Asymptotic behaviour of equicoercive diffusion energies in dimension two

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Abstract

In this paper, we study the asymptotic behaviour of a given equicoercive sequence of diffusion energies F_n, $n \in \mathbb{N}$, defined in $L^2(\Omega)$, for a bounded open subset Ω of $\mathbb{R}^2$. We prove that, contrary to the three dimension (or greater), the Γ-limit of any convergent subsequence of F_n is still a diffusion energy. We also provide an explicit representation formula of the Γ-limit when its domains contains the regular functions with compact support in Ω. This compactness result is based on the uniform convergence satisfied by some minimizers of the equicoercive sequence F_n, which is specific to the dimension two. The compactness result is applied to the period framework, when the energy density is a highly oscillating sequence of equicoercive matrix-valued functions. So, we give a definitive answer to the question of the asymptotic behaviour of periodic conduction problems under the only assumption of equicoerciveness for the two-dimensional conductivity.

1 Introduction

This paper deals with the asymptotic behaviour of sequences of diffusion energies in a bounded open subset Ω of $\mathbb{R}^2$. The prototype of the diffusion energy is given by the following quadratic functional defined in $L^2(\Omega)$:

$$F_n(u) := \begin{cases} \displaystyle\int_\Omega A_n \nabla u \cdot \nabla u \, dx & \text{if } u \in H_0^1(\Omega). \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega), \end{cases} \qquad \text{for } n \in \mathbb{N}, \tag{1.1}$$

where A_n is a symmetric positive definite matrix-valued function in $L^\infty(\Omega)^{2\times 2}$.

The knowledge of the limit behaviour of F_n is crucial in the homogenization theory applied to conduction problems (see e.g. [1] for an introduction), since A_n then represents the conductivity matrix of a given heterogeneous medium. In this context, Spagnolo [23], with the G-convergence theory, and Murat & Tartar [25], [20], with the H-convergence theory, proved the compactness of the sequence F_n, when A_n is assumed to be both equicoercive and equibounded. A few times later, Buttazzo & Dal Maso [10] and Carbone & Sbordone [12] extended the result of compactness by only assuming that the sequence A_n is bounded and equiintegrable in $L^1(\Omega)^{2\times 2}$. At the same period, Fenchenko & Khruslov [15] showed that the equiintegrability condition cannot be relaxed since high conductivity regions in three dimension may induce nonlocal effects which correspond to a lack of compactness in the homogenization process (see also [2], [9], [4] for different approaches).

Nonlocal effects naturally appear in the limit behaviour of the diffusion energy. Indeed, using the Beurling-Deny [3] theory Mosco [18] proved in particular that any sequence F_n Γ-converges, up to a subsequence, for the strong topology of $L^2(\Omega)$ (see Definition 3.1) to a Dirichlet form (see Definition 3.3). According to the Beurling-Deny formula any Dirichlet form can always be split up into three terms: a strongly local form (the diffusion part), a local form and a nonlocal one. Inversely, Camar Eddine & Seppecher [11] proved that any Dirichlet form in $L^2_{\rm loc}(\mathbb{R}^3)$ can be obtained as the Γ-limit of a sequence of diffusion energies of type (1.1) with a suitable isotropic conductivity A_n.

The nonlocal effects obtained in the previous works are based on three-dimensional microstructures whose model example is a medium reinforced by a periodic lattice of high conductivity thin fibers. Then, it is natural to ask if the appearance of nonlocal effects is specific to the three dimension (or greater). Recently, in [6] for periodic microstructures and more generally in [7], we showed that the answer is positive. Assuming that the sequence A_n is both equicoercive and bounded in $L^1(\Omega)^{2\times 2}$, we proved that the Γ-limit of any sequence of type F_n is a strongly local Dirichlet form. Therefore, the dimension two preserves the compactness in the homogenization process. The proof in [6], [7] is based on two-dimensional div-curl type lemmas which extend the one of Murat & Tartar [26], [21]. Note that the equicoerciveness assumption is essential to obtain strongly convergent sequences in $L^2(\Omega)$. However, the use of div-curl lemmas is strictly limited to conductivity sequences which are bounded in $L^1(\Omega)^{2\times 2}$.

In this paper, we study the asymptotic behaviour of the sequence of diffusion energies (1.1), without assuming any boundedness assumption on A_n. Our approach is completely different of the one used in [6] or [7] for A_n bounded in $L^1(\Omega)^{2\times 2}$. The key-ingredient of the method is a uniform convergence result satisfied by some energy minimizers (see Section 2). More precisely, we prove (see Theorem 2.1) that for any bounded energy (with respect to F_n) sequence in $H^1(\Omega)$ which strongly converges in $L^2(\Omega)$ to a continuous function, there exists a smaller energy subsequence which strongly converges to the same limit in $L^\infty_{\rm loc}(\Omega)$. The proof of this result uses that the p-capicity, for $p \in (1,2)$, of a continuous curve is positive (see Lemma 2.8), which is specific to the dimension two. This combined with the continuity of the limit and the maximum principle allows us to construct a uniformly convergent sequence.

Up to our knowledge, the previous result provides new uniform estimates on solutions of uniformly elliptic partial differential equations without any control from above on the coefficients. Here, we give an example (see Corollary 2.5) of such an estimate for A-harmonic functions, where A is any uniformly elliptic (but not necessarily uniformly bounded) matrix-valued in $L^\infty(O)$, for a bounded open subset O of $\mathbb{R}^2$. This uniform estimate is used in the last section of the paper. More general cases are the subject of a work in progress [8].

On the other hand, thanks to the uniform convergence result of Section 2 and under the only assumption of equicoerciveness for A_n, we prove (see Section 3 and Theorem 3.6) that the diffusion energy F_n Γ-converges (up to a subsequence) for the strong topology of $L^2(\Omega)$ to a strongly local Dirichlet form F. Moreover, if the domain of the Γ-limit F contains the space $C^1_c(\Omega)$ of the C^1-regular functions with compact support in Ω, we obtain (see Theorem 3.4) the following representation formula

$$F(u) = \int_\Omega A\nabla u \cdot \nabla u \, d\mu, \quad \forall\, u \in C^1_c(\Omega), \tag{1.2}$$

where μ is a Radon measure on Ω and A a matrix-valued function in $L^\infty_\mu(\Omega)^{2\times 2}$. In other words, the sequence of the diffusion energies F_n is relatively compact for the $L^2(\Omega)$-strong Γ-convergence topology in the set of the uniformly coercive diffusion energies. In particular, the compactness result implies that the limit of the energy density $A_n\, dx$ is still a density of type $A\, d\mu$.

The compactness result of Section 3 has a remarkable application in the periodic homogenization framework. In this context, the conductivity A_n is a highly oscillating sequence defined by $A_n(x) := B_n(\frac{x}{\varepsilon_n})$, where B_n is an equicoercive sequence of $(0,1)^2$-periodic matrix-valued functions in $L^\infty(\mathbb{R}^2)^{2\times 2}$ and ε_n is a positive sequence converging to zero. Associated with B_n the constant matrix A_n^* (see formula (4.4)) obtained, for a fixed n, from the periodic homogenization of $B_n(\frac{x}{\varepsilon})$ as $\varepsilon \to 0$ (see e.g. [1]), plays a fundamental role in the homogenization process. Indeed, extending [6] we prove (see Theorem 4.1) that the asymptotic behaviour of the diffusion energies (1.1) is completely determined by the limit behaviour of the matrix A_n^*, according to the following alternative:

- if the spectral radius $\rho(A_n^*)$ of A_n^* is bounded, A_n^* converges, up to a subsequence, to a matrix A^* and the Γ-limit F of F_n satisfies (1.2) with the constant density $A^*\,dx$, in the whole space $H_0^1(\Omega)$;
- if $\rho(A_n^*)$ tends to $+\infty$, the domain of the Γ-limit reduces to $\{0\}$.

As an immediate consequence, the question on the asymptotic behaviour of the two-dimensional periodic conduction problem

$$\begin{cases} -\operatorname{div}\Big(B_n(\frac{x}{\varepsilon_n})\nabla u_n\Big) &= f \quad \text{in } \Omega \\ \qquad\qquad u_n &= 0 \quad \text{on } \partial\Omega, \end{cases} \qquad \text{for } f \in H^{-1}(\Omega), \tag{1.3}$$

is now definitively solved under the only assumption of equicoerciveness for B_n:

- if $\rho(A_n^*)$ is bounded, (1.3) converges, up to a subsequence, to the conduction problem with the constant conductivity $\lim\limits_{n\to+\infty} A_n^*$;
- if $\rho(A_n^*)$ tends to $+\infty$, the potential u_n of (1.3) strongly converges to zero in $H_0^1(\Omega)$.

The paper is organized as follows. Section 2 is devoted to the uniform convergence results and Section 3 to the Γ-convergence of sequences of diffusion energies of type (1.1). In Section 4 we apply the results of Section 3 to the periodic framework.

Notations

- $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ denotes the set of the positive integers;
- $a \vee b$, resp. $a \wedge b$, denotes the maximum, resp. the minimum, of $a, b \in \mathbb{R}$;
- $B(x_0, \delta)$ denotes the disk of center $x_0 \in \mathbb{R}^2$ and of radius $\delta > 0$;
- χ_E denotes the characteristic of the set E;
- $\exists \lim$ means that the limit does exist;
- Ω denotes an open subset of $\mathbb{R}^2$ and $\bar{\Omega}$ the closure of Ω in $\mathbb{R}^2$;
- $H_{\text{loc}}(\Omega)$ means locally in the space $H(\Omega)$;
- $C(\Omega)$ denotes the space of the continuous functions in Ω, $C_0(\Omega)$ the subspace of $C(\Omega)$ composed of the functions which are zero on the boundary of Ω, $C_c(\Omega)$ the subspace of $C_0(\Omega)$ composed of the functions with compact support in Ω, and $C_c^k(\Omega)$, for $k \in \mathbb{N} \cap \{+\infty\}$, the subspace of $C_c(\Omega)$ composed of the k-th continuously differentiable functions in Ω;
- $\mathcal{D}'(\Omega)$ denotes the set of the distributions on Ω;
- $\mathcal{M}(\Omega)$ denotes the set of the Radon measures on Ω;

- a sequence μ_n in $\mathcal{M}(\Omega)$ converges to $\mu \in \mathcal{M}(\Omega)$ in the weak $*$ sense of the measures in Ω if
$$\lim_{n\to+\infty} \int_\Omega \varphi \, d\mu_n = \int_\Omega \varphi \, d\mu, \quad \forall \varphi \in C_0(\Omega),$$
and the convergence is denoted by $\mu_n \rightharpoonup \mu$ in $\mathcal{M}(\Omega)*$;
- *q.e.* means quasi-everywhere in the sense of the 2-capacity in $\mathbb{R}^2$, and *a.e.* means everywhere in the sense of the Lebesgue measure in $\mathbb{R}^2$;
- for any $p \in (1,2)$ and for any subset E of $\mathbb{R}^2$, $C_p(E)$ denotes the p-capacity of E with respect to $\mathbb{R}^2$, which is defined by
$$C_p(E) := \inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^p \, dx \;:\; u \in D^{1,p}(\mathbb{R}^2),\ u \geq 1 \text{ a.e. in a neighbourhood of } E \right\},$$
where $D^{1,p}(\mathbb{R}^2)$ is the space of the functions u in $L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ such that $\nabla u \in L^2(\mathbb{R}^2)^2$.

