, Günther (1995) constructs a finite-dimensional system that has a set homeomorphic to X as an attractor, and on which the dynamics are trivial. If one could show that the flow could be altered to make X exponentially attracting, this would uncover many connectivity properties of the global attractor currently unknown.

Note, however, that one cannot show that *any* attractor is a retract. Günther & Segal (1993) remark that one can construct a flow for which the pseudo-arc (Bing, 1951) is the global attractor, and this set is *not* arcwise connected, and hence not a retract. The topological properties of the global attractor are therefore intimately related to the rates of attraction towards it and expansion within it.

CONCLUSION

We have shown that any trajectory of the dynamical system can be followed arbitrarily closely by a sequence of trajectories (which are longer as $t \to \infty$) on the global attractor. This result gives us some relevant information about how the dynamics on the attractor relates to the asymptotic behaviour of the evolution equation (2.1).

On the other hand, we have found a sufficient condition for an invariant exponential attractor to be asymptotically complete, so that we can apply this result directly to obtain asymptotic completeness for flow-normally hyperbolic global attractors. The result could also be applied to invariant inertial manifolds not given as graphs.

Developing the above ideas, it would be interesting to try to generalise persistence results for linearised hyperbolic sets to sets that are not manifolds using the *flow* normal hyperbolicity assumption, and to try to use the retraction arguments of section 5 to further investigate connectivity properties of the global attractor.

Finally, the non-applicability of these results to (3.3) highlights the importance of obtaining a smooth system along similar lines.

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