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DISTRIBUCIÓN DE ÁLGEBRAS DE LIE, MALCEV Y DE EVOLUCIÓN EN CLASES DE ISOTOPISMOS

Author
Óscar Jesús Falcón Ganfornina

Supervisor
Prof. Dr. Juan Núñez Valdés

Co-supervisor
Prof. Dr. Raúl Manuel Falcón Ganfornina

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UNIVERSIDAD DE SEVILLA
Facultad de Matemáticas
DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA

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Fdo.: Óscar Jesús Falcón Ganfornina.

V$^{\underline{o}}$. B$^{\underline{o}}$. LOS DIRECTORES

Fdo.: Juan Núñez Valdés.
Profesor Titular de Universidad
Dpto. Geometría y Topología
Universidad de Sevilla.

Fdo.: Raúl Manuel Falcón Ganfornina
Profesor Contratado Doctor Interino
Dpto. Matemática Aplicada I
Universidad de Sevilla.

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"It is possible, you know, to drift off to an unknown world and find happiness there. Maybe even more happiness than you've ever known before".

– John Boyne, The Terrible Thing That Happened to Barnaby Brocket

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Many years have passed since this work started being part of my life. No mathematical thesis was in my thoughts when I graduated in 2009. However, who could have guessed that a disappointment in a competitive exam would lead me to write these acknowledgments. The following lines, where teachers, colleagues, friends and family appear mixed up, are the ending to all these years.

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Contents

Introduction

In Mathematics, the definition of *isotopism* differs depending on whether one deals with a topological or an algebraic problem. In both cases, however, isotopisms are used to classify objects with some common properties in a more general way than isomorphisms can do. From a chronological point of view, the first time that appeared the term *isotopy* was related to the concept of *homotopy* in Topology at the beginning of 1900's. It was not until 1942, however, that Abraham Adrian Albert [13] introduced the concept of isotopy for the study and classification of non-associative algebras. Shortly after, Richard Bruck [60] generalized this concept from algebras to quasigroups.

Since the original manuscripts of Albert and Bruck, a widely range of mathematicians has dealt with isotopisms in order to study the classification of distinct types of algebraic and combinatorial structures, some of them with applications in other fields as Quantum Mechanics and Cryptography, amongst others. Nevertheless, in spite of its importance, there does not exist any survey in the literature that gathers together the origin and development of the theory of isotopisms. This constitutes the first goal of the current manuscript, which is conveniently developed in its first chapter. Our main goal consists, however, in dealing with isotopisms of distinct types of non-associative algebras, such as Lie, Malcev and evolution algebras, with special importance not only in terms of Mathematics, but also for their multiple applications in Natural Sciences and Engineering [215, 261]. Unlike of the distinct results that there exist on isotopisms of division, alternative or Jordan algebras, amongst others, there barely exists any result about isotopisms of the algebras on which this manuscript focuses. Particularly, there does not exist any result about their explicit distribution into isotopism classes. Throughout the manuscript, the determination of these distributions gives rise to common algebraic properties that enable us to gather together algebras of distinct isomorphism classes, which, at first sight, they seemed to be completely different. Particularly, we focus the majority of our results on algebras defined over finite fields.

The structure of the manuscript is the following

- In Chapter 1, we expose a brief survey that gathers together the origin and development of the theory of isotopisms of algebras, quasigroups and related structures.
- Chapter 2 consists of those basic concepts and results in Computational Algebraic Geometry and Graph Theory that we use throughout the manuscript in order to determine the distribution of distinct types of algebras into isotopism classes. Particularly, we describe a pair of graphs that enable us to define faithful functors between finite-dimensional algebras over finite fields and these types of graphs. Depending on the functor, we map isomorphic or isotopic algebras to isomorphic graphs. Reciprocally, any pair of isomorphic graphs is uniquely related to a pair of algebras so that there exists a multiplicative map between them.
- Chapter 3 deals with the distribution into isomorphism and isotopism classes of the set $\mathcal{P}_{n,q}$ of n-dimensional pre-filiform Lie algebras over the finite field $\mathbb{F}_q$, with q a power prime, and the set $\mathcal{F}_n(\mathbb{K})$ of n-dimensional filiform Lie algebras over a field $\mathbb{K}$. We prove in particular the existence of n isotopism classes over $\mathcal{P}_{n,q}$ and five isotopism classes in $\mathcal{F}_6(\mathbb{K})$, whatever the base field $\mathbb{K}$ is. We also determine the distribution of the set $\mathcal{F}_7(\mathbb{K})$ into isotopism and isomorphism classes.
- In Chapter 4, the set $\mathcal{M}_n(\mathbb{K})$ of n-dimensional Malcev magma algebras over a finite field $\mathbb{K}$ is identified with algebraic sets defined by zero-dimensional radical ideals for which the computation of their reduced Gröbner bases makes feasible their enumeration and distribution into isomorphism and isotopism classes. Based on this computation and the classification of Lie algebras over finite fields given by De Graaf [149] and Strade [289], we determine the mentioned distribution for Malcev magma algebras of dimension $n \leq 4$. We also prove that every 3-dimensional Malcev algebra is isotopic to a Lie magma algebra. For $n = 4$, this assertion only holds when the characteristic of the base field $\mathbb{K}$ is not two.
- Chapter 5 deals with the distribution of the set $\mathcal{E}_n(\mathbb{K})$ of n-dimensional evolution algebras over a field $\mathbb{K}$ into isomorphism and isotopism classes. These

algebras constitute a type of genetic algebras whose description has a certain similarity with that of pre-filiform Lie algebras, which are introduced in Chapter 3, and whose distribution into isotopism classes is uniquely related with mutations in non-Mendelian Genetics. We focus in particular on the two-dimensional case, which is related to the asexual reproduction processes of diploid organisms. Specifically, we determine the distribution of the set $\mathcal{E}_2(\mathbb{K})$ into four isotopism classes, whatever the base field $\mathbb{K}$ is, and we characterize its isomorphism classes.

At the end of the manuscript, after the bibliographic references, we expose an index with the main terms and a glossary with the main notations that we use throughout our study. Finally, let us remark that this manuscript is largely based upon the following papers

- [122] Falcón OJ., Falcón RM., Núñez J. Isotopism and isomorphism classes of certain Lie algebras over finite fields. *Results Math.* 2015. In press. DOI: 10.1007/s00025 -015-0502-y.
- [123] Falcón OJ., Falcón RM., Núñez J., Pacheco A., Villar MT. Classification of Filiform Lie Algebras up to dimension 7 Over Finite Fields, *An. Sti. U. Ovid. Co. Mat.*, 2016; **2**: In press.
- [124] Falcón OJ., Falcón RM., Núñez J. A computational algebraic geometry approach to enumerate Malcev magma algebras over finite fields. *Math. Meth. Appl. Sci.* In press, 2016.
- [125] Falcón OJ., Falcón RM., Núñez J. Isomorphism and isotopism classes of filiform Lie algebras of dimension up to seven. Submitted. Available on ArXiv 1510.07066.
- Falcón OJ., Falcón RM., Núñez J., Pacheco A., Villar MT. Computation of isotopisms of algebras over finite fields by means of graph invariants. Submitted.
- Falcón OJ., Falcón RM., Núñez J. Classification of diploid asexual organisms by means of strongly isotopic evolution algebras defined over any field. Submitted.

Further, the results here exposed have already been pointed out in the following national and international conferences

- Falcón OJ., Falcón RM., Núñez J., Pacheco A., Villar MT. A faithful functor among algebras and graphs. 16^{th} *International Conference Computational and Mathematical Methods in Science and Engineering CMMSE 2016.* Rota, 2016.

- Falcón OJ., Falcón RM., Núñez J. Gene mutations in evolution algebras by means of strong isotopisms. 5^{th} *European Seminar on Computing.* Pilsen, Czech Republic, 2016.

- Falcón OJ., Falcón RM., Núñez J. Classifications of evolution algebras over finite fields. *III International School On Computer Algebra and its Applications.* Sevilla, 2016.

- Falcón OJ., Falcón RM., Núñez J. Distribution of low-dimensional Malcev algebras over finite fields into isomorphism and isotopism classess. 15^{th} *International Conference Computational and Mathematical Methods in Science and Engineering CMMSE 2015.* Rota, 2015.

- Falcón OJ., Falcón RM. and Núñez J. Isotopisms of Lie algebras. *Congreso de la RSME 2015.* Granada, 2015.

- Falcón OJ., Falcón RM. and Núñez J. Isotopismos de álgebras de Lie filiformes sobre cuerpos finitos. *Encuentro Andaluz de Matemática Discreta.* Sevilla, 2013.

- Falcón OJ., Falcón RM., Núñez J., Pacheco A. and Villar MT. Álgebras de Lie filiformes de baja dimensión sobre cuerpos finitos. *II Congresos de Jóvenes Investigadores.* Sevilla, 2013.

Chapter 1

A brief survey on isotopisms

Since the original manuscript of Albert [13] in 1942, a widely range of mathematicians has dealt with isotopisms of algebras in order to classify and enumerate distinct types of algebraic and combinatorial structures, some of them with applications in other fields as Quantum Mechanics and Cryptography, amongst others. In spite of its importance, there does not exist in the literature any survey that gathers together the origin and development of this theory. This chapter constitutes, therefore, a first approach in this regard.

1.1 The origin of the concept

The concept of isotopy of algebras was introduced by Albert [13] as a generalization of the concept of isomorphism that makes possible to gather together non-isomorphic algebras. He realized that the set of linear transformations generated by the right and left multiplication spaces of a non-associative algebra satisfies certain properties that are equivalent to many of the already known properties of associative algebras. Keeping this in mind, Albert generalized the concept of isomorphism of algebras in the next way[1].

Definition 1.1.1. *Two algebras $(A, \cdot)$ and $(A', \circ)$ over the same base field $\mathbb{K}$ are said to be* isotopic *if there exist three nonsingular linear maps f, g and h between A*

[1]Remark that, throughout the chapter, we follow the usual notation that is currently used and hence, this can differ from those to which we refer.

and A' such that

$$f(u) \circ g(v) = h(u \cdot v), \text{ for all } u, v \in A. \tag{1.1}$$

The triple (f, g, h) is called an isotopy *or* isotopism *between both algebras A and A'. It is an* autotopy *or* autotopism *if both algebras coincide. It is also said that A is an* isotope *of A' or that both algebras are* isotopic*. If $f = g = h$, then this constitutes an* isomorphism *(an* automorphism *if both algebras coincide).*

The set of isotopisms between two algebras is endowed of group structure with the componentwise composition of linear maps. This gives rise to the so-called isotopism groups *,* isomorphism groups*,* autotopism groups *and* automorphism groups *between algebras.*

This notion of isotopy of algebras was conceptually based on that of isotopy in Topology. Albert himself indicated that *the concept of isotopy was suggested by the work of Norman Steenrod who, in his study of homotopy groups in Topology, was led to study isotopy of division algebras*. Both of them, Albert and Steenrod, were appointed as assistant professors at the University of Chicago, where they coincided in the period 1939-1942 at the Department of Mathematics. Both had Salomon Lefschetz as common mentor: Lefschetz had introduced Albert to the theory of Riemann matrices during his postdoctoral year at Princenton in 1928-29 and had been the Ph.D. advisor of Steenrod, who defended his dissertation *"Universal Homology Groups"* in 1936. The relation between Homology Theory and division algebras, which had been during ten years one of the main goals of Albert, contributed even more to put his attention on the results of Steenrod in Topology. Let us review briefly this history.

1.1.1 Antecedents in Topology

The term *Analysis Situs*, firstly used by Gottfried Leibniz in the 18^{th} century, was the expression chosen by Henri Poincaré [252] in 1895 to entitle the article in which he would establish the fundamentals of Topology. Even if the term *topology* was introduced by Johann Benedict Listing in the 19^{th} century, it was Solomon Lefschetz in 1934 who introduced the current use of this term. In his original manuscript, Poincaré introduced in particular the term *homology* to deal with the relations and identities that exist among the manifolds that compose the boundary of a higher-dimensional manifold.[2] Based on this idea, he introduced the concept of *simplicial complex* by considering manifolds as generalized polyhedra consisting in n-dimensional cells. Besides, he defined the *fundamental group* of a manifold in a base point as the set of *contours* or paths that start and end in that base point. The product of two contours was just defined as the contour constituted by the former followed by the latter. As a consequence, the fundamental group was not commutative, unlike the algebraic relation that he defined to deal with the homologies of a manifold. Even a first tentative to establish an equivalence relation among contours was then exposed, it was in his *Fifth supplement to Analysis Situs* [253], published in 1904, when Poincaré indicated that two such contours are equivalent if there exists a continuous deformation between them without leaving the manifold. Although he did not expose the idea with a formal terminology, it was a first step towards the current concept of homotopy.[3] There were, however, Max Dehn and Poul Heegaard who introduced in Topology the terms homotopy and isotopy in their article *Analysis Situs* [97] of 1907, published in the *Enzyklopädie der Mathematischen Wissenschaften.* Nevertheless, the meanings that they gave to both concepts differ from the current ones and it is not until 1908 and 1911-12 that Heinrich Tietze [302] and Luitzen Brouwer [57, 58] introduced the ideas on which are based, respectively, the current definitions of isotopy and homotopy, which we expose in the following

Definition 1.1.2. *Two continuous maps f and g between two topological spaces X and Y are said to be* homotopic *if there exists a continuous map $H : X \times [0,1] \to Y$, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$, for all $x \in X$. The map H is then called a* homotopy *from f to g. It is said to be an* isotopy *if $H_t(x) = H(x,t)$ is an embedding for all $t \in [0,1]$.*

It was also Brouwer [58] who introduced in 1912 the notion of *homotopy class* by indicating that *"two transformations belong to the same class if they can be transformed continuously into each other"*. Immediately after, he proved that two continuous maps of the two-dimensional sphere $\mathbb{S}^2$ into itself belongs to the same class if and only if both maps have the same *degree*. Nowadays, it is equivalent to say that both maps are homologically equivalent. It was indeed the first time that homotopy and homology theories were explicitly connected.

[2] For a further analysis on the origin and development of the concept of manifold, we refer to the comprehensive survey of Scholz [279].

[3] The origin and development of the concept of homotopy is perfectly analyzed in the comprehensive survey of Ria Vanden Eyden [121].

The next stage of the history is reached in 1916, when the American Mathematical Society invited Oswald Veblen to give the *Cambridge Colloquium Lectures* of that year. Interested in topological questions since his paper of 1905 on the Jordan curve theorem [305], Veblen chose to lecture on Analysis Situs, as an *"introduction to the problem of discovering n-dimensional manifolds and characterizing them by means of invariants"*. The publication of these lectures in 1922 [306] constituted a *"systematic treatise on the elements of Analysis Situs"*, where the author revised, summarized and presented in a formal and comprehensive way all the concepts and results that were known on the subject. In particular, *"following the nomenclature introduced in the Dehn-Heegaard article"*, he gave in the fifth chapter a formal definition of the isotopy and homotopy of two continuous maps on a polyhedron, where one can already observe the fundamentals of Definition 1.1.2. Further, he exposed how homotopic 1-cells in a polyhedron determine the elements of the fundamental group, analyzed the homologies of a polyhedron and mentioned the term *homology group* as the commutative group whose identities correspond to these homologies. It was, however, Noether who realized in 1926 the importance of Group Theory in homology [167, 221]. Her ideas on the subject constituted the beginning of the theory on homology groups, whose fundamentals were developed by Leopold Vietoris [310], Heinz Hopf [169], Walther Mayer [220], Pável Alexandroff [23] and Eduard Čech [84].

In the *International Congress of Mathematicians* that was held in 1932, in Zürich, Čech introduced the concept of n^{th} *homotopy group* as the set of homotopy classes from the n-sphere $\mathbb{S}^n$ to a given topological space. That set could be endowed with group structure and coincided with the fundamental group when $n = 1$. Nevertheless, Alexandroff and Hopf observed that the group was Abelian for all $n > 1$ and thought that the concept did not provide any advantage with respect to that of homology group. Due to it, the work of Čech was only published as a paragraph of six lines [83] in the Proceedings of the Congress. Three years later, however, Hurewicz published four notes [177, 178, 179, 180] in which he rediscovered the concept of higher homotopy group and exposed the main properties, which would constitute in the future the base of a fundamental tool in Topology and Geometry. In the second note, he mentioned the independent work of Čech. Hurewicz also introduced [177] the notation $\pi_n(Y)$ to denote the n^{th} homotopy group of a given topological space Y. For each homotopy class of $\pi_n(Y)$, he indicated the existence of a base point $x_0 \in \mathbb{S}^n$ and a base point $y_0 \in Y$ such that $f(x_0) = y_0$, for all map f in the class. The set $\pi_n(Y)$ could be then endowed with group structure in the following way: Given two maps $f, g \in \pi_n(Y)$, the product fg is defined by mapping the equator of $\mathbb{S}^n$ to the base point y_0. The northern (respectively, southern) hemisphere is then mapped to $\mathbb{S}^n$ by collapsing the equator to a point and then using f (respectively, g) to map $\mathbb{S}^n$ to Y. In 1940, Hopf [172] proved that if there exists a continuous odd map of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ into $\mathbb{S}^{n-1}$, then n is a power of 2. As a consequence, he proved that the dimension of any real division algebra must be a power of 2. Moreover, every finite-dimensional real commutative division algebra is either 1- or 2-dimensional.

1.1.2 Antecedents in Algebra

The first time that Albert focused his attention on non-associative algebras was in 1934, when he studied the algebra $\mathfrak{M}_3^8$ of all three rowed Hermitian matrices with elements in the real non-associative algebra of Cayley numbers. That algebra was a singular case of the family of non-associative algebras used by Pascual Jordan, John Von Neumann and Eugene Wigner [192] in order to generalize quantum mechanics. In any case, the interest of Albert to classify algebras dates from 1927-28, when he defended his Master's Thesis *"A Determination of All Associative Algebras in Two, Three, and Four Units"* and his Ph. D. Thesis *"Algebras and their Radicals and Division Algebras"*. In the latter, under the supervision of Leonard Dickson as advisor, Albert obtained in the second half of his dissertation a classification of 16-dimensional associative division algebras on which he also based his first major paper [5]. Recall that a *division algebra* is an algebra where every nonzero element has a multiplicative inverse. In particular, the only one-dimensional division algebra over a given field is the field itself. The classification of associative division algebras was a main problem at that moment, because, according to the structure theorems established by Joseph H. M. Wedderburn [314], the structure of linear associative algebras is based on this classification. In order to better understand the origin of the theory of isotopisms of algebras, it is interesting to outline some remarkable aspects about the history of the classification of division algebras and the research of Albert on this topic. For a more comprehensive study on subject, we refer to the recent survey of Fenster [142] and the reference therein exposed.

Already in 1878, Ferdinand Frobenius [144] showed that quaternions and its subalgebras of reals and complex numbers constitute the only associative division algebras in the field of real numbers. Twenty years later, Hurwitz [181] proved that every real division algebra endowed with a non-degenerate quadratic form must be one-, two-, four- or eight-dimensional. At that moment, the only known non-associative division algebra was the Cayley algebra. However, it would not be until 1923 that it was published a paper of Hurwitz [182] where he proved that every real division algebra endowed with a non-degenerate quadratic is associative or the Cayley algebra.

In 1905, Wedderburn [313] proved that any finite associative division algebra is commutative and hence, a field. Some months later, in 1906, Dickson presented in the nineteenth regular meeting of the Chicago Section of the American Mathematical Society a contribution ([284], p. 442) where he constructed the first known family of associative division algebras of dimension a perfect square greater than one, which would subsequently become called *cyclic* due to the fact that any such an algebra over a field F contains a maximal commutative subfield S such that the Galois group $G(S/F)$ is cyclic. It was also Dickson [99, 100, 101] who constructed in the period 1906-08 the first known non-associative division algebras apart from the Cayley algebra. Apart from the mentioned result of Hurtwitz [182], during the next thirty years, the research community would mainly focused on associative division algebras and it would not be until 1935 that Dickson [105] resumed a methodological study of non-associative division algebras.

In 1914, Dickson [102] published a extended paper related to the previously mentioned contribution [284] where he studied the necessary conditions to construct these cyclic division algebras and focused on those ones of dimension 4 and 9. His theoretical results were immediately generalized for any dimension by Wedderburn [315], who also defined a *normal* division algebra as that one such that its only commutative elements are those of the ground field. This is equivalent to say that the algebra coincides with its center and hence, that the algebra is *central*. He observed that the dimension of a normal division algebra is a perfect square and that every cyclic division algebra is normal. Later, in 1921, Wedderburn [316] proved that any division algebra is the direct product of a field and a normal division algebra and that every nine-dimensional division algebra is cyclic. These results would be reviewed and extended by Dickson in the second appendix of his recognised book *Algebras and their Arithmetics* [103], published in 1923, where he determined all nine-dimensional associative division algebras. Four years later, Dickson, supported by Wedderburn, published the book *Algebren und ihre Zahlentheorie* [104], a revised and extended translation of his previously mentioned book, where he also determined all four-dimensional associative division algebras and proved that all of them are cyclic. Once one-, four- and nine-dimensional associative division algebras were determined, the next goal was to study those ones of dimension 16. It was achieved by Albert in his Ph. D. Thesis, where he classified all normal division algebras of such a dimension. Shortly after, he proved [5] that all these algebras belong to a family of division algebras, constructed by Francesco Cecioni [85], that are based on a non-cyclic abelian equation of degree four. Further, in 1930, Albert [6] proved that every 16-dimensional normal division algebra is cyclic and thus, the search of non-cyclic division algebras continued. In fact, it was not until 1932 that Albert [9] constructed, for the first time, non-cyclic normal division algebras. These algebras were based on the more general family of 16-dimensional associative algebras over a function field that had been previously studied by Richard Brauer [54] in 1930. After that, Albert would focus on the classification of normal division algebras over infinite modular fields. Specifically, he determined [10, 11] all normal division algebras of degree two, three and four over a field of characteristic two, three and two, respectively.

Brauer together with Herman Hasse and Emmy Noether constituted the German triumvirate that encouraged representation theory to deal with the arithmetics of linear algebras developed by Wedderburn and Dickson. Already in 1925, even before of the publication of the mentioned *Algebren und ihre Zahlentheorie*, which was a main first contact of many German mathematicians with the theory of the two American authors, Noether had exposed, for the first time, the possible use of representation theory to explain the structure theorems of Wedderburn [258]. With respect to the classification of division algebras, Brauer [53] reduced in 1929 the problem of determining all normal division algebras of order n^2 to the case where n is a power of a prime, and Hasse, who had began to investigate the structure of division algebras over p-adic fields in 1929-30, proved [162] in 1931 that every normal division algebra over a p-adic number field is cyclic. Further, in 1932, Brauer, Hasse and Noether [55] proved a result that had been conjectured by Dickson [103] in 1923 and had become a primordial goal for all the researchers on the subject. This was that every normal division algebra over an algebraic number field is cyclic. This result would be fundamental not only in the study and classification of associative (division) algebras, but also in that of non-associative ones. In this regard, Dickson [105] indicated in 1935 about that result that *this perfection of the theory of associative algebras justifies attention to non-associative algebras*. In his article, Dickson [105] defined new four-dimensional non-associative division algebras.

In the same year 1932, an alternative proof of Dickson's conjecture was published [22] by Albert together with Hasse, with whom Albert had started to keep correspondence since 1931. The controversial around both proofs [141, 258] would mark the character of Albert with respect to the representation theory. Although he recognized its importance and indicated [8] that that theory should be better known in America[4], it is also true that he always would explicitly expose in his subsequent work the advantages of using the methodology developed by Wedderburn, Dickson and himself in comparison of that of German authors. This fact can be clearly observed in distinct footnotes and comments of his articles [7, 22] and books [12]. In any case, what is important for the origin of the theory of isotopisms of algebras is the fact that, from that moment on, Albert would keep an eye on the theories developed by German mathematicians, who were indeed who introduced the concept of isotopy in Topology and who would apply Homology Theory in order to deal with the problem of the classification of division algebras.

1.2 The fundamentals of the theory

The fundamentals of the theory of isotopisms was originally developed not only by Albert, but also by the American mathematician Richard Bruck, who generalized in [60] the concept of isotopism from algebras to quasigroups. We review here some of the basic concepts and results that both authors exposed in their respective works.

An isotopism (f, g, h) between two algebras that are based on the same underlying set A of vectors is called *principal* if its third component h is the identity map on A. These two algebras are then said to be *principal isotopic*. To be (principal) isotopic constitutes an equivalence relation among algebras that enables us to distribute any set of algebras into *(principal) isotopism classes*. Albert proved that every isotope of an algebra is isomorphic to a principal isotope of the algebra. Principal isotopisms become, therefore, a useful way to distribute algebras according to their isotopism classes in those cases in which isomorphism implies equivalence. They also constitute a source for the study of possible isotopism invariants. Thus, for instance, Albert proved that isotopisms preserve right divisors of zero, zero algebras and simple algebras, whereas principal isotopisms preserve ideals. He also proposed necessary and sufficient conditions under which commutativity and alternativity of algebras are preserved by principal isotopisms. Albert focused then on isotopisms of algebras with unit elements (*unital algebras*), division algebras and Lie algebras. Thus, he proved that

[4]It was not until 1932 that Hasse [163] published the first paper in English where he reviewed the main results related to representation theory and its application in the theory of arithmetics of algebras.

- Every finite-dimensional unital algebra has a principal isotope which is simple and has neither left nor right ideals.
- A unital algebra A is associative if and only if all its isotopic unital algebras are associative and isomorphic to A.

Albert himself observed that this last result is not true in general if the algebras do not have unit elements. Thus, for instance, the three-dimensional algebra of basis $\{e_1, e_2, e_3\}$ that is linearly described by its non-zero products $e_1e_2 = -e_2e_1 = e_3$ is isotopic but not isomorphic to the algebra with the same basis such that $e_1e_1 = e_3$.

With respect to division algebras, Albert proved that

- Every division algebra is isotopic to a unital division algebra.
- Every real division algebra of order $n > 1$ is isotopic to a division algebra with unit element e and containing an element b such that $b^2 = -e$.
- Every real division algebra of order $n > 2$ is central simple.
- Every absolute-valued real finite-dimensional algebra has dimension 1, 2, 4 or 8 over the real field $\mathbb{R}$ and is either $\mathbb{R}$, the complex field $\mathbb{C}$, the real quaternion algebra $\mathbb{H}$, the real Cayley algebra $\mathbb{O}$, or a principal isotope of $\mathbb{H}$ and $\mathbb{O}$ defined by the product $xy = f(x)g(y)$, where f and g are orthogonal linear transformations and either f^{-1} is not a right multiplication or g^{-1} is not a left multiplication.

Finally, Albert also considered in his original manuscript the question as to whether principal isotopisms preserve Lie algebras. In this regard, he proved that

Lemma 1.2.1 ([13]). *A principal isotope A of a Lie algebra A' with respect to an isotopism (f, g, ϵ) is a Lie algebra if and only if the following two conditions hold for all $u, v, w \in A$.*

i. $f(u)g(v) = -f(v)g(u)$.

ii. $f(f(u)g(v))g(w) - f(f(u)g(w))g(v) - f(u)g(f(v)g(w)) = 0$. □

Shortly after, Bruck [61] introduced the concept of *isotopically simple algebra* as a simple algebra such that all their isotopic algebras are simple. He delved further into the study of isotopisms of division algebras, simple algebras and Lie algebras. With respect to the latter, he proved the next result.

Theorem 1.2.2. *[61] The following assertions hold.*

i. The Lie algebra of order $n(n-1)/2$, consisting of all skew-symmetric matrices, over any subfield of the field of all reals, under the multiplication $A \circ B = AB - BA$, is isotopically simple.

ii. The Lie algebra of order $n(n-1)$, consisting of all skew-hermitian matrices in any field $R(i)$ (where R is a subfield of the reals and $i^2 = -1$), under the multiplication $A \circ B = AB - BA$, is an isotopically simple algebra over R. □

1.2.1 Isotopisms of quasigroups

In 1943-44, Albert [14, 16] together with Bruck [60] extended the concept of isotopism from algebras to quasigroups. The term *quasigroup* was introduced in 1937 by Bernard Haussmann and Øystein Ore [164] to denote a nonempty set Q endowed with a product $\cdot$, such that if any two of the three symbols u, v and w in the equation $u \cdot v = w$ are given as elements of Q, then the third is uniquely determined as an element of Q. Every associative quasigroup is a *group* and every quasigroup endowed with unit element is a *loop*.

It was Bruck [61] who put together both theories on isotopisms of algebras and quasigroups. He introduced the concept of *quasigroup algebra* related to a quasigroup $(Q, \cdot)$ as an algebra of basis $\{e_u \mid u \in Q\}$ over a base field $\mathbb{K}$ such that $e_u e_v = h_{u,v} e_{uv}$, for all $u, v \in Q$, where $h_{u,v}$ is a non-zero element of $\mathbb{K}$. If $h_{u,v} = 1$, for all $u, v \in Q$, then the algebra was called a *quasigroup ring*. This was the starting point to generalize the concept of isotopism from algebras to quasigroups. In fact, the definition is completely similar: Two quasigroups $(Q, \cdot)$ and $(Q', \circ)$ are said to be *isotopic* if there exist three bijections f, g and h between Q and Q' such that $f(u) \circ g(v) = h(u \cdot v)$, for all $u, v \in Q$.

Unlike Albert, who focused on the study of isotopisms of algebras, Bruck dealt with isotopisms of distinct types of quasigroups like those ones endowed with the inverse property, totally symmetric quasigroups, Moufang loops or abelian quasigroups, amongst others. He focused on particular on isotopisms of loops [62, 63, 64, 65, 66], which were also considered by Albert himself [14, 16]. The next assertions hold in this regard.

- Every quasigroup is isotopic to a loop.
- A loop is isotopic to a group if and only they are isomorphic.
- Two groups are isotopic if and only if they are isomorphic.
- Every loop isotopic to a simple loop is simple.
- Isotopic loops have isomorphic centers.
- Every loop isotopic to a loop Q is isomorphic to a loop having precisely the same normal divisors as Q.
- Every loop isotopic to a loop with the inverse property is a Moufang quasigroup.
- Every abelian quasigroup is isotopic to an abelian group.

Before of the original works of Albert and Bruck, the underlying idea of isotopisms of quasigroups was already known for Latin squares. A *Latin square* of order n is an $n \times n$ array with elements chosen from a set of n symbols, such that each symbol occurs precisely once in each row and each column. This constitutes, therefore, the multiplication table of a finite quasigroup of n elements. In practice, the set of symbols of any Latin square $L = (l_{ij})$ of order n is usually considered to be the set $[n] = \{1, \ldots, n\}$. The *orthogonal array representation* of L is defined as the set $O(L) = \{(i, j, l_{ij}) \in [n]^3\}$. Isotopisms of quasigroups are, therefore, equivalent to permutations of rows, columns and symbols of Latin squares. These permutations give rise to new Latin squares (and thus, to new quasigroups), which are said to be *isotopic* to the initial one. Specifically, if S_n denotes the symmetric group on the set $[n]$, then the isotopic Latin square of L according to $\Theta = (\alpha, \beta, \gamma) \in S_n^3$ is denoted by L^Θ and satisfies that $O(L^\Theta) = \{(\alpha(i), \beta(j), \gamma(p_{i,j})) \mid (i, j, p_{ij}) \in O(L)\}$. Fisher and Yates [143] and Norton [239] called them *transformations* of a Latin square.

Another concept that was inherited from the theory of Latin squares to that of isotopisms of algebras and quasigroups was derived from the notion of adjugate:

Let π be a permutation in S_3 and let L be a Latin square of order n. Fisher and Yates [143] and Norton [239] defined the *adjugate* L^π as the Latin square of order n whose orthogonal array representation is the set $O(L^\pi) = \{(l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)}) \mid (l_1, l_2, l_3) \in O(L)\}$. Thus, there exist six adjugates related to each Latin square: $L^{\mathrm{Id}} = L$, $L^{(12)} = L^t$, $L^{(13)}$, $L^{(23)}$, $L^{(123)}$ and $L^{(132)}$. Norton [239] called *specie* to any transformation of an adjugate of a Latin square. Nowadays, the adjugates of a Latin square are called *conjugates* or *parastrophes*. The permutation π is called *parastrophism* and the composition of an isotopism and a parastrophism of Latin squares is called a *paratopism*. Hence, species of Latin squares are currently called *paratopes*. To be isotopic, parastrophic or paratopic are equivalence relations among Latin squares.

The concept of conjugacy of algebras was already considered by Shaw [282] in 1915: Let A be an algebra over a base field $\mathbb{K}$ of basis $\{e_1, \ldots, e_n\}$, which is described by the non-zero products $e_i e_j = \sum_{k=1}^{n} a_{ijk} e_k$, for all $i, j \leq n$ and some numbers $a_{ijk} \in \mathbb{K}$ that are called the *structure constants* of the algebra A. If these structure constants are interchanged by a common permutation of their indices, that is, if each structure constant a_{ijk} is replaced by $a_{ijk}, a_{jik}, a_{ikj}, a_{kij}, a_{jki}$ or a_{kji}, then the resulting algebra was called *parastrophic* or *conjugate* of the algebra A. Each one of the six possible changes of indices determines a *parastrophism* of algebras. It was Bruck [61] who introduced the problem of whether these six parastrophic algebras are isotopic or not. The same problem was proposed almost simultaneously by Etherington [118], who illustrated it with several types of algebras and quasigroups.

1.3 The development of the theory

Since the manuscripts of Albert and Bruck, a wide amount of authors have dealt with the distribution into isotopism classes of distinct types of algebras, quasigroups and related combinatorial structures. Let us finish this section by pointing out some of the references in this regard.

1.3.1 Division algebras

Recall that a *division algebra* is an algebra in which left- and right- division is possible. We have already exposed the interest that Albert had on division algebras and the fact that he introduced the concept of isotopisms in order to deal with the problem of classifying division algebras. This interest was further extended to the rest of the scientific community once Albert [19] proved in 1960 that two division algebras are isotopic if and only if their corresponding projective planes are isomorphic. Thus, for instance, at the beginning of the 1960s, Albert himself [18, 19, 20, 21] and Hughes [174, 175] studied the autotopism groups of certain types of non-associative division algebras that coordinatize finite projective planes. Shortly after, Sandler [275, 276] generalized these algebras and discussed the question of distributing them into isotopism classes. Particularly, he indicated that two such division algebras are related to isomorphic projective planes if and only if the algebras are isotopic. Together with Oehmke, Sandler [243] focused in particular on the study of the autotopism group of Jordan division algebras. In 1975, Kaplansky [193] conjectured that every three-dimensional non-associative division algebra is isotopic to generalized twisted field. This was shortly after proved by Menichetti [234]. In 1981, Benkart et al. [36, 45] gathered together some results and possible applications in Physics on isotopisms of real division algebras. More recently, Darpö and Dieterich [93] in 2007 used the isotopes of $\mathbb{C}$ to define an isotopism among real commutative division algebras that becomes an equivalence of categories. Schwarz [280] in 2010 classified small, non-associative division algebras up to isotopy. He reduced the classification problem to a case study solved by means of computation. In 2011, Deajim and Grant [95] dealt with the distribution of three-dimensional non-associative division algebras over p-adic fields under isotopism classes. Finally, in 2015, Darpö and Pérez Izquierdo [94] have imposed certain conditions to the group of autotopisms of a non-associative division algebra under which this algebra is isotopic to a classical real division algebra. This gives rise to a distribution into isomorphism classes of this type of algebras.

1.3.2 Semifields

An algebraic structure whose isotopisms have widely been analyzed and whose definition is similar to that of division algebra is the so-called (pre)semifield. Specifically, a *presemifield* is a set endowed with an abelian addition with unit element 0 and a distributive multiplication for which the left- and right-division are always possible. If the multiplication has unit element, then this is a *semifield*. Two (pre)semifields $(S,+,\cdot)$ and $(S',+,\circ)$ are isotopic if and only if their multiplicative structures $(S,\cdot)$ and $(S',\circ)$ are isotopic. The existence of both additive and multiplicative units, together with the distributive property, makes possible to consider any semifield as a *planar ternary ring* in the sense introduced by Hall [155, 156]. In 1965, Knuth [194] generalized the concept of isotopism from seminets to ternary rings. Specifically, given two ternary rings T and T', a triple (F, G, H) of one-one mappings between them is called an *isotopism* if $H(0) = 0$ and $(F(a)+G(b)) \times H(c) = H((a+b) \times c)$, for all $a, b, c \in T$. A generalization of this definition was exposed by Zotov [320] in 1976. Knuth described a simple method to construct all ternary rings that are isotopic to a given one. He also described a method to generate 24 semifields from any given one. In 1974, Soubeyran [290] described some derivable semifields that are not isotopic to any of the derivable semifields of Knuth. Besides, this result has been recently taken into account by Ball and Lavrauw [35], who have proved that there are at most five non-isotopic semifields among the 24 semifields described by Knuth. This last author also obtained the upper bound of $(n-1)^2$ elements as the number of non-isomorphic ternary rings that are isotopic to a given ternary ring of n elements. Besides, Knuth obtained necessary and sufficient conditions under which the isotopic ternary rings are related to isomorphic projective planes. Finally, he dealt with the possibility of considering non-linear isotopisms for constructing semifields. In 2008, Lavrauw gave a one-to-one correspondence between the isotopism classes of a finite semifield and the orbits of the action a subgroup of index two of the automorphism group of a Segre variety on subspaces of maximum dimension skew to a determinantal hypersurface.

In 1972, Ganley [145] gave a characterization of finite semifields that are isotopic to a commutative semifield. This was generalized for *quasifields*[5] by Jha [190] in 2005. In 1998, Spille and Pieper-Seier [291] dealt with the distribution of commutative semifields into strongly isotopism classes[6]. Coulter and Henderson [88] continued with the topic and obtained certain conditions under which two commutative presemifield are strongly isotopic. They also proved that every commutative presemifield of odd order can be uniquely described by a planar polynomial. This enables them to classify all planar functions describing presemifields isotopic to a finite field or to Albert's commutative twisted fields. This line of research continues currently active, as we can observe in the references [68, 229, 319].

We finish this part by indicating several works about the explicit construction and distribution of finite semifields into isotopism classes on which distinct authors have been focusing in the last years. Some references in this regard are [34, 207,

[5]Quasifields are defined in a similar way to semifields, but only the left- or right-distributive property is imposed.

[6]A *strong isotopism* consists of an isotopism of the form (f, f, h), where the first and second component coincide.

208, 209, 210, 211, 227, 228, 259, 260, 317].

1.3.3 Alternative algebras

An algebra $(A,\cdot)$ is said to be *alternative* if it holds the identities $(u,u,v)=0=(u,v,v)$, where (u,v,w) denotes the *associator product* $(u\cdot v)\cdot w-u\cdot(v\cdot w)$, for all $u,v,w\in A$. In general, isotopisms do not preseve alternativity. In 1966, Schaffer [277] proved that this assertion is, however, true whenever the algebras are unitary. A similar result had been already proved by Bruck [63] for loops. In 1971, McCrimmon [230] defined the u,v-*homotope* of $(A,\cdot)$ as the algebra $(A,\cdot_{u,v})$ such that $x\cdot_{u,v}y=(x\cdot u)\cdot(v\cdot y)$, for all $x,y\in A$. This constitutes a generalization of the notion of isotopism, which had been in fact already considered by Albert himself [13] for general linear algebras, by Bruck [63] for loops and, indeed, by Schaffer [277] for alternative algebras. In the particular case in which the algebra $(A,\cdot)$ is unitary and the two elements u and v are regular, the algebra $(A,\cdot_{u,v})$ is isotopic to $(A,\cdot)$. McCrimmon [230] proved that the u,v-homotope of an alternative algebra is also alternative and that every isotopism of an unitary alternative algebra $(A,\cdot)$ can be reduced to an u,v-homotopism. The isotopic alternative algebra is also unitary in this case and its unit is the inverse of $u\cdot v$. More recent works on this topic have been developed by Allison [24], Babikov [33] and Pchelintsev [246]. The first author focused on isotopisms of alternative algebras with involution. Particularly, it is proved that simple alternative algebras are isomorphic if and only if they are isotopic. The second one analyzed some conditions under which two $u,1$-isotopic alternative algebras are isomorphic, whereas the latter proved that $u,1$-isotopisms of unitary alternative algebras preserve primality.