2 Uniform convergence results

2.1 Statement of the results

Let Ω be an open subset of $\mathbb{R}^2$. In this section, we consider a given sequence of symmetric matrix-valued functions $A_n \in L^\infty(\Omega)^{2\times 2}$, $n \in \mathbb{N}$, which satisfies the following equicoerciveness property in Ω

$$\exists\, \alpha > 0 \qquad \text{such that} \qquad \forall\, n \in \mathbb{N},\ \forall\, \xi \in \mathbb{R}^2, \quad A_n\xi \cdot \xi \geq \alpha |\xi|^2 \quad \text{a.e. in } \Omega. \tag{2.1}$$

For any function $u \in H^1(\Omega) \cap C(\Omega)$, we will study some questions related to the existence of sequences u_n in $H^1(\Omega)$ which both converge uniformly to u in Ω and satisfy the following minimization property

$$\exists \lim_{n\to+\infty} \int_\Omega A_n \nabla u_n \cdot \nabla u_n \, dx \leq \liminf_{n\to+\infty} \int_\Omega A_n \nabla v_n \cdot \nabla v_n \, dx,$$

for any sequence v_n in $H^1(\Omega)$ (some boundary conditions can be added), which strongly converges to u in $L^2(\Omega)$. Our main result in this way is the following theorem:

Theorem 2.1. *Let u be a function in $H^1(\Omega) \cap C(\Omega)$ and let $\hat{u}_n$ be a sequence $H^1(\Omega)$ which strongly converges to u in $L^2(\Omega)$ and satisfies*

$$\exists \lim_{n\to+\infty} \int_\Omega A_n \nabla \hat{u}_n \cdot \nabla \hat{u}_n \, dx < +\infty.$$

Then, up to a subsequence of n, still denoted by n, there exists $u_n \in H^1(\Omega)$ which satisfies

$$\limsup_{n\to+\infty} \int_\Omega A_n \nabla u_n \cdot \nabla u_n \, dx \leq \lim_{n\to+\infty} \int_\Omega A_n \nabla \hat{u}_n \cdot \nabla \hat{u}_n \, dx, \tag{2.2}$$

$$\text{and} \qquad u_n \longrightarrow u \quad \text{strongly in } L^\infty_{\text{loc}}(\Omega). \tag{2.3}$$

Moreover, if the support of u is contained in a compact set K of Ω, then we can take u_n such that $u_n = 0$ q.e. in $\Omega \setminus K$, for any $n \in \mathbb{N}$.

Remark 2.2. If in Theorem 2.1 the sequence $\hat{u}_n$ is in $H^1_0(\Omega)$ and u in $H^1_0(\Omega) \cap C_0(\Omega)$, then we can choose u_n in $H^1_0(\Omega)$, which strongly converges to u in $L^\infty(\Omega)$. To this end, it is enough to consider a bounded open set $\tilde{\Omega}$ such that $\bar{\Omega} \subset \tilde{\Omega}$ and to apply the second part of Theorem 2.1 to the sequences $\tilde{u}_n$ and $\tilde{A}_n$ defined by

$$\tilde{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \tilde{\Omega} \setminus \Omega, \end{cases} \qquad \text{and} \qquad \tilde{A}_n(x) := \begin{cases} A_n(x) & \text{if } x \in \Omega \\ I_2 & \text{if } x \in \tilde{\Omega} \setminus \Omega. \end{cases}$$

Corollary 2.3. *Consider $\hat{u}_n \in H^1(\Omega)$ and $u \in H^1(\Omega) \cap C(\Omega)$ such that*

$$\hat{u}_n \rightharpoonup u \quad \textit{weakly in } H^1_{\rm loc}(\Omega) \qquad \textit{and} \qquad {\rm div}\,(A_n \nabla \hat{u}_n) = 0 \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.4}$$

Then, we have the following uniform convergence

$$\hat{u}_n \longrightarrow u \quad \textit{strongly in } C(\Omega). \tag{2.5}$$

Remark 2.4. Note that in Corollary 2.3 each function u_n is continuous in Ω by the De Giorgi-Stampacchia theorem (see e.g. [16] Chapter 8).

Corollary 2.5. *For any open subset Ω of $\mathbb{R}^2$ and any compact subset K of Ω, there exists a constant $C > 0$ which only depends on Ω and K such that, for any matrix-valued function $A \in L^\infty(\Omega)^{2\times 2}$ satisfying the uniform coerciveness* (2.1) *and any function $u \in H^1(\Omega)$ solution of*

$$ {\rm div}\,(A\nabla u) = 0 \quad \textit{in } \mathcal{D}'(\Omega),$$

the following estimate holds true

$$\|u\|_{C(K)} \leq C\,\|u\|_{H^1(\Omega)}.$$

Remark 2.6. Corollaries 2.3 and 2.5 can be deduced from [8] where more general results, for non-necessarily homogeneous equations, are proved.

2.2 Proof of the results

Let us now give the proof of the uniform convergence results stated in the previous section. We will need the two following lemmas:

Lemma 2.7. *Let O be a bounded open subset of $\mathbb{R}^2$ and let $u \in H^1(O) \cap C(\bar{O})$. Denote $M := \max\limits_{\partial O} u$ and $m := \min\limits_{\partial O} u$.*
Then, for any function v such that $v - u$ belongs to $H^1_0(O)$, the functions $(v - M)^+$ and $(m - v)^+$ belong to $H^1_0(O)$.

Proof. Consider $\varphi_\varepsilon \in C^\infty_c(O)$, for $\varepsilon > 0$, which strongly converges to $v - u$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$. Since u is continuous in $\bar{O}$, the functions $U_\varepsilon := (u + \varphi_\varepsilon - M - \varepsilon)^+$ and $u_\varepsilon := (m - \varepsilon - u - \varphi_\varepsilon)^+$ have compact support in O. The functions U_ε and u_ε belong to $H^1(O)$, hence they also belong to $H^1_0(O)$. Therefore, using that U_ε and u_ε strongly converge respectively to $(v - M)^+$ and $(m - v)^+$ in $H^1(O)$, yields the result. □

Lemma 2.8. *For any $p \in (1, 2)$ and any continuous curve L of extremities a, b, we have*

$$C_p(L) \geq R_p\,|a - b|^{2-p}, \tag{2.6}$$

where $R_p > 0$ is the p-capacity of a unit segment in $\mathbb{R}^2$.

Proof. Using a translation, a rotation and a homothety, we can reduce the proof to the case where $a = (0, 0)$, $b = (1, 0)$. Then, consider a curve L of extremities $(0, 0)$, $(1, 0)$ and take a function $\varphi \in C^\infty_c(\mathbb{R}^2)$ such that $\varphi > \chi_L$. By the Pólya-Szegö inequality [22] extended to any power $p \geq 1$ (see e.g. [24] and Chapter I.4 of [19]), it is known that the Steiner symmetrization φ^* of φ with respect to $\{x_2 = 0\}$, defined by its level sets

$$\big\{(x_1, x_2) \in \mathbb{R}^2 : \ \varphi^*(x_1, x_2) > c\big\} = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ |x_2| < \frac{1}{2}\,\big|\{y \in \mathbb{R} : \varphi(x_1, y) > c\}\big|\right\},$$

belongs to $W_c^{1,p}(\mathbb{R}^2)$ and satisfies

$$\int_{\mathbb{R}^2} |\nabla\varphi^*|^p dx \le \int_{\mathbb{R}^2} |\nabla\varphi|^p dx.$$

Moreover, since L and φ are continuous, it is clear that $\varphi^* > 1$ in $[0,1] \times \{0\}$, hence

$$C_p([0,1] \times \{0\}) \le \int_{\mathbb{R}^2} |\nabla\varphi^*|^p dx \le \int_{\mathbb{R}^2} |\nabla\varphi|^p dx.$$

Taking the infimum in φ, we get the desired estimate (2.6). $\square$

Proof of Theorem 2.1. Using the density of $H^1(\Omega) \cap C^\infty(\Omega)$ in $H^1(\Omega)$ (see e.g. [16]), we can also assume that $\hat{u}_n$ is continuous in Ω. For any $\delta > 0$, we define Ω_δ by

$$\Omega_\delta := \{x \in \Omega : \ d(x, \partial\Omega) > \delta\}.$$

Since u is continuous in $\bar{\Omega}_\delta$, for any $l \in \mathbb{N}^*$, there exists $\delta_l > 0$, with $\lim\limits_{l\to+\infty} \delta_l = 0$, such that

$$|u(x) - u(y)| < \frac{1}{2l}, \quad \forall\, x, y \in \bar{\Omega}_{\delta_l}, \text{ with } |x - y| \le \delta_l. \tag{2.7}$$

Let $p \in (1, 2)$. Since $\hat{u}_n$ weakly converges in $H^1(\Omega)$, there exists (see e.g. [14]) a subsequence of $\hat{u}_n$, still denoted by $\hat{u}_n$, which converges to u C_p-quasi uniformly in every open set $\omega \subset \Omega$, with $\bar{\omega} \subset \Omega$ (ω can be chosen as Ω if Ω is smooth). Thus, we can choose this sequence in such a way that for any $l \in \mathbb{N}^*$, there exists a relatively closed subset K_l of Ω satisfying

$$C_p(\Omega \setminus K_l) < R_p\, \delta_l^{2-p}, \tag{2.8}$$

$$|\hat{u}_n(x) - u(x)| < \frac{1}{2l}, \quad \forall\, x \in \Omega_{\delta_l} \cap K_l, \ \forall\, n \ge l. \tag{2.9}$$

Then, we define u_n by

$$\begin{cases} u_n := \hat{u}_n \ \text{ in } K_l, \quad u_n - \hat{u}_n \in H_0^1(\Omega \setminus K_l), \\ \displaystyle\int_{\Omega\setminus K_l} A_n \nabla u_n \cdot \nabla u_n \, dx \le \int_{\Omega\setminus K_l} A_n \nabla v \cdot \nabla v \, dx, \quad \forall\, v, \ v - \hat{u}_n \in H_0^1(\Omega \setminus K_l). \end{cases} \tag{2.10}$$

Clearly, u_n satisfies (2.2). Let us prove that u_n strongly converges to u in $L^\infty_{\rm loc}(\Omega)$. To this end, we fix $\delta > 0$. We have

$$|u_n(x) - u(x)| = |\hat{u}_n(x) - u(x)| < \frac{1}{2l}, \quad \forall\, x \in \Omega_\delta \cap K_l, \ \forall\, n \ge l, \text{ with } \delta_l < \delta. \tag{2.11}$$

Consider a connected component O of $\Omega \setminus K_l$ such that $O \cap \Omega_\delta \ne \emptyset$. Since O is opened and connected, it is connected by curves. Thus, for any $y_1, y_2 \in O$, there exists a curve $L \subset O$ which connects y_1, y_2. By Lemma 2.8 and (2.8), we have

$$R_p\, |y_1 - y_2|^{2-p} \le C_p(L) \le C_p(O) \le C_p(\Omega \setminus K_l) \le R_p\, \delta_l^{2-p},$$

hence $\operatorname{diam}(O) \le \delta_l$. Then, taking l large enough such that $2\delta_l < \delta$, we get that $\bar{O} \subset \Omega_{\delta_l}$, and in particular, $\partial O \subset \Omega_{\delta_l} \cap K_l$. Denote

$$m_n := \min_{\partial O} \hat{u}_n \quad \text{and} \quad M_n := \max_{\partial O} \hat{u}_n.$$

By Lemma 2.7 $(u_n - M_n)^+$ and $(m_n - u_n)^-$ belong to $H_0^1(O)$, and by definition (2.10) u_n is A_n-harmonic in O. Then, the maximum principle yields

$$m_n \le u_n \le M_n, \ \text{ q.e. in } O, \ \forall\, n \in \mathbb{N}. \tag{2.12}$$

On the other hand, since $\partial O \subset \Omega_{\delta_l} \cap K_l$, we have by (2.9).

$$m_n \geq \min_{\partial O} u - \frac{1}{2l} \quad \text{and} \quad M_n \leq \max_{\partial O} u + \frac{1}{2l}, \qquad \forall\, n \geq l.$$

Moreover, $\bar{O} \subset \Omega_{\delta_l}$, $\operatorname{diam}(O) \leq \delta_l$ and (2.7) imply that

$$\min_{\partial O} u \geq u(x) - \frac{1}{2l} \quad \text{and} \quad \max_{\partial O} u \leq u(x) + \frac{1}{2l}, \qquad \forall\, x \in O,\ \forall\, n \geq l.$$

Therefore, (2.12) combined with the two previous estimates yields

$$|u_n - u| \leq \frac{1}{2l}, \quad \text{q.e. in } O,$$

hence the sequence u_n satisfies the uniform convergence (2.3).