1.3.4 Jordan algebras

A *Jordan algebra* is a commutative algebra A that holds the identity $u(u^2v)=u^2(uv)$, for all $u,v\in A$. Even if there exist distinct manuscripts that refer to isotopisms of Jordan algebras, it is Petersson [247] the only author who, based on the previously mentioned works of Oehmke and Sandler [243], has dealt with isotopisms of Jordan algebras by following the classical notion introduced by Albert. Specifically, Petersson proved that two isotopic finite-dimensional Jordan algebras of characteristic distinct from two, at least one of which is semisimple, are always isomorphic. The rest of references dealing with isotopisms of Jordan algebras refers to a slightly different notion of isotopism that was introduced by Jacobson [185] in 1962. This last author characterized unitary Jordan algebras by means of standard involutions based on what he called *a-isotopies*. As we have just mentioned, this notion of isotopism differs from that introduced by Albert. Specifically, given a regular element u of an unitary Jordan algebra $(A, \cdot)$, Jacobson defined the *u-isotopic* Jordan algebra $(A, \cdot_u)$ from the product $x \cdot_u y = (x \cdot u) \cdot y + (u \cdot y) \cdot x - (x \cdot y) \cdot u$, for all $x, y \in A$. He called this isotopy because the conceptual idea derives from that described by Albert himself [13]. Observe in this regard the similarity between this product and the previously mentioned homotopic product $\cdot_{u,1}$ described by McCrimmon [230] for unitary alternative algebras. In this last reference, McCrimmon himself studied the relation among the use of this homotopic product in the study of quadratic Jordan algebras. The same relation has been dealt with much more recently by Petersson [250], who has related both notions of isotopisms of Albert and Jacobson by means of what he has defined as the *structure group* of an alternative algebra.

In 1963, Jacobson studied the way in which u-isotopisms act on the generic norm of an unitary Jordan algebra. This last aspect was also considered by McCrimmon [232] and much more recently by Loos [216], who considered a similar question for generically algebraic Jordan algebras in order to compute the generic minimum polynomial of an u-isotope. McCrimmon [231] also proved that this type of isotopism preserves inner ideals of unitary Jordan algebras. In 1978, Petersson [249] gave sufficient conditions for two reduced exceptional simple Jordan algebras to be u-isotopic. Shortly after, Petersson together with Racine [251] proved that all u-isotopes of a first construction exceptional Jordan division algebra are isomorphic. They also asked for the question of whether two Albert algebras (a specific type of exceptional central simple Jordan algebras) are u-isotopic under certain invariants. This question was affirmatively answered by Thakur [299] in 1999.

1.3.5 Lie algebras

A *Lie algebra* is an anti-commutative algebra A that holds the so-called *Jacobi identity* $u(vw) + v(wu) + w(uv) = 0$, for all $u, v, w \in A$. There barely exists any result about isotopisms of Lie algebras apart from the results of Albert and Bruck that we have previously cited as Lemma 1.2.1 and Theorem 1.2.2. Thus, for instance, Jiménez-Gestal and Pérez-Iquierdo [191] have recently studied the underlying relation that exists among the isotopisms of a finite-dimensional real division algebra and the Lie algebra of its ternary derivations. More recently, Allison et al. [28, 29] have studied isotopisms of a type of graded Lie algebras called Lie tori, but their notion of isotopism differ from the classical notion of Albert.

1.3.6 Genetic algebras

In 1966, Bertrand [46] exposed some basic concepts and results that make possible to endow Genetics with an algebraic structure theory. Particularly, she dealt with the possibility of using isotopisms of distinct types of algebras. The same year, Holgate [168] dealt with special train algebras that can be used to describe certain modes of inheritance in terms of mixture of chromosome and chromatid segregation. He studied in particular the distribution of these algebras into isotopism classes. In 1985, Ringwood [254] exposed an outline with the distinct non-associative algebras that arise in Genetics. He indicated in particular how to use isotopisms of algebras in order to treat selection. Shortly after, in 1987, Campos and Holgate [73] realized the important role that isotopisms play in the study of genetic algebras. They proved in particular that those algebras related to a polyploidy and chromosome segregation which correspond to distinct mutation rates are related by principal isotopisms. Besides, algebras reflecting varying degrees of double reduction are special isotopisms, which preserve the powers of their corresponding nilideals. As a consequence, the analysis of evolutionary operators can be reduced to the study of one algebra of each isotopism class.

1.3.7 Other algebras

Other algebras whose isotopisms have been analyzed since the original manuscript of Albert are quasi-composition algebras [248, 288], absolute valued algebras [17, 91, 226, 257] and structural algebras [27, 25, 26], amongst others.

1.3.8 Quasigroups and related structures

Apart from Albert and Bruck, a pair of authors that were specially prolific in the initial stage of the theory of isotopisms of quasigroups were Sade and Belousov. The former exposed distinct results on isotopisms [263, 269], autotopisms [262, 267, 268, 270] and paratopisms [264, 265, 266, 271, 272] of quasigroups, whereas the latter adapted the concept of isotopism for quasifields [37], proved that all quasigroups with balanced identity are isotopic to groups, defined the concept of orthogonal and crossed isotopism [43] and focused on isotopisms and paratopisms of several types of quasigroups [38, 39, 40, 41, 42, 44].

Some other authors who contributed to the consolidation of this theory were Evans [119, 120], Stein [292, 293], Artzy [31, 32], Aczél [1, 2, 3], Osborn [245], Robinson [255, 256], Falconer [136, 137] and Lindner [213]. At the same time, the theory of isotopisms of quasigroups was being extended to other algebraic structures like semifields [194], semigroups of functions [217] or projective planes and ternary rings [69, 219, 294]. The theory has continued being matured until nowadays with new results on isotopisms of quasigroups [47, 184, 188, 237, 238, 281, 283, 287], Hadamard cocycles [86] or alternating forms [173]. An application of isotopic quasigroups in Cryptography has also been developed [151, 285, 307] to evaluate security of cryptographic systems.

Isotopism and paratopisms have also been used to study and classify distinct combinatorial structures like frequency squares [51], frequency cubes [52], William designs [298], Moufang nets [157] or hypercubes [224]. However, without any doubt, Latin squares continue being the combinatorial structures par excellence in the theory of isotopisms. In 1977, Dénes and Keedwell [98] gathered together all the known theory about Latin squares, in particular, those results related to isotopisms. Since then, isotopisms and paratopisms have been used to study parities [109, 189, 196], critical sets [4, 108], cycle switches [311], transversals [222] and symmetries [183, 295, 312] of Latin squares. The number of isotopism and conjugate classes of Latin squares has been obtained for order up to 11 [176, 195, 223]. The set of autotopisms of Latin squares have also been analyzed [59, 79, 128, 129, 130, 133, 134, 233, 286, 296], with possible applications in the study of secret sharing schemes in Cryptography [126, 127, 297]. More recently, it has been studied the set of autotopisms of partial Latin squares [131] and partial Latin rectangles [132, 135].

Chapter 2

Computing isotopisms of algebras

We expose in this chapter those results in Computational Algebraic Geometry and Graph Theory that we use throughout the manuscript in order to compute the isotopism classes of each type of algebra under consideration in the subsequent chapters. We describe in particular a pair of graphs that enable us to define faithful functors between finite-dimensional algebras over finite fields and these types of graphs. These functors map isomorphic and isotopic algebras to isomorphic graphs. Reciprocally, any pair of isomorphic graphs is uniquely related to a pair of algebras so that there exists a multiplicative map between them. Previously, we recall some basic concepts and expose some preliminary results on isotopisms of algebras.

2.1 Isotopisms of algebras

Let us gather together here some basic concepts and results on the theory of isotopisms of algebras that we use throughout the manuscript. We refer to the original paper of Albert [13] for more details about this topic.

Two n-dimensional algebras $(A, \cdot)$ and $(A', \circ)$ defined over the same field $\mathbb{K}$ are said to be *isotopic* if there exist three non-singular linear transformations f, g and h from A to A' such that

$$f(u) \circ g(v) = h(u \cdot v), \text{ for all } u, v \in A. \tag{2.1}$$

Hereafter, in order to simplify the notation and whenever no confusion arises, we do not write explicitly the products $\cdot$ and $\circ$. That is, we write the previous identity as $f(u)g(v) = h(uv)$, for all $u, v \in A$. The triple (f, g, h) is an *isotopism* between the algebras A and A'. If h is the identity transformation Id, then the isotopism is called *principal*. If $f = g$, then this is called a *strong isotopism* and the algebras are said to be *strongly isotopic*. If $f = g = h$, then the isotopism constitutes an *isomorphism*, which is denoted by f instead of (f, f, f). To be isotopic, strongly isotopic or isomorphic are equivalence relations among algebras. Hereafter, we denote these three relations, respectively, as $\sim$, $\simeq$ and $\cong$. Further, throughout the manuscript, any non-singular linear transformation between two n-dimensional algebras that is linearly defined from a permutation in the symmetric group S_n of the indices of their corresponding basis vectors is identified with this permutation. Thus, for instance, given two three-dimensional algebras of respective bases $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$, the isomorphism $(23) \in S_3$ is linearly defined from mapping e_1 to e'_1, e_2 to e'_3 and e_3 to e'_2.

Let A be an n-dimensional algebra over a field $\mathbb{K}$ and let $\{e_1, \ldots, e_n\}$ be a basis of this algebra. The *structure constants* of A are the numbers $c_{ij}^k \in \mathbb{K}$ such that

$$e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k, \text{ for } 1 \leq i, j \leq n. \tag{2.2}$$

Let A and A' be two n-dimensional isotopic algebras over the same field $\mathbb{K}$ and let $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ be their respective bases. Any isotopism (f, g, h) between both algebras is uniquely determined by their respective sets of structure constants c_{ij}^k and c'^k_{ij} and the corresponding entries of the nonsingular matrices $F = (f_{ij})$, $G = (g_{ij})$ and $H = (h_{ij})$ that are respectively related to the maps f, g and h. Here, $\alpha(e_i) = \sum_{j=1}^n \alpha_{ij} e'_j$, for each $\alpha \in \{f, g, h\}$. The next equalities follow in particular from the coefficients of each basis vector e_m in the expression $f(e_i)g(e_j) = h(e_i e_j)$.

$$\sum_{k,l=1}^{n} c'^m_{kl} f_{ik} g_{jl} = \sum_{s=1}^{n} c_{ij}^s h_{sm}, \text{ for all } i, j \leq n. \tag{2.3}$$

If the structure constants of an algebra are all of them zeros, then this algebra is called *abelian*.

Lemma 2.1.1. *The n-dimensional abelian algebra is not isotopic to any other n-dimensional algebra.*

Proof. Let (f, g, h) be an isotopism between an n-dimensional non-abelian algebra of basis $\{e_1, \ldots, e_n\}$ and the n-dimensional abelian algebra. Let $i, j \leq n$ be such that $e_i e_j \neq 0$. Then, $0 = f(e_i)g(e_j) = h(e_i e_j) \neq 0$, which is a contradiction. $\square$

Proposition 2.1.2. *There exists two isomorphism and isotopism classes of one-dimensional algebras: the abelian and that described by the product of basis vectors* $e_1e_1 = e_1$.

Proof. Every one-dimensional non-abelian algebra over a field $\mathbb{K}$ is described by a product of basis vectors $e_1e_1 = ae_1$, where $a \in \mathbb{K} \setminus \{0\}$. The linear transformation f that maps e_1 to ae_1 is an isomorphism between this algebra and that described by the product $e_1e_1 = e_1$. The result follows then from Lemma 2.1.1. $\square$

Let S be a vector subspace of an algebra A. The *left* and *right annihilators* of S in A are respectively defined as the sets

$$\mathrm{Ann}_{A^-}(S) = \{u \in A \mid uv = 0, \text{ for all } v \in S\}. \tag{2.4}$$

$$\mathrm{Ann}_{A^+}(S) = \{u \in A \mid vu = 0, \text{ for all } v \in S\}. \tag{2.5}$$

The intersection of both sets is called the *annihilator* of S in A. It is defined as

$$\mathrm{Ann}_{A}(S) = \{u \in A \mid uv = vu = 0, \text{ for all } v \in S\}. \tag{2.6}$$

Lemma 2.1.3. *Let (f, g, h) be an isotopism between two n-dimensional algebras A and A'. Let S be a vector subspace of A. Then,*

a) $f(\mathrm{Ann}_{A^-}(S)) = \mathrm{Ann}_{A'^-}(g(S))$. *Hence,* $\dim \mathrm{Ann}_{A^-}(S) = \dim \mathrm{Ann}_{A'^-}(g(S))$.

b) $g(\mathrm{Ann}_{A^+}(S)) = \mathrm{Ann}_{A'^+}(f(S))$. *Hence,* $\dim \mathrm{Ann}_{A^+}(S) = \dim \mathrm{Ann}_{A'^+}(f(S))$.

c) $f(\mathrm{Ann}_{A^-}(S)) \cap g(\mathrm{Ann}_{A^+}(S)) = \mathrm{Ann}_{A'}(f(S) \cap g(S))$.

Proof. Let us prove assertion (a). Assertion (b) follows similarly and assertion (c) is an immediate consequence of (a) and (b). Let $u \in g(S)$ and $v \in f(\mathrm{Ann}_{A^-}(S))$. Then, $vu = f(f^{-1}(v))g(g^{-1}(u)) = h(f^{-1}(v)g^{-1}(u)) = h(0) = 0$, because $g^{-1}(u) \in S$ and $f^{-1}(v) \in \mathrm{Ann}_{A^-}(S)$. Hence, $f(\mathrm{Ann}_{A^-}(S)) \subseteq \mathrm{Ann}_{A'^-}(g(S))$. Now, let $u \in$

$\mathrm{Ann}_{A'^-}(g(S))$ and $v \in S$. From the regularity of f, we have that $h(f^{-1}(u)v) = ug(v) = 0$. The regularity of h involves that $f^{-1}(u)v = 0$. Thus, $u \in f(\mathrm{Ann}_{A^-}(S))$ and hence, $\mathrm{Ann}_{A'^-}(g(S)) \subseteq f(\mathrm{Ann}_{A^-}(S))$. Finally, the dimension of $\mathrm{Ann}_{A'^-}(g(S))$ and that of $\mathrm{Ann}_{A^-}(S)$ coincide from the regularity of f. $\square$

Proposition 2.1.4. *Let (f, g, h) be an isotopism between two n-dimensional algebras A and A'. Then,*

a) $f(\mathrm{Ann}_{A^-}(A)) = \mathrm{Ann}_{A'^-}(A')$. *Hence,* $\dim \mathrm{Ann}_{A^-}(A) = \dim \mathrm{Ann}_{A'^-}(A')$.

b) $g(\mathrm{Ann}_{A^+}(A)) = \mathrm{Ann}_{A'^+}(A')$. *Hence,* $\dim \mathrm{Ann}_{A^+}(A) = \dim \mathrm{Ann}_{A'^+}(A')$.

c) $f(\mathrm{Ann}_{A^-}(A)) \cap g(\mathrm{Ann}_{A^+}(A)) = \mathrm{Ann}_{A'}(A')$.

Proof. The result follows straightforward from Lemma 2.1.3 and the regularity of f and g. $\square$

Hereafter, given a vector subspace S of an algebra A, we define the vector subspace $SA = \{uv \mid u \in S \text{ and } v \in A\}$. The *derived algebra* of the algebra A is then defined as the subalgebra

$$A^2 = AA = \{uv \mid u, v \in A\} \subseteq A. \tag{2.7}$$

Let $u \in A$. The *adjoint action* of u in A is the map $\mathrm{ad}_u : A \to A^2$ such that $\mathrm{ad}_u(v) = uv$, for all $v \in A$. Further, the *derived series* of the algebra A is defined as

$$\mathcal{C}_1(A) = A \supseteq \mathcal{C}_2(A) = A^2 \supseteq \ldots \supseteq \mathcal{C}_k(A) = (\mathcal{C}_{k-1}(A))^2 \supseteq \ldots \tag{2.8}$$

The algebra A is said to be *solvable* if there exists a positive integer m such that $\mathcal{C}_{m+1}(A) \equiv 0$. The smallest such an integer is called the *solvability index* of the algebra.

Similarly, the *lower central series* of the algebra A is defined as

$$\mathcal{C}^1(A) = A \supseteq \mathcal{C}^2(A) = A^2 \supseteq \ldots \supseteq \mathcal{C}^k(A) = (\mathcal{C}^{k-1}(A))A \supseteq \ldots \tag{2.9}$$

The algebra A is said to be *nilpotent* if there exists a positive integer m such that $\mathcal{C}^{m+1}(A) \equiv 0$. The smallest such an integer is called the *nil-index* of the algebra.

The solvability index, the nil-index together and the dimensions of each vector subspace $\mathcal{C}_k(A)$ and $\mathcal{C}^k(A)$ are all of them preserved by isomorphisms. Particularly, the *type* of an algebra A of nil-index m is defined as the sequence

$$\{\dim A/\mathcal{C}^2(A), \dim \mathcal{C}^2(A)/\mathcal{C}^3(A), \ldots, \dim \mathcal{C}^{m-1}(A)/\mathcal{C}^m(A)\}. \tag{2.10}$$

This is also preserved by isomorphisms of algebras. Nevertheless, for isotopisms we can only assure the next result.

Lemma 2.1.5. *Let (f, g, h) be an isotopism between two n-dimensional algebras A and A'. Then, $h(A^2) = A'^2$ and $\dim(A^2) = \dim(A'^2)$.*

Proof. The regularity of f and g involves that $f(A) = g(A) = A'$. Hence, $A'^2 = f(A)g(A) = h(A^2)$ and the result follows then from the regularity of h. □

Let us finish the section with the description of partial-magma algebras, a type of algebras whose distribution into isotopism and isomorphism classes is dealt with in distinct parts of this manuscript.

A *partial magma* is a finite set endowed with a partial binary operation. Throughout the manuscript we suppose this set to be $[n] = \{1, \ldots, n\}$ and we denote the operation as $\cdot$. In this case, n is the *order* of the partial magma. If the operation $\cdot$ is defined in all $[n] \times [n]$, then the pair $([n], \cdot)$ is a *magma*. Two (partial) magmas $([n], \cdot)$ and $([n], \circ)$ are said to be *isotopic* if there exist three permutations α, β and γ in the symmetric group S_n such that

$$\alpha(i) \circ \beta(j) = \gamma(i \cdot j), \text{ for all } i, j \leq n \text{ such that } i \cdot j \text{ exists.} \tag{2.11}$$

If $\alpha = \beta = \gamma$, then the (partial) magmas are said to be *isomorphic*. The triple (α, β, γ) constitute an *isotopism* of (partial) magmas (an *isomorphism* if $\alpha = \beta = \gamma$).

Similarly to the concepts of quasigroup algebra and quasigroup ring that was introduced by Bruck [60], and which we have already exposed in Chapter 1, we say that an n-dimensional algebra over a field $\mathbb{K}$ is a *partial-magma algebra* if there exists a basis $\{e_1,\ldots,e_n\}$ of the algebra and a partial magma $([n],\cdot)$ such that

$$e_ie_j = \begin{cases} c_{ij}e_{i\cdot j}, \text{ if } i\cdot j \text{ exists,}\\ 0, \text{ otherwise,}\end{cases} \tag{2.12}$$

for each pair of elements $i,j\leq n$ and some non-zero structure constant $c_{ij}\in\mathbb{K}\setminus\{0\}$. If all these non-zero structure constant are equal to 1, then the algebraic structure is called a *partial-magma ring*. We denote this algebra as $A^{\cdot}$ and we say that such an algebra is *based on* the partial magma $([n],\cdot)$. If the pair $([n],\cdot)$ is a magma, then the algebra is said to be a *magma algebra* (a *magma ring* if all the non-zero structure constants are 1). Partial-magma algebras constitute, therefore, a natural generalization of the concept of quasigroup algebra, once the condition of being based on a quasigroup is replaced by that of being based on a partial magma. In this regard, a *partial quasigroup* is defined as a partial magma such that, if the equations $ix=j$ and $yi=j$, with $i,j\in[n]$, have solutions for x and y in $[n]$, then these solutions are unique. This is a quasigroup if both equations have always unique solutions.

We prove in the next result that isotopic (isomorphic, respectively) partial magmas give rise to isotopic (isomorphic, respectively) partial-magma rings.

Lemma 2.1.6. *Two partial-magma rings are isotopic (isomorphic, respectively) if their respective partial magmas on which they are based are isotopic (isomorphic, respectively).*

Proof. Let $A^{\cdot}$ and A° be two partial-magma rings based, respectively, on two isotopic partial magmas $([n],\cdot)$ and $([n],\circ)$. Let $\{e_1,\ldots,e_n\}$ and $\{e'_1,\ldots,e'_n\}$ be the respective bases of the algebras $A^{\cdot}$ and A° and let (f,g,h) be an isotopism between the partial magmas $([n],\cdot)$ and $([n],\circ)$. For each $\alpha\in\{f,g,h\}$, let us define the map $\overline{\alpha}(e_i)=e'_{\alpha(i)}$. Then,

$$\overline{f}(e_i)\overline{g}(e_j)=e'_{f(i)}e'_{g(j)}=e'_{f(i)\circ g(j)}=e'_{h(i\cdot j)}=\overline{h}(e_{i\cdot j})=\overline{h}(e_ie_j).$$

From linearity, the triple $(\overline{f},\overline{g},\overline{h})$ determines, therefore, an isotopism between the algebras $A^{\cdot}$ and A°. If $f=g=h$, then this constitutes an isomorphism. □

The reciprocal of Lemma 2.1.6 is not true in general. Thus, for instance, the two partial magmas $([2],\cdot)$ and $([2],\circ)$ that are respectively described by the non-zero products $1 \cdot 1 = 1$ and $1 \circ 1 = 1 = 2 \circ 1$ are not isotopic. Nevertheless, the partial-magma rings $A^{\cdot}$ and A°, with respective bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$, are isotopic by means of the isotopism $(f, \mathrm{Id}, \mathrm{Id})$, where the linear transformation f is described by $f(e_2) = e'_2 - e'_1$.

The previous remark gives rise to the open problem of distributing certain types of partial-magma rings into isotopism and isomorphism classes for which the corresponding classification of partial magmas on which they are based is known. This is the case, for instance, of partial quasigroups, which we deal with throughout this chapter. Specifically, every partial quasigroup of order n constitutes the multiplication table of a *partial Latin square* of order n. That is, an $n \times n$ array in which each cell is either empty or contains one element chosen from the set $[n]$, such that each symbol occurs at most once in each row and in each column. If there are not empty cells, then this is a *Latin square*, which constitutes in turn the multiplication table of a quasigroup of order n. Every isotopism of a (partial) quasigroup is uniquely related to a permutation of the rows, columns and symbols of the corresponding (partial) Latin square. At the end of Chapter 1 we have already indicated that the distribution of Latin squares into isomorphism and isotopism classes is known for order up to 11 [176, 195, 223]. Nevertheless, it is only known the distribution of partial Latin squares into isotopism classes for order up to 6 [131, 135]. Throughout the next sections we study which ones of the known non-isotopic classes of partial Latin squares of a given order give rise to isotopic classes of partial-quasigroup rings. In this regard, observe that, for order 1, it is straightforward verified that there exists only two one-dimensional partial-quasigroup rings: the abelian and that one described by the product $e_1 e_1 = e_1$. From Lemma 2.1.1, they constitute distinct isotopism classes.

2.2 Computational Algebraic Geometry

We expose here some basic concepts and results on Computational Algebraic Geometry that we use throughout the manuscript to deal with the distribution of algebras into isotopism and isomorphism classes. For more detail about this topic we refer to the monographs of Cox, Little and O'Shea [89, 90].

2.2.1 Preliminaries

Let X and $\mathbb{K}[X]$ be, respectively, the set of n variables $\{x_1, \ldots, x_n\}$ and the related multivariate polynomial ring over a field $\mathbb{K}$, endowed with the *standard grading* induced by the degree of polynomials, that is, $\mathbb{K}[X] = \bigoplus_{0 \leq d} \mathbb{K}_d[X]$, where each $\mathbb{K}_d[X]$ is the set of homogeneous polynomials in $\mathbb{K}[X]$ of degree d. Any monomial of $\mathbb{K}[X]$ has the form $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ and can be identified with the lattice point $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$.

A *total order* $\leq$ on $\mathbb{K}[X]$ is a binary relation among the polynomials of $\mathbb{K}[X]$ such that, given three polynomials $p, q, r \in \mathbb{K}[X]$, it is verified that

- If $p \leq q$ and $q \leq p$, then $p = q$.
- If $p \leq q$ and $q \leq r$, then $p \leq r$.
- $p \leq q$ or $q \leq p$.

A *monomial term order* $\prec$ on $\mathbb{K}[X]$ is a total order on the set of monomials in $\mathbb{K}[X]$ such that

- $1 \prec \mathbf{x}^a$, for all $a \in \mathbb{N}^n \setminus \{\mathbf{0}\}$.
- If $\mathbf{x}^a \prec \mathbf{x}^b$ for some $a, b \in \mathbb{N}^n$, then $\mathbf{x}^{a+c} \prec \mathbf{x}^{b+c}$, for all $c \in \mathbb{N}^n$.

Thus, for instance, the *lexicographic order* $\prec_{\text{lex}}$ is a monomial term order defined on $\mathbb{K}[X]$ such that, given two monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ and $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} \ldots x_n^{b_n}$ in $\mathbb{K}[X]$, we have that $\mathbf{x}^{\mathbf{a}} \prec_{\text{lex}} \mathbf{x}^{\mathbf{b}}$ if there exists a positive integer $m \leq n$ such that $a_i = b_i$, for all $i \leq m$ and $a_m < b_m$.

A subset I of $\mathbb{K}[X]$ is called an *ideal* of $\mathbb{K}[X]$ if

- $0 \in I$.
- Given two polynomials $p, q \in I$, it is verified that $p + q \in I$.
- Given two polynomials $p \in I$ and $q \in R$, it is $p \cdot q \in I$.

The ideal I is said to be *prime* if the next two conditions hold.

- I is a proper subset of $\mathbb{K}[X]$.
- If $pq \in I$, then $p \in I$ or $q \in I$.

The *height* of a prime ideal I is defined as the supremum of all positive integer $n \in \mathbb{N}$ so that there is a chain $I_0 \subset \ldots I_n = I$ of distinct prime ideals. The *Krull dimension* of $\mathbb{K}[X]$ is the supremum of all the heights of all its prime ideals.

The *ideal generated by* a finite set of polynomials $p_1, \ldots, p_m \in \mathbb{K}[X]$ is defined as

$$\langle\, p_1, \ldots, p_m \rangle \;=\; \{p \in \mathbb{K}[X] \mid p = \sum_{i=1}^{m} q_i \cdot p_i, \;\text{ where } q_i \in \mathbb{K}[X], \;\text{ for all } i \leq m\}. \quad (2.13)$$

Two polynomials $p, q \in \mathbb{K}[X]$ are *congruent modulo* an ideal I of $\mathbb{K}[X]$ if $p - q \in I$. This is an equivalence relation. The *polynomial quotient ring* $\mathbb{K}[X]/I$ is then defined as the set of equivalence classes of $\mathbb{K}[X]$ with respect to this relation.

The *algebraic set* defined by an ideal I of $\mathbb{K}[X]$ is the set $\mathcal{V}(I)$ of common zeros of all the polynomials in I, that is,

$$\mathcal{V}(I) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n \mid p(a_1, \ldots, a_n) = 0, \;\text{ for all } p \in I\}. \quad (2.14)$$

The ideal I is *zero-dimensional* if $\mathcal{V}(I)$ is finite. It is *radical* if every polynomial $p \in \mathbb{K}[X]$ belongs to I whenever there exists a positive integer $m \in \mathbb{N}$ such that $p^m \in I$. Let $\prec$ be a monomial term order on the set of monomials in $\mathbb{K}[X]$. The largest monomial of a polynomial in I with respect to $\prec$ is its *leading monomial* $\mathrm{LM}(f)$, whose coefficient in f is the *leading coefficient* $\mathrm{LC}(f)$. The *leading term* of f is the product $\mathrm{LT}(f) = \mathrm{LC}(f) \cdot \mathrm{LM}(f)$. The ideal generated by all the leading monomials of I is the *initial ideal* $I_\prec$. Those monomials that are not in $I_\prec$ are called *standard monomials* of I. Regardless of the monomial term order, if the ideal I is zero-dimensional, then the Krull dimension of the polynomial quotient ring $\mathbb{K}[X]/I$ coincides with the number of standard monomials of I. This number is always greater than or equal to the number of points of $\mathcal{V}(I)$. The equality holds whenever I is radical.

The Krull dimension of $\mathbb{K}[X]/I$ and the points of $\mathcal{V}(I)$ can be completely determined by means of Gröbner bases. A *Gröbner basis* of I with respect to $\prec$ is any subset G of polynomials in I whose leading monomials generate the initial ideal

$I_{\prec}$. This is *reduced* if all its polynomials are monic and no monomial of a polynomial in G is generated by the leading monomials of the rest of polynomials in the basis. There exists only one reduced Gröbner basis of the ideal I, which becomes an optimal way to count their number of standard monomials. Its decomposition into finitely many disjoint subsets, each of them being formed by the polynomials of a triangular system of polynomial equations makes also possible to enumerate the elements of the algebraic set $\mathcal{V}(I)$ [166, 212, 235]. The reduced Gröbner basis of an ideal I can always be computed from Buchberger's algorithm [67]. Similar to the Gaussian elimination on linear systems of equations, this consists of a sequential multivariate division of polynomials, also called *reduction* or *normal form* computation, which is based on the construction of the so-called *S-polynomials* $\mathrm{S}(f,g) = \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))(f/\mathrm{LT}(f) - g/\mathrm{LT}(g))$, for all $f, g \in I$, where lcm denotes the least common multiple. In particular, a subset G of polynomials in I is a Gröbner basis if $\mathrm{S}(f,g)$ reduces to zero after division by the polynomials in G, for all $f, g \in G$. Buchberger's algorithm can be implemented on ideals defined over any field, but its involved exact arithmetic becomes faster on finite fields. It is due to the large numbers that appear in general as intermediate coefficients during the computation of the reduced Gröbner basis and which constitute a major factor in the computational cost of the algorithm. Derived from Buchberger's algorithm, a pair of more efficient direct methods are the algorithms F_4 and F_5 [138, 139] and the algorithm *slimgb* [56]. The latter is based on F_4 and reduces the computation time and the memory usage by keeping small coefficients and short polynomials during the sequential division of polynomials. All these algorithms are more efficient over the rational field or a finite field.

In any case, the computation of a reduced Gröbner basis is always extremely sensitive to the number of variables [160, 161, 204, 205, 206]. Particularly, Lakshman described a radical basis algorithm that enables us to ensure the next result that follows straightforward from Theorem 3 in [204].

Theorem 2.2.1. *The reduced Gröbner basis of any radical zero-dimensional ideal defined over the rational field $\mathbb{Q}$ under any monomial term order can be computed in polynomial time $d^{O(n)}$, where d is the maximal degree of the polynomials of the ideal and n is the number of variables.* □

With respect to the complexity time that is required to compute a reduced Gröbner basis over a finite field $\mathbb{F}_q$, with q a prime power, the next result was proved by Gao [146].

Theorem 2.2.2 ([146], Proposition 4.1.1)**.** *The complexity time that is required by the Buchberger's algorithm in order to compute the reduced Gröbner bases of an ideal* $\langle\, p_1, \ldots, p_m, p_1^q - p_1, \ldots, p_m - p_m\,\rangle$ *defined over a polynomial ring* $\mathbb{F}_q[x_1, \ldots, x_n]$*, where* $p_1, \ldots, p_m$ *are polynomials given in sparse form and have longest length* l*, is* $q^{O(n)} + O(m^2 l)$*. Here, sparsity refers to the number of monomials.* □

2.2.2 Enumeration of algebras

The concepts and results on Computational Algebraic Geometry that we have just exposed can be straightforward implemented to enumerate n-dimensional algebras over a field $\mathbb{K}$ and distribute them into isotopism and isomorphism classes. In order to deal with the problem of enumerating n-dimensional algebras, let us define the set of variables

$$\mathfrak{C}_n = \{\mathfrak{c}_{ij}^k \mid i, j, k \leq n\}, \tag{2.15}$$

These variables play the role of the structure constants of an n-dimensional algebra $\mathfrak{A}$ over $\mathbb{K}[\mathfrak{C}_n]$, with basis $\{e_1, \ldots, e_n\}$, such that

$$e_i e_j = \sum_{k=1}^{n} \mathfrak{c}_{ij}^k e_k, \text{ for all } i, j \leq n. \tag{2.16}$$

The algebraic structure related to any ideal I of $\mathbb{K}[\mathfrak{C}_n]$ is then uniquely related to a set of n-dimensional algebras over $\mathbb{K}$ whose structure constants constitute zeros of any polynomial of such an ideal. Specifically, each zero $(c_{11}^1, \ldots, c_{nn}^n) \in \mathcal{V}(I)$ constitutes the structure constants of an n-dimensional algebra, with basis $\{e_1, \ldots, e_n\}$, such that $e_i e_j = \sum_{k=1}^n c_{ij}^k e_k$, for all $i, j \leq n$. Keeping this in mind, distinct ideals of the polynomial ring $\mathbb{K}[\mathfrak{C}_n]$ can be described in order to determine the set of n-dimensional algebras with a given property. In the subsequent chapters, we describe distinct ideals in this regard in order to deal with finite-dimensional Lie algebras, Malcev algebras and evolution algebras. In case of dealing with finite fields, the computation of the reduced Gröbner basis of each one of these ideals enables us to enumerate explicitly the corresponding algebras. In order to illustrate this fact with an example, we focus now on the particular case of dealing with finite-dimensional partial-magma algebras over a finite field.

Theorem 2.2.3. *The set of* n*-dimensional partial-magma algebras over the finite field* $\mathbb{F}_q$*, with* q *a prime power, is identified with the algebraic set defined by the next* *ideal of* $\mathbb{F}_q[\mathfrak{C}_n]$

$$I = \langle\, \mathfrak{c}_{ij}^{k}\mathfrak{c}_{ij}^{k'} \mid i,j,k,k' \leq n; k < k' \,\rangle.$$

Besides,

$$|\mathcal{V}(I)| = \dim_{\mathbb{F}_q}(\mathbb{F}_q[\mathfrak{C}_n]/I).$$

Proof. The generators of the ideal I involve each zero $(c_{11}^1, \ldots, c_{nn}^n) \in \mathcal{V}(I)$ to constitute the structure constants of an n-dimensional partial-magma algebra. Further, since the field $\mathbb{F}_q$ is finite, the ideal I is zero-dimensional and the algebraic set $\mathcal{V}(I)$ is a finite subset of $\mathbb{F}_q^{n^3}$. From Proposition 2.7 of [89], the ideal I is also radical, because, for each $i,j,k \leq n$, the unique monic generator of $I \cap \mathbb{F}_q[\mathfrak{c}_{ij}^k]$ is the polynomial $(\mathfrak{c}_{ij}^k)^q - \mathfrak{c}_{ij}^k$, which is intrinsically included in each ideal of $\mathbb{F}_q[\mathfrak{C}_n]$ and is square-free. As a consequence, the number of zeros in the algebraic set $\mathcal{V}(I)$ coincides with the Krull dimension of the quotient ring $\mathbb{F}_q[\mathfrak{C}_n]/\mathfrak{I}_n$ over $\mathbb{F}_q$. □

Corollary 2.2.4. *Let* $\mathbb{F}_q$ *be a finite field, with* q *a prime power. The time complexity that is required by the Buchberger's algorithm in order to compute the reduced Gröbner basis of the ideal in Theorem 2.2.3 is* $q^{O(n^3)} + O(n^8)$.

Proof. The result follows straightforward from Theorem 2.2.2, once we observe that all the generators of the ideal in Theorem 2.2.3 are sparse in $\mathbb{F}_q[\mathfrak{C}_n]$. More specically, here, the number of variables is n^3, the number of generators of the ideal under consideration that are not of the form $(\mathfrak{c}_{ij}^k)^q - \mathfrak{c}_{ij}^k$ is $\frac{n^3(n-1)}{2}$ and the maximal length of these generators is 1. □

In the development of this manuscript, computations of reduced Gröbner bases, algebraic sets and Krull dimensions are done by means of the open computer algebra system for polynomial computations Singular [96]. All the procedures that are described throughout the manuscript in this regard have been included in the library *isotopism.lib*, which is available online at

`http://personales.us.es/raufalgan/LS/isotopism.lib`.

Their correctness and termination are based on those of the algorithm *slimgb* [56] for the computation of reduced Gröbner bases and hence, on those of Buchberguer's algorithm [67]. All the computations that are exposed throughout the manuscript are implemented in a system with an *Intel Core i7-2600, with a 3.4 GHz processor and 16 GB of RAM*. Particularly, the enumeration of distinct types of n-dimensional algebras over a given finite field has been implemented in the procedure *algebra*, whose pseudocode is described in Algorithm 1.

Algorithm 1 Enumeration of finite-dimensional algebras of a certain type and given structure constants.

procedure ALGEBRA$(n, q, C, \mathrm{alg}, \mathrm{opt})$)
    The base ideal I is initialized depending on the argument *alg*.
    for $i \leftarrow 1, size(C)$ **do**
        $I = I + (\mathfrak{c}_{C_{i1}C_{i2}}^{C_{i3}} - C_{i4})$;
    end for
    $I = slimgb(I)$;
    if opt $= 1$ **then**
        return $|\mathcal{V}(I)|$
    else
        if opt $= 2$ **then**
            return $A \in \mathcal{V}(I)$
        else
            if opt $= 3$ **then**
                return $\mathcal{V}(I)$
            end if
        end if
    end if
end procedure

This procedure *algebra* receives as input

- The dimension n of the required algebras.
- The order q of the finite field.
- A list C formed by tuples (i, j, k, c_{ij}^k) that indicates some non-zero structure constants that must contain the required algebras.
- A list *alg* of positive integers that enables us to select the type of algebra in which we are interested. Particularly, the types that have been implemented are

1. Partial-magma algebras.
2. Lie algebras.
3. Malcev algebras.
4. Evolution algebras.

- A positive integer *opt* that enables us to select the output that generates the procedure. Particularly, the procedure indicates
  - the number of algebras that satisfy the imposed conditions, whenever *opt* = 1.
  - the structure constants of an algebra verifying the imposed conditions, whenever *opt* = 2.
  - the complete list of algebras verifying the imposed conditions, whenever *opt* = 3.