Now, assume that the support of u is contained in a compact subset K of Ω, and consider an open set $\tilde{\Omega}$ which contains K and is strictly contained in Ω. For any $\varepsilon > 0$, let S_ε be the function defined by $S_\varepsilon(s) := (s - \varepsilon \operatorname{sgn}(s)))\chi_{\{|s|>\varepsilon\}}$, for $s \in \mathbb{R}$. Since u_n strongly converges to u in $L^\infty(\tilde{\Omega})$, the sequence $\varepsilon_n := \|u_n - u\|_{L^\infty(\tilde{\Omega})}$ tends to zero. Therefore, the sequence $\tilde{u}_n := \chi_{\tilde{\Omega}} S_{\varepsilon_n}(u_n)$ satisfies conditions (2.2), (2.3) and vanish q.e. in $\Omega \setminus K$. □

Proof of Corollary 2.3. Since $\hat{u}_n$ satisfies $\operatorname{div}(A_n \nabla \hat{u}_n) = 0$ in $\mathcal{D}'(\Omega)$, it is Hölder continuous in Ω by the De Giorgi-Stampacchia theorem. Then, the argument used in the proof of Theorem 2.1 proves that the sequence u_n defined by (2.10) strongly converges to u in $L^\infty_{\text{loc}}(\Omega)$. However, we have $u_n = \hat{u}_n$ by construction, which yields the desired result. □

Proof of Corollary 2.5. We reason by contradiction. If the result does not hold true, then, for any $n \in \mathbb{N}$, there exist $u_n \in H^1(\Omega)$, $A_n \in L^\infty(\Omega)^{2\times 2}$ and $\gamma_n > 0$ such that

$$A_n\xi \cdot \xi \geq \gamma_n |\xi|^2, \quad \forall\, \xi \in \mathbb{R}^2, \quad \text{a.e. } x \in \Omega,$$

$$\text{and} \qquad \|u_n\|_{C(K)} > n\|u_n\|_{H^1(\Omega)}. \tag{2.13}$$

Up to replace A_n by A_n/γ_n and u_n by $u_n/\|u_n\|_{C(K)}$, we can assume that $\gamma_n = 1$ and $\|u_n\|_{C(K)} = 1$. Then, A_n is equicoercive and by (2.13) u_n strongly converges to zero in $H^1(\Omega)$. Therefore, by Corollary 2.3 u_n converges uniformly to zero in K, in contradiction with $\|u_n\|_{C(K)} = 1$. □

3 Γ-limit of equicoercive diffusion energies

3.1 Γ-convergence and Dirichlet forms

In this section we first recall the definition of the De Giorgi Γ-convergence and some of its properties which will be used in the sequel. We refer to [13] for an exhaustive presentation of the Γ-convergence.

Definition 3.1. A sequence of functionals $F_n : L^2(\Omega) \longrightarrow [0, +\infty]$ is said to Γ-converge to $F : L^2(\Omega) \longrightarrow [0, +\infty]$ for the strong topology of $L^2(\Omega)$ if, for any u in $L^2(\Omega)$,

(i) the Γ-liminf inequality holds

$$\forall\, u_n \longrightarrow u \ \text{ strongly in } L^2(\Omega), \quad F(u) \leq \liminf_{n\to+\infty} F_n(u_n), \tag{3.1}$$

(*ii*) the Γ-limsup inequality holds

$$\exists\, \bar{u}_n \longrightarrow u \text{ strongly in } L^2(\Omega), \quad F(u) = \lim_{n\to+\infty} F_n(\bar{u}_n). \tag{3.2}$$

Any sequence satisfing (3.2) will be called a recovery sequence for F_n, of limit u.

In the sequel, we will always consider the Γ-convergence with respect to the strong topology of L^2. Consequently, this topology will be not necessarily mentionned.

Properties 3.2.
a) Since $L^2(\Omega)$ is separable, any sequence of functionals $F_n : L^2(\Omega) \longrightarrow [0,+\infty]$ has a subsequence which Γ-converges with respect to the strong topology of $L^2(\Omega)$.
b) Let $F_n : L^2(\Omega) \longrightarrow [0,+\infty]$ be a sequence of quadratic forms which Γ-converges to F. Then, F is a quadratic form on $L^2(\Omega)$ which is semi-lower continuous with respect to the strong topology of $L^2(\Omega)$.
c) Let $F_n : L^2(\Omega) \longrightarrow [0,+\infty]$ be a sequence of quadratic forms which Γ-converges to F. Let Φ_n, Φ be the polar forms respectively associated with F_n, F on their domains. Then, for any $u \in L^2(\Omega)$, with $F(u) < +\infty$, a sequence u_n in $L^2(\Omega)$ is a recovery sequence (3.2) *for F_n, of limit u, if and only if*

$$\forall\, v_n \longrightarrow v \ \textit{strongly in } L^2(\Omega), \ \textit{with } F_n(v_n) \le c, \quad \lim_{n\to+\infty} \Phi_n(u_n, v_n) = \Phi(u,v), \tag{3.3}$$

or equivalently, (3.3) *with $v = 0$.*

Now, we recall some notions about Dirichlet forms, which will be used in the statement of Theorem 3.8. We refer to [18] for more details in connection with the Γ-convergence.

Definition 3.3. Let X be a Hausdorff, separable, locally compact space, and let m be a σ-finite nonnegative Radon measure on X. Let H be the space $L^2_m(X)$ endowed with its Hilbert norm $\|\cdot\|_H$. Let $F : H \longrightarrow [0,+\infty]$ be a quadratic form of domain $D(F) := \{u \in H : \ F(u) < +\infty\}$, whose polar form Φ is a bilinear form defined in $D(F) \times D(F)$.

(*i*) The form F is said to be *closed* if it is semi-lower continuous with respect to the norm $\|\cdot\|_H$. The form F is said to be *closable* if there exists an extension $\tilde{F}$ of F in H such that $D(F) \subset D(\tilde{F})$. The *closure* of a closable form is its smallest closed extension in H.

(*ii*) The form F is said to be *Markovian* if

$$\forall\, u \in D(F), \qquad v := (u \vee 0) \wedge 1 \in D(F) \quad \text{and} \quad F(v) \le F(u).$$

(*iii*) A *Dirichlet form* on H is a closed Markovian quadratic form defined in H.

(*iv*) The form F is said to be *regular* if there exists a subset of $D(F) \cap C_0(X)$, which is dense both in $C_0(X)$ with the uniform norm and in $D(F)$ with the norm $(F + \|\cdot\|_H)^{1/2}$.

(*v*) The form F is said to be *local* if

$$\Phi(u,v) = 0, \quad \forall\, u, v \in D(F), \text{ with } \operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset.$$

The form F is said to be *strongly local* if

$$\Phi(u,v) = 0, \quad \forall\, u, v \in D(F), \text{ with } u = \text{cst in } \operatorname{supp}(v). \tag{3.4}$$

Thanks to the Beurling-Deny theory [3] any regular Dirichlet form F on $L^2_m(X)$ can be split up on its domain into three specific forms: a strongly local form F_d, a local form and a nonlocal one. More precisely, the following representation formula holds for any $u \in D(F)$,

$$F(u) = F_d(u) + \int_X u^2(x)\, k(dx) + \iint_{X\times X\setminus \text{diag}} \big(u(x) - u(y)\big)^2\, j(dx,dy), \tag{3.5}$$

where F_d is called the diffusion part of F, k the killing measure and j the jumping measure.

3.2 Statement of the results

As in Section 2, let us consider a bounded open subset Ω of $\mathbb{R}^2$, and a sequence of symmetric matrix-valued functions $A_n \in L^\infty(\Omega)^{2\times 2}$ which satisfy (2.1). For any $n \in \mathbb{N}$ and any open subset ω of Ω, we define the quadratic form $F_n(\cdot,\omega)$ in $L^2(\omega)$ by

$$F_n(u,\omega) := \begin{cases} \displaystyle\int_\omega A_n\nabla u\cdot\nabla u\,dx & \text{if } u \in H^1_0(\omega). \\ +\infty & \text{if } u \in L^2(\omega)\setminus H^1_0(\omega). \end{cases} \tag{3.6}$$

The form $F_n(\cdot,\Omega)$ is simply denoted by F_n.

Assume that F_n Γ-converges to some quadratic form F for the topology of $L^2(\Omega)$, which holds true for a subsequence in virtue of Properties 3.2 *a*). Since F_n is clearly Markovian, the properties (3.1), (3.2) of the Γ-convergence imply that F is also Markovian. Moreover, thanks to Properties 3.2 *b*) F is closed. Therefore, F is a Dirichlet form in the sense of Definition 3.3 (*iii*). The following result gives a necessary and sufficient condition to have F regular with $C^1_c(\Omega)\subset D(F)$. When this condition is satisfied, F is a strongly local (3.4) Dirichlet form whose integral representation for regular functions is independent of the domain.

Theorem 3.4. *The domain $D(F)$ of F contains $C^1_c(\Omega)$ if and only if, for any $x_0 \in \Omega$, there exists $\delta > 0$, two functions w^1, w^2 in $C^1(B(x_0,\delta))$ and two sequences w^1_n, w^2_n in $H^1(B(x_0,\delta))$, $n \in \mathbb{N}$, such that*

$$\begin{cases} B(x_0,\delta)\subset\Omega, & \\ \nabla w^1(x_0), \nabla w^2(x_0) \text{ are linearly independent}, & \\ w^i_n \longrightarrow w^i \text{ strongly in } L^2\left(B(x_0,\delta)\right), & \text{for } i=1,2, \\ A_n\nabla w^i_n\cdot\nabla w^i_n \text{ is bounded in } L^1\left(B(x_0,\delta)\right), & \text{for } i=1,2. \end{cases} \tag{3.7}$$

Assume that $C^1_c(\Omega)$ is contained in $D(F)$. Then, there exist a nonnegative Radon measure μ on Ω and a nonnegative matrix-valued function A in $L^\infty_\mu(\Omega)^{2\times 2}$ such that the regular part A^r of $Ad\mu$ with respect to the Lebesgue measure satisfies

$$A^r\xi\cdot\xi \geq \alpha|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^2, \ \text{a.e. in } \Omega, \tag{3.8}$$

and such that, for any open set ω of Ω, the Γ-limit $F(\cdot,\omega)$ of $F_n(.,\omega)$ with respect to the strong topology of $L^2(\omega)$ does exist on $C^1_c(\omega)$ and reads as

$$F(u,\omega) = \int_\omega A\nabla u\cdot\nabla u\,d\mu, \quad \forall\,u\in C^1_c(\omega). \tag{3.9}$$

Moreover, for any $u \in C^1_c(\omega)$ and any $u_n \in H^1_0(\omega)$ which strongly converges to u in $L^2(\omega)$ and such that $F_n(u_n,\omega)$ tends to $F(u,\omega)$, the sequence $A_n\nabla u_n\cdot\nabla u_n$ converges to $A\nabla u\cdot\nabla u\,d\mu$ in the weak $*$ sense of the measures in ω.

Remark 3.5. In the second part of Theorem 3.4 the Γ-convergence of $F_n(\cdot,\omega)$ to $F(\cdot,\omega)$ holds true up to a subsequence which does depend on the open set ω. However, the integral representation (3.9) of $F(u,\omega)$, which is valid on $C^1_c(\omega)$, is independent of ω.

In fact, the integral expression (3.9) holds true for any $u\in C^1_0(\omega)$, with $A\nabla u\cdot\nabla u\in L^1_\mu(\omega)$. Indeed, it is easy to check that these functions can be approximated by functions in $C^1_c(\omega)$ in the strong topology of $D(F(.,\omega))$.

Theorem 3.4 provides an integral representation of F, assuming that $D(F)$ contains $C^1_c(\Omega)$. The following result gives a corrector result:

Theorem 3.6. *Assume that $D(F)$ contains $C_c^1(\Omega)$. Then, there exists a sequence (w_n^1, w_n^2) in $H_0^1(\Omega;\mathbb{R}^2)\cap L^\infty(\Omega;\mathbb{R}^2)$ which strongly converges to the identity in $L^\infty_{\rm loc}(\Omega;\mathbb{R}^2)$, and which satisfies, for any $i\in\{1,2\}$ and any compact subset K of Ω,*

$$\limsup_{n\to+\infty}\int_K A_n\nabla w_n^i\cdot\nabla w_n^i\,dx<+\infty,$$

$$\lim_{n\to+\infty}\int_K A_n\nabla w_n^i\cdot\nabla v_n\,dx=0,\qquad \begin{cases}\forall\, v_n\in H_0^1(\Omega),\ v_n=0 \ q.e.\ in\ \Omega\setminus K,\\ v_n\to 0\ in\ L^2(\Omega),\ A_n\nabla v_n\cdot\nabla v_n\ bounded\ in\ L^1(\Omega).\end{cases}$$

Let ω be open subset of Ω and let $u\in C_c^1(\omega)$. Assume that the sequence $F_n(\cdot,\omega)$ Γ-converges to $F(\cdot,\omega)$ for the strong topology of $L^2(\omega)$. Then, for any recovery sequence $u_n\in H_0^1(\omega)$ which is zero q.e. outside a compact subset of ω, which strongly converges to u in $L^2(\omega)$ and such that $F_n(u_n,\omega)$ tends to $F(u,\omega)$, we have

$$\lim_{n\to+\infty}\int_\omega A_n\left(\nabla u_n-\sum_{i=1}^2\partial_i u\nabla w_n^i\right)\cdot\left(\nabla u_n-\sum_{i=1}^2\partial_i u\nabla w_n^i\right)dx=0. \tag{3.10}$$

Theorems 3.4 and 3.6 are consequences of the following lemma:

Lemma 3.7. *Assume that there exist $x_0\in\Omega$, $\delta>0$, w^1,w^2 in $H^1(B(x_0,\delta))\cap C^1(B(x_0,\delta))$ and two sequences w_n^1,w_n^2 in $H^1(B(x_0,\delta))$, $n\in\mathbb{N}$, which satisfy (3.7).*
Then, there exists $\varepsilon\in(0,\delta)$, a nonnegative bounded Radon measure μ on $B(x_0,\delta)$ and a nonnegative matrix-valued function A in $L^\infty_\mu(B(x_0,\varepsilon))^{2\times 2}$, such that for any open subset ω of Ω, with $B(x_0,\varepsilon)\cap\omega\neq\emptyset$, the sequence $F_n(\cdot,\omega)$ defined by (3.6) Γ-converges for the strong topology of $L^2(\omega)$ (up to a subsequence of n, still denoted by n, which depends on ω) to the Dirichlet form $F(\cdot,\omega)$ which satisfies the following properties:

(i) The space $C_c^1(B(x_0,\varepsilon)\cap\omega))$ is contained in $D(F(\cdot,\omega))$, and

$$F(u,\omega)=\int_{B(x_0,\varepsilon)\cap\omega}A\nabla u\cdot\nabla u\,d\mu,\quad \forall\, u\in C_c^1(B(x_0,\varepsilon)\cap\omega). \tag{3.11}$$

(ii) For any $u\in D(F(.,\omega))\cap C_0^1(\omega)$, and any sequence u_n in $H_0^1(\omega)$, which strongly converges to u in $L^2(\omega)$ and such that $F_n(u_n,\omega)$ tends to $F(u,\omega)$, the sequence $A_n\nabla u_n\cdot\nabla u_n$ converges to $A\nabla u\cdot\nabla u\,d\mu$ in the weak $*$ sense of the measures in $B(x_0,\varepsilon)\cap\omega$.

Theorem 3.4 refers to the case where $D(F)$ contains $C_c^1(\Omega)$. In the general case we obtain a characterization of the Γ-limit of $F_n(\cdot,\omega)$ (3.6) but not explicit like the representation formula (3.9). In fact, we have the following abstract result:

Theorem 3.8. *Let $\mathcal{R}$ be the equivalence relation defined in $\bar\Omega$ by*

$$x\,\mathcal{R}\,y\ \Leftrightarrow\ u(x)=u(y),\quad \forall\, u\in D(F)\cap C_0(\Omega),$$

and note that any function $u\in D(F)\cap C_0(\Omega)$ defines a function in $\bar\Omega/\mathcal{R}$, still denoted by u. The set $\bar\Omega/\mathcal{R}$ is endowed with the smallest topology $\mathcal{T}$ such that the functions in $D(F)\cap C_0(\Omega)$ are continuous for $\mathcal{T}$. Then, $\bar\Omega/\mathcal{R}$ is a Hausdorff, separable and compact topological space.

We denote by Ω^* the set $\bar\Omega/\mathcal{R}$ without the class containing the elements of $\partial\Omega$. Then, Ω^* is locally compact and the set $D(F)\cap C_0(\Omega)$ is a dense separating subalgebra of $C_0(\Omega^*)$, which allows us to define a bounded Radon measure m on Ω^* by

$$\int_{\Omega^*}u\,dm=\int_\Omega u\,dx,\quad \forall\, u\in D(F)\cap C_0(\Omega).$$

Then, the restriction of the Γ-limit F to $D(F)\cap C_0(\Omega)$ is a closable Markovian form in $L^2_m(\Omega^*)$, which is strongly local. Its closure F^* is a strongly local Dirichlet form.

Remark 3.9. By Theorem 3.8 the Dirichlet form F^* is a diffusion. Denote by $\mathcal{M}^*$ the space of the Radon measures on Ω^*. Then, following Mosco [18] there exists a bilinear form $\nu : D(F^*) \times D(F^*) \longrightarrow \mathcal{M}^*$, such that

$$F^*(u) = \int_{\Omega^*} d\nu(u,u), \quad \forall\, u \in D(F^*).$$

For any $u \in D(F^*)$, the measure $\nu(u,u)$ is nonnegative and depends locally on u. That is, if $u_1, u_2 \in D(F^*)$ agree in an open subset G of Ω^*, then $\nu(u_1,u_1) = \nu(u_2,u_2)$ on G. Moreover, the measure ν satisfies several properties which are detailed in [18], such as the Leibnitz rule, the chain rule and the truncation principle.

On the other hand, by the Stone-Weierstrass theorem, $D(F) \cap C_0(\Omega)$ is dense in $C_0(\Omega)$ if and only if, for any $x \in \Omega$, the class of x by the relation $\mathcal{R}$ reduces to $\{x\}$. In this case, the sets Ω and Ω^* may be identified. However, even under the assumption $\Omega \equiv \Omega^*$, we cannot express F^* more precisely, since we do not know the exact composition of the domain $D(F)$. We also refer to [18] (p. 192) for an explicit treatment of the diffusion part of a Dirichlet form.

3.3 Proof of the results

Proof of Lemma 3.7. First, since by Properties 3.2 $a)$ the sequence F_n^* defined by

$$F_n^*\left(u, B(x_0,\delta)\right) := \begin{cases} \displaystyle\int_{B(x_0,\delta)} A_n \nabla u \cdot \nabla u \, dx & \text{if } u \in H^1(B(x_0,\delta)) \\ +\infty & \text{if } u \in L^2(B(x_0,\delta)) \setminus H^1(B(x_0,\delta)), \end{cases}$$

Γ-converges, up a subsequence, for the strong topology of $L^2(B(x_0,\delta))$, we may choose w_n^1, w_n^2 which satisfy (3.7) with limits w^1, w^2, as recovery sequences for F_n^*. Then, thanks to Properties 3.2 $c)$ the sequence w_n^i, for $i \in \{1,2\}$, satisfies

$$\begin{cases} \displaystyle\lim_{n\to+\infty} \int_{B(x_0,\delta)} A_n \nabla w_n^i \cdot \nabla v_n \, dx = 0, \quad \forall\, v_n \in H^1(B(x_0,\delta)), \\ v_n \to 0 \text{ strongly in } L^2(B(x_0,\delta)), \quad A_n \nabla v_n \cdot \nabla v_n \text{ bounded in } L^1(B(x_0,\delta)), \end{cases} \tag{3.12}$$

and by Theorem 2.1 w_n^i strongly converges to w^i in $L^\infty_{\text{loc}}(B(x_0,\delta))$.

On the other hand, we define the Radon measure μ on $B(x_0,\delta)$ and the matrix-valued A in $L^\infty_\mu(\Omega)^{2\times 2}$ by the following weak $*$ convergences which hold true up to a subsequence,

$$\begin{cases} A_n \nabla w_n^1 \cdot \nabla w_n^1 + A_n \nabla w_n^2 \cdot \nabla w_n^2 \rightharpoonup \mu & \text{in } \mathcal{M}\left(\bar{B}(x_0,\delta)\right) *, \\ A_n \nabla w_n^i \cdot \nabla w_n^j \, dx \rightharpoonup A \nabla w^i \cdot \nabla w^j \, d\mu & \text{in } \mathcal{M}\left(\bar{B}(x_0,\delta)\right) *, \quad i,j \in \{1,2\}. \end{cases} \tag{3.13}$$

Since $\nabla w^1(x_0), \nabla w^2(x_0)$ are linearly independent, there exists $\varepsilon \in (0,\delta)$ and an open subset O of $\mathbb{R}^2$, such that $\nabla w^1, \nabla w^2$ are linearly independent in $B(x_0,\varepsilon)$ and the function $w := (w_1, w_2)$ is one-to-one from $B(x_0,\varepsilon)$ onto O, with C^1 inverse.