In order to check the efficiency of the procedure *algebra* and Algorithm 1, we deal with the computation of the number of n-dimensional partial-magma algebras over the finite field $\mathbb{F}_q$. The description of this type of algebra involves this cardinality to be equal to $(n(q-1))^{n^2}$. Table 2.1 expose the run time and memory usage that are required to deal with small orders. Both measures of computation efficiency fit positive exponential models even for dimension $n = 4$. Nevertheless, the main goal of any study of algebras is not the computation of all these algebras, but only their distribution into isomorphism and isotopism classes. The next subsection indicates how to use Computational Algebraic Geometry in this regard.

2.2.3 Classification of algebras

The distribution of finite-dimensional algebras over finite fields into isotopism and isomorphism classes can also be analyzed by making use of Computational Algebraic Geometry. To this end, we define the next three set of variables

$$\mathfrak{F}_n = \{\mathfrak{f}_{ij} \mid i,j \leq n\}, \tag{2.17}$$

$$\mathfrak{G}_n = \{\mathfrak{g}_{ij} \mid i,j \leq n\}, \tag{2.18}$$

$$\mathfrak{H}_n = \{\mathfrak{h}_{ij} \mid i,j \leq n\}. \tag{2.19}$$

n	q	Number of partial-magma algebras	Run time	Used memory
2	2	81	0 s	0 MB
	3	625	0 s	0 MB
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	101	1632240801	0 s	0 MB
3	2	262144	0 s	0 MB
	3	40353607	0 s	0 MB
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	101	20281424743202871242701	0 s	0 MB
4	2	152587890625	823 s	0 MB
	3	-	>2 hours	-

Table 2.1: Computation of n-dimensional partial-magma algebras over the finite field $\mathbb{F}_q$.

The variables of these three sets play the respective role of the entries in the nonsingular matrices related to a possible isotopism between two n-dimensional algebras over $\mathbb{K}$ of respective structure constants c_{ij}^k and c'^k_{ij}. Specifically, similarly to (2.3), we have that

$$\sum_{k,l=1}^{n} \mathfrak{f}_{ik}\mathfrak{g}_{jl}c'^m_{kl} = \sum_{s=1}^{n} c_{ij}^s \mathfrak{h}_{sm}, \text{ for all } i,j,m \leq n. \tag{2.20}$$

The next results follow similarly to Theorem 2.2.3 and constitute the fundamentals on which we base the distribution of finite-dimensional algebras over finite fields into isotopism and isomorphism classes that are exposed in the subsequent chapters.

Theorem 2.2.5. *The isomorphism group between two n-dimensional algebras A and A' over a finite field $\mathbb{F}_q$, with q a prime power, respective basis $\{e_1,\ldots,e_n\}$ and $\{e'_1,\ldots,e'_n\}$ and respective structure constants c_{ij}^k and c'^k_{ij}, is identified with the algebraic set defined by the next ideal of $\mathbb{F}_q[\mathfrak{F}_n]$*

$$I^{\mathrm{Isom}}_{A,A'} = \langle \sum_{k,l=1}^{n} \mathfrak{f}_{ik}\mathfrak{f}_{jl}c'^m_{kl} - \sum_{s=1}^{n} c_{ij}^s \mathfrak{f}_{sm} \mid i,j,m \leq n \rangle + \langle \det(F)^{q-1} - 1 \rangle,$$

where F denotes the matrix of entries $\{\mathfrak{f}_{ij} \mid i,j \leq n\}$. Besides,

$$|\mathcal{V}(I^{\text{Isom}}_{A,A'})| = \dim_{\mathbb{F}_q}(\mathbb{F}_q[\mathfrak{F}_n]/I^{\text{Isom}}_{A,A'}).$$

□

Corollary 2.2.6. *The complexity time that is required by the Buchberger's algorithm in order to compute the reduced Gröbner basis of the ideal in Theorem 2.2.5 is $q^{O(n^2)} + O(n^6 n!)$.* □

Theorem 2.2.7. *The isotopism group between the two algebras of Theorem 2.2.5 is identified with the algebraic set defined by the next ideal of $\mathbb{F}_q[\mathfrak{F}_n \cup \mathfrak{G}_n \cup \mathfrak{H}_n]$*

$$I^{\text{Isot}}_{A,A'} = \langle \sum_{k,l=1}^{n} \mathfrak{f}_{ik}\mathfrak{g}_{jl}c'^{m}_{kl} - \sum_{s=1}^{n} c^{s}_{ij}\mathfrak{h}_{sm} \mid i,j,m \leq n \rangle + \langle \det(M)^{q-1} - 1 \mid M \in \{F,G,H\} \rangle,$$

where F, G and H denote, respectively, the matrices of entries $\{\mathfrak{f}_{ij} \mid i,j \leq n\}$, $\{\mathfrak{g}_{ij} \mid i,j \leq n\}$ and $\{\mathfrak{h}_{ij} \mid i,j \leq n\}$. Besides,

$$|\mathcal{V}(I^{\text{Isot}}_{A,A'})| = \dim_{\mathbb{F}_q}(\mathbb{F}_q[\mathfrak{F}_n \cup \mathfrak{G}_n \cup \mathfrak{H}_n]/I^{\text{Isot}}_{A,A'}).$$

□

Corollary 2.2.8. *The complexity time that is required by the Buchberger's algorithm in order to compute the reduced Gröbner basis of the ideal in Theorem 2.2.7 is $q^{O(3n^2)} + O(n^6 n!)$.* □

We have implemented Theorems 2.2.5 and 2.2.7 in the procedure *isoAlg*, which has been included in the previously mentioned library *isotopism.lib*. Having as output the number of isomorphisms or that of isotopisms between two given n-dimensional algebras A and A' over the finite field $\mathbb{F}_q$, with q a prime power, this procedure receives as input

1. The dimension n of both algebras.
2. The order q of the finite field.

3. A list $C1$ formed by tuples (i, j, k, c_{ij}^k) that indicates the non-zero structure constants of the algebra A.

4. A list $C2$ formed by tuples $(i, j, k, {c'}_{ij}^k)$ that indicates the non-zero structure constants of the algebra A'.

5. A positive integer $opt \leq 2$ that enables us to use the ideal $I_{A,A'}^{\mathrm{Isot}}$ if $opt = 1$, or the ideal $I_{A,A'}^{\mathrm{Isom}}$ if $opt = 2$.

Example 2.2.9. *We have made use of the procedure* isoAlg *to determine the distribution of two-dimensional partial quasigroup rings over the finite field $\mathbb{F}_2$ into isotopism and isomorphism classes. In this example we focus in particular on the pair of partial quasigroup rings that are respectively related to the partial Latin squares*

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \quad \textit{and} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

These two partial Latin squares are not isotopic because isotopisms preserve the number of filled cells. Nevertheless, their related partial quasigroup rings over $\mathbb{F}_2$, with respective bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ and which are respectively described by the products

$$\begin{cases} e_1e_1 = e_1, \\ e_1e_2 = e_2 = e_2e_1. \end{cases} \quad \textit{and} \quad \begin{cases} e'_1e'_1 = e'_1 = e'_2e'_2, \\ e'_1e'_2 = e'_2 = e'_2e'_1. \end{cases}$$

are isotopic. To see it, we made use of the procedure isoAlg *with the parameters*

$$n = 2, \quad q = 2, \quad C1 = \{(1,1,1,1), (1,2,2,1), (2,1,2,1)\},$$

$$C2 = \{(1,1,1,1), (1,2,2,1), (2,1,2,1), (2,2,1,1)\}, \quad \mathrm{opt} = 1.$$

In 0 *seconds, our system computes the existence of four isotopisms between these two partial quasigroup rings. One of this isotopisms is, for instance, the isomorphism f related to the matrix*

$$F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

that is, such that $f(e_1) = e'_1$ and $f(e_2) = e'_1 + e'_2$. In fact, if we consider the parameter opt $=$ *2, the procedure* isoAlg *ensures us the existence of f as the unique possible isomorphism.* $\triangleleft$

In practice, in those cases in which the run time required for the computations involved in Theorems 2.2.5 and 2.2.7 becomes excessive, it is recommendable to eliminate the generators of the corresponding ideal that are referred to the determinants of the matrices F, G and H. This reduces the complexity time to $q^{O(n^2)} + O(n^8)$ and $q^{O(3n^2)} + O(n^8)$, respectively, and gives enough information to analyze a case study on which base the possible isomorphisms and isotopisms between two given algebras. The next example illustrates this fact by focusing on the possible isotopisms that there exist over any field between the two partial-quasigroup rings that appear in Example 2.2.9.

Example 2.2.10. *The implementation of the procedure* isoAlg *enables us to ensure that, whatever the base field is, the reduced Gröbner basis of the ideal in Theorem 2.2.7 related to the isotopism group between the two partial-quasigroup rings of Example 2.2.9 holds in particular that*

$$\begin{cases} 2\mathfrak{h}_{22}^3 = 0, \\ \mathfrak{h}_{21}^2 + \mathfrak{h}_{22}^2 = 0. \end{cases}$$

If the characteristic of the base field is not two, then $\mathfrak{h}_{21} = \mathfrak{h}_{22} = 0$. This involves H to be singular and hence, these two partial-quasigroup rings are not isotopic. Otherwise, it is straightforward verified that the linear transformation f that is indicated in Example 2.2.9 constitutes an isomorphism between both rings for every base field of characteristic two. $\lhd$

Theorem 2.2.11. *The set of two-dimensional non-abelian partial-quasigroup rings is distributed into six isotopism classes.*

Proof. A case study based on a similar reasoning to that exposed in the previous example enables us to ensure the result. In particular, if the characteristic of the base field is not two, then the six isotopism classes under consideration are those related to the next partial Latin squares of order 2

1	

1	
	1

1	
2	

1	2

1	2
2	

1	2
2	1

Otherwise, if the characteristic of the base field is two, then the isotopism classes related to the last two partial Latin squares coincide. In this case, the next partial

Latin square corresponds to the sixth isotopism class

1	
	2

If the characteristic of the base field is not two, the partial-quasigroup ring related to this partial Latin square is isotopic to that related to the unique Latin square of the previous list. □

Theorems 2.2.5 and 2.2.7 can also be used to determine the distribution into isotopism and isomorphism classes of a given set of finite-dimensional algebras over a finite field. Algorithm 2 is introduced in this regard.

Algorithm 2 Computation of the isomorphism (isotopism, respectively) classes of a set of finite-dimensional algebras over a finite field.

Require: A set S of n-dimensional algebras over a finite field.
Ensure: C, the set of isomorphism (isotopism, respectively) classes of S.

$C = \emptyset$.
while $S \neq \emptyset$ **do**
  Take $A \in S$.
  $S := S \setminus \{A\}$.
  $C := C \cup \{A\}$.
  for $A' \in S$ **do**
    if $|\mathcal{V}(I_{A,A'}^{\text{Isom}})| > 0$ ($|\mathcal{V}(I_{A,A'}^{\text{Isot}})| > 0$, respectively) **then**
      $S := S \setminus \{A'\}$.
    end if
  end for
end while
return C.

Example 2.2.12. *It is known [131] that there are* 2*,* 8 *and* 81 *distinct isotopism classes of partial Latin squares of respective orders* 1 *to* 3*. We have made use of the procedure* isoAlg *to determine in particular those distinct isotopism classes that give rise to isotopic partial-quasigroup rings over the finite field* $\mathbb{F}_2$*. The run time has been* 761 *seconds in our system. Specifically, we have obtained that there exist* 2*,* 7 *and* 72 *distinct isotopism classes of partial-quasigroup rings of respective dimensions* 1 *to* 3*. The implementation of Algorithm 2 enables us to determine those isotopism* *classes of partial Latin squares that give rise to the same isotopism class of partial-quasigroup rings. Order* 1 *is immediate. The only isotopism class that disappears from partial Latin squares of order* 2 *to two-dimensional partial-quasigroup rings is that exposed in Example 2.2.9. Finally, for order* 3*, the next nine pairs of non-isotopic partial Latin squares give rise to isotopic partial-quasigroup rings*

$$\begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&&\\ \hline &&\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&1&\\ \hline &&\\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&&\\ \hline &&1\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&1&\\ \hline &&1\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&&\\ \hline &&3\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&1&\\ \hline &&3\\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&\\ \hline &1&\\ \hline 3&&\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&1&\\ \hline 3&&\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&\\ \hline &1&3\\ \hline &&\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&1&\\ \hline &&\\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&\\ \hline &1&\\ \hline 3&&2\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&1&\\ \hline 3&&1\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&&3\\ \hline &&1\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&1&3\\ \hline &&1\\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&\\ \hline &1&3\\ \hline 3&&\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&\\ \hline 2&1&3\\ \hline 3&&\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&\\ \hline &1&3\\ \hline 3&&2\\ \hline\end{array}\quad \textit{and} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&1&\\ \hline 3&&1\\ \hline\end{array}$$

$\triangleleft$

In practice, it is interesting to introduce some isomorphism and isotopism invariants of algebras that enable us to reduce the number of reduced Gröbner bases to be computed in the development of Algorithm 2. To this end, we describe in the next section a pair of graphs, whose isomorphism invariants give rise to isotopism and isomorphism invariants of algebras.

2.3 Faithful functors among algebras and graphs

Graph Theory has revealed to be an interesting tool to deal with distinct aspects on the study of algebras. Thus, for instance, we can mention the so-called Dynkin diagrams for simple finite dimensional Lie algebras [75, 158]. More recently, Carriazo et al. [74] proposed to use weighted digraphs that can be identified with certain families of Lie algebras (see also [80]). There also exist some studies in which Lie algebras are associated with distinct types of graphs. For example, Dani and Mainkar [92] defined a class of nilpotent Lie algebras related to a type of graph for which it has resulted that two such Lie algebras are isomorphic if and only if their associated graphs are equivalent [218]. Graph Theory has also been used to study and classify other types of algebras as Leibniz algebras [81], finitely generated algebras [82] or evolution algebras [114, 241, 242]

Nevertheless, to the best of the author knowledge, the problem of identifying a faithful functor that relates the category of algebras with that of graphs remains still open. Both categories are referred with respect to their corresponding isomorphisms among algebras and graphs. Based on a proposal of McKay et al. [223] for identifying isomorphisms of Latin squares[1] with isomorphism of vertex-colored graphs, we describe here a pair of families of graphs that enable us to find a faithful functor between finite-dimensional algebras over finite fields and these graphs. Previously, let us recall some basic concepts on Graph Theory.

A *graph* is a pair $G = (V, E)$ formed by a set V of points or *vertices* and a set E of lines or *edges* formed by subsets of two vertices of V. The *degree* of a vertex $v \in V$ is the number $d(v)$ of edges containing this vertex. A graph is said to be *vertex-colored* if there exists a partition into color sets of its set of vertices. The color of a vertex v is denoted as $\mathrm{color}(v)$. An *isomorphism* between two vertex-colored graphs $G = (V, E)$ and $G' = (V', E')$ is any bijective map f between the set of vertices V and V' that preserves collinearity and such that $\mathrm{color}(f(v)) = \mathrm{color}(v)$, for all $v \in V$.

Let $L = (l_{ij})$ be a Latin square of order n that is the multiplication table of a quasigroup $([n], \cdot)$. McKay et al. [223] defined the vertex-colored graph $G_2(L)$ with

[1]Recall that a Latin square of order n is an $n \times n$ array with elements chosen from a set of n symbols, such that each symbol occurs precisely once in each row and each column.

$n^2 + 3n$ vertices

$$\{r_i \mid i \leq n\} \cup \{c_i \mid i \leq n\} \cup \{s_i \mid i \leq n\} \cup \{t_{ij} \mid i, j \leq n\},$$

where each of the four subsets (related to the rows (r_i), columns (c_i), symbols (s_i) and cells (t_{ij}) of the Latin square L) has a different color, and $3n^2$ edges

$$\{r_i t_{ij}, c_j t_{ij}, s_{i \cdot j} t_{ij} \mid i, j \leq n\}\}.$$

They also defined the vertex-colored graph $G_1(L)$ from the graph $G_2(L)$ by adding 3 additional vertices $\{R, C, S\}$ and $3n$ additional edges $\{Rr_i, Cc_i, Ss_i \mid i \leq n\}$. Here, there are three colors: one for $\{R, C, S\}$, one for $\{r_i, c_i, s_i \mid i \leq n\}$ and one for the rest of vertices. Finally, they defined the vertex-colored graph $G_3(L)$ from the graph $G_2(L)$ by adding $3n$ additional edges $\{r_i c_i, c_i s_i, r_i s_i \mid i \leq n\}$. Here, the color of the vertices coincides with those of $G_1(L)$. These authors proved (Theorem 6 in [223]) that two Latin squares L_1 and L_2 of the same order are paratopic (respectively, isotopic or isomorphic) if and only if the graphs $G_1(L_1)$ and $G_1(L_2)$ (respectively, $G_2(L_1)$ and $G_2(L_2)$, and $G_3(L_1)$ and $G_3(L_2)$) are isomorphic. Figure 2.1 shows an example of the three graphs related to the next Latin square of order 2.

$$L = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

We have used distinct styles ($\circ$, $\blacktriangle$, $\blacktriangleright$, $\blacktriangleleft$ and $\bullet$) in the vertices of the graphs to represent their colors.

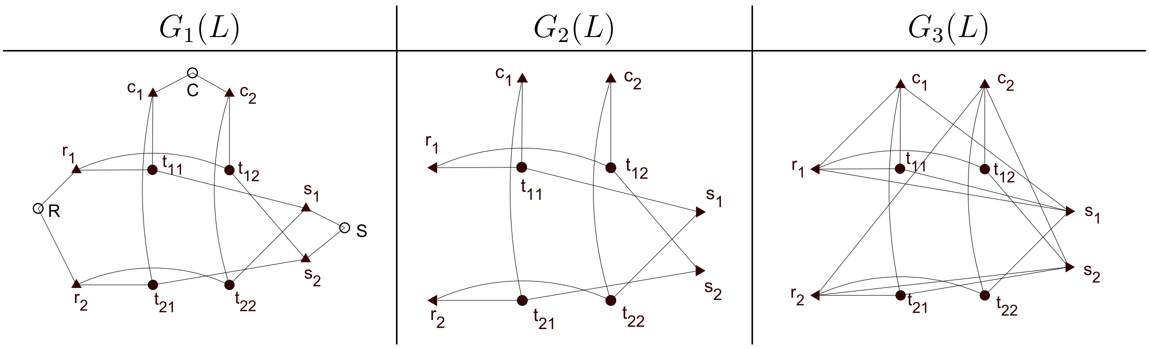

Figure 2.1: Graphs related to a Latin square of order 2.

Based on the proposal of McKay et al. for Latin squares, we describe now a pair of graphs that are uniquely related to a finite-dimensional algebra over a finite field that enable us to ensure that any two isotopic or isomorphic algebras map to two isomorphic graphs. To this end, let A be an n-dimensional algebra over a finite field $\mathbb{K}$. Firstly, we define the vertex-colored graph $G_1(A)$ with four maximal monochromatic subsets

$$\begin{cases} R_A = \{r_u \mid u \in A \setminus \mathrm{Ann}_{A^-}(A)\}, \\ C_A = \{c_u \mid u \in A \setminus \mathrm{Ann}_{A^+}(A)\}, \\ S_A = \{s_u \mid u \in A^2 \setminus \{0\}\}, \\ T_A = \{t_{u,v} \mid u, v \in A, uv \neq 0\}. \end{cases}$$

and edges

$$\{r_u t_{u,v}, c_v t_{u,v}, s_w t_{u,v} \mid u, v, w \in A, uv = w \neq 0\}.$$

From this graph we also define the vertex-colored graph $G_2(A)$ by adding the edges

$$\{r_u c_u, \mid u \in A \backslash \mathrm{Ann}_A(A)\} \cup \{c_u s_u \mid u \in A^2 \backslash \mathrm{Ann}_{A^+}(A)\} \cup \{r_u s_u \mid u \in A^2 \backslash \mathrm{Ann}_{A^-}(A)\}.$$

Just as an example, Figure 2.2 shows the two graphs that are related to any n-dimensional Lie algebra over the finite field $\mathbb{F}_2$, with basis $\{e_1, \ldots, e_n\}$, that is described by the non-zero product $e_1 e_2 = e_1$.

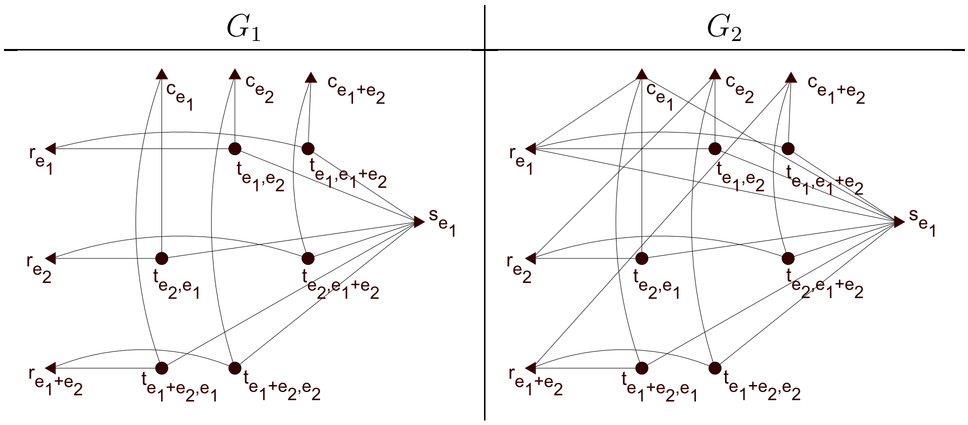

Figure 2.2: Graphs related to the Lie algebra $e_1 e_2 = e_1$ over $\mathbb{F}_2$.

Lemma 2.3.1. *Let A be an n-dimensional algebra over a finite field $\mathbb{K}$. Then,*

a) If the algebra A is abelian, then both graphs $G_1(A)$ and $G_2(A)$ are empty.

b) The graph $G_1(A)$ does not contain triangles.

c) In both graphs $G_1(A)$ and $G_2(A)$,

- *The number of vertices is*

$$|A \setminus \mathrm{Ann}_{A^-}(A)| + |A \setminus \mathrm{Ann}_{A^+}(A)| + |A^2| + |\{(u,v) \in A \times A \mid uv \neq 0\}| - 1.$$

- *$d(t_{u,v}) = 3$, for all $u, v \in A$ such that $uv \neq 0$.*

d) In the graph $G_1(A)$,

- *$d(r_u) = |A \setminus \mathrm{Ann}_{A^+}(\{u\})|$, for all $u \in A \setminus \mathrm{Ann}_{A^-}(A)$.*
- *$d(c_u) = |A \setminus \mathrm{Ann}_{A^-}(\{u\})|$, for all $u \in A \setminus \mathrm{Ann}_{A^+}(A)$.*
- *$d(s_u) = \sum_{v \in A} |\mathrm{ad}_v^{-1}(u)|$, for all $u \in A^2 \setminus \{0\}$.*

e) In the graph $G_2(A)$,

- *$d(r_u) = |A \setminus \mathrm{Ann}_{A^+}(\{u\})| + \mathbf{1}_{A \setminus \mathrm{Ann}_{A^-}(A)}(u) + \mathbf{1}_{A^2}(u)$, for all $u \in A \setminus \mathrm{Ann}_{A^-}(\{u\})$.*
- *$d(c_u) = |A \setminus \mathrm{Ann}_{A^-}(\{u\})| + \mathbf{1}_{A \setminus \mathrm{Ann}_{A^+}(A)}(u) + \mathbf{1}_{A^2}(u)$, for all $u \in A \setminus \mathrm{Ann}_{A^+}(\{u\})$.*
- *$d(s_u) = \mathbf{1}_{A \setminus \mathrm{Ann}_{A^-}(A)}(u) + \mathbf{1}_{A \setminus \mathrm{Ann}_{A^+}(A)}(u) + \sum_{v \in A} |\mathrm{ad}_v^{-1}(u)|$, for all $u \in A^2 \setminus \{0\}$.*

Here, $\mathbf{1}$ denotes the characteristic function.

Proof. All the assertions follow straightforward from the definition of the graphs $G_1(A)$ and $G_2(A)$. $\square$

Proposition 2.3.2. *Let A be an n-dimensional Lie algebra over a finite field $\mathbb{K}$. Then,*

a) The number of edges of its related graph $G_1(A)$ is

$$\sum_{u \notin \mathrm{Ann}_{A^-}(A)} (|A \setminus \mathrm{Ann}_{A^+}(\{u\})| + \sum_{v \in A^2 \setminus \{0\}} |\mathrm{ad}_u^{-1}(v)|) + \sum_{u \notin \mathrm{Ann}_{A^+}(A)} |A \setminus \mathrm{Ann}_{A^-}(\{u\})|.$$

b) The number of edges of its related graph $G_2(A)$ coincides with those of $G_1(A)$ plus

$$|A \setminus \mathrm{Ann}_A(A)| + |A^2 \setminus \mathrm{Ann}_{A^-}(A)| + |A^2 \setminus \mathrm{Ann}_{A^+}(A)|.$$

Proof. The result follows straightforward from the First Theorem of Graph Theory (see [159]) and assertions (b–d) in Lemma 2.3.1. □

Theorem 2.3.3. *Let A and A' be two n-dimensional algebras over a finite field $\mathbb{K}$. Then,*

a) If both algebras are isotopic, then their corresponding graphs $G_1(A)$ and $G_1(A')$ are isomorphic. Reciprocally, if the graphs $G_1(A)$ and $G_1(A')$ are isomorphic, then there exist three bijective maps f, g and h between A and A' such that $f(u)g(v) = h(uv)$.

b) If both algebras are isomorphic, then their corresponding graphs $G_2(A)$ and $G_2(A')$ are also isomorphic. Reciprocally, if the graphs $G_2(A)$ and $G_2(A')$ are isomorphic, then there exists a multiplicative bijective map between the algebras A and A', that is, a bijective map $f : A \to A'$ so that $f(u)f(v) = f(uv)$, for all $u, v \in A$.

Proof. Let (f, g, h) be an isotopism between the algebras A and A'. We define the map α between $G_1(A)$ and $G_1(A')$ such that

$$\begin{cases} \alpha(r_u) = r_{f(u)}, \text{ for all } u \in A \setminus \mathrm{Ann}_{A^-}(A), \\ \alpha(c_u) = c_{g(u)}, \text{ for all } u \in A \setminus \mathrm{Ann}_{A^+}(A), \\ \alpha(s_u) = s_{h(u)}, \text{ for all } u \in A^2 \setminus \{0\}, \\ \alpha(t_{u,v}) = t_{f(u),g(v)}, \text{ for all } u, v \in A \text{ such that } uv \neq 0. \end{cases}$$

The description of both graphs $G_1(A)$ and $G_1(A')$, together with Proposition 2.1.4, Lemma 2.1.5 and the regularity of f, g and h, involves α to be an isomorphism between these two vertex-colored graphs, that is, α is a well-defined bijection between the vertices of $G_1(A)$ and $G_1(A')$ that preserves collinearity and the color of the vertices. The same map α constitutes an isomorphism between the graphs $G_2(A)$ and $G_2(A')$ in case of being $f = g = h$, that is, if the algebras A and A' are isomorphic.

Reciprocally, let α be an isomorphism between the graphs $G_1(A)$ and $G_1(A')$. Collinearity involves this isomorphism to be uniquely determined by its restriction to $R_A \cup C_A \cup S_A$. Specifically, the image of each vertex $t_{u,v} \in T_A$ by means of α is uniquely determined by the corresponding images of r_u, c_v and s_{uv}. Let β and β' the respective bases of the algebras A and A' and let $\pi : A \to A'$ be the natural map that preserves the components of each vector with respect to the mentioned bases.

That is, $\pi((u_1,\ldots,u_n)_\beta)=(u_1,\ldots,u_n)_{\beta'}$, for all $u_1,\ldots,u_n\in\mathbb{K}$. Let us define three maps f, g and h from A to A' such that

$$f(u)=\begin{cases}\pi(u), \text{ for all } u\in \mathrm{Ann}_{A^-}(A),\\ v, \text{ otherwise, where } v\in A \text{ is such that } \alpha(r_u)=r_v.\end{cases}$$

$$g(u)=\begin{cases}\pi(u), \text{ for all } u\in \mathrm{Ann}_{A^+}(A),\\ v, \text{ otherwise, where } v\in A \text{ is such that } \alpha(c_u)=c_v.\end{cases}$$

$$h(u)=\begin{cases}\pi(u), \text{ for all } u\in (A\setminus A^2)\cup\{0\},\\ v, \text{ otherwise, where } v\in A \text{ is such that } \alpha(s_u)=s_v.\end{cases}$$

From Proposition 2.1.4 and Lemma 2.1.5, these three maps are bijective. Let $u,v\in A$. If $u\in\mathrm{Ann}_{A^-}(A)$ or $v\in\mathrm{Ann}_{A^+}(A)$, then there does not exist the vertex $t_{u,v}$ in the graph $G_1(A)$. Since α preserves collinearity, there does not exist the vertex $t_{f(u),g(v)}$ in the graph $G_1(A')$, which means that $f(u)\in\mathrm{Ann}_{A'^-}(A')$ or $g(v)\in\mathrm{Ann}_{A'^+}(A')$. In any case, we have that $f(u)g(v)=0=h(uv)$. Finally, if $u\notin\mathrm{Ann}_{A^-}(A)$ and $v\notin\mathrm{Ann}_{A^+}(A)$, then the vertex $t_{u,v}$ connects the vertices r_u, c_v and s_{uv} in the graph $G_1(A)$. Now, the isomorphism α maps this vertex $t_{u,v}$ in $G_1(A)$ to a vertex $t_{u',v'}$ in $G_2(A)$ that is connected to the vertices $r_{u'}$, $c_{v'}$ and $s_{u'v'}$. Again, since α preserves collinearity, it is $f(u)=u'$, $g(v)=v'$ and, finally, $h(uv)=f(u)g(v)$.

In case of being α an isomorphism between the graphs $G_2(A)$ and $G_2(A')$ it is enough to consider $f=g=h$ in the previous description. This is well-defined because of the new edges that are included to the graphs $G_1(A)$ and $G_1(A')$ in order to define, respectively, the graphs $G_2(A)$ and $G_2(A')$. Similarly to the previous reasoning, these edges involve the multiplicative character of the bijective map f, that is, $f(u)g(v)=h(uv)$, for all $u,v\in A$. □

Theorem 2.3.3 enables us to determine non-isomorphic algebras from their corresponding non-isomorphic graphs. Thus, for instance, it is known that the n-dimensional algebra over the finite field $\mathbb{F}_2$, with $n\geq 3$, described by the product $e_1e_2=e_3$ is not isomorphic to the n-dimensional algebra over $\mathbb{F}_2$ described by the product $e_1e_2=e_1$. This follows straightforward from the fact that the corresponding graph G_2 related to the former coincides with that associated with the latter, which is shown in Figure 2.2 (right), up to the vertex s_{e_1}, which becomes s_{e_3}, and the two edges $r_{e_1}s_{e_1}$ and $c_{e_1}s_{e_1}$, which disappear. Both graphs are, therefore, non-isomorphic and hence, the algebras are neither isomorphic. As we see in the next chapter, these two algebras are, however, isotopic. Even if this does not constitute a necessary condition, it is straightforward verified that their corresponding graphs G_1 are isomorphic. That graph shown in Figure 2.2 (left) is indeed the graph G_1 corresponding to the algebra described by the product $e_1e_2 = e_1$. In order to compute the graphs G_1 and G_2 related to a given algebra over a finite field, we have implemented the procedure *isoGraph* in the library *isotopism.lib*. Having as output a sequence with the number of vertices of each color, the number of edges and that of triangles of the graph under consideration, this procedure receives as input

1. The dimension n of the algebra.
2. The order q of the finite field.
3. A list C formed by tuples (i, j, k, c_{ij}^k) that indicates the non-zero structure constants of the algebra.
4. A positive integer $opt \leq 2$ that enables us to deal with the graph G_1 of the algebra if $opt = 1$, or the graph G_2 if $opt = 2$.

We have also implemented the auxiliary procedure *Prod* that outputs the list of polynomials that constitute the coefficient of each basis vector in the product of two arbitrary vectors of the algebra A. Its pseudocode is described in Algorithm 3.

Algorithm 3 Polynomials related to the product of two vectors in an algebra.

```
procedure PROD(u, v)
    for k ← 1, n do
        for i ← 1, n do
            for j ← i, n do
                L_k ← L_k + u_i v_j 𝔠^k_ij;
            end for
        end for
    end for
    return {L_1, ..., L_n}
end procedure
```

We finish the chapter with the implementation of the previous procedures into an illustrative example that focuses on those graphs G_1 and G_2 related to the set of non-abelian partial-quasigroup rings over a finite field that are based on the known distribution of partial Latin squares of order $n \leq 3$ into isotopism classes. Thus, for instance, Tables 2.2 and 2.3 show the isomorphism invariants of graphs that are related to the bi-dimensional case over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$, respectively. Partial Latin squares are written row after row in a single line, with empty cells represented by zeros. For each isotopism class we indicate the sequence with the number of vertices of each color, the number of edges and that of triangles of the corresponding graphs G_1 and G_2. Observe the coherence that exists among both tables and the remarks exposed in the proof of Theorem 2.2.11.

	G_1 & G_2	G_1	G_2	
Partial Latin square	Vertices	Edges	Edges	Triangles
10 00	(2,2,1,4)	12	16	7
10 01	(3,3,1,6)	18	23	7
10 02	(3,3,3,7)	21	30	16
10 20	(3,2,3,6)	18	25	12
12 00	(2,3,3,6)	18	25	12
12 20	(3,3,3,8)	24	33	13
12 21	(3,3,3,8)	24	33	13

Table 2.2: Graph invariants for the graphs G_1 and G_2 related to two-dimensional non-abelian partial-quasigroup rings over the finite field $\mathbb{F}_2$.

	G_1 & G_2	G_1	G_2	
Partial Latin square	Vertices	Edges	Edges	Triangles
10 00	(6,6,2,36)	108	118	20
10 01	(8,8,2,48)	144	156	22
10 02	(8,8,8,56)	168	192	48
10 20	(8,6,8,48)	144	164	42
12 00	(6,8,8,48)	144	164	42
12 20	(8,8,8,60)	180	204	38
12 21	(8,8,8,56)	168	192	48

Table 2.3: Graph invariants for the graphs G_1 and G_2 related to two-dimensional non-abelian partial-quasigroup rings over the finite field $\mathbb{F}_3$.

The implementation of the procedure *isoGraph* enables us to reduce the cost of computation that is required for computing all the reduced Gröbner bases involved in Algorithm 2. Specifically, in the seventh line of that algorithm, it is only required to compute the Krull dimension of those ideals $I^{\mathrm{isot}}_{A,A'}$ ($I^{\mathrm{isom}}_{A,A'}$, respectively) for which the isomorphism invariants of the corresponding graphs $G_1(A)$ and $G_1(A')$ ($G_2(A)$ and $G_2(A')$, respectively) coincide. This implementation enables us, for instance, to reduce the run time that is required to determine the distribution of n-dimensional non-abelian partial-quasigroup rings over the finite field $\mathbb{F}_2$ into isotopism classes, for $n \leq 3$, from 761 seconds in Example 2.2.12 to 30 seconds. This last run time includes the extra 9 seconds of computation that is required for computing the isotopism invariants that we have just exposed in Table 2.2 and those exposed in Table 2.4. The latter correspond to the isomorphism invariants of the graph G_1 related to each one of the 80 distinct isotopism classes of non-empty partial Latin squares of order 3.

Partial Latin square	Vertices	Edges
100 000 000	(4,4,1,16)	48
120 000 000	(4,6,3,24)	72
123 000 000	(4,7,7,28)	84
100 200 000	(6,4,3,24)	72
100 010 000	(6,6,1,24)	72
100 020 000	(6,6,3,28)	84
120 200 000	(6,6,3,32)	96
120 210 000	(6,6,3,32)	96
120 000 300	(6,6,6,32)	96
120 000 310	(6,6,6,36)	108
120 001 000	(6,7,3,32)	96
120 012 000	(6,7,3,36)	108
120 003 000	(6,7,7,34)	102
120 000 302	(6,7,7,36)	108
123 200 000	(6,7,7,36)	108
120 013 000	(6,7,7,38)	114
123 210 000	(6,7,7,38)	114
123 230 000	(6,7,7,40)	120
123 231 000	(6,7,7,40)	120
100 200 300	(7,4,7,28)	84
100 200 010	(7,6,3,32)	96
120 200 010	(7,6,3,36)	108
100 200 030	(7,6,7,34)	102
120 030 300	(7,6,7,36)	108
120 200 300	(7,6,7,36)	108
120 010 300	(7,6,7,38)	114
120 210 300	(7,6,7,38)	114
120 230 300	(7,6,7,40)	120
120 230 310	(7,6,7,40)	120
100 010 001	(7,7,1,28)	84
100 010 002	(7,7,3,34)	120
120 001 002	(7,7,3,36)	108
120 200 002	(7,7,3,36)	108
120 200 001	(7,7,3,38)	114
120 210 001	(7,7,3,38)	114
120 201 010	(7,7,3,40)	120
120 201 012	(7,7,3,40)	120
100 020 003	(7,7,7,37)	111
120 002 003	(7,7,7,38)	114
120 002 300	(7,7,7,38)	114
120 003 300	(7,7,7,38)	114
120 001 300	(7,7,7,39)	117
120 200 003	(7,7,7,40)	120
120 200 302	(7,7,7,40)	120
120 210 003	(7,7,7,40)	120
123 010 001	(7,7,7,40)	120
123 200 300	(7,7,7,40)	120
120 001 302	(7,7,7,41)	123
120 001 310	(7,7,7,41)	123
120 201 300	(7,7,7,41)	123
123 200 010	(7,7,7,41)	123
120 003 310	(7,7,7,42)	126
120 010 301	(7,7,7,42)	126
120 010 302	(7,7,7,42)	126
120 012 300	(7,7,7,42)	126
120 013 300	(7,7,7,42)	126
120 200 013	(7,7,7,42)	126
120 203 001	(7,7,7,42)	126
120 203 300	(7,7,7,42)	126
123 010 300	(7,7,7,42)	126
031 302	(7,7,7,42)	126
120 210 301	(7,7,7,42)	126
120 213 001	(7,7,7,42)	126
120 213 300	(7,7,7,42)	126
120 001 312	(7,7,7,43)	129
120 201 302	(7,7,7,43)	129
120 231 300	(7,7,7,43)	129
123 231 312	(7,7,7,43)	129
120 003 312	(7,7,7,44)	132
120 013 301	(7,7,7,44)	132
120 013 302	(7,7,7,44)	132
120 200 312	(7,7,7,44)	132
120 203 301	(7,7,7,44)	132
123 210 301	(7,7,7,44)	132
123 031 310	(7,7,7,45)	135
123 200 312	(7,7,7,45)	135
123 230 310	(7,7,7,45)	135
123 012 230	(7,7,7,46)	138
123 210 031	(7,7,7,46)	138
123 201 312	(7,7,7,46)	138

Table 2.4: Graph invariants for the graphs G_1 related to three-dimensional non-abelian partial-quasigroup rings over the finite field $\mathbb{F}_2$.

Chapter 3

Isotopisms of filiform Lie Algebras

Once we have exposed in the previous chapter those results in Computational Algebraic Geometry and Graph Theory that enable us to deal with the distribution of finite-dimensional algebras into isotopism classes, the rest of the manuscript focuses on three families of algebras whose distribution into isotopism classes has not been enough studied in the literature. This chapter deals in particular with the first family of algebras to be considered in this regard. We step forward in the study of isotopisms of Lie algebras in general and, more specifically, on that of pre-filiform and filiform Lie algebras.