Proof of (i). Let us consider an open subset ω of Ω, with $\omega \cap B(x_0,\varepsilon) \neq \emptyset$, and a new subsequence of n (which depends on ω), still denoted by n, such that $F_n(\cdot,\omega)$ Γ-converges to $F(\cdot,\omega)$ for the strong topology of $L^2(\omega)$.

Then, for a given function $u \in C_c^1(B(x_0,\varepsilon) \cap \omega)$, we may define the function

$$R := u(w^{-1}) \in C_c^1\left(w(B(x_0,\varepsilon) \cap \omega)\right),$$

so that $u = R(w)$ in $B(x_0,\varepsilon) \cap \omega$. Set $w_n := (w_n^1, w_n^2)$. Due to the uniform convergence of w_n^i, and to the compactness of $\operatorname{supp}(R)$ in $w(B(x_0,\varepsilon) \cap \omega)$, the function $R(w_n)$ belongs to $H_0^1(B(x_0,\varepsilon)\cap\omega)$. Thus, denoting by u_n the extension of $R(w_n)$ by zero outside of $B(x_0,\varepsilon)\cap\omega$, u_n is a sequence in $H_0^1(\omega)$, which strongly converges to u in $L^\infty(\omega)$. On the other hand, using that

$$F_n(u_n,\omega)=\sum_{i,j=1}^{2}\int_{B(x_0,\varepsilon)\cap\omega}A_n\left(\partial_i R(w_n)\nabla w_n^i\right)\cdot\left(\partial_j R(w_n)\nabla w_n^j\right)dx \tag{3.14}$$

is bounded, we deduce that u belongs to $D(F(.,\omega))$. Moreover, formula (3.14) combined with (3.13) and the uniform convergence of $\partial_i R(w_n)$, yields

$$\lim_{n\to+\infty}F_n(u_n,\omega)=\int_{B(x_0,\varepsilon)\cap\omega}A\nabla u\cdot\nabla u\,d\mu.$$

Therefore, it remains to prove that

$$F(u,\omega)=\lim_{n\to+\infty}F_n(u_n,\omega), \tag{3.15}$$

in order to obtain the desired formula (3.11).

First, note that Remark 2.2 implies that the Γ-limit of $F_n(\cdot,\omega)$ for the strong topology of $L^2(\omega)$ agrees with the Γ-limit for the strong topology of $L^\infty(\omega)$ over $H_0^1(\omega)\cap C_0(\omega)$. Therefore, thanks to Properties 3.2 *c*), to prove (3.15) it is enough to check that for any sequence v_n in $H_0^1(\omega)$, which strongly converges to zero in $L^\infty(\omega)$ and such that $A_n\nabla v_n\cdot\nabla v_n$ is bounded in $L^1(\omega)$,

$$\lim_{n\to+\infty}\int_\omega A_n\nabla u_n\cdot\nabla v_n\,dx=\lim_{n\to+\infty}\sum_{i=1}^{2}\int_{B(x_0,\varepsilon)\cap\omega}A_n\left(\partial_i R(w_n)\nabla w_n^i\right)\cdot\nabla v_n\,dx=0. \tag{3.16}$$

For such a sequence v_n, we consider, for $\rho>0$, $R_\rho\in C_c^2\left(w(B(x_0,\varepsilon)\cap\omega)\right)$ such that

$$\|R-R_\rho\|_{C^1(w(B(x_0,\varepsilon)\cap\omega))}<\rho.$$

For $i\in\{1,2\}$, we start from the equality

$$\begin{aligned}\int_{B(x_0,\varepsilon)\cap\omega}A_n(\partial_i R(w_n)\nabla w_n^i)\cdot\nabla v_n\,dx&=\int_{B(x_0,\varepsilon)\cap\omega}A_n(\partial_i R_\rho(w_n)\nabla w_n^i)\cdot\nabla v_n\,dx\\&+\int_{B(x_0,\varepsilon)\cap\omega}A_n\big((\partial_i R(w_n)-\partial_i R_\rho(w_n))\nabla w_n^i\big)\cdot\nabla v_n\,dx.\end{aligned} \tag{3.17}$$

The second term of the right-hand side of (3.17) clearly satisfies

$$\limsup_{n\to+\infty}\left|\int_{B(x_0,\varepsilon)\cap\omega}A_n\big(\partial_i(R(w_n)-R_\rho(w_n))\nabla w_n^i\big)\cdot\nabla v_n\,dx\right|\le C\rho.$$

For the first term, defining $r_n\in H_0^1(\Omega)$ as the extension of $\partial_i R_\rho(w_n)$ by zero outside $B(x_0,\varepsilon)\cap\omega$, we have

$$\begin{aligned}&\int_{B(x_0,\varepsilon)\cap\omega}A_n(\partial_i R_\rho(w_n)\nabla w_n^i)\cdot\nabla v_n\,dx\\&=\int_\omega A_n\nabla w_n^i\cdot\nabla\left(r_n v_n\right)dx-\int_{B(x_0,\varepsilon)\cap\omega}A_n\nabla w_n^i\cdot\nabla(\partial_i R_\rho(w_n))\,v_n\,dx,\end{aligned}$$

where by (3.12) and the uniform convergence of v_n to zero, the right-hand side tends to zero. Therefore, we obtain, for any $\rho>0$ and $i\in\{1,2\}$,

$$\limsup_{n\to+\infty}\left|\int_\Omega A_n\left(\partial_i R(w_n)\nabla w_n^i\right)\cdot\nabla v_n\,dx\right|\le C\rho,$$

which proves (3.16) and thus (3.15).

Proof of (ii). Let us consider $u \in D(F(.,\omega)) \cap C_0^1(\omega)$ and a sequence u_n in $H_0^1(\omega)$, which strongly converges to u in $L^2(\omega)$ and such that $F_n(u_n, \omega)$ tends to $F(u, \omega)$. Thanks to Remark 2.2 there exists another sequence $\hat{u}_n$ satisfying the same properties but which also strongly converges to u in $L^\infty(\omega)$. Since u_n and $\hat{u}_n$ are recovery sequences (3.2) for $F_n(\cdot, \omega)$, with the same limit u, Properties 3.2 $c)$ implies that

$$\lim_{n \to +\infty} \int_\omega A_n \nabla(u_n - \hat{u}_n) \cdot \nabla(u_n - \hat{u}_n)\, dx = 0,$$

hence the weak $*$ limits of $A_n \nabla u_n \cdot \nabla u_n$ and $A_n \nabla \hat{u}_n \cdot \nabla \hat{u}_n$ in the sense of the measures in ω coincide. So, replacing u_n by $\hat{u}_n$ we can assume that u_n strongly converges to u in $L^\infty(\omega)$.

For $\varphi \in C_c^1(B(x_0, \varepsilon) \cap \omega)$, we take $v \in C_c^1(B(x_0, \varepsilon))$ such that $v = u$ in $\operatorname{supp}(\varphi)$. Then, we consider two sequences φ_n, v_n in $H_0^1(\omega)$, which strongly converge in $L^\infty(\omega)$ respectively to φ, v and satisfy

$$\lim_{n \to +\infty} F_n(\varphi_n, \omega) = F(\varphi, \omega) \quad \text{and} \quad \lim_{n \to +\infty} F_n(v_n, \omega) = F(v, \omega). \tag{3.18}$$

Using the second result of Theorem 2.1 we may also choose φ_n such that $\operatorname{supp}(\varphi_n)$ is contained in $\operatorname{supp}(\varphi)$. We have

$$\begin{aligned} &\int_\omega A_n \nabla u_n \cdot \nabla u_n\, \varphi\, dx = \int_\omega A_n \nabla u_n \cdot \nabla u_n\, (\varphi - \varphi_n)\, dx \\ &+ \int_\omega A_n \nabla(u_n - v_n) \cdot \nabla u_n\, \varphi_n\, dx + \int_\omega A_n \nabla v_n \cdot \nabla(u_n - v_n)\, \varphi_n\, dx \\ &+ \int_\omega A_n \nabla v_n \cdot \nabla(v_n \varphi_n)\, dx - \frac{1}{2} \int_\omega A_n \nabla v_n^2 \cdot \nabla \varphi_n\, dx. \end{aligned} \tag{3.19}$$

The first term of the right-hand side of (3.19) tends to zero thanks to the uniform convergence of φ_n. For the second term, we use

$$\begin{aligned} \int_\omega A_n \nabla(u_n - v_n) \cdot \nabla u_n\, \varphi_n\, dx \; &= \int_\omega A_n \nabla((u_n - v_n)\varphi_n) \cdot \nabla u_n\, dx \\ &- \int_\omega A_n \nabla \varphi_n \cdot \nabla u_n\, (u_n - v_n)\, dx. \end{aligned} \tag{3.20}$$

Combining the convergence of $F_n(u_n, \omega)$ to $F(u, \omega)$, the strong convergence of $(u_n - v_n)\varphi_n$ to zero in $L^2(\omega)$ and the boundedness of $A_n \nabla\big((u_n - v_n)\varphi_n\big) \cdot \nabla\big((u_n - v_n)\varphi_n\big)$ in $L^1(\omega)$, we obtain by (3.3) that the first term of the right-hand side of (3.20) tends to zero. The second term also tends to zero thanks to the uniform convergence of $u_n - v_n$ to zero in $\operatorname{supp}(\varphi) \supset \operatorname{supp}(\varphi_n)$. Therefore, the second term of the right-hand side of (3.19) tends to zero. Similarly, the third term of the right-hand side of (3.19) converges to zero.
For the fourth term of the right-hand side of (3.19) we apply Properties 3.2 $c)$ to v_n which is a recovery sequence by (3.18). Then, since v and φ have support in $B(x_0, \varepsilon) \cap \omega$ and $v = u$ in $\operatorname{supp} \varphi$, the part (i) of Lemma 3.7 implies that

$$\lim_{n \to +\infty} \int_\omega A_n \nabla v_n \cdot \nabla(v_n \varphi_n)\, dx = \int_\omega A \nabla u \cdot \nabla(u \varphi)\, d\mu.$$

Similarly, since φ_n is a recovery sequence by (3.18), we have

$$\lim_{n \to +\infty} \int_\omega A_n \nabla v_n^2 \cdot \nabla \varphi_n\, dx = \int_\omega A \nabla u^2 \cdot \nabla \varphi\, d\mu.$$

Finally, passing to the limit in the right-hand side of (3.19) thanks to the previous convergences, yields

$$\lim_{n\to+\infty}\int_\omega A_n\nabla u_n\cdot\nabla u_n\,\varphi\,dx=\int_\omega A\nabla u\cdot\nabla u\,\varphi\,d\mu,\quad \forall\,\varphi\in C_c^1(B(x_0,\varepsilon)\cap\omega),$$

which concludes the proof of Lemma 3.7. □

Proof of Theorem 3.4. If $C_c^1(\Omega)$ is contained in $D(F)$, then it is clear that for any $x_0\in\Omega$, there exists $\delta>0$, two functions w^1,w^2 in $C^1(B(x_0,\delta))$ and two sequences w_n^1,w_n^2 in $H^1(B(x_0,\delta))$ which satisfy (3.7).