3.1 Isotopisms of Lie algebras

We expose here some basic concepts and results on isotopisms of Lie algebras that we use throughout the chapter. Particularly, we introduce two new series of isotopism invariants that play an important role not only in the distribution of filiform Lie algebras, but also in that of Malcev algebras in the next chapter. For more details about the fundamentals of Lie algebras we refer to the monograph of Varadarajan [304].

An algebra A is said to be a *Lie algebra* if

- It is anticommutative, that is, $uv = -vu$, for all $u, v \in A$.
- It holds the so-called *Jacobi identity*

$$J(u,v,w) = u(vw) + v(wu) + w(uv) = 0, \text{ for all } u, v, w \in A. \tag{3.1}$$

The *centralizer* of a subset S of a Lie algebra A is the vector subspace

$$\mathrm{Cen}_A(S) = \{u \in A \mid uv = 0, \text{ for all } v \in S\} \subseteq A. \tag{3.2}$$

The *center* of the Lie algebra A is defined as its own centralizer

$$Z(A) = \mathrm{Cen}_A(A). \tag{3.3}$$

Particularly, a Lie algebra A is abelian if and only if $Z(A) = A$. Observe that the concept of centralizer of a Lie algebra coincides with that of annihilator of an algebra. Hence, similarly to Lemma 2.1.3 and Proposition 2.1.4, the next results hold.

Lemma 3.1.1. *Let (f, g, h) be an isotopism between two Lie algebras A and A'. Let S be a subset of L. Then,*

a) $f(\mathrm{Cen}_A(S)) = \mathrm{Cen}_{A'}(g(S))$.

b) $g(\mathrm{Cen}_A(S)) = \mathrm{Cen}_{A'}(f(S))$.

c) $\dim(\mathrm{Cen}_A(S)) = \dim(\mathrm{Cen}_{A'}(f(S))) = \dim(\mathrm{Cen}_{A'}(g(S)))$. □

Proposition 3.1.2. *Let A and A' be two isotopic Lie algebras. Then, $f(Z(A)) = Z(A')$ and $\dim(Z(A)) = \dim(Z(A'))$.* □

These results that we have just exposed enable us to introduce here two new series of isotopism invariants. Let n be the dimension of the Lie algebra A. For each positive integer $m \leq n$, we define

$$d_m(A) := \min\{\dim \mathrm{Cen}_A(S) \mid S \text{ is an } m\text{-dimensional vector subspace of } A\}. \tag{3.4}$$

$$D_m(A) := \max\{\dim \mathrm{Cen}_A(S) \mid S \text{ is an } m\text{-dimensional vector subspace of } A\}. \tag{3.5}$$

We prove that both values are preserved by isotopisms.

Proposition 3.1.3. *Let A and A' be two isotopic n-dimensional Lie algebras and let $m \leq n$ be a positive integer. Then, $d_m(A) = d_m(A')$ and $D_m(A) = D_m(A')$.*

Proof. Let (f, g, h) be an isotopism between A and A' and let S be an m-dimensional vector subspace of A such that $d_m(A) = \dim \mathrm{Cen}_A(S)$. The regularity of g involves the set $g(S)$ to be m-dimensional. Besides, from Lemma 3.1.1, $\mathrm{Cen}_{A'}(g(S)) = f(\mathrm{Cen}_A(S))$. Hence, $d_m(A) \geq d_m(A')$. The equality follows similarly from the isotopism (f^{-1}, g^{-1}, h^{-1}) between A' and A. The invariance of D_m holds analogously. □

3.2 Pre-filiform Lie algebras

Let n and p respectively be a positive integer and a prime. This section deals with the distribution into isomorphism and isotopism classes of the set $\mathcal{P}_{n,q}$ of Lie algebras over the finite field $\mathbb{F}_q$, where q is a prime power, such that there exists a *natural basis* $\{e_1, \ldots, e_n\}$ such that the algebra is described by the non-zero products

$$e_i e_n \in \langle\, e_1, \ldots, e_{n-1} \,\rangle. \tag{3.6}$$

Any Lie algebra in $\mathcal{P}_{n,q}$ is uniquely described, therefore, by a tuple $T = (\mathfrak{t}_1, \ldots, \mathfrak{t}_{n-1}) \in \langle\, e_1, \ldots, e_{n-1} \,\rangle^{n-1}$ such that $\mathfrak{t}_i = e_1 e_i$, for all $i < n$. We call T the *structure tuple* of the Lie algebra. We denote this algebra by A_T. Besides, from here on, the set of structure tuples of Lie algebras in $\mathcal{P}_{n,q}$ is denoted as $\mathcal{T}_{n,q}$. This coincides with the $(n-1)$-dimensional $\mathbb{F}_q$-vector space with components in $\langle\, e_1, \ldots, e_{n-1} \,\rangle$.

The importance of this family of Lie algebras comes from the similarity that exists among the structure constants of its elements and those of a filiform Lie algebra, which are described in the next subsection. In this regard, the set $\mathcal{P}_{n,q}$ is considered as a precursor of the set of n-dimensional filiform Lie algebras over the finite field $\mathbb{F}_q$. Due to it, we call them *pre-filiform Lie algebras*. These algebras were introduced by Boza et al. [48], who identified each one of the Lie algebras of $\mathcal{P}_{n,2}$ with a directed pseudo-graph. This enabled them to determine the distribution of such Lie algebras into isomorphism classes for $n \leq 5$. The case $q = 3$ and $n < 5$ was similarly determined in [50, 240]. Both distributions are respectively exposed in Tables 3.1 and 3.2, where every class is enumerated according to the original notation that was used in [50, 240]. Its corresponding representative algebra has been conveniently chosen to agree with the results that are exposed in this section. In both tables, the algebras are ordered according to the isomorphism invariants of their corresponding graphs G_1 and G_2, which were introduced in Chapter 2. Observe that the majority of these invariants characterizes the corresponding isomorphism class. Further, it is remarkable the existence of at least n isotopism classes in each one of the cases under consideration. In the next subsections we expose distinct results that enable us to deal with the distribution of pre-filiform Lie algebras of higher orders, not only into isomorphism classes, but also into isotopism classes. In this last regard we prove that there exist exactly n isotopism classes of n-dimensional pre-filiform Lie algebras, whatever the finite field is.

			G_1 & G_2	G_1	G_2					G_1 & G_2	G_1	G_2	
n	A_T	T	Vertices	Edges	Edges	Triangles	n	A_T	T	Vertices	Edges	Edges	Triangles
2	$\mathfrak{h}_2^1$	(0)	(0,0,0,0)	0	0	0	5	$\mathfrak{h}_5^5$	$(e_2, e_1+e_2, 0, 0)$	(28,28,3,576)	1728	1762	3
	$\mathfrak{h}_2^2$	(e_1)	(3,3,1,6)	18	23	5		$\mathfrak{h}_5^4$	$(e_2, e_1, 0, 0)$	(28,28,3,576)	1728	1762	35
3	$\mathfrak{h}_3^1$	$(0, 0)$	(0,0,0,0)	0	0	0		$\mathfrak{h}_5^8$	$(e_1, e_2, 0, 0)$	(28,28,3,576)	1728	1762	99
	$\mathfrak{h}_3^3$	$(e_2, 0)$	(6,6,1,24)	72	78	0		$\mathfrak{h}_5^{18}$	$(e_2, e_1+e_2, e_4, 0)$	(30,30,7,672)	2016	2058	6
	$\mathfrak{h}_3^2$	$(e_1, 0)$	(6,6,1,24)	72	80	9		$\mathfrak{h}_5^{19}$	$(e_2, e_3, e_4, 0)$	(30,30,7,672)	2016	2058	6
	$\mathfrak{h}_3^5$	(e_2, e_1+e_2)	(7,7,3,36)	108	121	3		$\mathfrak{h}_5^{13}$	$(e_2, e_4, e_3, 0)$	(30,30,7,672)	2016	2058	38
	$\mathfrak{h}_3^4$	(e_2, e_1)	(7,7,3,36)	108	121	11		$\mathfrak{h}_5^{15}$	$(e_2, e_1, e_4, 0)$	(30,30,7,672)	2016	2058	38
	$\mathfrak{h}_3^6$	(e_1, e_2)	(7,7,3,36)	108	121	27		$\mathfrak{h}_5^{14}$	$(e_1, e_2, e_4, 0)$	(30,30,7,672)	2016	2058	102
4	$\mathfrak{h}_4^1$	$(0, 0, 0)$	(0,0,0,0)	0	0	0		$\mathfrak{h}_5^{16}$	$(e_2, e_2+e_3, e_1, 0)$	(30,30,7,672)	2016	2060	7
	$\mathfrak{h}_4^3$	$(e_2, 0, 0)$	(12,12,1,96)	288	300	0		$\mathfrak{h}_5^{17}$	$(e_2, e_1+e_3, e_1, 0)$	(30,30,7,672)	2016	2060	7
	$\mathfrak{h}_4^2$	$(e_1, 0, 0)$	(12,12,1,96)	288	302	17		$\mathfrak{h}_5^{11}$	$(e_2, e_3, e_1, 0)$	(30,30,7,672)	2016	2060	39
	$\mathfrak{h}_4^6$	$(e_2, e_3, 0)$	(14,14,3,144)	432	450	2		$\mathfrak{h}_5^{10}$	$(e_2, e_1, e_3, 0)$	(30,30,7,672)	2016	2060	103
	$\mathfrak{h}_4^7$	$(e_1, e_3, 0)$	(14,14,3,144)	432	450	18		$\mathfrak{h}_5^{12}$	$(e_2, e_1+e_3, e_3, 0)$	(30,30,7,672)	2016	2060	103
	$\mathfrak{h}_4^5$	$(e_2, e_1+e_2, 0)$	(14,14,3,144)	432	452	3		$\mathfrak{h}_5^{20}$	$(e_1, e_2, e_3, 0)$	(30,30,7,672)	2016	2060	231
	$\mathfrak{h}_4^4$	$(e_2, e_1, 0)$	(14,14,3,144)	432	452	19		$\mathfrak{h}_5^{25}$	(e_2, e_3, e_4, e_1+e_3)	(31,31,15,720)	2160	2221	15
	$\mathfrak{h}_4^8$	$(e_1, e_2, 0)$	(14,14,3,144)	432	452	51		$\mathfrak{h}_5^{26}$	$(e_2, e_2+e_3, e_4, e_1+e_3)$	(31,31,15,720)	2160	2221	15
	$\mathfrak{h}_4^{12}$	(e_2, e_2+e_3, e_1)	(15,15,7,168)	504	533	7		$\mathfrak{h}_5^{27}$	(e_2, e_4, e_1+e_2, e_3)	(31,31,15,720)	2160	2221	15
	$\mathfrak{h}_4^{13}$	(e_2, e_1+e_3, e_1)	(15,15,7,168)	504	533	7		$\mathfrak{h}_5^{28}$	(e_2, e_2+e_4, e_1, e_3)	(31,31,15,720)	2160	2221	15
	$\mathfrak{h}_4^{10}$	(e_2, e_3, e_1)	(15,15,7,168)	504	533	23		$\mathfrak{h}_5^{31}$	$(e_2, e_1+e_2, e_3+e_4, e_3)$	(31,31,15,720)	2160	2221	15
	$\mathfrak{h}_4^9$	(e_2, e_1, e_3)	(15,15,7,168)	504	533	55		$\mathfrak{h}_5^{22}$	(e_2, e_3+e_4, e_3, e_1)	(31,31,15,720)	2160	2221	47
	$\mathfrak{h}_4^{11}$	(e_2, e_1+e_3, e_3)	(15,15,7,168)	504	533	23		$\mathfrak{h}_5^{23}$	(e_2, e_3, e_4, e_1)	(31,31,15,720)	2160	2221	47
	$\mathfrak{h}_4^{14}$	(e_1, e_2, e_3)	(15,15,7,168)	504	533	119		$\mathfrak{h}_5^{29}$	(e_2, e_2+e_3, e_1, e_4)	(31,31,15,720)	2160	2221	47
5	$\mathfrak{h}_5^1$	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	0	0	0		$\mathfrak{h}_5^{30}$	(e_2, e_3, e_1+e_2, e_4)	(31,31,15,720)	2160	2221	47
	$\mathfrak{h}_5^3$	$(e_2, 0, 0, 0)$	(24,24,1,384)	1152	1176	0		$\mathfrak{h}_5^{21}$	(e_2, e_1, e_4, e_3)	(31,31,15,720)	2160	2221	111
	$\mathfrak{h}_5^2$	$(e_1, 0, 0, 0)$	(24,24,1,384)	1152	1178	33		$\mathfrak{h}_5^{24}$	(e_2, e_1+e_3, e_3, e_4)	(31,31,15,720)	2160	2221	111
	$\mathfrak{h}_5^9$	$(e_3, e_4, 0, 0)$	(28,28,3,576)	1728	1756	0		$\mathfrak{h}_5^{32}$	(e_2, e_3, e_1, e_4)	(31,31,15,720)	2160	2221	111
	$\mathfrak{h}_5^6$	$(e_2, e_3, 0, 0)$	(28,28,3,576)	1728	1760	2		$\mathfrak{h}_5^{33}$	(e_2, e_1, e_3, e_4)	(31,31,15,720)	2160	2221	239
	$\mathfrak{h}_5^7$	$(e_1, e_3, 0, 0)$	(28,28,3,576)	1728	1760	34		$\mathfrak{h}_5^{34}$	(e_1, e_2, e_3, e_4)	(31,31,15,720)	2160	2221	495

Table 3.1: Graph invariants for the graphs G_1 and G_2 related to Lie algebras in $\mathcal{P}_{n,2}$, for $n \leq 5$.

3.2.1 Isotopisms classes of $\mathcal{P}_{n,q}$.

In order to determine the distribution of $\mathcal{P}_{n,q}$ into isotopism classes, we focus on the structure tuples of this kind of Lie algebras.

Lemma 3.2.1. *Let T and T' be two structure tuples in $\mathcal{T}_{n,q}$ that are equal up to permutation of their components and relabeling of the sub-indices of the basis vectors of $\mathcal{P}_{n,q}$. The Lie algebras A_T and $A_{T'}$ are strongly isotopic.*

Proof. Suppose $T = (\sum_{j=1}^{n-1} t_{1j}e_j, \ldots, \sum_{j=1}^{n-1} t_{(n-1)j}e_j)$ and $T' = (\sum_{j=1}^{n-1} t'_{1j}e_j, \ldots, \sum_{j=1}^{n-1} t'_{(n-1)j}e_j)$. From the hypothesis, there exist two permutations $\alpha, \beta \in S_{n-1}$

			G_1 & G_2	G_1	G_2					G_1 & G_2	G_1	G_2	
n	A_T	T	Vertices	Edges	Edges	Triangles	n	A_T	T	Vertices	Edges	Edges	Triangles
2	$\mathfrak{g}_2^1$	(0)	(0,0,0,0)	0	0	0	4	$\mathfrak{g}_4^5$	$(e_2, e_1+e_2, 0)$	(78,78,8,5184)	15552	15646	8
	$\mathfrak{g}_2^2$	(e_1)	(8,8,2,48)	144	156	18		$\mathfrak{g}_4^6$	$(e_2, 2e_1, 0)$	(78,78,8,5184)	15552	15646	8
3	$\mathfrak{g}_3^1$	$(0,0)$	(0,0,0,0)	0	0	0		$\mathfrak{g}_4^7$	$(e_2, 2e_1+e_2, 0)$	(78,78,8,5184)	15552	15646	116
	$\mathfrak{g}_3^3$	$(e_2, 0)$	(24,24,2,432)	1296	1320	0		$\mathfrak{g}_4^{10}$	$(e_1, e_2, 0)$	(78,78,8,5184)	15552	15646	440
	$\mathfrak{g}_3^2$	$(e_1, 0)$	(24,24,2,432)	1296	1324	38		$\mathfrak{g}_4^{18}$	(e_3, e_2+e_3, e_1+e_2)	(80,80,26,5616)	16848	16980	26
	$\mathfrak{g}_3^6$	$(e_2, 2e_1)$	(26,26,8,576)	1728	1770	8		$\mathfrak{g}_4^{19}$	$(e_3, e_2+e_3, e_1+e_2+e_3)$	(80,80,26,5616)	16848	16980	26
	$\mathfrak{g}_3^5$	(e_2, e_1+e_2)	(26,26,8,576)	1728	1770	8		$\mathfrak{g}_4^{20}$	(e_3, e_2+e_3, e_1+2e_2)	(80,80,26,5616)	16848	16980	26
	$\mathfrak{g}_3^7$	$(e_2, 2e_1+e_2)$	(26,26,8,576)	1728	1770	44		$\mathfrak{g}_4^{21}$	$(e_3, e_2+e_3, e_1+2e_2+2e_3)$	(80,80,26,5616)	16848	16980	26
	$\mathfrak{g}_3^4$	(e_2, e_1)	(26,26,8,576)	1728	1770	80		$\mathfrak{g}_4^{12}$	(e_2, e_1+e_2, e_3)	(80,80,26,5616)	16848	16980	134
	$\mathfrak{g}_3^8$	(e_1, e_2)	(26,26,8,576)	1728	1770	152		$\mathfrak{g}_4^{13}$	(e_3, e_2, e_1+2e_3)	(80,80,26,5616)	16848	16980	134
4	$\mathfrak{g}_4^1$	$(0,0,0)$	(0,0,0,0)	0	0	0		$\mathfrak{g}_4^{15}$	$(e_2, 2e_1, e_3)$	(80,80,26,5616)	16848	16980	134
	$\mathfrak{g}_4^3$	$(e_2, 0, 0)$	(72,72,2,3888)	11664	11736	0		$\mathfrak{g}_4^{17}$	$(e_3, e_2, 2e_1+e_2+2e_3)$	(80,80,26,5616)	16848	16980	134
	$\mathfrak{g}_4^2$	$(e_1, 0, 0)$	(72,72,2,3888)	11664	11740	110		$\mathfrak{g}_4^{14}$	(e_3, e_2, e_1+e_2)	(80,80,26,5616)	16848	16980	242
	$\mathfrak{g}_4^8$	$(e_2, e_3, 0)$	(78,78,8,5184)	15552	15642	6		$\mathfrak{g}_4^{16}$	$(e_3, e_2, 2e_1+2e_3)$	(80,80,26,5616)	16848	16980	458
	$\mathfrak{g}_4^9$	$(e_1, e_3, 0)$	(78,78,8,5184)	15552	15642	114		$\mathfrak{g}_4^{11}$	(e_2, e_1, e_3)	(80,80,26,5616)	16848	16980	566
	$\mathfrak{g}_4^4$	$(e_2, e_1, 0)$	(78,78,8,5184)	15552	15646	8		$\mathfrak{g}_4^{22}$	$(e_1, e_2, 2e_3)$	(80,80,26,5616)	16848	16980	566

Table 3.2: Graph invariants for the graphs G_1 and G_2 related to Lie algebras in $\mathcal{P}_{n,3}$, for $n \leq 4$.

such that $t'_{\alpha(i)\beta(j)} = t_{ij}$, for all $i, j < n$. It is then enough to define by linearity the strong isotopism (f, f, h) from A_T to $A_{T'}$ such that $f(e_n) = h(e_n) = e_n$, $f(e_i) = e_{\alpha(i)}$ and $h(e_i) = e_{\beta(i)}$, for all $i < n$. Then,

$$f(e_i)f(e_n) = e_{\alpha(i)}e_n = \sum_{j=1}^{n-1} t'_{\alpha(i)j}e_j = \sum_{j=1}^{n-1} t'_{\alpha(i)\beta(j)}e_{\beta(j)} = \sum_{j=1}^{n-1} t_{ij}e_{\beta(j)} = h(e_ie_n),$$

for all $i < n$. $\square$

Example 3.2.2. *From Lemma 3.2.1, we have, for instance, that the non-isomorphic Lie algebras $\mathfrak{h}_4^4$, $\mathfrak{h}_4^6$, $\mathfrak{h}_4^7$ and $\mathfrak{h}_4^8$ in $\mathcal{P}_{4,2}$ (see Table 3.1) are pairwise strongly isotopic. Similarly, the non-isomorphic Lie algebras $\mathfrak{h}_4^{11}$ and $\mathfrak{h}_4^{13}$ are also strongly isotopic.* $\lhd$

Proposition 3.2.3. *Let T be a structure tuple in $\mathcal{T}_{n,q}$. There always exists a structure tuple $T' = (\sum_{j=1}^{n-1} t'_{1j}e_j, \ldots, \sum_{j=1}^{n-1} t'_{(n-1)j}e_j) \in \mathcal{T}_{n,q}$ such that $A_{T'}$ is strongly isotopic to A_T and the next two conditions hold*

a) If $t'_{ii} = 0$ for some $i \geq 1$, then $t'_{jk} = 0$, for all $j, k \geq i$.

b) If $t'_{ii} \neq 0$ for some $i \geq 1$, then $t'_{ij} = 0$, for all $j \neq i$.

Proof. Let $T = (\sum_{j=1}^{n-1} t_{1j}e_j, \ldots, \sum_{j=1}^{n-1} t_{(n-1)j}e_j) \in \mathcal{T}_{n,q}$. As a first step in the construction of the required structure tuple T', let us consider $T' = T$. From Lemma 3.2.1, any permutation of the components of T' and any relabeling of the indices of the basis vectors of $\mathcal{P}_{n,q}$ give rise to a new structure tuple of a Lie algebra in $\mathcal{P}_{n,q}$ that is strongly isotopic to A_T. Keeping this in mind, we can modify T' so that, if $t'_{ii} = 0$, for some $i < n$, then

- $t'_{ji} = 0$, for all $j > i$. Otherwise, we rearrange conveniently from the i^{th} to the $(n-1)^{th}$ components of T'.
- $t'_{ij} = 0$, for all $j > i$. Otherwise, we permute conveniently the indices of the basis vectors $e_i, \ldots, e_{n-1}$.

Condition (a) in the statement holds then from the combination of these two assumptions. Now, in order to obtain condition (b), we modify T' so that, for each $i < n$ such that $t'_{ii} \neq 0$, we define by linearity the strong isotopism $(\mathrm{Id}, \mathrm{Id}, h)$ from $A_{T'}$ in such a way that $h(e_i) = e_i - \frac{1}{t'_{ii}}(\sum_{j=1}^{i-1} t'_{ij}e_j - \sum_{j=i+1}^{n-1} t'_{ij}e_j)$ and $h(e_j) = e_j$, for all $j \neq i$. Then,

$$e_ie_n = \mathrm{Id}(e_i)\mathrm{Id}(e_n) = h(e_ie_n) = h(\sum_{j=1}^{n-1} t'_{ij}e_j) = \sum_{j=1}^{i-1} t'_{ij}e_j + t'_{ii}h(e_i) + \sum_{j=i+1}^{n-1} t'_{ij}e_j = t'_{ii}e_i$$

and condition (b) holds. □

Example 3.2.4. *Proposition 3.2.3 involves that every Lie algebra in $\mathcal{P}_{n,q}$ is strongly isotopic to a Lie algebra whose structure tuple has triangular form. Thus, for instance, let us consider the Lie algebra in $\mathcal{P}_{4,2}$ with structure tuple $(e_3, e_3 + e_4, e_2)$. From Lemma 3.2.1, this is strongly isotopic to the Lie algebra in $\mathcal{P}_{4,2}$ with structure tuple $(e_1, e_2 + e_3, e_2)$, once we permute the first and the third components in T and relabel e_2, e_3 and e_4 by e_1, e_2 and e_3, respectively. This is, in turn, strongly isotopic to the Lie algebra in $\mathcal{P}_{4,2}$ with structure tuple $(e_1, e_2, e_2 - e_3)$, once we consider the strong isotopism $(\mathrm{Id}, \mathrm{Id}, h)$, where h is linearly defined in such a way that $h(e_2) = e_2 - e_3$ and $h(e_i) = e_i$, for all $i \in \{1, 3\}$. This structure tuple is already given in triangular form, because*

$$\begin{cases} e_1e_4 = e_1, \\ e_2e_4 = e_2, \\ e_3e_4 = e_2 - e_3. \end{cases}$$

In the next results, we see indeed that every Lie algebra in $\mathcal{P}_{n,q}$ is strongly isotopic to a Lie algebra whose structure tuple has diagonal form. Thus, for instance, our *last Lie algebra is strongly isotopic to the Lie algebra in $\mathcal{P}_{4,2}$ with structure tuple (e_1, e_2, e_3), once we consider the strong isotopism $(\mathrm{Id}, \mathrm{Id}, h')$, where h' is linearly defined in such a way that $h(e_3) = -e_3 + e_2$ and $h(e_i) = e_i$, for all $i \in \{1, 2\}$. We have the diagonal form*

$$\begin{cases} e_1e_4 &= e_1, \\ e_2e_4 &= \quad e_2, \\ e_3e_4 &= \qquad e_3. \end{cases}$$

$\triangleleft$

Lemma 3.2.5. *Let $T = (\mathfrak{t}_1, \ldots, \mathfrak{t}_{n-1})$ be a structure tuple in $\mathcal{T}_{n,q}$ and let i, j be two distinct positive integers less than n. Let $T' = (\mathfrak{t}_1, \ldots, \mathfrak{t}_{i-1}, a\mathfrak{t}_i + b\mathfrak{t}_j, \mathfrak{t}_{i+1}, \ldots, \mathfrak{t}_{n-1}) \in \mathcal{T}_{n,q}$, for some $a, b \in \mathbb{F}_q$ such that $a \neq 0$. Then, A_T is strongly isotopic to $A_{T'}$.*

Proof. It is enough to define by linearity the principal strong isotopism (f, f, Id) from A_T to $A_{T'}$ such that $f(e_i) = \frac{1}{a}(e_i - be_j)$ and $f(e_k) = e_k$, for all $k \in [n]\backslash\{i\}$. In particular,

$$e_i \circ e_n = (e_i - be_j + be_j) \circ e_n = f(ae_i + be_j) \circ f(e_n) = (ae_i + be_j) \cdot e_n = ae_i \cdot e_n + be_j \cdot e_n,$$

where, in order to avoid confusion, we have denoted by $\cdot$ and $\circ$ the respective products in A_T and $A_{T'}$. $\square$

Theorem 3.2.6. *There exist n isotopism classes in $\mathcal{P}_{n,q}$.*

Proof. Let A be a Lie algebra in $\mathcal{P}_{n,q}$. From Proposition 3.2.3, we can find a structure tuple in triangular form $T = (t_{11}e_1, \sum_{j=1}^{2} t_{2j}e_j, \ldots, \sum_{j=1}^{n-1} t_{(n-1)j}e_j) \in \mathcal{T}_{n,q}$ satisfying conditions (a) and (b) of that result and so that $A_T \simeq A$. From condition (a), if $t_{11} = 0$, then A_T is the abelian Lie algebra. Otherwise, we can made use of Lemma 3.2.5 to determine a second structure tuple $T' = (e_1, t_{22}e_2, \sum_{j=2}^{3} t_{3j}e_j, \ldots, \sum_{j=2}^{n-1} t_{(n-1)j}e_j) \in \mathcal{T}_{n,q}$, where the unique component with a non-zero coefficient in e_1 is the first one, and such that $A_{T'} \simeq A_T \simeq A$.

Now, again from condition (a), if $t'_{22} = 0$, then $T' = (e_1, 0, \ldots, 0)$. Otherwise, from a similar reasoning to that which we have just done to define T', we find a structure tuple $T'' = (e_1, e_2, t_{33}e_3, \sum_{j=3}^{4} t_{4j}e_j, \ldots, \sum_{j=3}^{n-1} t_{(n-1)j}e_j) \in \mathcal{T}_{n,q}$ such that $A_{T''} \simeq A$. We repeat this reasoning with the rest of components of our structure tuple and hence, we observe that any non-abelian Lie algebra in $\mathcal{P}_{n,q}$ is strongly isotopic to a Lie algebra with structure tuple $T_m = (e_1, \ldots, e_m, 0, \ldots, 0) \in \mathcal{T}_{n,q}$, for some $m < n$.

Since $Z(A_{T_m}) = \langle e_{m+1}, \ldots, e_{n-1} \rangle$, for all $m < n$, Proposition 3.1.2 involves A_{T_m} not to be isotopic to A_{T_l}, for any two distinct positive integers $m, l < n$. Therefore, the n-dimensional abelian Lie algebra, together with the $n-1$ Lie algebras A_{T_m}, with $m < n$, determines the set of (strongly) isotopism classes of $\mathcal{P}_{n,q}$. $\square$

Table 3.3 shows the distribution into isotopism classes of the isomorphism classes which appear in Tables 3.1 and 3.2.

p	n	Isotopism classes				
2	2	$\mathfrak{h}_2^1$	$\mathfrak{h}_2^2$			
	3	$\mathfrak{h}_3^1$	$\mathfrak{h}_3^2 \sim \mathfrak{h}_3^3$	$\mathfrak{h}_3^4 \sim \mathfrak{h}_3^5 \sim \mathfrak{h}_3^6$		
	4	$\mathfrak{h}_4^1$	$\mathfrak{h}_4^2 \sim \mathfrak{h}_4^3$	$\mathfrak{h}_4^4 \sim \ldots \sim h_4^8$	$\mathfrak{h}_4^9 \sim \ldots \sim \mathfrak{h}_4^{14}$	
	5	$\mathfrak{h}_5^1$	$\mathfrak{h}_5^2 \sim \mathfrak{h}_5^3$	$\mathfrak{h}_5^4 \sim \ldots \sim \mathfrak{h}_5^9$	$\mathfrak{h}_5^{10} \sim \ldots \sim \mathfrak{h}_5^{20}$	$\mathfrak{h}_5^{21} \sim \ldots \sim \mathfrak{h}_5^{34}$
3	2	$\mathfrak{g}_2^1$	$\mathfrak{g}_2^2$			
	3	$\mathfrak{g}_3^1$	$\mathfrak{g}_3^2 \sim \mathfrak{g}_3^3$	$\mathfrak{g}_3^4 \sim \ldots \sim \mathfrak{g}_3^8$		
	4	$\mathfrak{g}_4^1$	$\mathfrak{g}_4^2 \sim \mathfrak{g}_4^3$	$\mathfrak{g}_4^4 \sim \ldots \sim \mathfrak{g}_4^{10}$	$\mathfrak{g}_4^{11} \sim \ldots \sim \mathfrak{g}_4^{22}$	

Table 3.3: Isotopism classes of $\mathcal{P}_{n,q}$.

3.2.2 Isomorphisms classes of $\mathcal{P}_{n,q}$

We focus now on the distribution of $\mathcal{P}_{n,q}$ into isomorphism classes. Since, from Lemma 2.1.1, the abelian Lie algebra constitutes itself an isomorphism class, we analyze here the non-abelian case. For each positive integer $m < n$, let $\mathcal{P}_{n,q;m}$ be the set of pre-filiform Lie algebras in $\mathcal{P}_{n,q}$ with an $(n-m-1)$-dimensional center. This set is invariant by isomorphisms due to the fact that centers of Lie algebras are preserved by isomorphisms. From a convenient change of basis, we can focus, therefore, on the subset $\mathcal{T}_{n,q;m}$ of structure tuples $(\sum_{j=1}^{n-1} t_{1j}e_j, \ldots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \ldots, 0) \in \mathcal{T}_{n,q}$ so that the $m \times (n-1)$ matrix of coefficients (t_{ij}) is regular.

Lemma 3.2.7. *Let $T = (\mathfrak{t}_1, \ldots, \mathfrak{t}_m, 0, \ldots, 0)$ be a structure tuple in $\mathcal{T}_{n,q;m}$. Given $k \in \mathbb{F}_q \setminus \{0\}$, let kT be the structure tuple $(k\mathfrak{t}_1, \ldots, k\mathfrak{t}_m, 0, \ldots, 0) \in \mathcal{T}_{n,q;m}$. Then, the Lie algebras A_{kT} and A are isomorphic.*

Proof. Let $k \in \mathbb{F}_q \setminus \{0\}$. In order to prove that $A_{kT} \cong A$, it is enough to define by linearity the isomorphism f such that $f(e_i) = e_i$ if $i < n$ and $f(e_n) = \frac{1}{k}e_n$. Then,

$$e_i \circ e_n = (ke_i) \circ (\frac{1}{k}e_n) = f(ke_i) \circ f(e_n) = kf(e_i \cdot e_n) = k(e_i \cdot e_n),$$

for all $i < n$, where the last equality follows from the fact of being $A^2 \subseteq \mathcal{T}_{n,q}$. In order to avoid confusion, we have denoted by $\cdot$ and $\circ$ the respective products in A_T and $A_{T'}$. □

Example 3.2.8. *From Lemma 3.2.7, we have, for instance, that the Lie algebra* $\mathfrak{g}_3^8 \in \mathcal{P}_{3,3;2}$ *in Table 3.2, with structure tuple* $(e_1, e_2) \in \mathcal{T}_{3,3;2}$ *is isomorphic to the Lie algebra in* $\mathcal{P}_{3,3;2}$ *of structure tuple* $(2e_1, 2e_2) \in \mathcal{T}_{3,3;2}$. ◁

Lemma 3.2.9. *Let* $T = (\sum_{j=1}^{n-1} t_{1j}e_j, \dots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \dots, 0) \in \mathcal{T}_{n,q;m}$ *be such that there exists a positive integer* $i \leq m$ *such that* $t_{ii} \neq 0$. *Then, the Lie algebra* $(A_T, \cdot)$ *is isomorphic to a Lie algebra* $(A', \circ)$ *in* $\mathcal{P}_{n,q;m}$ *such that*

$$e_j \circ e_n = \begin{cases} \sum_{k \leq m} t_{ik}e_k, & \text{if } j = i, \\ e_j \cdot e_n, & \text{otherwise.} \end{cases}$$

Proof. It is enough to define by linearity the isomorphism f from A_T to A' such that $f(e_i) = e_i - \frac{1}{t_{ii}} \sum_{k>m} t_{ik}e_k$ and $f(e_j) = e_j$, for all $j \neq i$. Then, $e_j \circ e_n = f(e_j) \circ f(e_n) = e_j \cdot e_n$ if $j \neq i$. Otherwise,

$$e_i \circ e_n = f(e_i + \frac{1}{t_{ii}} \sum_{k>m} t_{ik}e_k) \circ f(e_n) = f((e_i + \frac{1}{t_{ii}} \sum_{k>m} t_{ik}e_k) \cdot e_n) =$$

$$= f(e_i \cdot e_n) = \sum_{k \neq i} t_{ik}e_k + t_{ii}f(e_i) = \sum_{k \leq m} t_{ik}e_k.$$

□

Example 3.2.10. *From Lemma 3.2.9, we have, for instance, that the Lie algebra in* $\mathcal{P}_{4,3;2}$, *with structure tuple* $(e_1 + e_3, e_2, 0) \in \mathcal{T}_{4,3;2}$, *is isomorphic to a Lie algebra whose structure tuple does not have a non-zero structure constant as coefficient of* e_3. *Specifically, if we follow the proof of Lemma 3.2.9, we obtain that this is isomorphic to the Lie algebra* $\mathfrak{g}_4^{10} \in \mathcal{T}_{4,3;2}$ *in Table 3.2, with structure tuple* $(e_1, e_2, 0) \in \mathcal{T}_{4,3;2}$. ◁

Proposition 3.2.11. *Let* $T = (\sum_{j=1}^{n-1} t_{1j}e_j, \ldots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \ldots, 0)$ *be a structure tuple in* $\mathcal{T}_{n,q;m}$ *and let* $j_1 = \min\{j < n \mid t_{1j} > 0\}$*. Then, the Lie algebra* $(A_T, \cdot)$ *is isomorphic to a Lie algebra* $(A, \circ)$ *in* $\mathcal{P}_{n,q;m}$ *such that*

$$e_1 \circ e_n = \begin{cases} e_1, & \text{if } e_1 \cdot e_n \in \langle\, e_1 \,\rangle, \\ e_2, & \text{if there exists } j \in \{2, \ldots, m\} \text{ such that } t_{1j} \neq 0, \\ e_{m+1}, & \text{if } j_1 > m. \end{cases}$$

Proof. We can suppose that $t_{1j_1} = 1$. Otherwise, consider the Lie algebra $A_{\frac{1}{t_{1j_1}}T}$, which is isomorphic to A_T from Lemma 3.2.7. Hence, $t_{11} \in \{0, 1\}$. Let us study each case separately. Firstly, suppose $t_{11} = 1$. Then,

- If $e_1 \cdot e_n = e_1$, then it is enough to consider $A = A_T$.
- If $j_1 = 1$ and there exists $j > 1$ such that $t_{1j} \neq 0$, then we can suppose, from Lemma 3.2.9, that $j \leq m$. It is then enough to define by linearity the isomorphism f such that $f(e_i) = e_i$ if $i \neq j$ and $f(e_j) = \frac{1}{t_{1j}}(e_j - e_1 - \sum_{k>j} t_{1k}e_k)$. The Lie algebra $(A', *)$ isomorphic to A_T by means of the isomorphism f verifies that
$$e_1 * e_n = f(e_1) * f(e_n) = f(e_1 \cdot e_n) = e_j.$$
The Lie algebra $(A, \circ)$ isomorphic to A' with respect to the isomorphism $(2j)$, which switches the basis vectors e_2 and e_j, verifies that $e_1 \circ e_n = e_2$.

Now, suppose $t_{11} = 0$ and hence, $j_1 \neq 1$. Let us define by linearity the isomorphism f such that $f(e_i) = e_i$ if $i \neq j_1$ and $f(e_{j_1}) = e_{j_1} - \sum_{k>j_1} t_{1k}e_k$. The Lie algebra $(A', *)$ isomorphic to A_T with respect to f verifies that

$$e_1 * e_n = f(e_1) * f(e_n) = f(e_1 \cdot e_n) = f(e_{j_1}) + \sum_{k>j_1} t_{1k}e_k = e_{j_1}.$$

If $j_1 \in \{2, \ldots, m\}$, then we can use the isomorphism defined in the previous case in order to obtain a Lie algebra $(A, \circ)$ isomorphic to A_T and such that $e_1 \circ e_n = e_2$. Analogously, if $j_1 > m$, then it is enough to consider the isomorphism $(j_1 m)$ that switches the basis vectors e_{j_1} and e_m. $\square$

The previous result can be used in particular to determine the distribution of $\mathcal{P}_{n,q;1}$ into isomorphism classes, for all positive integer $n \geq 2$ and q a power prime.

Theorem 3.2.12. *Let q be a power prime. There exist only one non-abelian isomorphism class in $\mathcal{P}_{2,q;1}$. This is determined by the structure tuple (e_1) in $\mathcal{T}_{2,q;1}$.*

Proof. The result follows straightforward from Proposition 3.2.11. □

Theorem 3.2.13. *Let n be a positive integer greater than 2 and let q be a power prime. There exist two isomorphism classes in $\mathcal{P}_{n,q;1}$. They are determined by the structure tuples $(e_1, 0, \ldots, 0)$ and $(e_2, 0, \ldots, 0)$ in $\mathcal{T}_{n,q;1}$.*

Proof. From Proposition 3.2.11, any element in $\mathcal{P}_{n,q;1}$ is isomorphic to a Lie algebra with structure tuple $T_1 = (e_1, 0, \ldots, 0)$ or $T_2 = (e_2, 0, \ldots, 0)$ in $\mathcal{T}_{n,q;1}$. Since isomorphisms of Lie algebras preserve nilpotency, the algebras A_{T_1} or A_{T_2} are not isomorphic, because the latter is nilpotent, but the former is not. □

Proposition 3.2.11 is also useful to find a representative class in $\mathcal{P}_{n,q;m}$, for $m > 1$, with a structure tuple having the number of addends in its first component as small as possible. Nevertheless, unlike Theorem 3.2.13, we cannot assure that any two Lie algebras in $\mathcal{P}_{n,q;m}$, with $m > 1$ and structure tuples starting with e_1 and e_2, respectively, are not isomorphic. Thus, for instance, the Lie algebras $A_{(e_1,e_1,0)}$ and $A_{(e_2,e_2,0)}$ in $\mathcal{P}_{3,2;2}$ are isomorphic by means of the isomorphism (12) that switches the basis vectors e_1 and e_2. The next result determines explicitly those isomorphism classes in $\mathcal{P}_{n,q;m}$ that do not have any representative Lie algebra with a structure tuple of first component equal to e_2. It is the case, for instance, of the isomorphism class $\mathfrak{h}_5^7$ in Table 3.1, with structure tuple $(e_1, e_3, 0, 0)$.