Inversely, if condition (3.7) is satisfied, Lemma 3.7 proves that for any $x_0\in\Omega$, there exist $\varepsilon>0$, a nonnegative bounded Radon measure $\hat\mu$ and a nonnegative matrix-valued function $\hat A$ in $L^\infty_{\hat\mu}(B(x_0,\varepsilon))^{2\times 2}$ such that (3.11) holds true for any open subset ω of Ω and for the Γ-limit $F(\cdot,\omega)$ of any convergent Γ-subsequence of $F_n(\cdot,\omega)$ in $L^2(\omega)$, with $C_c^1(B(x_0,\varepsilon))\subset D(F(\cdot,\omega))$. ¿From the covering of Ω by the disks $B(x_0,\varepsilon)$, we can deduce the existence of $x_i\in\Omega$, $\varepsilon_i>0$ and $\varphi_i\in C_c^\infty(B(x_i,\varepsilon_i))$, $i\in\mathbb{N}^*$, such that any compact subset of Ω only intersects a finite number of $B(x_i,\varepsilon_i)$ and $\sum_{i\in\mathbb{N}^*}\varphi_i(x)=1$ in Ω. Then, considering the Radon measure μ_i and the matrix-valued function A_i associated with each disk $B(x_i,\varepsilon_i)$, for $i\in\mathbb{N}^*$, according to the procedure of Lemma 3.7 combined with a diagonal extraction, we define the measure μ by

$$\mu(B):=\sum_{i=1}^{\infty}\mu_i\left(B(x_i,\varepsilon_i)\cap B\right),\quad \forall\,B\text{ Borel subset of }\Omega,$$

and, using the Radon-Nikodym theorem we define the matrix-valued measure $A\,d\mu$ by

$$\int_B A\,d\mu:=\sum_{i\in\mathbb{N}^*}\int_{B(x_i,\varepsilon_i)\cap B}A^i\varphi_i\,d\mu_i\quad \forall\,B\text{ Borel set, with }\bar B\subset\Omega.$$

Now, let us consider an open subset ω of Ω and $u\in C_c^1(\omega)$. We have

$$u=\sum_{i=1}^{\infty}u\varphi_i,$$

where the sum carries on a finite set of indexes i. Since by Lemma 3.7 each function $u\varphi_i$ is in $D(F(\cdot,\omega))$, the function u also belongs to $D(F(\cdot,\omega))$. Let u_n be a sequence in $H_0^1(\omega)$, which strongly converges to u in $L^2(\omega)$ and such that $F_n(u_n,\omega)$ tends to $F(u,\omega)$. By the part (ii) of Lemma 3.7 the sequence $A_n\nabla u_n\cdot\nabla u_n$ converges to $A^i\nabla u\cdot\nabla u\,d\mu_i$ in the weak ∗ sense of the measures in $B(x_i,\varepsilon_i)\cap\omega$, for each $i\in\mathbb{N}^*$. Then, since for any $\varphi\in C_c^1(\omega)$, $\operatorname{supp}(\varphi)\cap\operatorname{supp}(\varphi_i)\neq\emptyset$ only for a finite set of indexes i, we have

$$\begin{aligned}&\lim_{n\to+\infty}\int_\omega A_n\nabla u_n\cdot\nabla u_n\,\varphi\,dx=\lim_{n\to+\infty}\sum_{i=1}^{\infty}\int_\omega\varphi_i\,A_n\nabla u_n\cdot\nabla u_n\,\varphi\,dx\\&=\sum_{i=1}^{\infty}\int_\omega\varphi_i\,A^i\nabla u\cdot\nabla u\,\varphi\,d\mu_i=\int_\omega A\nabla u\cdot\nabla u\,\varphi\,d\mu.\end{aligned}\tag{3.21}$$

This proves that $A_n\nabla u_n\cdot\nabla u_n$ weakly converges to $A\nabla u\cdot\nabla u\,d\mu$ in $\mathcal{M}(\omega)*$. Thanks to second result of Theorem 2.1 we can choose u_n such that $\operatorname{supp}(u_n)\subset\operatorname{supp}(u)$. Then, taking in (3.21) $\varphi\in C_c^1(\omega)$ such that $\varphi=1$ in $\operatorname{supp}(u)$, we get

$$\begin{aligned}&F(u,\omega)=\lim_{n\to+\infty}F_n(u_n,\omega)=\lim_{n\to+\infty}\int_\omega A_n\nabla u_n\cdot\nabla u_n\,dx=\lim_{n\to+\infty}\int_\omega A_n\nabla u_n\cdot\nabla u_n\,\varphi\,dx\\&=\int_\omega A\nabla u\cdot\nabla u\,\varphi\,d\mu=\int_\omega A\nabla u\cdot\nabla u\,\varphi\,d\mu=\int_\omega A\nabla u\cdot\nabla u\,d\mu,\end{aligned}$$

which proves (3.9). Moreover, by (2.1) the sequence u_n converges weakly to u in $H_0^1(\Omega)$, and for any $u \in C_c^1(\omega)$, we have

$$\alpha \int_\Omega |\nabla u|^2\,dx \le \alpha \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^2\,dx \le \lim_{n\to+\infty} F_n(u_n,\omega) = F(u,\omega) = \int_\Omega A\nabla u\cdot\nabla u\,d\mu.$$

This implies (3.8) (see e.g. [13] Lemma 22.5 p. 234) and concludes the proof. □

Proof of Theorem 3.6. Similarly to the proof of Theorem 3.4 we consider $\varphi^j \in C_c^1(\Omega)$, for $j \in \mathbb{N}$, which gives a partition of the unity associated with a locally finite covering of Ω. For any $i \in \{1,2\}$ and any $j \in \mathbb{N}$, we consider $z_n^{i,j} \in H_0^1(\Omega)$, with $z_n^{i,j} = 0$ q.e. outside supp(φ^j), such that $z_n^{i,j}$ strongly converges to $x_i\varphi^j$ in $L^\infty(\Omega)$ and $F_n(z_n^{i,j})$ tends to $F(x_i\varphi^j)$. Then, for $i \in \{1,2\}$, we define

$$w_n^i := \sum_{j\in J_n} z_n^{i,j}, \quad \text{where} \quad J_n := \left\{ j\in\mathbb{N} :\ \operatorname{supp}(\varphi_j)\cap\{x\in\Omega : \operatorname{dist}(x,\partial\Omega) < \tfrac{1}{n}\} = \emptyset \right\}.$$

The sequences w_n^i clearly satisfy the conditions (3.7) of Theorem 3.6.

Now, consider an open subset ω of Ω, a function $u \in C_c^1(\omega)$, and assume that $F_n(\cdot,\omega)$ Γ-converges to $F(\cdot,\omega)$ for the strong topology of $L^2(\omega)$. Since the sequence $w_n := (w_n^1, w_n^2)$ strongly converges to the identity in $L^\infty_{\text{loc}}(\Omega)$, the argument used in the proof of Lemma 3.7 shows that the sequence $F_n(u(w_n),\omega)$ converges to $F(u,\omega)$. So, for any recovery sequence u_n in $H_0^1(\omega)$, which strongly converges to u in $L^2(\omega)$ and such that $F_n(u_n,\omega)$ tends to $F(u,\omega)$, Properties 3.2 c) implies that

$$\lim_{n\to+\infty} \int_\omega A_n\nabla(u_n - u(w_n))\cdot\nabla(u_n - u(w_n))\,dx = 0. \tag{3.22}$$

On the other hand, since w_n uniformly converges to the identity and $u \in C_c^1(\omega)$, we also have

$$\begin{aligned}&\lim_{n\to+\infty} \int_\omega A_n \left(\nabla[u(w_n)] - \sum_{i=1}^2 \partial_i u \nabla w_n^i\right)\cdot\left(\nabla[u(w_n)] - \sum_{i=1}^2 \partial_i u\nabla w_n^i\right) dx \\ &= \lim_{n\to+\infty} \sum_{i,k=1}^2 \int_\omega A_n\nabla w_n^i\cdot\nabla w_n^k\,(\partial_i u(w_n) - \partial_i u)\,(\partial_k u(w_n) - \partial_k u)\,dx = 0,\end{aligned}$$

which combined with (3.22) yields the desired limit (3.10). □

Proof of Theorem 3.8. For $x \in \bar\Omega$, we denote by $[x]$ its class in $\bar\Omega/\mathcal{R}$. The class containing the elements of $\partial\Omega$ (note that all the elements of $\partial\Omega$ are in relation by $\mathcal{R}$) is denoted by $[\partial]$. A basis for the topology $\mathcal{T}$ is given by the subsets of $\bar\Omega/\mathcal{R}$ of the form

$$\bigcap_{i=1}^n u_i^{-1}\big((s_i-\varepsilon_i, s_i+\varepsilon_i)\big), \tag{3.23}$$

with $u_1,\dots,u_n \in D(F)\cap C_0(\Omega)$, $s_1,\dots,s_n \in \mathbb{R}$, $\varepsilon_1,\dots,\varepsilon_n > 0$ and $n \in \mathbb{N}^*$. Then, by definition of $\mathcal{R}$, for any $x,y \in \bar\Omega$ with $[x] \ne [y]$, there exists $u \in D(F)\cap C_0(\Omega)$ such that $u(x) \ne u(y)$. Therefore, $\mathcal{T}$ is Hausdorff.

Let $x \in \bar\Omega$ and let V be a neighbourhood of $[x]$ of type (3.23). By the density of $\mathbb{Q}^2$ in $\mathbb{R}^2$ and the continuity of the functions u_i in (3.23), there exists $z \in \mathbb{Q}^2\cap\bar\Omega$ such that $[z] \in V$. Therefore, the classes of the elements in $\mathbb{Q}^2\cap\bar\Omega$ are dense in $\bar\Omega/\mathcal{R}$, which implies the separability of $\mathcal{T}$.

By the definition of $\mathcal{T}$, for any open set $\mathcal{O}$ of $\bar\Omega/\mathcal{R}$, the set $\{x\in\Omega : [x]\in\mathcal{O}\}$ is an open subset of $\bar\Omega$. By considering the complementary the same property holds true for the closed sets. Using the compactness of $\bar{\Omega}$, the characterization of the open sets implies that $\bar{\Omega}/\mathcal{R}$ is compact and thus, the set $\Omega^* := \bar{\Omega}/\mathcal{R} \setminus \{[\partial]\}$ is locally compact.

Using that the Γ-limit F of F_n (3.6) is closed for the strong topology of $L^2(\Omega)$ and the definition of the measure m, the restriction of F to $D(F) \cap C_0(\Omega)$ is a closable form in $L^2_m(\Omega^*)$, whose closure is denoted by F^*. Let us prove that F^* is a regular Dirichlet form according to Definition 3.3.

Set $H(s) := (s \vee 0) \wedge 1$, for $s \in \mathbb{R}$. Then, for any $u \in D(F)$ and any recovery sequence $u_n \in H^1_0(\Omega)$ associated with u and F_n by (3.2), we have

$$F(H(u)) \leq \liminf_{n\to+\infty} F_n(H(u_n)) \leq \lim F_n(u_n) = F(u).$$

In particular, this holds for any $u \in D(F) \cap C_0(\Omega)$, hence the restriction of F to $D(F) \cap C_0(\Omega)$ is Markovian.