Theorem 3.2.14. *Let T be a structure tuple in $\mathcal{T}_{n,q;m}$ such that A_T is not isomorphic to a Lie algebra in $\mathcal{P}_{n,q;m}$ with product $e_1e_n = e_2$. Then, there exists a non-negative integer $m' \leq m$ with $2m - m' < n$ such that A_T is isomorphic to a Lie algebra of structure tuple $(e_1, t_2e_2, \ldots, t_{m'}e_{m'}, e_{m+1}, \ldots, e_{2m-m'}, 0, \ \ldots, 0) \in \mathcal{T}_{n,q;m}$, where $t_2 \leq \ldots \leq t_{m'}$.*

Proof. Suppose $T = (\sum_{j=1}^{n-1} t_{1j}e_j, \ldots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \ldots, 0) \in \mathcal{T}_{n,q;m}$. The hypothesis is only possible if $t_{ij} = 0$, for all $i, j \leq m$ such that $i \neq j$. Otherwise, we consider the isomorphism $(1i) \in S_n$ that switches the two basis vectors e_1 and e_i. This gives rise to a Lie algebra in $\mathcal{P}_{n,q;m}$ whose structure tuple has a non-zero addend $t_{ij}e_j$ in its first component. Proposition 3.2.11 involves this algebra to be isomorphic to a Lie algebra in $\mathcal{P}_{n,q;m}$ with product $e_1e_n = e_2$, which is a contradiction with the hypothesis. Hence, we have in A_T that

$$e_ie_n = t_{ii}e_i + \sum_{j>m+1} t_{ij}e_j, \text{ for all } i \leq m.$$

Let us define by linearity the isomorphism f such that

$$f(e_i) = \begin{cases} \frac{1}{t_{ii}}(e_i - \sum_{j>m+1} t_{ij}e_j), \text{ if } t_{ii} \neq 0, \\ e_i, \text{ otherwise.} \end{cases}$$

The Lie algebra $(A, \cdot)$ isomorphic to A_T with respect to f verifies that

$$e_i \cdot e_n = \begin{cases} t_{ii}e_i, \text{ if } t_{ii} \neq 0, \\ \sum_{j>m+1} t_{ij}e_j, \text{ otherwise.} \end{cases}$$

Let m' be the number of positive integers $i \leq m$ such that $t_{ii} \neq 0$. Since $A \in \mathcal{P}_{n,q;m}$, there exists $(m - m')$ distinct positive integers $i_1, \ldots, i_{m-m'} \leq m$ and $(m - m')$ distinct positive integers $j_1, \ldots, j_{m-m'} \in \{m + 1, \ldots, n - 1\}$ such that $t_{i_kj_k} \neq 0$, for all $k \leq m - m'$. Let us define by linearity the isomorphism g such that

$$g(e_i) = \begin{cases} e_i, \text{ if } i \notin \{i_1, \ldots, i_{m-m'}\}, \\ \frac{1}{t_{i_kj_k}}(e_{j_k} - \sum_{j \neq j_k} t_{i_kj}e_j), \text{ otherwise.} \end{cases}$$

From Lemma 3.2.7, any relabeling of the basis vectors of the Lie algebra that is isomorphic to A by means of g gives rise to a new isomorphic Lie algebra. This relabeling can be done in such a way that we find $m - 1$ numbers $t_2, \ldots, t_{m'}$, $t_{m+1}, \ldots, t_{2m-m'} \in \mathbb{F}_q \backslash \{0\}$, such that $t_2 \leq \ldots \leq t_{m'}$ so that the Lie algebra of structure tuple $T' = (e_1, t_2e_2, \ldots, t_{m'}e_{m'}, t_{m+1}e_{m+1}, \ldots, t_{2m-m'}e_{2m-m'}, 0, \ldots, 0) \in \mathcal{T}_{n,q;m}$ is isomorphic to A. In particular, it must be $2m - m' < n$. Let us define by linearity the isomorphism h such that

$$h(e_i) = \begin{cases} \frac{1}{t_i}e_i, \text{ if } i \in \{m + 1, \ldots, 2m - m'\}, \\ e_i, \text{ otherwise.} \end{cases}$$

The Lie algebra that is isomorphic to $A_{T'}$, and hence to A_T, by means of h has the required structure tuple $(e_1, t_2e_2, \ldots, t_{m'}e_{m'}, e_{m+1}, \ldots, e_{2m-m'}, 0, \ldots, 0) \in \mathcal{T}_{n,q;m}$. $\square$

Once we have identified the isomorphism classes of $\mathcal{P}_{n,q;m}$ of Theorem 3.2.14, we focus our study on those Lie algebras with structure tuple $T = (e_2, \sum_{j=1}^{n-1} t_{2j}e_j, \dots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \dots, 0) \in \mathcal{T}_{n,q;m}$. We can suppose that $t_{i2} = 0$, for all $i > 2$. Otherwise, we define by linearity the isomorphism f such that $f(e_j) = e_j$ if $j \neq i$ and $f(e_i) = e_i + t_{i2}e_1$.

Proposition 3.2.15. *Let $T = (e_2, \sum_{j=1}^{n-1} t_{2j}e_j, \dots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \dots, 0)$ be a structure tuple in $\mathcal{T}_{n,q;m}$ and let $j_2 = \min\{j < n \mid j \neq 2 \text{ and } t_{2j} \neq 0\}$. Then, the Lie algebra A_T is isomorphic to a Lie algebra $(A, \cdot)$ in $\mathcal{P}_{n,q;m}$ such that $e_1 \cdot e_n = e_2$ and*

$$e_2 \cdot e_n = \begin{cases} e_1 + t_{22}e_2, & \text{if } j_2 = 1, \\ t_{22}e_2 + e_3, & \text{if } 3 \leq j_2 \leq m, \\ t_{22}e_2 + e_{m+1}, & \text{if } j_2 > m. \end{cases}$$

Proof. The result follows similarly to Proposition 3.2.11. □

Similarly to what happens for e_1e_n, the majority of the isomorphism classes of $\mathcal{P}_{n,q;m}$ can be rewritten in such a way that the first component of their structure tuples is e_2. To do it, we give a previous definition. Let m' be a positive integer less than $n-m$ and let us consider the structure tuples $T = (\sum_{j=1}^{n-1} t_{1j}e_j, \dots, \sum_{j=1}^{n-1} t_{m'j}e_j, 0, \dots, 0)$ and $T' = (\sum_{j=1}^{n-m'-1} t'_{1j}e_j, \dots, \sum_{j=1}^{n-m'-1} t'_{(m-m')j}e_j, 0, \dots, 0)$ in $\mathcal{T}_{n,q;m'}$ and $\mathcal{T}_{n-m',q;m-m'}$, respectively. We define the Lie algebra $A_{T\oplus T'} \in \mathcal{F}^p_{n,m}$ of structure tuple

$$T \oplus T' = (\sum_{j=1}^{n-1} t_{1j}e_j, \dots, \sum_{j=1}^{n-1} t_{m'j}e_j, \sum_{j=1}^{n-m'-1} t'_{1j}e_{m'+j}, \dots, \sum_{j=1}^{n-m'-1} t'_{m'j}e_{m'+j}, 0, \dots, 0).$$

Theorem 3.2.16. *Let T be a structure tuple in $\mathcal{T}_{n,q;m}$ of the form $(e_1, \sum_{j=1}^{n-1} t_{2j}e_j, \dots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \dots, 0)$. Then, there exists a structure tuple $T' \in \mathcal{T}_{n-1,q;m-1}$ such that $A_T \cong A_{T_1} \oplus A_{T'}$, where $T_1 = (e_1, 0, \dots, 0) \in \mathcal{T}_{n,q;1}$.*

Proof. We can suppose that $t_{i1} = 0$, for all $i > 1$. Otherwise, we define by linearity the isomorphism f such that $f(e_j) = e_j$ if $j \neq i$ and $f(e_i) = e_i + t_{i1}e_1$. Hence, it is enough to consider the structure tuple

$$T' = (\sum_{j=1}^{n-2} t_{2(j+1)}e_j, \dots, \sum_{j=1}^{n-2} t_{m(j+1)}e_j, 0, \dots, 0) \in \mathcal{T}_{n-1,q;m-1}.$$

□

From Theorem 3.2.16, the isomorphism classes of $\mathcal{P}_{n,q;m}$ having a structure tuple of the form $(e_1, \sum_{j=1}^{n-1} t_{2j}e_j, \ldots, \sum_{j=1}^{n-1} t_{mj}e_j, 0, \ldots, 0)$ are uniquely determined by the structure tuples of $\mathcal{P}_{n-1,q;m-1}$. We can suppose that $t_{i2} = 0$, for all $i \geq 2$. Otherwise, we define by linearity the isomorphism f such that $f(e_i) = e_i + t_{i2}e_1$ and $f(e_j) = e_j$ if $j \neq i$. It is enough to distinguish the next three cases, from which we would follow a similar reasoning for subsequent components

- **Case 1**. $t_{i1} = 0$, for all $i \geq 2$.
- **Case 2**. $t_{21} \neq 0$.
- **Case 3**. $t_{21} = 0$ and there exists $i > 2$ such that $t_{i1} \neq 0$.

We have made use of all the precedent results in order to determine the distribution of $\mathcal{P}_{5,3}$ and $\mathcal{P}_{n,5}$, for $n \leq 4$, into isomorphism classes, which we expose, respectively, in Tables 3.4 and 3.5. The possible isomorphisms among distinct pre-filiform Lie algebras of both sets have been determined by means of the implementation of Algorithm 2 and both procedures *isoAlg* and *isoGraph* in SINGULAR, which were all of them introduced in Chapter 2. Besides, in order to compute each distribution, we have made an exhaustive search in those structure tuples of the corresponding sets $\mathcal{T}_{n,q;m}$ that satisfy the results that have been exposed throughout this section. More specifically, the computation focuses on those structure tuples whose first component is e_1, e_2 or e_{m+1}. The run time that is required to compute both distributions is 1457 seconds in our system, with a mean time of 1 second to compute the reduced Gröbner basis related to the set of isomorphisms between each pair of pre-filiform Lie algebras under consideration. A random search among pairs of pre-filiform Lie algebras in $\mathcal{P}_{5,5}$ enables us to ensure that the bottleneck of this last computation occurs when we deal with this set, for which storage memory seems to be a problem for the required computation. Further work to reduce the cost of computation is, therefore, required in this regard to determine the exact distribution into isomorphism classes of pre-filiform Lie algebras of higher orders.

3.3 Filiform Lie algebras over finite fields

This section deals with the distribution of n-dimensional filiform Lie algebras into isomorphism and isotopism classes. For $n \leq 6$, this distribution is explicitly obtained over any field. For $n = 7$, this is determined over algebraically closed fields and over

A_T	T	A_T	T	A_T	T
$\mathfrak{g}_5^1$	$(0,0,0,0)$	$\mathfrak{g}_5^2$	$(e_2,0,0,0)$	$\mathfrak{g}_5^3$	$(e_1,0,0,0)$
$\mathfrak{g}_5^4$	$(e_3,e_4,0,0)$	$\mathfrak{g}_5^5$	$(e_2,e_3,0,0)$	$\mathfrak{g}_5^6$	$(e_1,e_3,0,0)$
$\mathfrak{g}_5^7$	$(e_2,e_1+e_2,0,0)$	$\mathfrak{g}_5^8$	$(e_2,e_1,0,0)$	$\mathfrak{g}_5^9$	$(e_2,2e_1,0,0)$
$\mathfrak{g}_5^{10}$	$(e_2,2e_1+e_2,0,0)$	$\mathfrak{g}_5^{11}$	$(e_1,e_2,0,0)$	$\mathfrak{g}_5^{12}$	$(e_1,e_3,e_2+e_3,0)$
$\mathfrak{g}_5^{13}$	$(e_1,e_3,e_2+2e_3,0)$	$\mathfrak{g}_5^{14}$	$(e_2,e_1+e_3,e_1+2e_2,0)$	$\mathfrak{g}_5^{15}$	$(e_1,2e_3,e_2,0)$
$\mathfrak{g}_5^{16}$	$(e_1,2e_3,e_2+2e_3,0)$	$\mathfrak{g}_5^{17}$	$(e_1+e_3,2e_3,e_2+2e_3,0)$	$\mathfrak{g}_5^{18}$	$(e_2,e_4,e_3,0)$
$\mathfrak{g}_5^{19}$	$(e_2,e_2+e_4,e_3,0)$	$\mathfrak{g}_5^{20}$	$(e_2,2e_2+e_4,e_3,0)$	$\mathfrak{g}_5^{21}$	$(e_2,e_2+e_4,e_2+e_3,0)$
$\mathfrak{g}_5^{22}$	$(e_2,e_3,e_1+2e_3,0)$	$\mathfrak{g}_5^{23}$	$(e_2+e_3,e_3,e_1,0)$	$\mathfrak{g}_5^{24}$	$(e_2+e_3,e_3,e_1+e_3,0)$
$\mathfrak{g}_5^{25}$	$(e_2+2e_3,e_3,e_1+2e_3,0)$	$\mathfrak{g}_5^{26}$	$(e_2,e_3,e_4,0)$	$\mathfrak{g}_5^{27}$	$(e_2,e_3+e_4,e_2+e_4,0)$
$\mathfrak{g}_5^{28}$	$(e_2,e_3,2e_2+e_4,0)$	$\mathfrak{g}_5^{29}$	$(e_2,e_1,e_3,0)$	$\mathfrak{g}_5^{30}$	$(e_1,e_2,e_3,0)$
$\mathfrak{g}_5^{31}$	(e_4,e_3,e_2,e_1)	$\mathfrak{g}_5^{32}$	(e_4,e_3,e_2,e_1+e_4)	$\mathfrak{g}_5^{33}$	(e_4,e_3,e_2+e_4,e_1)
$\mathfrak{g}_5^{34}$	$(e_4,e_3+e_4,e_2+e_4,e_1)$	$\mathfrak{g}_5^{35}$	$(e_4,e_3,e_2,2e_1)$	$\mathfrak{g}_5^{36}$	$(e_4,e_3,e_2,2e_1+e_4)$
$\mathfrak{g}_5^{37}$	$(e_4,e_3,e_2+2e_4,2e_1+e_4)$	$\mathfrak{g}_5^{38}$	$(e_4,e_3,e_2+e_3,e_1+e_4)$	$\mathfrak{g}_5^{39}$	$(e_4,e_3,e_2+e_3,e_1+2e_4)$
$\mathfrak{g}_5^{40}$	$(e_4,e_3,e_2+e_3+e_4,e_1+e_4)$	$\mathfrak{g}_5^{41}$	$(e_4,e_3,e_2+e_3,2e_1)$	$\mathfrak{g}_5^{42}$	$(e_4,e_3,e_2+e_3,2e_1+e_4)$
$\mathfrak{g}_5^{43}$	$(e_4,e_3,e_2+e_3,2e_1+2e_4)$	$\mathfrak{g}_5^{44}$	$(e_4,2e_3,e_2,2e_1)$	$\mathfrak{g}_5^{45}$	$(e_4,2e_3,e_2,2e_1+e_4)$
$\mathfrak{g}_5^{46}$	$(e_4,2e_3,e_2+2e_4,2e_1)$	$\mathfrak{g}_5^{47}$	$(e_4,2e_3,e_2+e_3,2e_1+e_4)$	$\mathfrak{g}_5^{48}$	$(e_4,2e_3,e_2+e_3+2e_4,2e_1+e_4)$
$\mathfrak{g}_5^{49}$	$(e_4,2e_3+2e_4,e_2+e_3+2e_4,2e_1+e_4)$	$\mathfrak{g}_5^{50}$	(e_2,e_4,e_1,e_3+e_4)	$\mathfrak{g}_5^{51}$	(e_2,e_4,e_1+e_2,e_3)
$\mathfrak{g}_5^{52}$	$(e_2,2e_4,e_1,e_3+e_4)$	$\mathfrak{g}_5^{53}$	$(e_2,e_4,e_1+e_2,2e_3)$	$\mathfrak{g}_5^{54}$	$(e_2,2e_2+e_4,e_1+e_2,2e_3)$
$\mathfrak{g}_5^{55}$	(e_2,e_4,e_1+e_4,e_3)	$\mathfrak{g}_5^{56}$	$(e_2,e_4,e_1+e_4,e_3+e_4)$	$\mathfrak{g}_5^{57}$	$(e_2,e_2+e_4,e_1+e_2,e_2+e_3)$
$\mathfrak{g}_5^{58}$	$(e_2,2e_2+e_4,e_1+e_2,e_2+e_3)$	$\mathfrak{g}_5^{59}$	$(e_2,2e_4,e_1+e_4,e_3+e_4)$	$\mathfrak{g}_5^{60}$	$(2e_2,e_4,e_1+e_2,e_2+e_3)$
$\mathfrak{g}_5^{61}$	(e_2,e_4,e_1+2e_4,e_3)	$\mathfrak{g}_5^{62}$	$(e_2,e_4,e_1+e_2,2e_2+e_3)$	$\mathfrak{g}_5^{63}$	$(e_2,e_2+e_4,e_1+e_2,2e_2+e_3)$
$\mathfrak{g}_5^{64}$	$(e_2,2e_2+e_4,e_1+e_2,2e_2+e_3)$	$\mathfrak{g}_5^{65}$	$(e_2,e_2+2e_4,e_1,e_2+e_3)$	$\mathfrak{g}_5^{66}$	$(e_2,2e_4,e_1+2e_2,e_2+e_3)$
$\mathfrak{g}_5^{67}$	$(e_2,e_2+2e_4,e_1+2e_2,e_2+e_3)$	$\mathfrak{g}_5^{68}$	(e_4,e_2,e_3,e_1)	$\mathfrak{g}_5^{69}$	(e_4,e_2,e_3,e_1+e_4)
$\mathfrak{g}_5^{70}$	(e_4,e_2,e_3,e_1+2e_4)	$\mathfrak{g}_5^{71}$	$(e_4,e_2,e_3,2e_1)$	$\mathfrak{g}_5^{72}$	$(e_4,e_2,e_3,2e_1+2e_4)$
$\mathfrak{g}_5^{73}$	(e_1,e_2,e_3,e_4)				

Table 3.4: Distribution of $\mathcal{P}_{5,3}$ into isomorphism classes.

finite fields. Previously, let us recall some preliminary results on this family of Lie algebras.

3.3.1 Preliminaries on filiform Lie algebras

An n-dimensional nilpotent Lie algebra A with maximum nil-index n is called *filiform*. In this case, dim $\mathcal{C}^k(A) = n - k$, for all $k \in \{2, \ldots, n\}$. Filiform Lie algebras constitute the most structured subset of nilpotent Lie algebras and have a large number of applications in Applied Mathematics, Engineering and Physics [147, 148]. They were introduced formally by Vergne [308, 309] in the late 1960s, although Umlauf had already used them as an example in his thesis [303].

The distribution into isomorphism classes of n-dimensional filiform Lie algebras over the complex field is known for $n \leq 12$ [49, 140], whereas it is only known for nilpotent Lie algebras over the complex field of dimension $n \leq 7$ [30]. More recently, some authors have dealt with the classification of n-dimensional nilpotent

A_T	T	A_T	T	A_T	T			
2	$\mathfrak{f}_2^1$	(0)		$\mathfrak{f}_2^2$	(e_1)	3	$\mathfrak{f}_3^1$	$(0,0)$
	$\mathfrak{f}_3^2$	$(e_2,0)$		$\mathfrak{f}_3^3$	$(e_1,0)$		$\mathfrak{f}_3^4$	(e_2,e_1)
	$\mathfrak{f}_3^5$	$(e_2,2e_1)$		$\mathfrak{f}_3^6$	(e_2,e_1+e_2)		$\mathfrak{f}_3^7$	$(e_2,2e_1+e_2)$
	$\mathfrak{f}_3^8$	(e_2,e_1+2e_2)		$\mathfrak{f}_3^9$	$(e_2,2e_1+2e_2)$		$\mathfrak{f}_3^{10}$	(e_1,e_2)
4	$\mathfrak{f}_4^1$	$(0,0,0)$		$\mathfrak{f}_4^2$	$(e_2,0,0)$		$\mathfrak{f}_4^3$	$(e_1,0,0)$
	$\mathfrak{f}_4^4$	$(e_2,e_1,0)$		$\mathfrak{f}_4^5$	$(e_2,2e_1,0)$		$\mathfrak{f}_4^6$	$(e_2,e_1+e_2,0)$
	$\mathfrak{f}_4^7$	$(e_2,2e_1+e_2,0)$		$\mathfrak{f}_4^8$	$(e_2,e_1+2e_2,0)$		$\mathfrak{f}_4^9$	$(e_2,2e_1+2e_2,0)$
	$\mathfrak{f}_4^{10}$	$(e_2,e_3,0)$		$\mathfrak{f}_4^{11}$	$(e_2,e_2+e_3,0)$		$\mathfrak{f}_4^{12}$	$(e_1,e_2,0)$
	$\mathfrak{f}_4^{13}$	(e_2,e_3,e_1)		$\mathfrak{f}_4^{14}$	(e_1,e_3,e_2+e_3)		$\mathfrak{f}_4^{15}$	(e_1,e_3,e_2+2e_3)
	$\mathfrak{f}_4^{16}$	(e_1,e_3,e_2+3e_3)		$\mathfrak{f}_4^{17}$	(e_1,e_3,e_2+4e_3)		$\mathfrak{f}_4^{18}$	(e_2,e_1,e_2+e_3)
	$\mathfrak{f}_4^{19}$	$(e_2,2e_3,2e_1)$		$\mathfrak{f}_4^{20}$	$(e_1,2e_3,e_2+e_3)$		$\mathfrak{f}_4^{21}$	$(e_1,2e_3,e_2+2e_3)$
	$\mathfrak{f}_4^{22}$	$(e_1,2e_3,e_2+3e_3)$		$\mathfrak{f}_4^{23}$	$(e_1,2e_3,e_2+4e_3)$		$\mathfrak{f}_4^{24}$	$(e_2,3e_3,3e_1)$
	$\mathfrak{f}_4^{25}$	$(e_1,3e_3,e_2+e_3)$		$\mathfrak{f}_4^{26}$	$(e_1,3e_3,e_2+3e_3)$		$\mathfrak{f}_4^{27}$	$(e_1,3e_3,e_2+4e_3)$
	$\mathfrak{f}_4^{28}$	$(e_1,4e_3,e_2+e_3)$		$\mathfrak{f}_4^{29}$	$(e_1,4e_3,e_2+2e_3)$		$\mathfrak{f}_4^{30}$	$(e_1,4e_3,e_2+4e_3)$
	$\mathfrak{f}_4^{31}$	$(e_2,4e_1+2e_2,4e_2+e_3)$		$\mathfrak{f}_4^{32}$	(e_2,e_3,e_1+e_3)		$\mathfrak{f}_4^{33}$	(e_2,e_3,e_1+2e_3)
	$\mathfrak{f}_4^{34}$	(e_2,e_2+e_3,e_1+e_2)		$\mathfrak{f}_4^{35}$	$(e_2,3e_2+e_3,e_1+e_2)$		$\mathfrak{f}_4^{36}$	$(e_2,2e_3,e_1+3e_2)$
	$\mathfrak{f}_4^{37}$	$(e_2,2e_2+2e_3,e_1+3e_2)$		$\mathfrak{f}_4^{38}$	$(e_2,4e_2+2e_3,e_1+3e_2)$		$\mathfrak{f}_4^{39}$	(e_2,e_2+e_3,e_1+2e_2)
	$\mathfrak{f}_4^{40}$	$(e_2,4e_2+e_3,e_1+2e_2)$		$\mathfrak{f}_4^{41}$	$(e_2,2e_3,e_1+e_2)$		$\mathfrak{f}_4^{42}$	(e_1,e_2,e_3)

Table 3.5: Distribution of $\mathcal{P}_{n,5}$ into isomorphism classes, for $n \leq 4$.

Lie algebras over finite fields $\mathbb{F}_q$, with q a power prime. Specifically, Schneider [278] obtained the number of isomorphism classes over the finite field $\mathbb{F}_2$, for $n \leq 9$, and over $\mathbb{F}_3$ and $\mathbb{F}_5$, for $n \leq 7$. The classification of six-dimensional nilpotent Lie algebras over a field of characteristic distinct from two was determined by de Graaf [150] and over any arbitrary field by Cicalò et al. [87]. With respect to the classification of filiform Lie algebras over $\mathbb{F}_q$, Schneider obtained in particular that there exist six six-dimensional filiform Lie algebras over $\mathbb{F}_2$ and five over $\mathbb{F}_3$ and $\mathbb{F}_5$; whereas there exist 15 seven-dimensional filiform Lie algebras over $\mathbb{F}_2$, 11 over $\mathbb{F}_3$ and 13 over $\mathbb{F}_5$.

Hereafter, we denote by $\mathcal{F}_n(\mathbb{K})$ the set of n-dimensional filiform Lie algebras over a base field $\mathbb{K}$. The only algebra in $\mathcal{F}_2(\mathbb{K})$ is the abelian. Let $n > 2$ and let A be a filiform Lie algebra in $\mathcal{F}_n(\mathbb{K})$. Its type is $\{2, 1, \ldots, 1\}$. Besides, $D_1(A) = n$ and $d_n(A) = 1$, where d_n and D_1 refer to the isotopism invariants that were respectively described in (3.4) and (3.5). The next two numbers are isomorphism invariants in $\mathcal{F}_n(\mathbb{K})$ [112, 113]

$$z_1(A) := \max\{k \in \mathbb{N} \mid \operatorname{Cen}_A(\mathcal{C}^{n-k+2}(A)) \supset A^2\}. \tag{3.7}$$

$$z_2(A) := \max\{k \in \mathbb{N} \mid \mathcal{C}^{n-k+1}(A) \text{ is abelian}\}. \tag{3.8}$$

It is always possible to find a basis $\{e_1, \ldots, e_n\}$ of the algebra A that is *compatible* *with respect to its lower central series*, that is, such that

$$\mathcal{C}^2(A) = \langle e_2, \ldots, e_{n-1}\rangle,\ \mathcal{C}^3(A) = \langle e_2, \ldots, e_{n-2}\rangle, \ldots, \mathcal{C}^{n-1}(A) = \langle e_2\rangle,\ \mathcal{C}^n(A) = 0.$$

Vergne [308, 309] proved the existence of an *adapted basis* of A whenever the characteristic of the base field $\mathbb{K}$ is zero. This is a compatible basis of A with respect to its lower central series such that

$$\begin{cases} e_1e_i = e_{i-1}, \text{ for all } i \in \{3, \ldots, n\}, \\ e_3e_n = 0. \end{cases} \tag{3.9}$$

It is not always possible, however, to find an adapted basis if the characteristic of the base field is not zero. An alternative basis is required in this case in order to facilitate the distribution of filiform Lie algebras into isomorphism and isotopism classes. Let $\{e_1, \ldots, e_n\}$ be a compatible basis with respect to the lower central series of A. The set $\{e_1, \mathrm{ad}_{e_1}^{n-1}(e_n), \ldots, \mathrm{ad}_{e_1}(e_n)\}$ also constitutes a basis of A. For each positive integer $i < n$, let us relabel the vector $\mathrm{ad}_{e_1}^{i}(e_n)$ as e_{n-i+1}. We can then suppose that $e_3e_n = 0$. Otherwise, it must be $e_3e_n = ae_2$, for some $a \in \mathbb{K} \setminus \{0\}$ and then, it is enough to replace the vector e_n by $-ae_1 + e_n$. We call the new basis a *filiform basis* of A. Unlike adapted bases, a filiform basis can not be compatible with respect to the lower central series of the algebra. In fact, this compatibility would involve the filiform basis to be an adapted basis. To prove this, observe assertion (a) in the next result.

Lemma 3.3.1. *Let $\{e_1, \ldots, e_n\}$ be a filiform basis of an algebra $A \in \mathcal{F}_n(\mathbb{K})$. Then,*

a) $e_1e_i = e_{i-1}$, for all $i \in \{3, \ldots, n\}$.

b) $Z(A) = \langle e_2\rangle$.

c) $e_ie_j \in \langle e_2, \ldots, e_{n-1}\rangle$, whenever $3 \leq i, j \leq n$.

d) $e_3u = 0$, for all $u \in A^2$.

e) $D_{n-1}(A) = \dim(\mathrm{Cen}(\langle e_2, \ldots, e_n \rangle))$.

Proof. (a)-(c) follow straightforward from the definition of the basis. Now, the compatibility with respect to the lower central series of A of the basis from which our filiform basis is derived involves that $e_3u \in \langle e_2 \rangle$, for all $u \in A^2$. (d) follows then from

(a) and (b), and the fact of being $J(e_1, e_3, u) = 0$, for all $u \in A$. Finally, (e) follows from the description of the filiform basis and the fact of being $\mathrm{Cen}(\langle e_1 \rangle) = \langle e_2 \rangle$, due to assertions (a) and (b). $\square$

A filiform Lie algebra A is said to be *model* if the only nonzero products between the elements of a compatible basis with respect to its lower central series are

$$e_1 e_i = e_{i-1} \text{ for all } i \in \{3, \ldots, n\}. \tag{3.10}$$

Observe that the n-dimensional model algebra is not isomorphic to any other algebra of the same dimension, because it is the only filiform Lie algebra for which the isomorphism invariant z_1 does not exist. This result also holds for isotopisms.

Proposition 3.3.2. *The n-dimensional model algebra is not isotopic to any other filiform Lie algebra of the same dimension.*

Proof. Let A be the n-dimensional model algebra of filiform basis $\{e_1, \ldots, e_n\}$. Then, $D_m(A') = n - 1$, for all $m \in \{2, \ldots, n-1\}$, because the centralizer of the m-dimensional vector subspace that is generated by the m basis vectors $e_2, \ldots, e_{2+m-1}$ is the $(n-1)$-dimensional vector subspace generated by $e_2, \ldots, e_n$. Besides, the centralizer cannot be n-dimensional because of the non-zero structure constants related to the basis vector e_1. Now, let A' be a non-model n-dimensional filiform Lie algebra of filiform basis $\{e'_1, \ldots, e'_n\}$. From assertion (e) in Lemma 3.3.1, the value of $D_{n-1}(A')$ coincides with the dimension of the centralizer of the vector subspace generated by the basis vectors $e'_2, \ldots, e'_n$. This dimension is less than $n-1$ because e'_1 cannot be in this centralizer and there exists a pair of positive integers $i, j \leq n$, with $i \neq 1 \neq j$, such that $e'_i e'_j$ is not zero. Hence, $D_{n-1}(A') < n - 1$. The result follows then from Proposition 3.1.3. $\square$

The only isotopism (isomorphism) class of the set of filiform Lie algebras of dimension $n \leq 4$ corresponds to the model algebra of such a dimension. For $n = 5$, there exist two isomorphism classes of filiform Lie algebras: the model algebra and that having an adapted basis satisfying the product $e_4 e_5 = e_2$. Both classes determine indeed distinct isotopism classes, because the isotopism invariant D_4 is equal to 4 for the model algebra, but it is equal to 2 for the second exposed algebra. For higher dimensions, the distribution into isomorphism and isotopism classes requires a more detailed study of the corresponding structure constants. The cases $n \in \{6, 7\}$

are analyzed in the next two subsections. The next results on adapted bases of a filiform Lie algebra are useful in our study.

Lemma 3.3.3. *Let f be an isomorphism between two isomorphic n-dimensional filiform Lie algebras A and A'. If $\{e_1, \ldots, e_n\}$ is an adapted basis of A with respect to its lower central series, then $f(e_i) \in A'^{n-i+1}$, for all $i \in \{2, \ldots, n-1\}$.*

Proof. This result follows straightforward from the compatibility of any adapted basis with respect to the lower central series of the algebra. □

Proposition 3.3.4. *Under the hypothesis of Lemma 3.3.3, let $F = (f_{ij})$ be the regular matrix related to the isomorphism f between the filiform Lie algebras A and A'. Then,*

a) $f_{ij} = 0$, *for all* $i, j < n$ *such that* $1 < i < j$.

b) $f_{21} = f_{23} = \ldots = f_{2n} = 0$.

c) $f_{n1} = 0$.

d) $f_{22} = f_{11}f_{33}$.

e) $f_{(n-1)(n-1)} = f_{11}f_{nn}$.

Proof. Let $\{e'_1, \ldots, e'_n\}$ be an adapted basis of the algebra A'. Let us prove each assertion separately.

a) This follows straightforward from Lemma 3.3.3.

b) From Proposition 3.1.2, since $Z(A) = \langle e_2 \rangle$, it must be $f(e_2) \in Z(A') = \langle e'_2 \rangle$ and hence, (b) holds.

c) From Proposition 3.1.2, $f(e_3) \in \langle e_2, e_3 \rangle$. Then, $0 = f(0) = f(e_3e_n) = f(e_3)f(e_n)$ $= -f_{33}f_{n1}e'_2$. The result follows, therefore, from the regularity of the matrix F and assertion (a).

The other two assertions follow, respectively, from the fact of being $f(e_i) = f(e_1e_{i+1}) = f(e_1)f(e_{i+1})$, for $i \in \{2, n-1\}$. □

Since $A^2 = \langle e_2, \ldots, e_{n-1}\rangle$, the numbers f_{12} and f_{n2} do not have any influence on the isomorphism f. We can suppose, therefore, that $f_{12} = f_{n2} = 0$. This fact, together with Lemma 3.3.3 and Proposition 3.3.4, involve that

$$F = \begin{pmatrix} f_{11} & 0 & f_{13} & \cdots & f_{1(n-1)} & f_{1n} \\ 0 & f_{11}f_{33} & 0 & \cdots & 0 & 0 \\ 0 & f_{32} & f_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_{(n-1)2} & f_{(n-1)3} & \cdots & f_{(n-1)(n-1)} & 0 \\ 0 & 0 & f_{n3} & \cdots & f_{n(n-1)} & f_{nn} \end{pmatrix} \tag{3.11}$$

3.3.2 Classification of six-dimensional filiform Lie algebras.

The distribution of the set $\mathcal{F}_6(\mathbb{K})$ into isomorphism classes is already known [87, 150, 278]. It is interesting, however, to deal again with this distribution in order to expose in detail the algebraic methodology related to the new isotopism invariants. Higher dimensions are similarly dealt with, but they require more extensive and tedious computation. The next result is fundamental in our study and follows straightforward from a case study based on the Jacobi identity in $\mathcal{F}_6(\mathbb{K})$.

Lemma 3.3.5. *Let $A \in \mathcal{F}_6(\mathbb{K})$. Then, there exist three numbers $a, b, c \in \mathbb{K}$ and a filiform basis of A such that*

$$A \cong A_{abc} := \begin{cases} e_1e_i = e_{i-1}, \ \textit{for all } i \in \{3,4,5,6\}, \\ e_4e_5 = ae_2, \\ e_4e_6 = be_2 + ae_3, \\ e_5e_6 = ce_2 + be_3 + ae_4. \end{cases}$$

□

The filiform basis of A_{abc} constitutes indeed an adapted basis, because it is compatible with respect to the lower central series of the algebra.

Hereafter, given an algebra A and a positive integer $i \in \mathbb{N}$, we denote by $A^{(i)}$ the quotient algebra $A^{(i-1)}/Z(A^{(i-1)})$, where we consider $A^{(0)} = A$. Particularly, two algebras A and A' are isomorphic only if the algebras $A^{(i)}$ and $A'^{(i)}$ are isomorphic.

Our filiform Lie algebra A_{abc} holds then the next isotopism invariants

$$\begin{cases} D_5(A_{000}) = 5, \\ \text{If } c \neq 0, \text{ then } D_5(A_{00c}) = 3, \\ \text{If } b \neq 0, \text{ then } D_5(A_{0bc}) = 2 = D_4({A_{0bc}}^{(1)}), \\ \text{If } a \neq 0, \text{ then } D_5(A_{abc}) = 2 > 1 = D_4({A_{abc}}^{(1)}). \end{cases} \tag{3.12}$$

Proposition 3.1.3 involves, therefore, the existence of at least four isotopism (and hence, isomorphism) classes in $\mathcal{F}_6(\mathbb{K})$. Since the model algebra A_{000} constitutes an isomorphism class by itself, we focus on the rest of cases.

Proposition 3.3.6. *Let $c, \gamma \in \mathbb{K}\backslash\{0\}$. Then, $A_{00c} \cong A_{00\gamma}$.*

Proof. It is enough to consider the isomorphism f that is linearly defined from $f(e_1) = e_1$ and $f(e_i) = \frac{c}{\gamma} e_i$, for all $i > 1$. □

The remaining cases require an analysis of the entries of the matrix $F = (f_{ij})$ of an isomorphism between two algebras A_{abc} and $A_{\alpha\beta\gamma}$ in $\mathcal{F}_6(\mathbb{K})$, with respective adapted bases $\{e_1, \ldots, e_6\}$ and $\{e'_1, \ldots, e'_6\}$. Its form has been exposed in (3.11). The central rows of F are determined by using the entries of their first and last rows.

Lemma 3.3.7. *The entries of the matrix F hold the next equalities.*

a) $f_{44} = (f_{11} - \alpha f_{16}) f_{55}$.

b) $f_{33} = (f_{11} - \alpha f_{16}) f_{44}$.

c) $f_{54} = (f_{11} - \alpha f_{16}) f_{65} + \alpha f_{15} f_{66}$.

d) $f_{43} = (f_{11} - \alpha f_{16}) f_{54} - \beta f_{55} f_{16}$.

e) $f_{32} = f_{11} f_{43} - (\alpha f_{15} + \beta f_{16}) f_{44}$.

f) $f_{53} = (f_{11} - \alpha f_{16}) f_{64} + \alpha f_{14} f_{66} + \beta (f_{15} f_{66} - f_{65} f_{16})$.

g) $f_{42} = f_{11} f_{53} - (\alpha f_{15} + \beta f_{16}) f_{54} + (\alpha f_{14} - \gamma f_{16}) f_{55}$.

h) $f_{52} = f_{11} f_{63} + \alpha (f_{14} f_{65} - f_{64} f_{15}) + \beta (f_{14} f_{66} - f_{64} f_{16}) + \gamma (f_{15} f_{66} - f_{65} f_{16})$.

Proof. All the equalities follow from the fact that $f(e_i) = f(e_1e_{i+1}) = f(e_1)\, f(e_{i+1})$, for all $i \in \{2, \ldots, n-1\}$. $\square$

Since F is a regular matrix, its determinant is distinct from zero. Lemma 3.3.7 involves then that

$$f_{11}f_{66}(f_{11} - \alpha f_{16}) \neq 0. \tag{3.13}$$

This condition, together with the fact of being f an isomorphism, involves the rest of constraints for the entries of F. We have used the procedure *isoAlg* in SINGULAR, which was described in Chapter 2, in order to determine the reduced Gröbner basis related to the ideal of polynomials generated by all these constraints. This enables us to ensure the next conditions

$$af_{11} = \alpha(f_{66} + af_{16}), \tag{3.14}$$

$$b(f_{11} - \alpha f_{16})^2 = \beta(f_{66} + af_{16}) = 0, \tag{3.15}$$

$$\begin{aligned} &f_{11}f_{66}(cf_{11}(f_{11} - \alpha f_{16})^2 + b\beta(\alpha f_{16} - 2f_{11})f_{16} + 2a\alpha f_{14} - a\gamma f_{16} \\ &+2\alpha f_{64} - \gamma f_{66}) - \alpha f_{65}^2(f_{11} - \alpha f_{16}) - \alpha^2(af_{15} + 2f_{65})f_{15}f_{66} = 0. \end{aligned} \tag{3.16}$$

Proposition 3.3.8. *Let A_{0bc} and $A_{0\beta\gamma}$ be two algebras in $\mathcal{F}_6(\mathbb{K})$ such that $b \neq 0 \neq \beta$. They are isomorphic whenever the characteristic of the base field $\mathbb{K}$ is not two. Otherwise, $A_{0bc} \cong A_{011}$ if and only if $c \neq 0$. If $c = 0$, then $A_{0b0} \cong A_{010}$.*

Proof. Impose $a = A = 0$ in Equations (3.13 – 3.16). Then,

$$\begin{cases} bf_{11}^2 = \beta f_{66}, \\ cf_{11}^4 - 2\beta^2 f_{16}f_{66} = \gamma f_{11}f_{66}. \end{cases}$$

Take $f_{66} = bf_{11}^2/\beta$. If the characteristic of the base field $\mathbb{K}$ is not two, then it is enough to consider $f_{16} = (\beta cf_{11}^2 - b\gamma f_{11})/2\beta^2 b$ in order to define the required isomorphism. If the characteristic of $\mathbb{K}$ is two, then $cf_{11} = b\gamma/\beta$ and hence, $c = 0$ if and only if $\gamma = 0$. As a consequence, A_{010} is not isomorphic to A_{011}. Further, if $c \neq 0 \neq C$, then we take $f_{11} = b\gamma/\beta c$ in order to define the isomorphism between A_{0bc} and $A_{0\beta\gamma}$. $\square$

Equation (3.16) can be used to fix an entry in F that does not appear in Equations (3.13 – 3.15). If $a \neq 0 \neq \alpha$, then the variable f_{64} can be isolated in Equation (3.16) whenever the characteristic of the base field $\mathbb{K}$ is not two, or the variable f_{65}, otherwise. The following result holds from Condition (3.13) and the isotopism invariants (3.12).

Theorem 3.3.9. *The next assertions hold in* $\mathcal{F}_6(\mathbb{K})$.

a) If $a \neq 0 \neq \alpha$, *then* $A_{a0c} \cong A_{\alpha 0 \gamma}$.

b) If $a \neq 0 \neq \alpha$ *and* $b \neq 0 \neq \beta$, *then* $A_{abc} \cong A_{\alpha\beta\gamma}$.

c) None of the algebras of (a) is isomorphic to one of (b). □

The previous results establish the following distribution of the set $\mathcal{F}_6(\mathbb{K})$ into isomorphism and isotopism classes.

Theorem 3.3.10. *If the characteristic of the base field* $\mathbb{K}$ *is two, then the set* $\mathcal{F}_6(\mathbb{K})$ *is distributed into six isomorphism classes, which are described by the filiform algebras*

$$A_{000}, A_{001}, A_{010}, A_{011}, A_{100} \text{ and } A_{110}.$$

Otherwise, if the characteristic of $\mathbb{K}$ *is not two, then the distribution of the set* $\mathcal{F}_6(\mathbb{K})$ *into isomorphism classes coincides with the previous list up to the algebra* A_{011}, *which is isomorphic to* A_{010}. □

Proposition 3.3.11. *The set* $\mathcal{F}_6(\mathbb{K})$ *is distributed into five isotopism classes, which are described by the filiform algebras*

$$A_{000}, A_{001}, A_{010}, A_{100} \text{ and } A_{110}.$$

Proof. From Theorem 3.3.10 and the isotopism invariants (3.12), there exist at least four isotopism classes in $\mathcal{F}_6(\mathbb{K})$. We observe that $A_{011} \sim A_{010}$ by the strong isotopism (f, f, h), where f and h are linearly defined from

$$f(e_4) = e_4 - e_3 \text{ and } f(e_i) = e_i, \text{ if } i \neq 4,$$

$$h(e_3) = e_3 - e_2 \text{ and } h(e_i) = e_i, \text{ if } i \neq 3.$$

On the other hand, a simple case study on possible isotopisms between the Lie algebras A_{100} and A_{110} determines that these two algebras are not isotopic, whatever the ground field is. Therefore, there exist exactly five isotopism classes in $\mathcal{F}_6(\mathbb{K})$. □

3.3.3 Classification of seven-dimensional filiform Lie algebras.

The study of the Jacobi identity related to seven-dimensional filiform bases enables us to ensure the next result.

Lemma 3.3.12. *Let $A \in \mathcal{F}_7(\mathbb{K})$. Then, there exist a tuple $(\alpha, \beta, \gamma, \delta, \epsilon, \varphi) \in \mathbb{K}^6$ and a filiform basis $\{e_1, \ldots, e_7\}$ of A such that*

$$\begin{cases} e_1e_i = e_{i-1}, \ \textit{for all } i \in \{3,4,5,6,7\}, \\ e_4e_5 = \alpha e_2, \\ e_4e_6 = \beta e_2 + \alpha e_3, \\ e_4e_7 = \gamma e_2 + \beta e_3 + \alpha e_4, \\ e_5e_6 = \delta e_2 + \beta e_3 + \alpha e_4, \\ e_5e_7 = \epsilon e_2 + (\gamma+\delta)e_3 + 2\beta e_4 + 2\alpha e_5, \\ e_6e_7 = \varphi e_2 + \epsilon e_3 + (\gamma+\delta)e_4 + 2\beta e_5 + 2\alpha e_6, \end{cases}$$

where $3\alpha^2 = 0$, $5\alpha\beta = 0$ and $2\beta^2 + 3\alpha\delta - 2\alpha\gamma = 0$. □

We distinguish three cases depending on the characteristic of the base field.

Characteristic distinct from two and three

Let $\mathbb{K}$ be a field whose characteristic is distinct from two and three. From Lemma 3.3.12, any algebra $A \in \mathcal{F}_7(\mathbb{K})$ has an adapted basis $\{e_1, \ldots, e_7\}$ such that

$$A \cong A_{abcd} := \begin{cases} e_1e_i = e_{i-1}, \ \text{for all } i \in \{3,4,5,6,7\}, \\ e_4e_7 = ae_2, \\ e_5e_6 = be_2, \\ e_5e_7 = ce_2 + (a+b)e_3, \\ e_6e_7 = de_2 + ce_3 + (a+b)e_4, \end{cases}$$

for some tuple $(a, b, c, d) \in \mathbb{K}^4$. The next isotopism invariants hold

$$\begin{cases} D_6(A_{0000}) = 6, \\ \text{If } d \neq 0, \text{ then } D_6(A_{000d}) = 4, \\ \text{If } c \neq 0, \text{ then } D_6(A_{00cd}) = 3 = D_5({A_{00cd}}^{(1)}), \\ \text{If } b \neq 0, \text{ then } D_6(A_{0bcd}) = 3 > 2 = D_5({A_{0bcd}}^{(1)}), \\ \text{If } 0 \neq a \neq -b, \text{ then } D_6(A_{abcd}) = 2 = D_5({A_{abcd}}^{(1)}), \\ \text{If } a \neq 0, \text{ then } D_6(A_{a(-a)0d}) = 2 < 5 = D_5({A_{a(-a)0d}}^{(1)}), \\ \text{If } a \neq 0 \neq c, \text{ then } D_6(A_{a(-a)cd}) = 2 < 3 = D_5({A_{a(-a)cd}}^{(1)}). \end{cases} \tag{3.17}$$

Proposition 3.1.3 involves the existence of at least seven isotopism (and hence, isomorphism) classes in $\mathcal{F}_7(\mathbb{K})$. Let A_{abcd} and $A_{\alpha\beta\gamma\delta}$ be two isomorphic algebras in $\mathcal{F}_7(\mathbb{K})$ and let $F = (f_{ij})$ be the matrix related to an isomorphism f between them. Since the bases of both algebras are adapted, the form of the matrix F is that exposed in (3.11). Its central rows are determined by its first and last rows. In particular,

$$f_{ii} = f_{11}^{7-i} f_{77}, \text{ for all } i \in \{2, \ldots, 6\}.$$

Since F is a regular matrix, its determinant is distinct from zero and hence,

$$f_{11} f_{77} \neq 0. \tag{3.18}$$

Then, from the definition of isomorphism,

$$a f_{11}^2 = \alpha f_{77}, \tag{3.19}$$

$$b f_{11}^2 = \beta f_{77}, \tag{3.20}$$

$$c f_{11}^4 = 2(\alpha + \beta)^2 f_{17} f_{77} + \gamma f_{11} f_{77}, \tag{3.21}$$

$$\begin{gathered} d f_{11}^5 f_{77} - (3\alpha + 2\beta) c f_{11}^3 f_{17} f_{77} + a\alpha^2 f_{11} f_{17}^2 f_{77} + \\ (bA^2 + a\alpha\beta + b\alpha\beta) f_{11} f_{17}^2 f_{77} - 2(\alpha\gamma + \beta\gamma) f_{27} f_{77}^2 - \\ \beta f_{11} f_{76}^2 + 2\beta f_{11} f_{75} f_{77} = \delta f_{11} f_{77}^2. \end{gathered} \tag{3.22}$$

Proposition 3.3.13. *Let $\mathbb{K}$ be a field whose characteristic is distinct from two and three. Two non-model algebras A_{00cd} and $A_{00\gamma\delta}$ in $\mathcal{F}_7(\mathbb{K})$ are isomorphic if and only if one of the following conditions holds*

a) $c = \gamma = 0$ and $d \neq 0 \neq \delta$.

b) $d = \delta = 0$ *and* $c \neq 0 \neq \gamma$.

c) The numbers c, γ, d *and* δ *are all of them distinct from zero.*

Proof. Impose $a = \alpha = b = \beta = 0$ in Equations (3.18 – 3.22). Then, $cf_{11}^3 = \gamma f_{77}$ and $df_{11}^4 = \delta f_{77}$. It is then enough to impose $f_{77} = df_{11}^4/\delta$ if the first condition holds or $f_{77} = cf_{11}^3$ if the second one holds. In the third case, $f_{77} = df_{11}^4/\delta = cf_{11}^3/\gamma$ and we impose $f_{11} = c\delta/\gamma d$. □

Proposition 3.3.14. *Let* $\mathbb{K}$ *be a field whose characteristic is distinct from two and three. Let* A_{0bcd} *and* $A_{0\beta\gamma\delta}$ *be two non-model algebras in* $\mathcal{F}_7(\mathbb{K})$. *If* $b \neq 0 \neq \beta$, *then both algebras are isomorphic.*

Proof. Impose $a = \alpha = 0$ in Equations (3.18 – 3.22). Then,

$$\begin{cases} bf_{11}^2 = \beta f_{77}, \\ cf_{11}^4 = 2\beta^2 f_{17} f_{77} + \gamma f_{11} f_{77}, \\ df_{11}^5 f_{77} = 2\beta(cf_{11}^3 f_{17} f_{77} + \gamma f_{17} f_{77}^2 - f_{11} f_{75} f_{77}) + \beta f_{11} f_{76}^2 + \delta f_{11} f_{77}^2. \end{cases}$$

Take $f_{77} = bf_{11}^2/\beta$. It is enough to isolate f_{17} and f_{75} in the second and third equations, respectively, in order to obtain our isomorphism. □

Proposition 3.3.15. *Let* $\mathbb{K}$ *be a field whose characteristic is distinct from two and three. Let* $a, c, d \in \mathbb{K}$ *be such that* $a \neq 0$. *The next assertions hold in* $\mathcal{F}_7(\mathbb{K})$.

a) If $4ad = 5c^2$, *then* $A_{a0cd} \cong A_{1000}$.

b) If $4ad \neq 5c^2$ *and* $4ad - 5c^2$ *is a perfect square in* $\mathbb{K}$, *then* $A_{a0cd} \cong A_{1001}$.

c) Let q *be a non-perfect square in* $\mathbb{K}$. *If* $\mathbb{K}$ *is finite,* $4ad \neq 5c^2$ *and* $4ad - 5c^2$ *is a non-perfect square in* $\mathbb{K}$, *then* $A_{a0cd} \cong A_{100q}$.

Proof. Let A_{a0cd} and $A_{\alpha 0\gamma\delta}$ be two algebras in $\mathcal{F}_7(\mathbb{K})$ such that $a \neq 0 \neq A$. Impose $b = \beta = 0$ in Equations (3.18 – 3.22). Then,

$$\begin{cases} af_{11}^2 = \alpha f_{77}, \\ cf_{11}^4 = 2\alpha^2 f_{17} f_{77} + \gamma f_{11} f_{77}, \\ df_{11}^5 = 3\alpha c f_{11}^3 f_{17} - a\alpha^2 f_{11} f_{17}^2 + 2\alpha\gamma f_{17} f_{77} + \delta f_{11} f_{77} \end{cases}$$

Hence, $f_{77} = af_{11}^2/\alpha$. Then, $f_{17} = (cf_{11}^4 - \gamma f_{11} f_{77})/2\alpha^2 f_{77} = (\alpha c f_{11}^2 - a\gamma f_{11})/2a\alpha^2$ and thus,

$$(4ad - 5c^2)\alpha^2 f_{11}^2 = (4\alpha\delta - 5\gamma^2)a^2.$$

The algebras A_{a0cd} and $A_{\alpha 0\gamma\delta}$ are, therefore, isomorphic if and only if one of the following conditions is satisfied

i. $4ad = 5c^2$ and $4\alpha\delta = 5\gamma^2$.

ii. $4ad \neq 5c^2$, $4\alpha\delta \neq 5\gamma^2$ and $(4\alpha\delta - 5\gamma^2)/(4ad - 5c^2)$ is a perfect square in $\mathbb{K}$.

Assertions (a) and (b) follow then immediately from (i) and (ii). Finally, let us suppose that $\mathbb{K}$ is a finite field $\mathbb{F}_q$ and let q be a non-perfect square in $\mathbb{F}_q$. Every perfect square r in $\mathbb{F}_q$ is uniquely related to a non-perfect square s in $\mathbb{F}_q$ such that $r = q/s$. Such a relation is 1-1, because $\mathbb{F}_q$ is a finite field that contains exactly $(p-1)/2$ perfect squares distinct from zero. Assertion (c) follows then from (ii). $\square$

Proposition 3.3.16. *Let $\mathbb{K}$ be a field whose characteristic is distinct from two and three. Let A_{abcd} and $A_{\alpha\beta\gamma\delta}$ be two algebras in $\mathcal{F}_7(\mathbb{K})$ such that a, b, α and β are all of them distinct from zero. They are isomorphic if and only $a\beta = \alpha b$ and one of the following assertion is verified*

1. *$a + b \neq 0$.*

2. *If $a + b = 0$, then $c = 0 = \gamma$ or $c \neq 0 \neq \gamma$.*

Proof. Isolate f_{75} from Equation (3.22). Since $\alpha \neq 0$, we have from (3.19) that $f_{77} = af_{11}^2/\alpha = bf_{11}^2/\beta$ and hence, $a\beta = \alpha b$. From Equation (3.21),

$$a\alpha c f_{11}^2 - a^2\gamma f_{11} = 2(a+b)^2\alpha^2 f_{17}.$$

If $a + b \neq 0$, then a possible isomorphism can be determined once f_{17} is isolated from the previous equation. Otherwise, it is $\alpha c f_{11} = a\gamma$ and hence, there exists an isomorphism between A_{abcd} and $A_{\alpha(\alpha b/a)\gamma\delta}$ if and only if $c = \gamma = 0$ or $c \neq 0 \neq \gamma$. In the last case, it is enough to consider $f_{11} = a\gamma/\alpha c$. $\square$

The previous results establish the next distribution of $\mathcal{F}_7(\mathbb{K})$ into isomorphism and isotopism classes.

Theorem 3.3.17. *Let $\mathbb{K}$ be a field whose characteristic is distinct from two and three. Then,*

a) If $\mathbb{K}$ is algebraically closed, then the isomorphism classes of $\mathcal{F}_7(\mathbb{K})$ are

$$\{A_{0000}, A_{0001}, A_{0010}, A_{0011}, A_{0100}, A_{1001}, A_{1(-1)10}\} \cup \{A_{1b00} \mid b \in \mathbb{K}\}.$$

b) If $\mathbb{K} = \mathbb{F}_q$, then the isomorphism classes of $\mathcal{F}_7(\mathbb{K})$ are

$$\{A_{0000}, A_{0001}, A_{0010}, A_{0011}, A_{0100}, A_{1001}, A_{1(-1)10}, A_{100q}\} \cup \{A_{1b00} \mid b \in \mathbb{K}\},$$

where q is a non-perfect square of $\mathbb{F}_q$. □

Theorem 3.3.18. *Let $\mathbb{K}$ be a field whose characteristic is distinct from two and three. If $\mathbb{K}$ is algebraically closed or $\mathbb{K} = \mathbb{F}_q$, then there exist eight isotopism classes in $\mathcal{F}_7(\mathbb{K})$*

$$\{A_{0000}, A_{0001}, A_{0010}, A_{0100}, A_{1000}, A_{1100}, A_{1(-1)00}, A_{1(-1)10}\}.$$

Proof. In order to prove the result, we establish distinct strong isotopisms (f, f, h) among the isomorphism classes in $\mathcal{F}_7(\mathbb{K})$. Each isotopism is described by means of those basis vectors that are not preserved by the transformations f and h.

a) $A_{0011} \simeq A_{0010}$ by means of the strong isotopism (f, f, h) such that

$$f(e_4) = e_4 - e_3 \quad \text{and} \quad h(e_3) = e_3 - e_2.$$

b) $A_{10cd} \simeq A_{1000}$, for all $c, d \in \mathbb{K}$, by means of the strong isotopism (f, f, h) such that

$$f(e_5) = e_5 + (c^2 - d)e_3, \qquad h(e_3) = e_3 - ce_2,$$
$$h(e_4) = e_4 - ce_3 + (c^2 - d)e_2 \quad \text{and} \quad h(e_5) = e_5 - ce_4.$$

c) $A_{1b00} \simeq A_{1100}$, for all $b \in \{2, \ldots, q-2\}$, by means of the strong isotopism (f, f, h) such that

$$f(e_3) = f_{44}e_3, \quad f(e_4) = f_{44}e_4, \quad f(e_5) = f_{55}e_5, \quad f(e_6) = f_{66}e_6,$$
$$h(e_2) = f_{44}e_2, \quad h(e_3) = f_{44}e_3, \quad h(e_4) = f_{55}e_4, \quad h(e_5) = f_{66}e_5,$$

with

$$f_{44} = \frac{4}{(b+1)^2} f_{66}, \quad f_{55} = \frac{2}{b+1} f_{66}, \quad f_{66} = \frac{2b}{b+1}.$$

A simple analysis on possible isotopisms among the Lie algebras A_{1100}, $A_{1(-1)00}$ and $A_{1(-1)10}$ indicates that they determine distinct isotopism classes. The result holds then from the isotopism invariants (3.17). □

Characteristic two

Let $\mathbb{K}$ be a field of characteristic two. From Lemma 3.3.12, every algebra $A \in \mathcal{F}_7(\mathbb{K})$ has an adapted basis $\{e_1, \ldots, e_7\}$ such that

$$A \cong A_{abcde} := \begin{cases} e_1e_i = e_{i-1}, \text{ for all } i \in \{3,4,5,6,7\}, \\ e_4e_6 = ee_2, \\ e_4e_7 = ae_2 + ee_3, \\ e_5e_6 = be_2 + ee_3, \\ e_5e_7 = ce_2 + (a+b)e_3, \\ e_6e_7 = de_2 + ce_3 + (a+b)e_4, \end{cases}$$

for some tuple $(a,b,c,d,e) \in \mathbb{K}^5$. The structure constants of A_{abcd0} coincide with those of A_{abcd} in the previous case and hence, their isotopism invariants coincide. The next isotopism invariants correspond to the case $e \neq 0$:

$$\begin{cases} D_6(A_{abcde}) = 2, \\ D_5({A_{abcde}}^{(1)}) = 1, \\ D_4({A_{a(-a)cde}}^{(2)}) = 4, \\ \text{If } a \neq -b, \text{ then } D_4({A_{abcde}}^{(2)}) = 2. \end{cases} \tag{3.23}$$

The first two isotopism invariants, together with (3.17), involve that two algebras A_{abcde} and A_{ABCD0} in $\mathcal{F}_7(\mathbb{K})$ are not isotopic if $e \neq 0$. Hence, we can focus on the distribution into isotopism and isomorphism classes of A_{abcde} with $e \neq 0$. A similar reasoning to that exposed for fields of characteristic distinct from two and three enables us to ensure the next result.

Theorem 3.3.19. *Let $\mathbb{K}$ be a field of characteristic two. If $\mathbb{K}$ is algebraically closed or finite, then*

a) There exist 15 *isomorphism classes in $\mathcal{F}_7(\mathbb{K})$:*

$$\{A_{00000}, A_{00001}, A_{00010}, A_{00100}, A_{00110}, A_{01000}, A_{01001}, A_{01100},$$
$$A_{10000}, A_{10001}, A_{10100}, A_{10110}, A_{11000}, A_{11001}, A_{11100}\}.$$

b) There exist 10 isotopism classes in $\mathcal{F}_7(\mathbb{K})$:

$$\{A_{00000}, A_{00001}, A_{00010}, A_{00100}, A_{01000}, A_{01001}, A_{10000}, A_{11000}, A_{11001}, A_{11100}\}.$$

□

Characteristic three

Let $\mathbb{K}$ be a field of characteristic three. From Lemma 3.3.12, any algebra A in $\mathcal{F}_7(\mathbb{K})$ has a filiform basis $\{e_1,\ldots,e_7\}$ such that

$$A \cong A_{abcd} := \begin{cases} e_1e_i = e_{i-1}, \text{ for all } i \in \{3,4,5,6,7\}, \\ e_4e_7 = ae_2, \\ e_5e_6 = be_2, \\ e_5e_7 = ce_2 + (a+b)e_3, \\ e_6e_7 = de_2 + ce_3 + (a+b)e_4, \end{cases}$$

or

$$A \cong A'_{abcd} := \begin{cases} e_1e_i = e_{i-1}, \text{ for all } i \in \{3,4,5,6,7\}, \\ e_4e_5 = ae_2, \\ e_4e_6 = ae_3, \\ e_4e_7 = ae_4, \\ e_5e_6 = be_2 + ae_4, \\ e_5e_7 = ce_2 + be_3 + 2ae_5, \\ e_6e_7 = de_2 + ce_3 + be_4 + 2ae_6, \end{cases}$$

for some tuple $(a,b,c,d) \in \mathbb{K}^4$. Observe that, if $a = 0$, then $A'_{0bcd} = A_{0bcd}$ and hence, we suppose $a \neq 0$ in A'_{abcd}. The structure constants and isotopism invariants of A_{abcd} coincide with those of the homonym algebra exposed in case of being $\mathbb{K}$ a field with characteristic distinct from two and three. Further, $D_6(A'_{abcd}) = 2 > 1 = D_5({A'_{abcd}}^{(1)})$ and hence, none pair of distinct algebras A_{abcd} and $A'_{\alpha\beta\gamma\delta}$ are isotopic if $a \neq 0 \neq \alpha$. Similarly to the previous cases, the next result holds.

Theorem 3.3.20. *Let $\mathbb{K}$ be a field of characteristic three. Then,*

a) If $\mathbb{K}$ is algebraically closed, then the isomorphism classes of $\mathcal{F}_7(\mathbb{K})$ are

$$\{A_{0000}, A_{0001}, A_{0010}, A_{0011}, A_{0100}, A_{1001}, A_{1(-1)10}, A'_{1000}\} \cup \{A_{1b00} \mid b \in \mathbb{K}\}.$$

b) If $\mathbb{K} = \mathbb{F}_q$, then the isomorphism classes of $\mathcal{F}_7(\mathbb{K})$ are

$$\{A_{0000}, A_{0001}, A_{0010}, A_{0011}, A_{0100}, A_{1001}, A_{1002}, A_{1(-1)10}, A'_{1000}\} \cup \{A_{1b00} \mid b \in \mathbb{K}\}.$$

c) If $\mathbb{K}$ is algebraically closed or finite, then the isotopism classes of $\mathcal{F}_7(\mathbb{K})$ are

$$\{A_{0000}, A_{0001}, A_{0010}, A_{0100}, A_{1000}, A_{1100}, A_{1(-1)00}, A_{1(-1)10}, A'_{1000}\}.$$

□

Chapter 4

Isotopisms of Malcev partial-magma algebras

Partial-magma algebras were introduced in Chapter 2, where we focused in particular on the distribution of finite-dimensional partial-quasigroup rings over finite fields. This chapter deals with the distribution into isotopism classes of the so-called *Malcev partial-magma algebras*, which are partial-magma algebras that also are Malcev algebras. The latter constitute a generalization of Lie algebras. This chapter is established, therefore, as a natural meeting point among the distinct results that we have exposed until now.

4.1 Preliminaries

We expose here some basic concepts and results on Malcev algebras that we use throughout the chapter. For more details about this topic we refer to the original articles of Malcev [225] and Sagle [274].

In 1955, Malcev [225] introduced the concept of Malcev algebra as the tangent algebra of a local analytic Moufang loop. It was Sagle [274] who, shortly after, presented Malcev algebras in a formal way as a generalization of Lie algebras. Specifically, a *Malcev algebra* A over a field $\mathbb{K}$ is an anticommutative algebra such that $u^2 = 0$, for all $u \in A$, and

$$((uv)w)u + ((vw)u)u + ((wu)u)v = (uv)(uw), \text{ for all } u, v, w \in A. \tag{4.1}$$

This is equivalent to the so-called *Malcev identity*

$$M(u,v,w) = J(u,v,w)u - J(u,v,uw) = 0, \text{ for all } u,v,w \in A, \tag{4.2}$$

where J denotes the Jacobian. If the base field is the finite field $\mathbb{F}_q$, with q a power prime and characteristic distinct from two, then both identities (4.1) and (4.2) are equivalent to the *Sagle identity*

$$S(u,v,w,y) = ((uv)w)y + ((vw)y)u + ((wy)u)v + ((yu)v)w - (uw)(vy) = 0, \tag{4.3}$$

for all $u,v,w,y \in A$. Unlike Malcev identity, which is not linear in its first argument $u \in A$, Sagle identity is linear in the four arguments $u,v,w,y \in A$ and invariant under cyclic permutations of the variables.

Every Lie algebra is, therefore, a Malcev algebra. If the Jacobi identity does not hold, then the Malcev algebra is said to have a *Jacobi anomaly*. In quantum mechanics, the existence of Jacobi anomalies in the underlying non-associative algebraic structure related to the coordinates and momenta of a quantum non-Hamiltonian dissipative system was already claimed by Dirac [107] in the process of taking Poisson brackets. In String Theory, for instance, one such an anomaly is involved by the non-associative algebraic structure that is defined by coordinates (x) and velocities or momenta (v) of an electron moving in the field of a constant magnetic charge distribution, at the position of the location of the magnetic monopole [214]. In particular, $J(v_1,v_2,v_3) = -\overrightarrow{\nabla} \cdot \overrightarrow{B}(x)$, where $\overrightarrow{\nabla} \cdot \overrightarrow{B}(x)$ denotes the divergence of the magnetic field $\overrightarrow{B}(x)$. The underlying algebraic structure constitutes a non-Lie Malcev algebra [152], with the commutation relations $[x_a,x_b] = 0$, $[x_a,v_b] = i\delta_{ab}$ and $[v_a,v_b] = i\epsilon_{abc}B_c(x)$, where $a,b,c \in \{1,2,3\}$, δ_{ab} denotes the Kronecker delta and ϵ_{abc} denotes the Levi-Civita symbol. If the magnetic field is proportional to the coordinates, the latter can be normalized and $B_c(x)$ can then be supposed to coincide with x_c. The resulting algebra is then called *magnetic* [153]. A generalization to electric charges has recently been considered [154] by defining the products $[x_a,x_b] = -i\epsilon_{abc}E_c(x,v)$, where the electric field E as well as the magnetic field B must depend not only on coordinates but also on velocities. Remark that both magnetic and electric algebras constitute partial-magma algebras.

A main open problem in the theory of Malcev algebras is their enumeration and distribution into isomorphism classes [198, 199, 273]. Over finite fields, this problem has already been dealt with for Lie algebras of dimension up to six. Particularly, De Graaf [149] made use of Gröbner bases and Computational Algebraic Geometry in order to determine the distribution of solvable Lie algebras of dimension up to four over any field, whereas Strade [289] obtained that of nonsolvable Lie algebras of dimension up to six over a finite field. The classification of nilpotent Lie algebras of dimension up to six over any field is also known [87]. The authors in this last reference indicated explicitly that some of their results were inspired by Gröbner basis computations.

4.2 Malcev partial-magma algebras

From here on, we denote by $\mathcal{M}_{n,q}$ the set of n-dimensional Malcev partial-magma algebras over the finite field $\mathbb{F}_q$, with q a power prime. Every algebra in this set is described by means of a basis of vectors satisfying the conditions exposed in the next result.

Lemma 4.2.1. *Let $n \geq 3$. Every n-dimensional non-abelian Malcev partial-magma algebra based on a partial-magma $([n], \cdot)$ holds, up to isomorphism, one of the next two non-isomorphic possibilities*

a) $e_1e_2 = e_2$, *or*

b) $e_1e_2 = e_3$ *and there does not exist a non-zero structure constant* c_{ij} *such that* $i \cdot j \in \{i, j\}$.

Proof. Since the algebra is not abelian, there exists at least one non-zero structure constant. If there exist two distinct positive integers $i, j \leq n$ such that $c_{ij} \neq 0$ and $i \cdot j = j$, then we consider the isomorphism that maps e_i and e_j to $c_{ij}e_1$ and e_2, respectively, and preserves the rest of basis vectors. We get in this way the product $e_1e_2 = e_2$. Otherwise, we take a non-zero structure constant c_{uv} and the isomorphism that maps e_u, e_v and $e_{u\cdot v}$ to $c_{uv}e_1$, e_2 and e_3, respectively, in order to get the product $e_1e_2 = e_3$. $\square$

Every Malcev algebra is *binary-Lie*, that is, any two of its elements generate a Lie subalgebra. As a consequence, every Malcev algebra of dimension $n \leq 3$ is a Lie algebra. Particularly, the only one-dimensional Malcev algebra is the abelian and the only two-dimensional non-abelian Malcev algebra is, up to isomorphism, the partial-magma algebra determined by the product of basis vectors $e_1e_2 = e_2$. The next theorem deals with the distribution of three-dimensional Malcev partial-magma algebras over a finite field into isomorphism classes. In the statement of the result, each isomorphism class is labeled according to the notation given by De Graaf [149] and Strade [289] in their respective classifications of solvable and non-solvable Lie algebras, but we have chosen for its description a basis that follows the conditions exposed in Lemma 4.2.1.

Theorem 4.2.2. *Every three-dimensional Malcev partial-magma algebra over the finite field $\mathbb{F}_q$, with $q = p^m$ a power prime, is isomorphic to exactly one of the next algebras*

a) The three-dimensional abelian Lie algebra L^1.

b) The solvable Lie algebras

- $L^2 : e_1e_2 = e_2$ *and* $e_1e_3 = e_3$.
- $L^3_0 : e_1e_2 = e_2$.
- $L^4_a : e_1e_2 = e_3$ *and* $e_1e_3 = ae_2$, *for all* $a \in \mathbb{F}_q$.
  Here, $L^4_a \cong L^4_b$ *if and only if there exists* $\alpha \in \mathbb{F}_q \setminus \{0\}$ *such that* $a = \alpha^2 b$.

c) The non-solvable Lie algebras

- $W(1;\underline{2})^{(1)} : e_1e_2 = e_2,\ e_1e_3 = e_3$ *and* $e_2e_3 = e_1$, *whenever* $p = 2$.
- $\mathfrak{sl}(2,\mathbb{F}_q) : e_1e_2 = e_2,\ e_1e_3 = -e_3$ *and* $e_2e_3 = e_1$, *whenever* $p \neq 2$.

Proof. The mentioned classifications of Lie algebras obtained by De Graaf and Strade enable us to ensure that each Lie algebra of the list constitutes an isomorphism class in $\mathcal{M}_{3,q}$. In order to ensure that there does not exist any other isomorphism class, it is required to prove that none of the next three-dimensional solvable Lie algebras is a partial-magma algebra

$$L^3_a : e_1e_2 = e_2 + ae_3 \text{ and } e_1e_3 = e_2, \text{ with } a \in \mathbb{F}_q \setminus \{0\}. \tag{4.4}$$

We prove this fact in Proposition 4.3.1. □

For dimension $n = 4$, there exists, up to isomorphism, a unique non-Lie Malcev algebra [200] over $\mathbb{F}_q$, with $q = p^m$ a power prime, whenever $p \notin \{2,3\}$. This coincides with the next solvable partial-magma algebra

$$M^0 : e_1e_2 = e_2,\ e_1e_3 = -e_3,\ e_1e_4 = -e_4 \text{ and } e_3e_4 = -e_2.$$

The distribution of $\mathcal{M}_{4,q}$ into isomorphism classes is exposed in the next result, which is again based on the classifications given by De Graaf and Strade.

Theorem 4.2.3. *Let $\mathbb{F}_q$ be a finite field, with $q = p^m$ a power prime. Every Malcev partial-magma algebra $A \in \mathcal{M}_{4,q}$ is isomorphic to exactly one of the next algebras*

a) The non-Lie Malcev algebra M^0.

b) The four-dimensional abelian Lie algebra M^1.

c) The solvable Lie algebras

- $M^2 : e_1e_2 = e_2,\ e_1e_3 = e_3$ *and* $e_1e_4 = e_4$.
- $M^3_{-1} : e_1e_2 = e_2,\ e_1e_3 = e_4$ *and* $e_1e_4 = e_3$.
- $M^3_0 : e_1e_2 = e_2,\ e_1e_3 = e_4$ *and* $e_1e_4 = e_4$.
- $M^4 : e_1e_2 = e_2$ *and* $e_1e_3 = e_2$.
- $M^5 : e_1e_2 = e_3$.
- $M^6_{0,0} : e_1e_2 = e_2,\ e_1e_3 = e_4$ *and* $e_1e_4 = e_2$.
- $M^7_{0,0} : e_1e_2 = e_3$ *and* $e_1e_3 = e_4$.
- $M^7_{a,0} : e_1e_2 = e_3,\ e_1e_3 = e_4$ *and* $e_1e_4 = ae_2$, *for all* $a \in \mathbb{F}_q \setminus \{0\}$.
  Here, $M^7_{a,0} \cong M^7_{b,0}$ *if and only if there exists* $\alpha \in \mathbb{F}_q \setminus \{0\}$ *such that* $a = \alpha^3 b$.
- $M^7_{0,a} : e_1e_2 = e_3,\ e_1e_3 = e_4$ *and* $e_1e_4 = ae_3$, *for all* $a \in \mathbb{F}_q \setminus \{0\}$.
  Here, $M^7_{0,a} \cong M^7_{0,b}$ *if and only if there exits* $\alpha \in \mathbb{F}_q \setminus \{0\}$ *such that* $a = \alpha^2 b$.
- $M^8 : e_1e_2 = e_2$ *and* $e_3e_4 = e_4$.
- $M^{11}_{1,0} : e_1e_2 = e_2,\ e_1e_4 = 1e_4,\ e_2e_4 = e_3$ *and* $e_3e_4 = e_2$, *whenever* $p = 2$.
- $M^{12} : e_1e_2 = e_2,\ e_1e_3 = 2e_3,\ e_1e_4 = e_4$ *and* $e_2e_4 = -e_3$, *whenever* $p \neq 2$.
- $M^{13}_0 : e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_1e_4 = e_2$ *and* $e_2e_4 = -e_3$.
- $M^{14}_a : e_1e_2 = e_3,\ e_1e_3 = ae_2$ *and* $e_2e_3 = e_4$, *for all* $a \in \mathbb{F}_q \setminus \{0\}$.
  Here, $M^{14}_a \cong M^{14}_b$ *if and only if there exists* $\alpha \in \mathbb{F}_q \setminus \{0\}$ *such that* $a = \alpha^2 b$.

d) The non-solvable Lie algebras over a field of characteristic two

- $W(1;\underline{2}) : e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_2e_3 = e_1$ *and* $e_3e_4 = e_2$.

- $W(1;\underline{2})^{(1)} \oplus Z(A) : e_1e_2 = e_2,\ e_1e_3 = e_3$ *and* $e_2e_3 = e_1$.

e) The non-solvable Lie algebra over a field of characteristic distinct from two

- $\mathfrak{gl}(2,\mathbb{F}_q) : e_1e_2 = e_2,\ e_1e_3 = -e_3,\ e_2e_3 = e_1,\ e_2e_4 = e_2$ *and* $e_3e_4 = -e_3$.

Proof. The classification given by De Graaf and Strade for solvable and non-solvable four-dimensional Lie algebras over finite fields implies that all these algebras constitute distinct isomorphism classes. In order to ensure that the distribution is exhaustive, it is required to prove that none of the next four-dimensional solvable Lie algebras is isomorphic to a partial-magma algebra

$$M_a^3 : e_1e_2 = e_2, e_1e_3 = e_4 \text{ and } e_1e_4 = -ae_3 + (a+1)e_4, \text{ with } a \not\in \{0,-1\}. \tag{4.5}$$

$$M_{a,b}^6 : e_1e_2 = e_2+ae_3+be_4, e_1e_3 = e_4 \text{ and } e_1e_4 = e_2, \text{ for all } (a,b) \in \mathbb{F}_q^2\setminus(0,0). \tag{4.6}$$

$$M_{a,a}^7 : e_1e_2 = e_3, e_1e_3 = e_4 \text{ and } e_1e_4 = ae_2 + ae_3, \text{ for all } a \in \mathbb{F}_q \setminus \{0\}. \tag{4.7}$$

$$\begin{array}{rc} M_{a_0}^9 : & e_1e_2 = e_2 + a_0e_3, e_1e_3 = e_2, e_2e_4 = e_2 \text{ and } e_3e_4 = e_3, \\ & \text{whenever p=2 and } a_0 \in \mathbb{F}_q \text{ is such that } T^2 - T - a_0 \\ & \text{has no root in } \mathbb{F}_q. \end{array} \tag{4.8}$$

$$M_a^{13} : e_1e_2 = e_2+ae_4, e_1e_3 = e_3, e_1e_4 = e_2 \text{ and } e_2e_4 = -e_3, \text{ for all } a \in \mathbb{F}_q\setminus\{0\}. \tag{4.9}$$

We prove this fact in Proposition 4.3.4. □

In order to finish the proofs of Theorems 4.2.2 and 4.2.3, we are going to identify the sets $\mathcal{L}_{n,q}$ and $\mathcal{M}_{n,q}$ of n-dimensional Lie and Malcev partial-magma algebras over the finite field $\mathbb{F}_q$, where $q = p^m$ is a power prime, with the algebraic set of an ideal of polynomials. Let X and $\mathbb{F}_q[X]$ respectively be the set of n^3 variables $\{\mathfrak{c}_{ij}^k \mid i,j,k \leq n\}$ and its related multivariate polynomial ring over $\mathbb{F}_q$. Let $\mathfrak{A}$ be the n-dimensional algebra over $\mathbb{F}_q[X]$ with basis $\{e_1,\ldots,e_n\}$ so that

$$e_ie_j = \sum_{k=1}^{n} \mathfrak{c}_{ij}^k e_k, \tag{4.10}$$

for all $i,j \leq n$. Let us also consider

i. The coefficient $\mathrm{l}_{ijkl} \in \mathbb{F}_q[X]$ of e_l in the Jacobi identity $J(e_i,e_j,e_k) = 0$, for all $i,j,k \leq n$.

ii. The coefficient $\mathrm{m}_{uijk} \in \mathbb{F}_q[X]$ of e_k in the Malcev identity $M(u, e_i, e_j) = 0$, for all $u \in \mathfrak{A}$ and $i, j \leq n$.

iii. The coefficient $\mathrm{s}_{ijklm} \in \mathbb{F}_q[X]$ of e_m in the Sagle identity $S(e_i, e_j, e_k, e_l) = 0$, for all $i, j, k, l \leq n$, whenever $p \neq 2$.