Since $D(F) \cap C_0(\Omega)$ is an algebra which separates points, the Stone-Weierstrass theorem shows that the functions of the form $u + c$, with $u \in D(F) \cap C_0(\Omega)$ and $c \in \mathbb{R}$, are dense in $C(\bar{\Omega}/\mathcal{R})$. Now, consider $v \in C_c(\Omega^*)$, $v_n \in D(F) \cap C_0(\Omega)$ and $c_n \in \mathbb{R}$, such that $v_n + c_n$ converges to v in $C(\bar{\Omega}/\mathcal{R})$. Since v and v_n vanish in $[\partial]$, the sequence c_n converges to zero and thus v_n converges to v in $C(\bar{\Omega}/\mathcal{R})$. This proves that $D(F) \cap C_0(\Omega)$ is dense in $C_0(\Omega^*)$, which implies that F^* is regular. Therefore, F^* is a regular Dirichlet form.

It remains to prove that F^* is strongly local, i.e., the polar form Φ of F satisfies (3.4). Let $u, v \in D(F) \cap C_0(\Omega)$ and $c \in \mathbb{R}$, such that $u = c$ constant in $\operatorname{supp}(v)$. First, we assume that $c \geq 0$. Taking into account Remark 2.2, we consider two recovery sequences u_n, v_n which strongly converge respectively to u, v in $L^\infty(\Omega)$ and such that $F_n(u_n), F_n(v_n)$ tend respectively to $F(u), F(v)$. We also choose v_n such that $\operatorname{supp}(v_n) \subset \operatorname{supp}(v)$. Let H_ε, for $\varepsilon > 0$, be the function defined in $\mathbb{R}$ by

$$H_\varepsilon(s) := \begin{cases} \frac{c}{c-\varepsilon}\, s & \text{if } s < c - \varepsilon \\ c & \text{if } c - \varepsilon \leq s \leq c + \varepsilon \\ s + \varepsilon & \text{if } s > c + \varepsilon, \end{cases} \qquad \text{if } \varepsilon < c,$$

(note that $H_\varepsilon(0) = 0$), and $H_\varepsilon(s) := (s - \varepsilon\,\mathrm{sgn}(s))\,\chi_{\{|s|>\varepsilon\}}$ if $c = 0$. The sequence $\varepsilon_n := \|u_n - u\|_{L^\infty(\Omega)}$ converges to zero. Then, the sequence $H_{\varepsilon_n}(u_n)$ satisfies the same properties than u_n, but we also have $H_{\varepsilon_n}(u_n) = c$ in $\operatorname{supp}(v) \supset \operatorname{supp}(v_n)$, for ε_n small enough. Therefore, the Properties 3.2 c) of the recovery sequence v_n yields

$$\Phi(u, v) = \lim_{n\to+\infty} \int_\Omega A_n \nabla \left(H_{\varepsilon_n}(u_n)\right) \cdot \nabla v_n \, dx = 0.$$

In the case $c < 0$, we simply use the equality $\Phi(u, v) = -\Phi(-u, v) = 0$. □

4 Application to the periodic case

4.1 Statement of the results

In this section we consider a sequence B_n of symmetric matrix-valued functions in $L^\infty(\mathbb{R}^2)^{2\times 2}$, which satisfies the following assumptions:

B_n is Y-periodic, where $Y := (0, 1)^2$, i.e.,

$$\forall n \in \mathbb{N},\ \forall \kappa \in \mathbb{Z}^2, \quad B_n(\cdot + \kappa) = B_n(\cdot) \quad \text{a.e. in } \mathbb{R}^2, \tag{4.1}$$

B_n is equicoercive in $\mathbb{R}^2$, i.e.,

$$\exists\, \alpha > 0 \qquad \text{such that} \qquad \forall n \in \mathbb{N},\ \forall \xi \in \mathbb{R}^2, \quad B_n\xi \cdot \xi \geq \alpha|\xi|^2 \quad \text{a.e. in } \mathbb{R}^2. \tag{4.2}$$

Let ε_n be a sequence of positive numbers which tends to 0. From the sequences B_n and ε_n we define the highly oscillating sequence of matrix-valued functions A_n by

$$A_n(x) := B_n\left(\frac{x}{\varepsilon_n}\right), \quad \text{a.e. } x \in \mathbb{R}^2. \tag{4.3}$$

In virtue of (4.1) and (4.2) A_n is an equicoercive sequence of ε_n-periodic matrix-valued functions in $L^\infty(\mathbb{R}^2)^{2\times 2}$. Let A_n^* be the constant matrix defined by

$$A_n^*\lambda\cdot\lambda := \min\left\{\int_Y B_n(y)(\lambda + \nabla\varphi(y))\cdot(\lambda + \nabla\varphi(y))\,dy \;:\; \varphi \in H^1_\#(Y)\right\}, \quad \lambda \in \mathbb{R}^2, \tag{4.4}$$

where $H^1_\#(Y)$ denotes the set of Y-periodic functions in $H^1_{\text{loc}}(\mathbb{R}^2)$. The matrix A_n^* is symmetric and positive definite with $A_n^* \geq \alpha\, I_2$. By the classical result of periodic homogenization (see e.g. [1]) A_n^*, for fixed n, is the homogenized matrix associated with the oscillating sequence $A_n(\frac{x}{\varepsilon})$ as ε tends to zero. Note that in the definition (4.3) of A_n the oscillations period ε_n depends on the sequence n.

In this periodic framework we are interested in the asymptotic behaviour of the diffusion energy F_n defined by

$$F_n(u) := \begin{cases} \displaystyle\int_\Omega A_n\nabla u\cdot\nabla u\,dx & \text{if } u \in H^1_0(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega)\setminus H^1_0(\Omega), \end{cases} \tag{4.5}$$

as well as the conduction problem

$$\begin{cases} -\operatorname{div}(A_n\nabla u_n) &= f \quad \text{in } \Omega \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

for a given f in $H^{-1}(\Omega)$.

The following result shows that the asymptotic behaviour of the diffusion energy F_n (4.5) only depends on the limit of the spectral radius $\rho(A_n^*)$ of the matrix A_n^* (4.4).

Theorem 4.1. *Let Ω be a bounded open set of $\mathbb{R}^2$. Consider a highly oscillating sequence of matrix-valued functions A_n satisfying* (4.1), (4.2) *and* (4.3)*. Then, we have the following alternative:*

If $\rho(A_n^*)$ is bounded, there exists a subsequence, still denoted by n, and a positive definite matrix A^* such that A_n^* (4.4) *converges to A^* in $\mathbb{R}^{2\times 2}$ and F_n* (4.5) *Γ-converges for the strong topology of $L^2(\Omega)$ to the quadratic form F associated with A^* by*

$$F(u) := \begin{cases} \displaystyle\int_\Omega A^*\nabla u\cdot\nabla u\,dx & \text{if } u \in H^1_0(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega)\setminus H^1_0(\Omega). \end{cases} \tag{4.7}$$

If $\rho(A_n^*)$ tends to $+\infty$, the sequence F_n Γ-converges for the strong topology of $L^2(\Omega)$ to the quadratic form F whose domain is

$$D(F) = \{0\}. \tag{4.8}$$

In term of the conduction problem (4.6) Theorem 4.1 implies the following result:

Corollary 4.2. *Let Ω be a bounded open set of $\mathbb{R}^2$. Consider a highly oscillating sequence of matrix-valued functions A_n satisfying* (4.1), (4.2) *and* (4.3)*. Then, we have the following alternative:*

If $\rho(A_n^*)$ is bounded, there exists a subsequence, still denoted by n, and a positive definite matrix A^* such that A_n^* (4.4) converges to A^* in $\mathbb{R}^{2\times 2}$ and, for any f in $H^{-1}(\Omega)$, the solution u_n of (4.6) weakly converges in $H_0^1(\Omega)$ to the solution u of the conduction problem

$$\begin{cases} -\operatorname{div}(A^*\nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

If $\rho(A_n^*)$ tends to $+\infty$, the sequence u_n strongly converges to 0 in $H_0^1(\Omega)$.

Proof. Corollary 4.2 is a immediate consequence of Theorem 4.1 using the fact that the solution u_n of (4.6) is the minimizer of the functional

$$u \in L^2(\Omega) \longmapsto \frac{1}{2} F_n(u) - \int_\Omega f\, u\, dx,$$

and the minimizers convergence property of the Γ-convergence (see e.g. Corollary 7.24 p. 84 of [13]). $\square$

Remark 4.3. Corollary 4.2 is an extension of the similar homogenization result obtained in [4] (by a complete different approach) under the assumption that the sequence of periodic matrix-valued B_n (4.3) is bounded in $L^1(Y)^{2\times 2}$. This condition is more restrictive since it is easy to check that the boundedness of B_n in $L^1(Y)^{2\times 2}$ implies the boundedness of $\rho(A_n^*)$. We can also build a periodic two-dimensional microstructure such that $\rho(A_n^*)$ is bounded while $\|B_n\|_{L^1(Y)^{2\times 2}}$ is not. Therefore, Corollary 4.2 provides a complete answer to the periodic homogenization of the conduction problems with equicoercive sequences of symmetric conductivities.

4.2 Proof of Theorem 4.1

The case where $\rho(A_n^*)$ is bounded

Let X_n^i, $i = 1, 2$, be the unique function in $H^1_\#(Y)$, with zero Y-average value, solution of

$$\forall \varphi \in H^1_\#(Y), \quad \int_Y B_n \nabla W_n^i \cdot \nabla\varphi \, dy = 0, \quad \text{where} \quad W_n^i(y) := y_i + X_n^i(y), \tag{4.10}$$

or equivalently,

$$\operatorname{div}\left(B_n\left(e_i + \nabla X_n^i\right)\right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{4.11}$$

where (e_1, e_2) denotes the canonic basis of $\mathbb{R}^2$. Let w_n^i be the highly oscillating sequence defined by

$$w_n^i(x) := \varepsilon_n\, W_n^i\left(\frac{x}{\varepsilon_n}\right) = x_i + \varepsilon_n\, X_n^i\left(\frac{x}{\varepsilon_n}\right), \quad \text{for } x \in \Omega. \tag{4.12}$$

By (4.11) and the definition (4.3) of A_n the function w_n^i is clearly A_n-harmonic. Moreover, by the Y-periodicity of $B_n\nabla W_n^i \cdot \nabla W_n^i$, by (4.10) and the definition (4.4) of A_n^*, we have for any bounded open subset ω of $\mathbb{R}^2$,

$$\int_\omega A_n \nabla w_n^i \cdot \nabla w_n^i \, dx \le c_\omega \int_Y B_n \nabla W_n^i \cdot \nabla W_n^i \, dy = c_\omega\, A_n^* e_i \cdot e_i \le c < +\infty.$$

This combined with the equicoerciveness of A_n implies that the sequence w_n^i is bounded in $H^1_{\rm loc}(\mathbb{R}^2)$ and thus weakly converges to x_i in $H^1_{\rm loc}(\mathbb{R}^2)$. Then, thanks to Corollary 2.3 the sequence w_n^i strongly converges to x_i in $L^\infty(\Omega)$. On the other hand, by the boundedness assumption on $\rho(A_n^*)$ the sequence A_n^* converges, up to a subsequence, to some constant matrix A^* in the space $\mathbb{R}^{2\times 2}$. Moreover, the ε_n-periodicity of ∇w_n^i implies that, for any $i, j = 1, 2$,

$$A_n \nabla w_n^i \cdot \nabla w_n^j = \left(B_n \nabla W_n^i \cdot \nabla W_n^j\right)\left(\frac{x}{\varepsilon_n}\right) \rightharpoonup A^* e_i \cdot e_j \quad \text{weakly in } \mathcal{M}(\mathbb{R}^2) * .$$