Finally, let us define the ideal in $\mathbb{F}_q[X]$

$$I = \langle\, \mathfrak{c}_{ii}^{j} \mid i,j \leq n \,\rangle + \langle\, \mathfrak{c}_{ij}^{k} - \mathfrak{c}_{ji}^{k} \mid i,j,k \leq n \,\rangle + \langle\, \mathfrak{c}_{ij}^{k}\mathfrak{c}_{ij}^{k'} \mid i,j,k,k' \leq n; k < k' \,\rangle. \quad (4.11)$$

The next results follow similarly to Theorem 2.2.3 and Corollary 2.2.4.

Theorem 4.2.4. *The sets $\mathcal{L}_{n,q}$ and $\mathcal{M}_{n,q}$ are respectively identified with the algebraic sets defined by the zero-dimensional radical ideals in $\mathbb{F}_q[X]$*

$$I_L = \langle\, \mathrm{l}_{ijkl} \mid i,j,k,l \leq n \,\rangle + I \qquad \textit{and} \qquad I_M = \langle\, \mathrm{m}_{uijk} \mid u \in \mathfrak{A}, i,j,k \leq n \,\rangle + I.$$

If the characteristic p of the base field is not two, then the set $\mathcal{M}_{n,q}$ is also identified with the algebraic set defined by the zero-dimensional radical ideal in $\mathbb{F}_q[X]$

$$I_S = \langle\, \mathrm{s}_{ijklm} \mid i,j,k,l,m \leq n \,\rangle + I.$$

Besides, $|\mathcal{L}_{n,q}| = \dim_{\mathbb{F}_q}(\mathbb{F}_q[X]/I_L)$ and $|\mathcal{M}_{n,q}| = \dim_{\mathbb{F}_q}(\mathbb{F}_q[X]/I_M)$. The latter coincides with $\dim_{\mathbb{F}_q}(\mathbb{F}_q[X]/I_S)$ if $p \neq 2$. □

Corollary 4.2.5. *Let $\mathbb{F}_q$ be a finite field, with q a prime power. The run time that is required by the Buchberger's algorithm in order to compute the reduced Gröbner bases of the ideals I_L, I_M and I_S in Theorem 4.2.4 are, respectively, $q^{O(n^3)} + O(n^8)$, $\max\{3, q\}^{O(n^3)} + O(q^{2n})$ and $q^{O(n^3)} + O(n^{10})$.* □

The previous result discards the use of the ideal I_M whenever the characteristic of the base field is not two. In order to compute the reduced Gröbner bases of the ideals in Theorem 4.2.4 and hence, their respective algebraic sets and Krull dimensions, we have implemented in the library *isotopism.lib* three subprocedures called *JacobiId*, *MalcevId* and *SagleId* and a main procedure called *MalcevAlg*. The subprocedure *JacobiId* outputs the list of polynomials corresponding to the coefficient of each basis vector in the Jacobi identity related to any three arbitrary vectors of the algebra A. Similar lists of polynomials are output by the rest of subprocedures. Their respective pseudocodes are described in Algorithms 4-6.

Algorithm 4 Polynomials related to the Jacobi identity in A.

procedure JacobiId(u, v, w)
  $L1 \leftarrow Prod(Prod(u, v), w)$;
  $L2 \leftarrow Prod(Prod(v, w), u)$;
  $L3 \leftarrow Prod(Prod(w, u), v)$;
  for $i \leftarrow 1, n$ **do**
    $L1_i \leftarrow L1_i + L2_i + L3_i$;
  end for
  return $L1$
end procedure

Algorithm 5 Polynomials related to the Malcev identity in A.

procedure MalcevId(u, v, w)
  $L1 \leftarrow Prod(JacobiId(u, v, w), u)$;
  $L2 \leftarrow JacobiId(u, v, Prod(u, w))$;
  for $i \leftarrow 1, n$ **do**
    $L1_i \leftarrow L1_i - L2_i$;
  end for
  return $L1$
end procedure

Algorithm 6 Polynomials related to the Sagle identity in A.

procedure SagleId(u, v, w, y)
  $L1 \leftarrow Prod(Prod(Prod(u, v), w), y)$;
  $L2 \leftarrow Prod(Prod(Prod(v, w), y), u)$;
  $L3 \leftarrow Prod(Prod(Prod(w, y), u), v)$;
  $L4 \leftarrow Prod(Prod(Prod(y, u), v), w)$;
  $L5 \leftarrow Prod(Prod(u, w), Prod(v, y))$;
  for $i \leftarrow 1, n$ **do**
    $L1_i \leftarrow L1_i + L2_i + L3_i + L4_i - L5_i$;
  end for
  return $L1$
end procedure

The effectiveness of all these procedures has been checked by computing the cardinalities that are exposed in Tables 4.1 and 4.2. The run time and memory usage are explicitly indicated in both tables. Both measures of computation efficiency fit positive exponential models even for dimension $n = 4$.

		Run time		Used memory	
q	$\lvert\mathcal{M}_{3,q}\rvert$	I_L	$I_{M/S}$	I_L	$I_{M/S}$
2	32	0 s	0 s	0 MB	0 MB
3	123	0 s	0 s	0 MB	0 MB
5	581	0 s	0 s	0 MB	0 MB
7	1,567	0 s	0 s	0 MB	0 MB
11	5,891	0 s	0 s	0 MB	0 MB
13	9,613	0 s	0 s	0 MB	0 MB
17	21,137	0 s	0 s	0 MB	0 MB
⋮	⋮	⋮	⋮	⋮	⋮
499	498,492,523	0 s	0 s	0 MB	0 MB

Table 4.1: Computation of $|\mathcal{M}_{3,q}|$.

q	$\lvert\mathcal{L}_{4,q}\rvert$	Run time	Used memory	$\lvert\mathcal{M}_{4,q}\rvert$	Run time	Used memory
2	853	0 s	0 MB	897	6 s	0 MB
3	7,073	1 s	0 MB	7,073	8 s	0 MB
5	89,185	11 s	0 MB	89,377	41 s	0 MB
7	445,537	20 s	3 MB	445,969	55 s	2 MB
11	3,803,041	91 s	17 MB	3,804,241	154 s	427 MB
13	8,412,193	183 s	676 MB	8,413,921	258 s	859 MB
17	30,247,297	595 s	1,2 GB	30,250,369	752 s	1,5 GB

Table 4.2: Computation of $|\mathcal{L}_{4,q}|$ and $|\mathcal{M}_{4,q}|$.

4.3 Malcev partial-magma algebras of small dimensions

We have implemented the procedures described in the previous section to determine the distribution of three- and four-dimensional Malcev partial-magma algebras into isomorphism and isotopism classes.

4.3.1 Three-dimensional Malcev partial-magma algebras

The next result finishes the proof of Theorem 4.2.2 about the distribution of $\mathcal{M}_{3,q}$ into isomorphism classes.

Proposition 4.3.1. *None of the algebras that are described in (4.4) is isomorphic to a Lie partial-magma algebra.*

Proof. Let $a \in \mathbb{F}_q \setminus \{0\}$. Let f be an isomorphism between the algebra L_a^3 and a Malcev partial-magma algebra A and let $F = (f_{ij})$ be the nonsingular matrix related to f. This algebra A should have a two-dimensional derived algebra and its solvability index should be 2. The implementation of Algorithm 1 into a case study that takes into account Lemma 4.2.1 and the Jacobi identity involves A to be isomorphic to one of the next two partial-magma algebras

a) $e_1e_2 = e_3$ and $e_1e_3 = \alpha e_2$.

b) $e_1e_2 = e_2$ and $e_1e_3 = \alpha e_3$.

In both cases, $\alpha \in \mathbb{F}_q \setminus \{0\}$. The implementation of the procedure *isoMalcev* enables us to compute for each case the corresponding reduced Gröbner basis in Theorem 4.2.4. In (a), the normal form of the polynomial that is related to the determinant of the corresponding matrix F modulo this reduced Gröbner basis is zero whatever the number α and the characteristic of the base field are. This means that F is a singular matrix, which is a contradiction with being f an isomorphism. Hence, L_a^3 is not isomorphic to a Malcev partial-magma algebra of type (a). On the other hand, in (b), the previous normal form is the polynomial $(1-\alpha)f_{11}f_{32}f_{33}$. The regularity of the matrix F involves that $\alpha \neq 1$. Besides, the reduced Gröbner basis involves the identity $a\alpha = a$. Since $\alpha \neq 1$, it must be $a = 0$, what is a contradiction with the hypothesis. Hence, L_a^3 is not isomorphic either to a Malcev partial-magma algebra of type (b). ☐

Even if no every three-dimensional Malcev algebra is isomorphic to a Lie partial-magma algebra, the next result shows that this statement is true in the case of dealing with isotopisms instead of isomorphisms.

Proposition 4.3.2. *Every three-dimensional Malcev algebra is isotopic to a Lie partial-magma algebra.*

Proof. The result holds from the fact that, for all $a \in \mathbb{F}_q \setminus \{0\}$, the Lie algebras L_a^3 and L^2 are isotopic by means of the strong isotopism $(\mathrm{Id}, \mathrm{Id}, h)$, where $h(e_1) = e_1$, $h(e_2) = e_3$ and $h(e_3) = -\frac{1}{a}(e_3 - e_2)$. □

The next result indicates the distribution of three-dimensional Malcev partial-magma algebras over a finite field into isotopism classes.

Theorem 4.3.3. *Let $\mathbb{F}_q$ be a finite field, with $q = p^m$ a prime power. There exist four isotopism classes of $\mathcal{M}_{3,q}$. They correspond to the abelian algebra and the algebras L^2, L_0^3 and*

- $W(1;\underline{2})^{(1)}$, *if* $p = 2$.
- $\mathfrak{sl}(2, \mathbb{F}_q)$, *otherwise.*

Proof. From Proposition 4.3.2, it is enough to study the distribution into isotopism classes of the Lie algebras in Theorem 4.2.2. Particularly, from Lemma 2.1.5, the algebras $W(1;\underline{2})^{(1)}$ and $\mathfrak{sl}(2, \mathbb{F}_q)$, with a three-dimensional derived algebra, are not isotopic to any other Malcev partial-magma algebra of a distinct isomorphism class. Similarly, the algebra L^2, with a two-dimensional derived algebra, can only be isotopic to L_a^4, with $a \in \mathbb{F}_q \setminus \{0\}$; whereas the algebra L_0^3, with an one-dimensional derived algebra, can only be isotopic to L_0^4. Specifically, it is straightforward verified that the triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_3$ and $h(e_3) = ae_2$ is a strong isotopism between L^2 and L_a^4, whereas the triple $(\mathrm{Id}, \mathrm{Id}, (23))$ is a strong isotopism between L_0^3 and L_0^4. □

Tables 4.3 and 4.4 show, respectively, some graph invariants for the graphs G_1 and G_2 that were described in Chapter 2 and that are related to the distribution of three-dimensional Lie partial-magma algebras over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$ into isotopism classes. The classification that is shown in both Tables is based on the articles of De Graaf [149] and Strade [289] about the distribution of solvable and non-solvable Lie algebras into isomorphism classes. Observe that the four isotopism classes that are exposed in Theorem 4.3.3 coincide exactly with the distinct sets of invariants that are exposed in Table 4.3.

	$\mathbb{F}_2$		$\mathbb{F}_3$	
A	Vertices	Edges	Vertices	Edges
Abelian	(0,0,0,0)	0	(0,0,0,0)	0
$e_1e_2 = e_3$	(6,6,1,24)	72	(24,24,2,432)	1296
$e_1e_2 = e_2$	(6,6,1,24)	72	(24,24,2,432)	1296
$e_1e_2 = e_3, e_1e_3 = -e_2$	-	-	(26,26,8,576)	1728
$e_1e_2 = e_3, e_1e_3 = e_2$	(7,7,3,36)	108	(26,26,8,576)	1728
$e_1e_2 = e_2, e_1e_3 = e_3$	(7,7,3,36)	108	(26,26,8,576)	1728
$e_1e_2 = e_2, e_1e_3 = -e_3, e_2e_3 = -e_1$	(7,7,7,42)	126	-	-
$e_1e_2 = e_2, e_1e_3 = -e_3, e_2e_3 = 2e_1$	-	-	(26,26,26,624)	1872

Table 4.3: Graph invariants for the graph G_1 related to each isomorphism class of three-dimensional Lie partial-magma algebras over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$.

	$\mathbb{F}_2$			$\mathbb{F}_3$		
A	Vertices	Edges	Triangles	Vertices	Edges	Triangles
Abelian	(0,0,0,0)	0	0	(0,0,0,0)	0	0
$e_1e_2 = e_3$	(6,6,1,24)	78	0	(24,24,2,432)	1320	0
$e_1e_2 = e_2$	(6,6,1,24)	80	9	(24,24,2,432)	1324	38
$e_1e_2 = e_3, e_1e_3 = -e_2$	-	-	-	(26,26,8,576)	1770	8
$e_1e_2 = e_3, e_1e_3 = e_2$	(7,7,3,36)	121	11	(26,26,8,576)	1770	80
$e_1e_2 = e_2, e_1e_3 = e_3$	(7,7,3,36)	121	27	(26,26,8,576)	1770	152
$e_1e_2 = e_2, e_1e_3 = -e_3, e_2e_3 = -e_1$	(7,7,7,42)	147	19	-	-	-
$e_1e_2 = e_2, e_1e_3 = -e_3, e_2e_3 = 2e_1$	-	-	-	(26,26,26,624)	1950	74

Table 4.4: Graph invariants for the graph G_2 related to each isomorphism class of three-dimensional partial-magma algebras over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$.

4.3.2 Four-dimensional Malcev partial-magma algebras

The next result concludes the proof of Theorem 4.2.3 about the distribution of four-dimensional Malcev partial-magma algebras into isomorphism classes.

Proposition 4.3.4. *None of the four-dimensional Lie algebras that are described in (4.5)–(4.9) is isomorphic to a Malcev partial-magma algebra.*

Proof. The implementation of Algorithm 1 into a case study that takes into account the derived series and centers of a Malcev algebra, together with the Jacobi identity, enables us to ensure that any possible four-dimensional Malcev partial-magma algebra with a two-dimensional derived algebra and solvability index 2 is isomorphic to exactly one of the algebras M_0^3, $M_{0,0}^6$, $M_{0,a}^7$ (with $a \in \mathbb{F}_q$) or M_0^{13}. Hence, the

Lie algebras $M^6_{0,b}$ (with $b \in \mathbb{F}_q \setminus \{0\}$) and $M^9_{a_0}$ are not isomorphic to any Malcev partial-magma algebra. Further, a similar case study enables us to ensure that none of the algebras M^3_a, $M^6_{a,b}$, $M^7_{a,a}$ and M^{13}_a, with $a, b \in \mathbb{F}_q$ such that $a \neq 0$, is isomorphic to a Malcev partial-magma algebra. This is due to the fact that any possible four-dimensional Malcev partial-magma algebra with a three-dimensional derived algebra is isomorphic to one of the algebras M^2, M^3_{-1} or $M^7_{a,0}$ if its solvability index is 2 or to $M^{11}_{1,0}$, M^{12} or M^{14}_a if its solvability index is 3. In both cases, $a \in \mathbb{F}_q \setminus \{0\}$. □

We finish our study by focusing on the distribution of four-dimensional Malcev partial-magma algebras into isotopism classes.

Proposition 4.3.5. *Let $\mathbb{F}_q$ be a finite field, with $q = p^m$ a prime power. If $p \neq 2$, then every four-dimensional Malcev algebra over $\mathbb{F}_q$ is isotopic to a Malcev partial-magma algebra. Otherwise, this assertion holds except for those Malcev algebras that are isomorphic to the Malcev algebra $M^9_{a_0}$.*

Proof. Let $\mathbb{F}_q$ be a finite field. The implementation of the procedure *isoMalcev* in Algorithm 2 gives us the next isotopisms among the Malcev partial-magma algebras described in Theorem 4.2.3 and those in (4.5)–(4.9). For each pair of isotopic algebras we show a strong isotopism (f, f, h), which is described by means of those basis vectors that are not preserved by the transformations f and h.

a) $M^2 \simeq M^3_a \simeq M^6_{b,c}$, for all $a, b, c \in \mathbb{F}_q$ such that $a \notin \{0, -1\}$ and $b \neq 0$. Here,

- The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_3) = e_4$ and $h(e_4) = -ae_3 + (a+1)e_4$ is a strong isotopism between M^2 and M^3_a, for all $a \in \mathbb{F}_q \setminus \{0, -1\}$.
- The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_2 + be_3 + ce_4$, $h(e_3) = e_4$ and $h(e_4) = e_2$ is a strong isotopism between M^2 and $M^6_{b,c}$, for all $b, c \in \mathbb{F}_q$ such that $b \neq 0$.

b) $M^0 \simeq M^{13}_a$, for all $a \in \mathbb{F}_q \setminus \{0\}$. Here, the triple (f, f, h) such that $f(e_2) = h(e_2) = e_3$, $f(e_3) = -e_4$, $f(e_4) = e_2 - e_4$, $h(e_3) = e_2$ and $h(e_4) = -ae_4$ is a strong isotopism between M^0 and M^{13}_a, for all $a \in \mathbb{F}_q \setminus \{0\}$.

A case study based on the same implementation of the procedure *isoMalcev* also enables us to ensure that, if the characteristic of the base field is two, then the Malcev algebra $M^9_{a_0}$ is not isotopic to any Malcev partial-magma algebra with a two-dimensional derived algebra and solvability index 3. □

The proof of Proposition 4.3.5 enables us to ensure that isotopisms do not preserve Jacobi anomalies. Specifically, we have proved that the non-Lie Malcev algebra M^0 is isotopic to any Lie algebra M_a^{13}, with $a \in \mathbb{F}_q \setminus \{0\}$. The next result holds straightforward.

Theorem 4.3.6. *Any four-dimensional Malcev algebra is isotopic to a Lie algebra.*

□

The distribution of four-dimensional Malcev algebras over finite fields into isotopism classes is exposed in the next final result.

Theorem 4.3.7. *Let $\mathbb{F}_q$ be a finite field, with q a power prime. There exist eight isotopism classes of $\mathcal{M}_{4,q}$. They correspond to the abelian algebra and the algebras M^0, M^2, M_0^3, M^4, M^8, M_0^{13} and M_1^{14}.*

Proof. From Proposition 4.3.5, it is enough to study the distribution into isotopism classes of the Lie partial-magma algebras in Theorem 4.2.3. This distribution has been obtained again from the implementation of the procedure *isoMalcev* in Algorithm 2. For each pair of isotopic algebras we show a strong isotopism (f, f, h), which is described by means of those basis vectors that are not preserved by the transformations f and h. Besides, each class is described according to the dimension of their derived algebras and centers and the isotopism invariants described in (3.4) and (3.5). Hereafter, we suppose $q = p^m$ to be a prime power.

a) $\dim(\mathcal{C}_2(M)) = 1$: $M^4 \simeq M^5$. Here, the triple $(f, f, (23))$ such that $f(e_3) = e_2 + e_3$ is a strong isotopism between both algebras.

b) $\dim(\mathcal{C}_2(M)) = 2$:

- $\dim(Z(M)) = 0$:
  - $d_3(M) = 0$: M_0^{13}.
  - $d_3(M) = 1$: M^8.
- $\dim(Z(M)) = 1$: $M_0^3 \simeq M_{0,a}^6 \simeq M_{0,b}^7$, for all $a, b \in \mathbb{F}_q$ such that $b \neq 0$. Here,
  - The triple (f, f, h) such that $f(e_4) = -e_2 + e_3 + e_4$ and $h(e_2) = e_2 - ae_4$ is a strong isotopism between M_0^3 and $M_{0,a}^6$, for all $a \in \mathbb{F}_q$.
  - The triple $(f, f, (23))$ such that $f(e_4) = e_3 + e_4$ is a strong isotopism between M_0^3 and $M_{0,b}^7$, for all $b \in \mathbb{F}_q \setminus \{0\}$.

c) $\dim(\mathcal{C}_2(M)) = 3$:

- $\dim(Z(M)) = 0$:
  - $D_3(M) = 1$: If $p = 2$, then $M^0 \simeq M^{11}_{1,0} \simeq W(1;\underline{2})$. Otherwise, $M^0 \simeq M^{12}$. Here,
    * If $p = 2$, then the triple $((14)(23), (14)(23), \mathrm{Id})$ is a strong isotopism between M^0 and $M^{11}_{1,0}$, whereas the triple $((1324), (1324), (143))$ is a strong isotopism between M^0 and $W(1;\underline{2})$.
    * If $p \neq 2$, then the triple (f, f, h) such that $f(e_2) = e_3$, $f(e_3) = -2e_4$, $f(e_4) = e_2 + e_4$, $h(e_2) = 2e_3$, $h(e_3) = 2e_4$ and $h(e_4) = -e_2 - e_4$ is a strong isotopism between M^0 and M^{12}.
  - $D_3(M) = 3$: $M^2 \simeq M^3_{-1} \simeq M^7_{a,a} \simeq M^7_{c,0}$, for all $a, c \in \mathbb{F}_q \setminus \{0\}$. Here,
    * The triple $(\mathrm{Id}, \mathrm{Id}, (34))$ is a strong isotopism between M^2 and M^3_{-1}.
    * The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_3$, $h(e_3) = e_4$ and $h(e_4) = ae_2 + ae_3$ is a strong isotopism between M^2 and $M^7_{a,a}$, for all $a \in \mathbb{F}_q \setminus \{0\}$.
    * The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_3$, $h(e_3) = e_4$ and $h(e_4) = ce_2$ is a strong isotopism between M^2 and $M^7_{c,0}$, for all $c \in \mathbb{F}_q \setminus \{0\}$.
- $\dim(Z(M)) = 1$: $M^{14}_1 \simeq M^{14}_a$, for all $a \in \mathbb{F}_q \setminus \{0\}$. If $p = 2$, then $M^{14}_1 \simeq W(1;\underline{2})^{(1)} \oplus Z(L)$. Otherwise, $M^{14}_1 \simeq \mathfrak{gl}(2, \mathbb{F}_q)$. Here,
  - The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = ae_2$ is a strong isotopism between M^{14}_1 and M^{14}_a, for all $\in \mathbb{F}_q \setminus \{0\}$.
  - If $p = 2$, then the triple $(\mathrm{Id}, \mathrm{Id}, (1432))$ is a strong isotopism between M^{14}_1 and $W(1;\underline{2})^{(1)} \oplus Z(L)$.
  - If $p \neq 2$, then the triple (f, f, h) such that $f(e_4) = e_1 + e_4$, $h(e_1) = e_4$, $h(e_2) = -e_3$, $h(e_3) = e_2$ and $h(e_4) = e_1$ is a strong isotopism between M^{14}_1 and $\mathfrak{gl}(2, \mathbb{F}_q)$.

$\square$

Chapter 5

Isotopisms of evolution algebras

The implementation of results, procedures and algorithms that have been described throughout this manuscript enables us to show the importance that isotopisms can have in other fields. In this regard, the final chapter of the manuscript deals with the set $\mathcal{E}_n(\mathbb{K})$ of n-dimensional evolution algebras over a field $\mathbb{K}$, whose description has a certain similarity with that of pre-filiform Lie algebras, which were introduced in Chapter 3, and whose distribution into isotopism classes is uniquely related with mutations in non-Mendelian Genetics. We focus on the two-dimensional case, which is related to the asexual reproduction processes of diploid organisms. We prove in particular the distribution of two-dimensional evolution algebras into four isotopism classes, whatever the base field is. After that, we deal with the distribution of two-dimensional evolution algebras over any base field into isomorphism classes.

5.1 Preliminaries

In this section we expose some basic concepts and results on Genetics and evolution algebras that are used throughout the chapter. For more details about these topics we refer, respectively, to the manuscripts of Wörz-Busekros [318] and Tian [300].

Let us start with some preliminary concepts in Genetics. A *gene* is the molecular unit of hereditary information. This consists of *deoxyribonucleic acid* (*DNA*), which contains in turn the code to synthesize proteins and determines each one of the attributes that characterize and distinguish each organism. Genes related to a given attribute can have alternative forms, which are called *alleles*. Thus, for instance, color of eyes are related to brown, green and blue alleles. Genes are disposed in *chromosomes*, which constitute long strands of DNA formed by ordered sequences of genes. The location of alleles related to a given attribute in a chromosome is its *locus*, which is preserved by inheritance. Chromosomes carry, therefore, the genetic code of any organism.

Chromosomes also play a main role in the process of reproduction, because the attributes that characterize the offspring are inherited from the alleles that are contained in the chromosomes of the parents. This inheritance depends on the type of organisms under consideration. Thus, for instance, *diploid* organisms carry a double set of chromosomes (one of each parent). They reproduce by means of sex cells or *gametes*, each of them carrying a single set of chromosomes. The fusion of two gametes of opposite sex gives rise to a *zygote*, which contains a double set of chromosomes. Each one of the attributes that characterize the new diploid organism is uniquely determined by the pair of alleles having the same loci in these two chromosomes. If A and a denote these two alleles, it is said that the new individual is of *zygotic type* Aa. If $A = a$, then the zygote is called *homozygous*. Otherwise, it is called *heterozygous*. There exist distinct laws that regulate, from a probabilistic point of view, the theoretical influence of each one of these two alleles in the final attribute inherited by the offspring. Thus, for instance, the laws of *simple Mendelian inheritance* indicate that for each pair of alleles related to a given attribute, the next generation will inherit with equal frequency both alleles.

Nonassociative algebras were introduced into Genetics by Etherington [115, 116, 117] in order to endow Mendel's laws with a mathematical formulation that enables us to deal with the sexual reproduction and the mechanism of inheritance of an organism by considering the fusion of gametes into a zygote as an algebraic multiplication whose structure constants determine the probability distribution of the gametic output. Specifically, if $\beta = \{e_1, \ldots, e_n\}$ constitutes the set of genetically distinct alleles that are related to a given attribute of a population, then a *genetic algebra* over a field $\mathbb{K}$ that is based on the set β is an n-dimensional algebra of basis β whose structure constants in each product $e_ie_j = \sum_{k=1}^{n} c_{ijk}e_k$ refer to the probability that an arbitrary gamete produced by an individual of zygotic type e_ie_j contains the allele e_k. Hence, $\sum_{k=1}^{n} c_{ijk} = 1$, for all $i, j \leq n$. Here, all zygotes have the same fertility and there is an absence of selection. Observe that nilpotency and solvability of genetic algebras characterize the disappearance of population in evolution processes [117].

Depending on some possible variations in the initial conditions, distinct types of genetic algebras can be defined. Thus, for instance, a *gametic algebra* is a finite-dimensional real genetic algebra, where each element $\sum_{i=1}^n a_i e_i$ that satisfies that $0 \leq a_i \leq 1$, for all $i \leq n$, and $\sum_{i=1}^n a_i = 1$, can represent a population, a single individual or a single gamete. For each $i \leq n$, the coefficient a_i constitutes, respectively, the percentage of frequency of the allele e_i in the corresponding population, individual or gamete. In simple Mendelian inheritance, for instance, we have that $e_i e_j = \frac{1}{2}(e_i + e_j)$, for any pair of alleles e_i and e_j. Observe that, if two gametes carry the same allele, then the offspring will inherit it. Particularly, in case of dealing with a diploid organism, the multiplication table of the corresponding two-dimensional gametic algebra of basis $\{e_1, e_2\}$ is, therefore,

	e_1	e_2
e_1	e_1	$\frac{1}{2}(e_1 + e_2)$
e_2	$\frac{1}{2}(e_1 + e_2)$	e_2

Holgate and Campos [73, 168] showed that certain known families of genetic algebras are isotopic. Particularly, they considered isotopisms of genetic algebras as a way to formulate mathematically the mutation of alleles in the inheritance process. In this regard, a second type of genetic algebra that is interesting to be considered here is that formed by *mutation algebras*. In these algebras, before of participating in the formation of a zygote, each allele e_i mutates to an allele e_j with probability m_{ij}. Hence, $0 \leq m_{ij} \leq 1$ and $\sum_{j=1}^n m_{ij} = 1$. Specifically, if $(A, \cdot)$ is a genetic algebra over a field $\mathbb{K}$ of basis $\beta = \{e_1, \ldots, e_n\}$ and $M = (m_{ij})$ is an $n \times n$ matrix with entries in $\mathbb{K}$, then we can define the mutation algebra $(A, \circ)$ of basis β such that $e_i \circ e_j = \sum_{k=1}^n m_{ik} e_k \cdot \sum_{l=1}^n m_{jl} e_l = \sum_{k,l=1}^n m_{ik} m_{jl} e_i \cdot e_k$. Mutation algebras constitute, therefore, principal isotopes of genetic algebras.

A third type of genetic algebras, whose study constitutes in fact the main goal of this chapter, is that formed by evolution algebras. In order to deal with asexual reproduction processes, Tian and Vojtechovsky [300, 301] introduced these algebras as a type of genetic algebra that makes possible to deal algebraically with the self-reproduction of alleles in non-Mendelian Genetics. Nowadays, these algebras also constitute a fundamental connection between algebra, dynamic systems, Markov processes, Knot Theory, Graph Theory and Group Theory (see [197, 300]). Specifically, an n-dimensional algebra over a field $\mathbb{K}$ is said to be an *evolution algebra* if it admits a *natural basis* $\{e_1, \ldots, e_n\}$ such that the next two conditions hold

- $e_ie_j = 0$, if $i \neq j$.
- $e_ie_i = \sum_{j=1}^{n} t_{ij}e_j$, for some structure constants $t_{i1}, \ldots, t_{in} \in \mathbb{K}$.

An evolution algebra is said to be *nondegenerate* if there does not exist any zero row in the quadratic matrix (t_{ij}) formed by its structure constants, that is, if $e_ie_i \neq 0$, for all $i \leq n$. Otherwise, this is said to be *degenerate*. Hereafter, the set of n-dimensional evolution algebras over a field $\mathbb{K}$ is denoted as $\mathcal{E}_n(\mathbb{K})$.

As a genetic algebra, each basis vector of an evolution algebra constitutes an allele; the product $e_ie_j = 0$, for $i \neq j$, represents uniparental inheritance; the product e_ie_i represents self-replication; and each structure constant t_{ij} constitutes the probability that the allele e_i becomes the allele e_j in the next generation. In any case, the theory of evolution algebras has been being developed in the last years with no probabilistic restrictions on the structure constants [70, 72, 76, 77, 110, 111, 201, 202, 203, 244]. An exhaustive list of papers related to this theory is exposed online in `https://www.math.nmsu.edu/~jtian/e- algebra/e-alg-index.htm`, which is currently maintained by Tian himself.

A main problem in the theory of n-dimensional evolution algebras is their distribution into isomorphism and isotopism classes. The former has already been dealt with for two-dimensional evolution algebras over the complex field. Thus, for instance, Camacho et al. [71] and Casas et al. [78] studied those evolution algebras in $\mathcal{E}_2(\mathbb{C})$ whose quadratic matrices are, respectively, Jordan matrices and upper triangular matrices. With respect to arbitrary fields, Hegazi and Abdelwahab [165] have recently classified nilpotent evolution algebras of dimension up to four. Particularly, the next result is known.

Theorem 5.1.1 ([78]). *Every two-dimensional non-abelian complex evolution algebra $A \in \mathcal{E}_2(\mathbb{C})$ is isomorphic to exactly one of the next algebras*

- $\dim A^2 = 1$:
  - $E_1 : e_1e_1 = e_1$.
  - $E_2 : e_1e_1 = e_2e_2 = e_1$.
  - $E_3 : e_1e_1 = e_1 + e_2$ *and* $e_2e_2 = -e_1 - e_2$.
  - $E_4 : e_1e_1 = e_2$.
- $\dim A^2 = 2$:

– $E_{5_{a,b}}$: $e_1e_1 = e_1 + ae_2$ *and* $e_2e_2 = be_1 + e_2$*, where* $a, b \in \mathbb{C}$ *are such that* $ab \neq 1$*. Here,* $E_{5_{a,b}} \cong E_{5_{b,a}}$*, for all* $a, b \in \mathbb{C}$*.*

– E_{6_a} : $e_1e_1 = e_2$ *and* $e_2e_2 = e_1 + ae_2$*, where* $a \in \mathbb{C}$*. If* $a, b \in \mathbb{C} \setminus \{0\}$*, then* $E_{6_a} \cong E_{6_b}$ *if and only if* $\frac{a}{b} = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$*, for some* $k \in \{0, 1, 2\}$*.* $\square$

To the best of the author knowledge, even if isotopisms have emerged as an interesting tool to simulate mutations in genetic algebras, they have not yet been considered in case of dealing with evolution algebras. The main goal of this chapter is to delve into this aspect. Particularly, the next section deals with the distribution of finite-dimensional evolution algebras over any base field into isotopism classes according to their structure tuples and to the dimension of their annihilators. After that, we focus on the corresponding distribution of two-dimensional evolution algebras over any base field into isomorphism classes.

5.2 Structure tuples and annihilators

In this section we analyze two aspects of evolution algebras that can be used in order to determine their distribution into isotopism and isomorphism classes: their structure tuples and their annihilators. Let us study each aspect separately.

5.2.1 Structure tuples of evolution algebras

Evolution algebras have a certain similarity with pre-filiform Lie algebras, which were introduced in Chapter 3. Particularly, the non-zero products of the basis vectors of an n-dimensional pre-filiform Lie algebra are of the form e_ie_n, with $i \leq n$, whereas those of an n-dimensional evolution algebra are of the form e_ie_i, with $i \leq n$. In this regard and similarly to what was exposed in the mentioned chapter, we introduce the concept of *structure tuple* of an evolution algebra in $\mathcal{E}_n(\mathbb{K})$ as the tuple $T = (\mathfrak{t}_1, \ldots, \mathfrak{t}_n)$ such that $\mathfrak{t}_i = e_ie_i$, for all $i \leq n$. We denote this evolution algebra by A_T. Besides, from here on, the set of structure tuples of evolution algebras in $\mathcal{E}_n(\mathbb{K})$ is denoted as $\mathcal{T}_n(\mathbb{K})$. This coincides with the n-dimensional $\mathbb{K}$-vector space with components in $\langle\, e_1, \ldots, e_n \,\rangle$. The next results follow analogously to Lemma 3.2.1 and Proposition 3.2.3, which were exposed in Chapter 3.

Lemma 5.2.1. *Let T and T' be two structure tuples in $\mathcal{T}_n(\mathbb{K})$ that are equal up to permutation of their components and basis vectors. The evolution algebras A_T and $A_{T'}$ are strongly isotopic.* $\square$

Example 5.2.2. *From Lemma 5.2.1, we have, for instance, that the evolution algebras E_1 and E_4 in Theorem 5.1.1 are strongly isotopic. Specifically, the triple* $(\mathrm{Id}, \mathrm{Id}, (12))$ *is a strong isotopism between both algebras.* $\triangleleft$

Proposition 5.2.3. *Let T be a structure tuple in $\mathcal{T}_n(\mathbb{K})$. There always exists a structure tuple $T' = (\sum_{j=1}^n t'_{1j}e_j, \ldots, \sum_{j=1}^n t'_{nj}e_j) \in \mathcal{T}_n(\mathbb{K})$ such that $A_{T'}$ is strongly isotopic to A_T and the next two conditions hold*

a) If $t'_{ii} = 0$ for some $i \geq 1$, then $t'_{jk} = 0$, for all $j, k \geq i$.

b) If $t'_{ii} \neq 0$ for some $i \geq 1$, then $t'_{ij} = 0$, for all $j \neq i$. $\square$

Example 5.2.4. *The proof of Proposition 5.2.3 follows similarly to that of Proposition 3.2.3, which was referred to pre-filiform algebras. By following the steps of that proof, we obtain, for instance, that the evolution algebra E_3 in Theorem 5.1.1 is strongly isotopic to the evolution algebra $A_{(e_1,-e_1)} \in \mathcal{E}_2(\mathbb{C})$ by means of the strong isotopism* $(\mathrm{Id}, \mathrm{Id}, h)$*, where $h(e_1) = e_1 - e_2$ and $h(e_2) = e_2$.*

Similarly, any evolution algebra $E_{5_{a,b}}$ in Theorem 5.1.1 is strongly isotopic to the algebra $E_{5_{0,0}}$. Specifically, if a and b are two complexes numbers such that $ab \neq 1$, then the triple $(\mathrm{Id}, \mathrm{Id}, h)$ *such that $h(e_1) = e_1 - ae_2$ and $h(e_2) = e_2$ is a strong isotopism between the evolution algebra $E_{5_{a,b}}$ and the evolution algebra A_T of structure tuple $T = (e_1, be_1 + (1 - ab)e_2) \in \mathcal{T}_2(\mathbb{C})$. Now, the triple* $(\mathrm{Id}, \mathrm{Id}, h')$ *such that $h'(e_2) = \frac{1}{1-ab}(e_2 - be_1)$ and $h'(e_1) = e_1$ is a strong isotopism between the evolution algebras A_T and $E_{5_{0,0}}$.*

Finally, we can prove that any evolution algebra E_{6_a} in Theorem 5.1.1 is also strongly isotopic to the algebra $E_{5_{0,0}}$. Specifically, if a is a complex number distinct of zero, then the triple $((12), (12), h'')$ *such that $h''(e_1) = e_1 - ae_2$ and $h''(e_2) = e_2$ is a strong isotopism between the evolution algebras E_{6_a} and $E_{5_{0,0}}$.* $\triangleleft$

We have mentioned that Lemma 5.2.1 and Proposition 5.2.3 follow similarly to Lemma 3.2.1 and Proposition 3.2.3. Nevertheless, evolution algebras do not hold a similar result to that exposed in Lemma 3.2.5, which is referred to the fact that structure tuples that are equal up to addition of their components give rise to strongly isotopic pre-filiform Lie algebras. Moreover, this result is not true even for isotopisms of evolution algebras. Thus, for instance, the evolution algebra $E_3 = A_{(e_1+e_2,-e_1-e_2)}$ in Theorem 5.1.1 is not isotopic to the evolution algebra $A_{(e_1+e_2,0)} \in \mathcal{E}_2(\mathbb{C})$, whose structure tuple results from that of E_3 after adding its first component to its second one. This follows from Proposition 2.1.4 and the fact of being $\dim \mathrm{Ann}_{E_3}(E_3) = 0 \neq 1 = \dim \mathrm{Ann}_{A_{(e_1+e_2,0)}}(A_{(e_1+e_2,0)})$.