So, as the gradients of the functions x_i, $i \in \{1, 2\}$, are independent at each point of Ω, the sequences w_n^i satisfy (3.12) for any open disk contained in Ω. Therefore, since w_n^i are A_n-harmonic and converge uniformly in Ω, the construction of Lemma 3.7 yields the measure μ and the matrix-valued A by

$$d\mu = (A^* e_1 \cdot e_1 + A^* e_2 \cdot e_2)\, dx \quad \text{and} \quad A e_i \cdot e_j \, d\mu = A^* e_i \cdot e_j \, dx,$$

hence $A^* = (A^* e_1 \cdot e_1 + A^* e_2 \cdot e_2)\, A$. Then, in virtue of Theorem 3.4 the Γ-limit F of the sequence F_n (4.5) satisfies

$$C_c^1(\Omega) \subset D(F) \qquad \text{and} \qquad F(u) = \int_\Omega A^* \nabla u \cdot \nabla u \, dx, \quad \forall\, u \in C_c^1(\Omega). \tag{4.13}$$

Let us conclude. On the one side, the equicoerciveness of A_n and the lower semicontinuity of the $H_0^1(\Omega)$-norm give $D(F) \subset H_0^1(\Omega)$. On the other side, the density of $C_c^1(\Omega)$ in $H_0^1(\Omega)$, combined with the fact that $D(F)$ is a Hilbert space and that from (4.13) a sequence of $C_c^1(\Omega)$ which strongly converges in $H_0^1(\Omega)$ also strongly converges in $D(F)$, we get $D(F) = H_0^1(\Omega)$ and equality (4.13) holds true in $H_0^1(\Omega)$. Note that F_n Γ-converges to F for the whole sequence such that A_n^* converges to A^* in $\mathbb{R}^{2\times 2}$.

The case where $\rho(A_n^*)$ tends to $+\infty$

We proceed by contradiction. We assume that the domain $D(F)$ of the Γ-limit F does not reduce to $\{0\}$. Then, we prove that $\rho(A_n^*)$ is necessarily bounded. To this end, we proceed in two steps. In the first step, we prove that there exists a continuous function in $D(F) \setminus \{0\}$. The second step is devoted to the proof of the boundedness of $\rho(A_n^*)$.

First step : $D(F) \cap C(\Omega) \neq \{0\}$.
Up to an extraction of a subsequence we can assume that the sequence F_n defined by (4.5) Γ-converges to some quadratic functional $F : L^2(\Omega) \longrightarrow [0, +\infty]$. The starting assumption is that $D(F) \neq \{0\}$. Let $u \in D(F) \setminus \{0\}$. By the equicoerciveness of A_n the function u belongs to $H_0^1(\Omega)$. There exists a sequence u_n in $H_0^1(\Omega)$ which strongly converges to u in $L^2(\Omega)$ and such that $F_n(u_n)$ tends to $F(u)$. Up to enlarge the domain Ω and to extend the functions of $H_0^1(\Omega)$ by 0 outside Ω, we may assume that the supports of u, u_n are contained in a fixed compact K of Ω.

Firstly, let us prove that, for any $\tau \in \mathbb{R}^2$ of small enough norm, the translated function $u(\cdot + \tau)$ belongs to $D(F)$ and $F(u(\cdot + \tau)) = F(u)$. We follow the procedure given in the proof of Theorem 24.1 of [13]. Let $\tau \in \mathbb{R}^2$ and let κ_n be a sequence in $\mathbb{Z}^2$ such that $\tau_n := \varepsilon_n \kappa_n$ tends to τ. If τ has a small enough norm, then we have $K - \tau_n \subset \Omega$ for any $n \in \mathbb{N}$. Then, using successively the fact that $u_n(\cdot + \tau_n)$ is equal to 0 in $\Omega \setminus (K - \tau_n)$, the change of variable $y = x + \tau_n$ and the ε_n-periodicity of A_n, we obtain

$$\begin{aligned} F(u_n(\cdot + \tau_n)) &= \int_{K-\tau_n} A_n(x) \nabla u_n(x + \tau_n) \cdot \nabla u_n(x + \tau_n)\, dx \\ &= \int_K A_n(y) \nabla u_n(y) \cdot \nabla u_n(y)\, dy = F_n(u_n). \end{aligned}$$

Moreover, the sequence $u_n(\cdot + \tau_n)$ strongly converges to $u(\cdot + \tau)$. Therefore, the Γ-liminf inequality implies that

$$F(u(\cdot + \tau)) \leq \liminf_{n\to+\infty} F_n(u_n(\cdot + \tau_n)) = \liminf_{n\to+\infty} F_n(u_n) = F(u),$$

which also yields $F(u) \leq F(u(\cdot + \tau - \tau)) \leq F(u(\cdot + \tau))$, and thus $F(u(\cdot + \tau)) = F(u)$.

Secondly, let δ be a small enough positive number and let v_δ be the function defined on Ω by

$$v_\delta(x) := \frac{1}{\delta^2} \int_{\delta Y} u(x + y)\, dy, \quad \text{for } x \in \Omega,$$

which is continuous on Ω. Let $(Q_k^j)_{1 \leq j \leq k}$, for $k \in \mathbb{N}^*$, be a covering of the set δY by k squares of side $\frac{\delta}{\sqrt{k}}$ and let y_k^j be the center of Q_k^j. Then, the sequence of convex combinations of translated of u defined by

$$v_\delta^k := \frac{1}{\delta^2} \sum_{j=1}^{k} |Q_k^j|\, u(\cdot + y_k^j)$$

strongly converges to v_δ in $L^2(\Omega)$ as $k \to +\infty$, for fixed δ. Then, the lower semi-continuity and the convexity of F yield

$$F(v_\delta) \leq \liminf_{k \to +\infty} F(v_\delta^k) \leq \liminf_{k \to +\infty} \frac{1}{\delta^2} \sum_{j=1}^{k} |Q_k^j|\, F\big(u(\cdot + y_k^j)\big) = F(u) < +\infty.$$

Therefore, v_δ belongs to $D(F) \cap C(\Omega)$. Since v_δ strongly converges to $u \neq 0$ in $L^2(\Omega)$, v_δ is a non-zero function in $D(F) \cap C(\Omega)$ for δ small enough, which concludes the first step.

Second step : Boundedness of $\rho(A_n^*)$.
Let v be a non-zero function in $D(F) \cap C(\Omega)$ and let v_n be a sequence in $H_0^1(\Omega)$ which strongly converges to v in $L^2(\Omega)$ and such that $F_n(v_n)$ tends to $F(v)$. By Theorem 2.1 the sequence v_n uniformly converges to v in Ω. Since v is a non-zero continuous function on Ω, the uniform convergence of v_n to v implies that there exists a non-empty open subset ω_0 of Ω and a constant $c_0 > 0$ such that

$$|v_n(x)| \geq c_0 \quad \text{a.e. } x \in \omega_0. \tag{4.14}$$

Let λ_n be a unit norm vector in $\mathbb{R}^2$ such that $A_n^* \lambda_n \cdot \lambda_n = \rho(A_n^*)$. Let w_n be the highly oscillating sequence defined by

$$w_n(x) := \lambda_n \cdot x + \varepsilon_n X_n\left(\frac{x}{\varepsilon_n}\right) \quad x \in \mathbb{R}^2, \quad \text{where} \quad X_n := (\lambda_n \cdot e_1)\, X_n^1 + (\lambda_n \cdot e_2)\, X_n^2 \tag{4.15}$$

and X_n^i, $i = 1, 2$, are the Y-periodic solutions of (4.11). Set $\tilde{Y} := (-\frac{1}{2}, \frac{3}{2})^2$. Since the function $W_n(y) := \lambda_n \cdot y + X_n(y)$ is B_n-harmonic in $\mathbb{R}^2$, by Corollary 2.5 there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$,

$$\|W_n\|_{L^\infty(Y)} \leq C\, \|W_n\|_{H^1(\tilde{Y})}.$$

Then, using the periodicity and the zero Y-average value of X_n, as well as the Poincaré-Wirtinger inequality yields

$$\begin{aligned}
\|X_n\|_{L^\infty(Y)} &\leq 1 + \|W_n\|_{L^\infty(Y)} \\
&\leq 1 + \|\lambda_n \cdot y\|_{L^\infty(\tilde{Y})} + C\, \|X_n\|_{H^1(\tilde{Y})} \leq 1 + \tfrac{3}{\sqrt{2}} + 4\, C\, \|X_n\|_{H^1(Y)} \\
&\leq C' + C'\, \|\nabla X_n\|_{L^2(Y)} \leq 2\, C' + C'\, \|\nabla W_n\|_{L^2(Y)}.
\end{aligned}$$

Moreover, the coerciveness of B_n and the definition (4.4) of A_n^* imply that

$$\alpha\, \|\nabla W_n\|_{L^2(Y)^2}^2 \leq \int_Y B_n \nabla W_n \cdot \nabla W_n\, dy = A_n^* \lambda_n \cdot \lambda_n = \rho(A_n^*),$$

which combined with the previous estimates gives

$$\|X_n\|_{L^\infty(Y)} \leq C' + \frac{C'}{\sqrt{\alpha}}\sqrt{\rho(A_n^*)}.$$

Therefore, by the Y-periodicity of X_n and the definition (4.15) of w_n there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$,

$$\|w_n\|^2_{L^\infty(\Omega)} \leq c + c\,\varepsilon_n^2\,\rho(A_n^*). \tag{4.16}$$

On the other hand, using the A_n-harmonicity of w_n (4.15) and the Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_\Omega A_n\nabla w_n \cdot \nabla w_n\, v_n^2\, dx &= -2\int_\Omega A_n\nabla w_n \cdot \nabla v_n\, v_n\, w_n\, dx \\ &\leq 2\left(\int_\Omega A_n\nabla w_n \cdot \nabla w_n\, v_n^2\, dx\right)^{\frac12}\left(\int_\Omega A_n\nabla v_n \cdot \nabla v_n\, w_n^2\, dx\right)^{\frac12},\end{aligned}$$

hence the inequality

$$\int_\Omega A_n\nabla w_n \cdot \nabla w_n\, v_n^2\, dx \leq 4\int_\Omega A_n\nabla v_n \cdot \nabla v_n\, w_n^2\, dx. \tag{4.17}$$

Let us conclude. On the one side, thanks to the uniform estimate (4.14) and the ε_n-periodicity of $A_n\nabla w_n \cdot \nabla w_n$ the left hand-side of (4.17) is bounded from below by a positive constant times

$$\int_{\omega_0} A_n\nabla w_n \cdot \nabla w_n\, dx \geq c_{\omega_0}\int_Y B_n\nabla W_n \cdot \nabla W_n\, dy = c_{\omega_0}\,\rho(A_n^*),$$

where c_{ω_0} is a positive constant only depending on ω_0. On the other side, thanks to the uniform estimate (4.16) combined with the boundedness of $F_n(v_n)$ the right hand-side of (4.17) is bounded from above by

$$c + c\,\varepsilon_n^2\,\rho(A_n^*).$$

Therefore, there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$,

$$\rho(A_n^*) \leq c + c\,\varepsilon_n^2\,\rho(A_n^*), \quad \text{with } \varepsilon_n \to 0,$$

which implies that $\rho(A_n^*)$ is bounded. The proof of Theorem 4.1 is done.

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