Let us finish this subsection by determining explicitly the distribution of the set $\mathcal{E}_2(\mathbb{C})$ into isotopism classes.

Proposition 5.2.5. *There exist four isotopism classes of two-dimensional complex evolution algebras. They correspond to the abelian algebra and the evolution algebras E_1, E_2 and $E_{5_{0,0}}$ in Theorem 5.1.1.*

Proof. In Examples 5.2.2 and 5.2.4 we have already seen that

- $E_1 \simeq E_4$.
- $E_3 \simeq A_{(e_1,-e_1)}$.
- $E_{5_{a,b}} \simeq E_{6_c}$, for all $a, b, c \in \mathbb{C}$ such that $ab \neq 1$ and $c \neq 0$.

Observe now that the triple (f, f, Id) such that $f(e_1) = -ie_2$ and $f(e_2) = e_1$ is a strong isotopism between the algebras E_2 and $A_{(e_1,-e_1)}$. Hence, $E_2 \simeq E_3$. It is enough to prove, therefore, that the four evolution algebras of the statement are not isotopic. From Lemma 2.1.1, we can focus on the three algebras E_1, E_2 and $E_{5_{0,0}}$. From Proposition 2.1.4, the former is not isotopic to E_2 or $E_{5_{0,0}}$, because $\dim \mathrm{Ann}_{E_1}(E_1) = 1 \neq 0 = \dim \mathrm{Ann}_{E_2}(E_2) = \dim \mathrm{Ann}_{E_{5_{0,0}}}(E_{5_{0,0}})$. Finally, from Lemma 2.1.5, the evolution algebras E_2 and $E_{5_{0,0}}$ are not isotopic, because $E_2^2 = \langle\, e_1 \,\rangle \subset \langle\, e_1, e_2 \,\rangle = E_{5_{0,0}}^2$. $\square$

The proof of Proposition 5.2.5 involves the fact that the existence of nilpotent elements of evolution algebras is not preserved by isotopisms. Thus, for instance, we have just seen that the evolution algebras E_2 and E_3 are isotopic. Nevertheless, even if the basis vector e_2 is nilpotent in E_2 (specifically, $e_2^3 = 0$), the evolution algebra E_3 does not have nilpotent elements.

5.2.2 Annihilators of evolution algebras

Let n be a positive integer and let m be a non-negative integer such that $m \leq n$. Let $\mathcal{E}_{n;m}(\mathbb{K})$ denote the subset of n-dimensional evolution algebras over a field $\mathbb{K}$ having an $(n-m)$-dimensional annihilator. The set $\{\mathcal{E}_{n;m}(\mathbb{K}) \mid 0 \leq m \leq n\}$ constitutes, therefore, a partition of the set $\mathcal{E}_n(\mathbb{K})$. The next results hold.

Lemma 5.2.6. *Every evolution algebra in $\mathcal{E}_{n;m}(\mathbb{K})$ is isomorphic to an n-dimensional evolution algebra with a natural basis $\{e_1, \ldots, e_n\}$ such that $e_ie_i \neq 0$ if and only if $i \leq m$.*

Proof. Let $A \in \mathcal{E}_{n;m}(\mathbb{K})$ be an evolution algebra of natural basis $\{e_1, \ldots, e_n\}$. From the description of the set $\mathcal{E}_{n;m}(\mathbb{K})$, there exists a subset $S = \{i_1, \ldots, i_m\} \subseteq [n]$ such that $e_ie_i \neq 0$ if and only if $i \in S$. It is then enough to consider the isomorphism that maps, respectively, the basis vectors $e_{i_1}, \ldots, e_{i_m}$ to $e_1, \ldots, e_m$ and preserves the rest of basis vectors. $\square$

Proposition 5.2.7. *Let m and m' be two distinct non-negative integers less than or equal to n. Then, none evolution algebra in $\mathcal{E}_{n;m}(\mathbb{K})$ is isotopic to an evolution algebra in $\mathcal{E}_{n;m'}(\mathbb{K})$.*

Proof. The result follows straightforward from Proposition 2.1.4 and the fact that $\mathrm{Ann}_{A^-}(A) = \mathrm{Ann}_{A^+}(A) = \mathrm{Ann}_A(A)$, for all $A \in \mathcal{E}_n(\mathbb{K})$. $\square$

The next result deals with the distribution of the set $\mathcal{E}_{n;m}(\mathbb{K})$ into isomorphism and isotopism classes, for all positive integer $n \in \mathbb{N}$ and $m \in \{0, 1, 2\}$. In the statement of the result we make use of the description of the algebras that were exposed in Theorem 5.1.1 with the exception of dealing here with n-dimensional evolution algebras over the field $\mathbb{K}$ instead of two-dimensional complex evolution algebras. Similar abuse of notation is done from here on in order to get a simple and coherent labeling of the evolution algebras that are exposed in this chapter.

Proposition 5.2.8. *The next assertions hold.*

a) The set $\mathcal{E}_{n;0}(\mathbb{K})$ is only formed by the n-dimensional abelian algebra.

b) Any evolution algebra in $\mathcal{E}_{1;1}(\mathbb{K})$ is isomorphic to the algebra E_1.

c) If $n > 1$, then any evolution algebra in $\mathcal{E}_{n;1}(\mathbb{K})$ is isomorphic to the algebra E_1 or to the algebra E_4.

d) Any evolution algebra in $\mathcal{E}_{n;1}(\mathbb{K})$ is isotopic to the algebra E_1.

e) Any evolution algebra in $\mathcal{E}_{2;2}(\mathbb{K})$ is isomorphic to an evolution algebra in $\mathcal{E}_{2;2}(\mathbb{K})$ with natural basis $\{e_1, e_2\}$ such that $e_1e_1 \in \{e_1, e_2, e_1 + e_2\}$.

f) Any evolution algebra in $\mathcal{E}_{n;2}(\mathbb{K})$ is isotopic to E_2 or $E_{5_{0,0}}$.

Proof. Let us prove each assertion separately.

a) This assertion follows straightforward from the definition of $\mathcal{E}_{n;0}(\mathbb{K})$.

b) This results follows from Proposition 2.1.2.

c) Let A be an n-dimensional evolution algebra in $\mathcal{E}_{n;1}(\mathbb{K})$ with a natural basis $\{e_1, \ldots, e_n\}$, which is described by its quadratic matrix of structure constants (t_{ij}). From Lemma 5.2.6, we can suppose that $t_{ij} = 0$, for all $i > 1$. Let $j_0 \leq n$ denote the minimum positive integer such that $t_{1j_0} \neq 0$. This exists because A is non-abelian. Besides, we can suppose that $j_0 \in \{1, 2\}$. Otherwise, it is enough to consider the isomorphism $(2j_0)$ that switches the basis vectors e_2 and e_{j_0}. Let us study both cases.

- If $j_0 = 1$, then the linear transformation f that is defined such that $f(e_1) = t_{11}e_1 - \frac{1}{t_{11}} \sum_{j=2}^{n} t_{1j}e_j$ and $f(e_i) = e_i$, for all $i > 1$, is an isomorphism between A and the evolution algebra E_1.
- If $j_0 = 2$, then the linear transformation f that is defined such that $f(e_2) = \frac{1}{t_{12}}(e_2 - \sum_{j=3}^{n} t_{1j}e_j)$ and $f(e_i) = e_i$, for all $i \neq 2$, is an isomorphism between A and the evolution algebra E_4.

d) If $n = 1$, then the result follows immediately from Proposition 2.1.2. Otherwise, it is enough to observe that the triple $(\mathrm{Id}, \mathrm{Id}, (12))$ is a strong isotopism between the evolution algebras E_1 and E_4 in assertion (c).

e) Let A be an evolution algebra in $\mathcal{E}_{2;2}(\mathbb{K})$ with natural basis $\{e_1, e_2\}$. From Lemma 5.2.6, we can suppose the existence of a pair $(a, b) \in \mathbb{K}^2 \setminus \{(0, 0)\}$ such that $e_1e_1 = ae_1 + be_2$. If $a \neq 0$, then the algebra A is isomorphic to the evolution algebra with the same natural basis such that $e_1e_1 = e_1$, whenever $b = 0$, and $e_1e_1 = e_1 + e_2$, whenever $b \neq 0$. To this end, it is enough to consider the isomorphism f such that $f(e_1) = ae_1$ and

$$f(e_2) = \begin{cases} e_2, & \text{if } b = 0, \\ \frac{a^2}{b} e_2, & \text{otherwise.} \end{cases}$$

If $a = 0$, then $b \neq 0$ and the algebra A is isomorphic to the evolution algebra with the same natural basis such that $e_1e_1 = e_2$ by means of the isomorphism that maps e_2 to be_2 and preserves the basis vector e_1.

f) Let A be an evolution algebra in $\mathcal{E}_{n;2}(\mathbb{K})$. From Proposition 5.2.3 and Lemma 5.2.6, we can suppose the existence of two elements $a, b \in \mathbb{K} \setminus \{0\}$ such that A is strongly isotopic to the evolution algebra in $\mathcal{E}_{n;2}(\mathbb{K})$ with structure tuple $T_1 = (ae_1, be_1, 0, \ldots, 0)$ or $T_2 = (ae_1, be_2, 0, \ldots, 0)$ in $\mathcal{T}_n(\mathbb{K})$. The evolution algebra A_{T_1} is isotopic to E_2 by means of the triple $(f, \mathrm{Id}, \mathrm{Id})$ such that $f(e_1) = ae_1$, $f(e_2) = be_2$ and $f(e_i) = e_i$, for all $i > 2$, whereas the evolution algebra A_{T_2} is strongly isotopic to $E_{5_{0,0}}$ by means of the triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_1) = \frac{1}{a}e_1$, $h(e_2) = \frac{1}{b}e_2$ and $h(e_i) = e_i$, for all $i > 2$. □

The next theorem, which follows straightforward from Proposition 5.2.8, generalizes Proposition 5.2.5 and determines explicitly the distribution of two-dimensional evolution algebras into isotopism classes, whatever the base field is.

Theorem 5.2.9. *There exist four isotopism classes of two-dimensional evolution algebras over any field. They correspond to the abelian algebra and the evolution algebras E_1, E_2 and $E_{5_{0,0}}$.* □

The distribution of two-dimensional evolution algebras over any base field into isomorphism classes requires, however, a further study, which constitutes the final part of this manuscript.

5.3 Isomorphism classes of the set $\mathcal{E}_2(\mathbb{K})$

Firstly, as a preliminary study, we focus on the finite field $\mathbb{K} = \mathbb{F}_q$, with q a prime power. Particularly, we have implemented the procedure *isoAlg* into Algorithm 2, both of them introduced in Chapter 2, in order to show in Tables 5.1–5.3 the distribution of the set $\mathcal{E}_2(\mathbb{F}_q)$ into isomorphism classes, for $q \in \{2,3,5,7\}$. In the first two tables we indicate for each class the isomorphism invariants of the graphs G_1 and G_2 that were also described in Chapter 2. Observe that the distribution of all these evolution algebras into the four isotopism classes that have been exposed in Theorem 5.2.9 are clearly identified by means of these invariants. In order to expose the efficiency of our procedure, we also expose in Table 5.4 the run time and usage memory that are required to compute each distribution.

The next result follows straightforward from the previous computation.

Theorem 5.3.1. *The sets $\mathcal{E}_2(\mathbb{F}_q)$, with $q \in \{2,3,5,7\}$, are respectively distributed into 9, 13, 23 and 38 isomorphism classes.* □

Observe in Table 5.1 that the distribution of the set $\mathcal{E}_2(\mathbb{F}_2)$ into nine isomorphism classes agrees with that corresponding to the set $\mathcal{E}_2(\mathbb{C})$, which was exposed in Theorem 5.1.1. Nevertheless, this does not hold for finite fields of higher orders. Thus, for instance, the evolution algebra $A_{(e_1,2e_1)}$, which has a one-dimensional derived algebra, is not isomorphic to any of the corresponding four evolution algebras E_1 to E_4 in $\mathcal{E}_2(\mathbb{F}_q)$, for $q > 2$. A further study that generalizes the result of Casas et al. [78] is then required for a general base field $\mathbb{K}$. Similarly to the results that have been exposed in previous chapters, we deal with this general case by making use of Computational Algebraic Geometry. From Proposition 5.2.8, we can focus on the distribution of the set $\mathcal{E}_{2,2}(\mathbb{K})$ into isomorphism classes and, more specifically, on those two-dimensional evolution algebras with natural basis $\{e_1, e_2\}$ such that $e_1e_1 \in \{e_1, e_2, e_1 + e_2\}$.

Let $A = A_{(ae_1+be_2,ce_1+de_2)}$ and $A' = A_{(\alpha e'_1+\beta e'_2,\gamma e'_1+\delta e'_2)}$ be two isomorphic two-dimensional evolution algebras in $\mathcal{E}_{2,2}(\mathbb{K})$ with respective natural bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$. Let f be an isomorphism between both algebras with a related non-singular matrix $F = (f_{ij})$ such that $f(e_i) = f_{i1}e'_1 + f_{i2}e'_2$, for all $i \in \{1,2\}$. The implementation of the procedure *isoAlg* enables us to ensure that, whatever the base field is, the reduced Gröbner basis of the ideal in Theorem 2.2.5 related to the isomorphism

		G_1 & G_2	G_1	G_2	
q	Structure tuple	Vertices	Edges	Edges	Triangles
2	$(0,0)$	(0,0,0,0)	0	0	0
	$(e_2,0)$	(2,2,1,4)	12	16	2
	$(e_1,0)$	(2,2,1,4)	12	18	7
	(e_1,e_1)	(3,3,1,6)	18	25	7
	(e_1+e_2,e_1+e_2)	(3,3,1,6)	18	25	7
	(e_2,e_1)	(3,3,3,7)	21	33	8
	(e_2,e_1+e_2)	(3,3,3,7)	21	33	8
	(e_1,e_1+e_2)	(3,3,3,7)	21	33	12
	(e_1,e_2)	(3,3,3,7)	21	33	16
3	$(0,0)$	(0,0,0,0)	0	0	0
	$(e_2,0)$	(6,6,2,36)	108	120	6
	$(e_1,0)$	(6,6,2,36)	108	124	20
	$(e_1,2e_1)$	(8,8,2,48)	144	160	18
	$(e_1+e_2,2e_1+2e_2)$	(8,8,2,48)	144	160	18
	(e_1,e_1)	(8,8,2,48)	144	164	22
	(e_2,e_1)	(8,8,8,56)	168	200	24
	(e_2,e_1+e_2)	(8,8,8,56)	168	200	24
	(e_2,e_1+2e_2)	(8,8,8,56)	168	200	24
	$(e_1+e_2,2e_1+e_2)$	(8,8,8,56)	168	200	24
	(e_1,e_1+e_2)	(8,8,8,56)	168	200	36
	$(e_1,2e_1+e_2)$	(8,8,8,56)	168	200	36
	(e_1,e_2)	(8,8,8,56)	168	200	48

Table 5.1: Graph invariants for the graphs G_1 and G_2 related to the isomorphism classes of the set $\mathcal{E}_2(\mathbb{F}_q)$, for $q \in \{2,3\}$.

	G_1 & G_2	G_1	G_2	
Structure tuple	Vertices	Edges	Edges	Triangles
$(0,0)$	(0,0,0,0)	0	0	0
$(e_2,0)$	(20,20,4,400)	1200	1240	20
$(e_1,0)$	(20,20,4,400)	1200	1248	64
(e_1,e_1)	(24,24,4,480)	1440	1488	60
$(e_1+e_2,4e_1+4e_2)$	(24,24,4,480)	1440	1488	60
$(e_1,2e_1)$	(24,24,4,480)	1440	1496	68
(e_2,e_1)	(24,24,24,544)	1632	1728	80
(e_2,e_1+e_2)	(24,24,24,544)	1632	1728	80
(e_2,e_1+2e_2)	(24,24,24,544)	1632	1728	80
(e_2,e_1+3e_2)	(24,24,24,544)	1632	1728	80
(e_2,e_1+4e_2)	(24,24,24,544)	1632	1728	80
(e_1+e_2,e_1+2e_2)	(24,24,24,544)	1632	1728	80
(e_1+e_2,e_1+3e_2)	(24,24,24,544)	1632	1728	80
(e_1+e_2,e_1+4e_2)	(24,24,24,544)	1632	1728	80
$(e_1+e_2,2e_1+e_2)$	(24,24,24,544)	1632	1728	80
$(e_1+e_2,3e_1+e_2)$	(24,24,24,544)	1632	1728	80
$(e_1+e_2,2e_1+3e_2)$	(24,24,24,544)	1632	1728	80
$(e_1+e_2,3e_1+2e_2)$	(24,24,24,544)	1632	1728	80
(e_1,e_1+e_2)	(24,24,24,544)	1632	1728	120
$(e_1,2e_1+e_2)$	(24,24,24,544)	1632	1728	120
$(e_1,3e_1+e_2)$	(24,24,24,544)	1632	1728	120
$(e_1,4e_1+e_2)$	(24,24,24,544)	1632	1728	120
(e_1,e_2)	(24,24,24,544)	1632	1728	160

Table 5.2: Graph invariants for the graphs G_1 and G_2 related to the isomorphism classes of the set $\mathcal{E}_2(\mathbb{F}_5)$.

Structure tuples			
$(0,0)$	$(e_2, 2e_1+e_2)$	(e_1+e_2, e_1+2e_2)	$(e_1+e_2, 3e_1+5e_2)$
$(e_1,0)$	$(e_2, 2e_1+3e_2)$	(e_1+e_2, e_1+3e_2)	$(e_1+e_2, 3e_1+6e_2)$
$(e_2,0)$	$(e_2, 3e_1+e_2)$	(e_1+e_2, e_1+4e_2)	$(e_1+e_2, 4e_1+3e_2)$
(e_1,e_1)	$(e_2, 3e_1+3e_2)$	(e_1+e_2, e_1+5e_2)	$(e_1+e_2, 4e_1+5e_2)$
$(e_1,2e_1)$	(e_1, e_1+e_2)	(e_1+e_2, e_1+6e_2)	$(e_1+e_2, 4e_1+6e_2)$
$(e_1,3e_1)$	(e_1, e_1+2e_2)	$(e_1+e_2, 2e_1+e_2)$	$(e_1+e_2, 6e_1+3e_2)$
(e_1,e_2)	(e_1, e_1+3e_2)	$(e_1+e_2, 2e_1+3e_2)$	$(e_1+e_2, 6e_1+5e_2)$
(e_2,e_1)	$(e_1, 3e_1+e_2)$	$(e_1+e_2, 2e_1+4e_2)$	$(e_1+e_2, 6e_1+6e_2)$
(e_2,e_1+e_2)	$(e_1, 3e_1+2e_2)$	$(e_1+e_2, 2e_1+5e_2)$	
(e_2,e_1+3e_2)	$(e_1, 3e_1+3e_2)$	$(e_1+e_2, 2e_1+6e_2)$	

Table 5.3: Distribution into isomorphism classes of the set $\mathcal{E}_2(\mathbb{F}_7)$.

q	Run time	Usage memory
2	0 seconds	0 MB
3	3 seconds	0 MB
5	38 seconds	80 MB
7	278 seconds	1360 MB

Table 5.4: Run time and memory usage that is required to compute the distribution of the set $\mathcal{E}_2(\mathbb{F}_q)$ into isomorphism classes, for $q \leq 7$.

group between the evolution algebras A and A' involves in particular that

$$\begin{cases} (ad-bc)f_{11}f_{21} = 0, \\ (ad-bc)f_{12}f_{22} = 0. \end{cases} \tag{5.1}$$

From the previous conditions, we can distinguish two cases depending on the fact of being $ad = bc$ or $ad \neq bc$. They refer, respectively, to two-dimensional evolution algebras with a one- or two-dimensional derived algebra. Recall in this regard that any isomorphism between two algebras preserves the dimension of their corresponding derived algebras. In the next two subsections we study each one of the two mentioned cases separately.

5.3.1 One-dimensional derived algebra ($ad = bc$)

In this subsection, f is an isomorphism of regular matrix $F = (f_{ij})$ between a pair of evolution algebras $A = A_{(ae_1+be_2,ce_1+de_2)}$ and $A' = A_{(\alpha e_1'+\beta e_2',\gamma e_1'+\delta e_2')}$ in $\mathcal{E}_{2,2}(\mathbb{K})$ such that $ad = bc$ and $\alpha\delta = \beta\gamma$. From assertion (e) in Proposition 5.2.8, we can suppose that $a, b, \alpha, \beta \in \{0, 1\}$. Firstly, suppose $A = A_{(e_1,ce_1)}$, with $c \in \mathbb{K} \setminus \{0\}$. Assertion (e) in Proposition 5.2.8 gives rise to the next case study

- **Case 1**. $A' = A_{(e_1',\gamma e_1')}$, with $\gamma \in \mathbb{K} \setminus \{0\}$.

  The identification of coefficients of a same basis vector in each one of the equalities $f(e_ie_j) = f(e_i)f(e_j)$, for all $i, j \leq 2$, involves that f is an isomorphism between the two algebras under consideration if and only if $f_{11}f_{21} = f_{12}f_{22} = 0$. The regularity of the matrix F involves that $f_{11} = f_{22} = 0$ or $f_{21} = f_{12} = 0$. In the first case, we obtain that f_{21} must be zero, what is a contradiction with the regularity of the matrix F. In the second case, we obtain that $c = \gamma f_{22}^2$. This fact enables us to ensure that $A_{(e_1,ce_1)} \cong A_{(e_1,cm^2e_1)}$, for all $c, m \in \mathbb{K} \setminus \{0\}$.

- **Case 2**. $A' = A_{(e_2',\delta e_2')}$, with $\delta \in \mathbb{K} \setminus \{0\}$.

  The computation of the corresponding reduced Gröbner basis, which has previously been mentioned, related to these assumptions enables us to ensure that

$$\begin{cases} f_{11} = f_{22} = 0, \\ f_{12} = 1/\delta, \\ f_{21}^2 = c/\delta. \end{cases} \tag{5.2}$$

  If we take $f_{21} = 1$, then we can ensure in particular that $A_{(e_2,ce_2)} \cong A_{(e_1,ce_1)}$, for all $c \in \mathbb{K} \setminus \{0\}$.

- **Case 3**. $A' = A_{(e_1'+e_2',\gamma(e_1'+e_2'))}$, with $\gamma \in \mathbb{K} \setminus \{0\}$.

  From the reduced Gröbner basis related to this case, we deduce that

$$\begin{cases} \gamma \neq -1, \\ f_{11} = f_{12} = 1/(\gamma + 1), \\ f_{21} = -\gamma f_{22}, \\ c = \gamma(\gamma + 1)^2 f_{22}^2. \end{cases} \tag{5.3}$$

  Particularly, the determinant of the matrix F coincides with f_{22}, which must be distinct of zero. As a consequence, $A_{(e_1+e_2,c(e_1+e_2))} \cong A_{(e_1,c(c+1)^2e_1)}$, for all $c \in \mathbb{K} \setminus \{0, -1\}$.

From the previous case study, the case $A = A_{(e_2,de_2)}$, with $d \in \mathbb{K}\setminus\{0\}$, can be referred to Case 1.2, because $A_{(e_2,de_2)} \cong A_{(e_1,de_1)}$. Besides, the case $A = A_{(e_1+e_2,c(e_1+e_2))}$, with $c \in \mathbb{K}\setminus\{0\}$, can be referred to Case 1.3 except for the case $c = -1$, that is, except for the evolution algebra $A_{(e_1+e_2,-(e_1+e_2))}$. The next results gather together what we have just exposed in the previous case study.

Proposition 5.3.2. *The next assertions hold in the set $\mathcal{E}_{2,2}(\mathbb{K})$.*

a) $A_{(e_1,ce_1)} \cong A_{(e_1,cm^2e_1)}$ *for all* $c, m \in \mathbb{K}\setminus\{0\}$.

b) $A_{(e_2,ce_2)} \cong A_{(e_1,ce_1)}$, *for all* $c \in \mathbb{K}\setminus\{0\}$.

c) $A_{(e_1+e_2,c(e_1+e_2))} \cong A_{(e_1,c(c+1)^2e_1)}$, *for all* $c \in \mathbb{K}\setminus\{0,-1\}$. □

Theorem 5.3.3. *Any two-dimensional evolution algebra in $\mathcal{E}_{2,2}(\mathbb{K})$ with a one-dimensional derived algebra is isomorphic to exactly one of the next algebras*

- $A_{(e_1,ce_1)}$, *with* $c \in \mathbb{K}\setminus\{0\}$. *Here,* $A_{(e_1,ce_1)} \cong A_{(e_1,\gamma e_1)}$ *if and only if* $\gamma = cm^2$ *for some* $m \in \mathbb{K}\setminus\{0\}$.
- $A_{(e_1+e_2,-e_1-e_2)}$. □

5.3.2 Two-dimensional derived algebra ($ad \neq bc$)

Let us focus now on the case in which the evolution algebra $A = A_{(ae_1+be_2,ce_1+de_2)} \in \mathcal{E}_{2,2}(\mathbb{K})$ is such that $ad \neq bc$.

Lemma 5.3.4. *Let $A = A_{(ae_1+be_2,ce_1+de_2)}$ be a two-dimensional evolution algebra in $\mathcal{E}_{2,2}(\mathbb{K})$ such that $ad \neq bc$. Then, any isomorphism from A to another evolution algebra in $\mathcal{E}_{2,2}(\mathbb{K})$, with related regular matrix $F = (f_{ij})$, holds that $f_{11} = f_{22} = 0$ or $f_{12} = f_{21} = 0$.*

Proof. The result follows straightforward from both conditions in (5.1) and the regularity of the matrix F. □

Let us study each case in Lemma 5.3.4 separately. Here, f is an isomorphism of regular matrix $F = (f_{ij})$ between a pair of evolution algebras $A = A_{(ae_1+be_2,ce_1+de_2)}$ and $A' = A_{(\alpha e'_1+\beta e'_2,\gamma e'_1+\delta e'_2)}$ in $\mathcal{E}_{2,2}(\mathbb{K})$, where $ad \neq bc$ and $\alpha\delta \neq \beta\gamma$.

1. **Case 1**. $f_{11} = f_{22} = 0$.

   Here, $f(e_1) = f_{12}e'_2$ and $f(e_2) = f_{21}e'_1$, where $f_{12} \neq 0 \neq f_{21}$. Similarly to the reasoning exposed in the case study of the previous subsection, the identification of coefficients of a same basis vector in the equalities $f(e_ie_j) = f(e_i)f(e_j)$, for all $i, j \leq 2$, involves that f is an isomorphism between the two algebras under consideration if and only if

   $$\begin{cases} a = \delta f_{12}, \\ b f_{21} = \gamma f_{12}^2, \\ c f_{12} = \beta f_{21}^2, \\ d = \alpha f_{21}. \end{cases} \tag{5.4}$$

   The regularity of the matrix F implies that, a coefficient a, b, c or d is zero in the structure tuple of the algebra A if and only if the respective coefficient δ, γ, β or α is zero in the structure tuple of A'. Now, assertion (e) in Proposition 5.2.8 enables us to focus on the following cases for the evolution algebras A and A' under the conditions of Lemma 5.3.4.

   - **Case 1.1**. $A = A_{(e_1, ce_1+de_2)}$ and $A' = A_{(e'_1, \delta e'_2)}$, where $d \neq 0 \neq \delta$. From (5.4), we have that

     $$\begin{cases} f_{12} = 1/\delta, \\ f_{21} = d, \\ c = 0. \end{cases} \tag{5.5}$$

     Hence, $A_{(e_1, de_2)} \cong A_{(e_1, e_2)}$, for all $d \in \mathbb{K} \setminus \{0\}$.
   - **Case 1.2**. $A = A_{(e_1, ce_1+de_2)}$ and $A' = A_{(e'_1+e'_2, \delta e'_2)}$, where $d \neq 0 \neq \delta$. From (5.4), we have that

     $$\begin{cases} f_{12} = 1/\delta, \\ f_{21} = d, \\ \delta = c/d^2. \end{cases} \tag{5.6}$$

     Hence, $A_{(e_1+e_2, de_2)} \cong A_{(e_1, de_1+e_2)}$, for all $d \in \mathbb{K} \setminus \{0\}$.
   - **Case 1.3**. $A = A_{(e_2, ce_1+de_2)}$ and $A' = A_{(e'_2, \gamma e'_1)}$, where $c \neq 0 \neq \gamma$. From (5.4), we have that

     $$\begin{cases} f_{21} = \gamma f_{12}^2, \\ c = \gamma^2 f_{12}^3, \\ d = 0. \end{cases} \tag{5.7}$$

Hence, $A_{(e_2,c^2m^3e_1)} \cong A_{(e_2,ce_1)}$, for all $c, m \in \mathbb{K} \setminus \{0\}$.

- **Case 1.4**. $A = A_{(e_2,ce_1+de_2)}$ and $A' = A_{(e'_1+e'_2,\gamma e'_1)}$, where c, d and γ are all of them distinct of zero. From (5.4), we have that

$$\begin{cases} f_{12} = d^2/c, \\ f_{21} = d, \\ c^2 = \gamma d^3. \end{cases} \tag{5.8}$$

Hence, $A_{(e_1+e_2,ce_1)} \cong A_{(e_2,\frac{1}{c}(e_1+e_2))}$, for all $c \in \mathbb{K} \setminus \{0\}$.

- **Case 1.5**. $A = A_{(e_1+e_2,ce_1+de_2)}$ and $A' = A_{(e'_1+e'_2,\gamma e'_1+\delta e'_2)}$, where $c \neq d$ and c, d, γ and δ are all of them distinct of zero. From (5.4), we have that

$$\begin{cases} f_{12} = 1/\delta, \\ f_{21} = d, \\ \gamma = c^2/d^3, \\ \delta = c/d^2. \end{cases} \tag{5.9}$$

Hence, $A_{(e_1+e_2,ce_1+de_2)} \cong A_{(e_1+e_2,\frac{c}{d^2}(\frac{c}{d}e_1+e_2))}$, for all $c, d \in \mathbb{K} \setminus \{0\}$ such that $c \neq d$.

2. **Case 2**. $f_{12} = f_{21} = 0$.

Here, $f(e_1) = f_{11}e'_1$ and $f(e_2) = f_{22}e'_2$, where $f_{11} \neq 0 \neq f_{22}$. Similarly to the previous case, the identification of coefficients of a same basis vector in the equalities $f(e_ie_j) = f(e_i)f(e_j)$, for all $i, j \leq 2$, involves that

$$\begin{cases} a = \alpha f_{11}, \\ f_{22}b = \beta f_{11}^2, \\ f_{11}c = \gamma f_{22}^2, \\ d = \delta f_{22}. \end{cases} \tag{5.10}$$

Again from the regularity of the matrix F, we have that a, b, c or d is zero if and only if α, β, γ or δ is zero, respectively. From assertion (e) in Proposition 5.2.8, we consider the following case study.

- **Case 2.1**. $A = A_{(e_1,ce_1+de_2)}$ and $A' = A_{(e'_1,\gamma e'_1+\delta e'_2)}$, where $d \neq 0 \neq \delta$. From (5.10), we have that

$$\begin{cases} f_{11} = 1, \\ f_{22} = d/\delta \\ c\delta^2 = d^2\gamma. \end{cases} \tag{5.11}$$

Hence, $A_{(e_1,ce_1+de_2)} \cong A_{(e_1,\gamma e_1+\delta e_2)}$ for all $c,d,\gamma,\delta \in \mathbb{K}$ such that $d \neq 0 \neq \delta$ and $c\delta^2 = d^2\gamma$.

- **Case 2.2**. $A = A_{(e_2,ce_1+de_2)}$ and $A' = A_{(e'_2,\gamma e'_1+\delta e'_2)}$, where $c \neq 0 \neq \gamma$. From (5.10), we have that

$$\begin{cases} f_{11}^3 = c/\gamma, \\ f_{22} = f_{11}^2, \\ d = \delta f_{22}. \end{cases} \tag{5.12}$$

Hence, $A_{(e_2,ce_1+de_2)} \cong A_{(e_2,\frac{c}{m^3}e_1+\frac{d}{m^2}e_2)}$ for all $c,d,m \in \mathbb{K} \setminus \{0\}$.

- **Case 2.3**. $A = A_{(e_1+e_2,ce_1+de_2)}$ and $A' = A_{(e'_1+e'_2,\gamma e'_1+\delta e'_2)}$, where $c \neq d$. From (5.10), we have that f is the trivial isomorphism and that A' coincides with A.

The next results gather together what we have just exposed in the previous case study.

Proposition 5.3.5. *The next assertions hold in the set $\mathcal{E}_{2,2}(\mathbb{K})$.*

a) $A_{(e_1,de_2)} \cong A_{(e_1,e_2)}$, *for all* $d \in \mathbb{K} \setminus \{0\}$.

b) $A_{(e_1+e_2,de_2)} \cong A_{(e_1,de_1+e_2)}$, *for all* $d \in \mathbb{K} \setminus \{0\}$.

c) $A_{(e_2,c^2m^3e_1)} \cong A_{(e_2,ce_1)}$, *for all* $c,m \in \mathbb{K} \setminus \{0\}$.

d) $A_{(e_1+e_2,ce_1)} \cong A_{(e_2,\frac{1}{c}(e_1+e_2))}$, *for all* $c \in \mathbb{K} \setminus \{0\}$.

e) $A_{(e_1+e_2,ce_1+de_2)} \cong A_{(e_1+e_2,\frac{c}{d^2}(\frac{c}{d}e_1+e_2))}$, *for all* $c,d \in \mathbb{K} \setminus \{0\}$ *such that* $c \neq d$.

f) $A_{(e_1,ce_1+de_2)} \cong A_{(e_1,\gamma e_1+\delta e_2)}$ *for all* $c,d,\gamma,\delta \in \mathbb{K}$ *such that* $d \neq 0 \neq \delta$ *and* $c\delta^2 = d^2\gamma$.

g) $A_{(e_2,ce_1+de_2)} \cong A_{(e_2,\frac{c}{m^3}e_1+\frac{d}{m^2}e_2)}$ *for all* $c,d,m \in \mathbb{K} \setminus \{0\}$. $\square$

Theorem 5.3.6. *Any two-dimensional evolution algebra in $\mathcal{E}_{2,2}(\mathbb{K})$ with a two-dimensional derived algebra is isomorphic to exactly one of the next algebras*

- $A_{(e_1,ce_1+de_2)}$, *with* $d \in \mathbb{K} \setminus \{0\}$. *Here,* $A_{(e_1,ce_1+de_2)} \cong A_{(e_1,\gamma e_1+\delta e_2)}$ *if and only if* $c\delta^2 = d^2\gamma$.

- $A_{(e_2,ce_1+de_2)}$, *with* $c \in \mathbb{K} \setminus \{0\}$. *Here,* $A_{(e_2,ce_1+de_2)} \cong A_{(e_2,\gamma e_1+\delta e_2)}$ *if and only if there exists an element* $m \in \mathbb{K} \setminus \{0\}$ *such that* $c = \gamma m^3$ *and* $d = \delta m^2$, *or,* $c = \gamma^2 m^3$ *and* $d = \delta = 0$.

- $A_{(e_1+e_2,ce_1+de_2)}$, *with* $c, d \in \mathbb{K} \setminus \{0\}$. *Here,* $A_{(e_1+e_2,ce_1+de_2)} \cong A_{(e_1+e_2,\gamma e_1+\delta e_2)}$ *if and only if* $\gamma = c^2/d^3$ *and* $\delta = c/d^2$. □

Conclusions and further works

This manuscript has dealt with distinct aspects of the theory of isotopisms of algebras. Particularly, we have focused on isotopisms of Lie, Malcev and evolution algebras, for which this theory had not been enough studied in the literature. Let us expose here some conclusions and further works that are deduced from the manuscript.

In Chapter 1 we have exposed a brief survey about the origin and development of the theory of isotopisms. A further development of each of the branches into which this theory subdivides is required as further work. In any case, the exposed survey constitutes a first attempt in the literature to introduce this theory from a general point of view.

In Chapter 2 we have exposed those results in Computational Algebraic Geometry and Graph Theory that we have used throughout the manuscript in order to compute the isotopism classes of each type of algebra under consideration in the subsequent chapters. We have described in particular a pair of graphs that enable us to define faithful functors between finite-dimensional algebras over finite fields and these graphs. The computation of isomorphism invariants of these graphs plays a remarkable role in the distribution of distinct families of algebras into isotopism and isomorphism classes. Some preliminary results have been exposed in this regard, particularly on the distribution of partial-quasigroup rings over finite fields. Based on the known classification of partial Latin squares into isotopism classes, further work is required to determine completely this distribution.

In Chapter 3 we have focused on the distribution into isomorphism and isotopism classes of two families of Lie algebras: the set $\mathcal{P}_{n,q}$ of n-dimensional pre-filiform Lie algebras over the finite field $\mathbb{F}_q$ and the set $\mathcal{F}_n(\mathbb{K})$ of n-dimensional filiform Lie algebras over a base field $\mathbb{K}$. Particularly, we have proved the existence of n isotopism classes in $\mathcal{P}_{n,q}$. We have also introduced two new series of isotopism invariants that have been used to determine the isotopism classes of the set $\mathcal{F}_n(\mathbb{K})$ for $n \leq 7$ over algebraically closed fields and finite fields. Higher dimensions can similarly be analyzed, although with more extensive case studies. Their computation is established as further work together with that of the corresponding distribution into isotopism classes of other families of Lie algebras, distinct from the filiform case.

In Chapter 4 we have defined distinct zero-dimensional radical ideals whose related algebraic sets are uniquely identified with the set $\mathcal{M}_n(\mathbb{K})$ of n-dimensional Malcev magma algebras over a finite field $\mathbb{K}$. The computation of their reduced Gröbner bases, together with the classification of Lie algebras over finite fields given by De Graaf [149] and Strade [289], has enabled us to determine the distribution of $\mathcal{M}_3(\mathbb{K})$ and $\mathcal{M}_4(\mathbb{K})$ not only into isomorphism classes, which is the usual criterion, but also into isotopism classes. Particularly, we have proved the existence of four isotopism classes in $\mathcal{M}_3(\mathbb{K})$ and eight isotopism classes in $\mathcal{M}_4(\mathbb{K})$. Besides, we have proved that every 3-dimensional Malcev algebra over any finite field and every 4-dimensional Malcev algebra over a finite field of characteristic distinct from two is isotopic to a Lie magma algebra. Keeping in mind the obtained results, the study of magma algebras by means of computational algebraic geometry constitutes a good first approach to the distribution of Malcev algebras over finite fields into isomorphism and isotopism classes. In this regard, the study of the sets $\mathcal{M}_5(\mathbb{K})$ and $\mathcal{M}_6(\mathbb{K})$ is established as a further work that complements the already known classification of 5- and 6-dimensional Malcev algebras in case of non-solvability [289] and nilpotency [87].

Finally, Chapter 5 has dealt with the set $\mathcal{E}_n(\mathbb{K})$ of n-dimensional evolution algebras over a field $\mathbb{K}$, whose distribution into isotopism classes is uniquely related with mutations in non-Mendelian Genetics. Particularly, we have focused on the two-dimensional case, which is related to the asexual reproduction processes of diploid organisms. We have proved that the set $\mathcal{E}_2(\mathbb{K})$ is distributed into four isotopism classes, whatever the base field is, and we have characterized its isomorphism classes. Similar case studies to those ones that have been exposed in this chapter are established as further work in order to deal with evolution algebras of dimension $n > 2$.

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Glossary

$A^{(i)}$	The quotient algebra $A^{(i-1)}/Z(A^{(i-1)})$, where $A^{(0)} = A$.
A_T	The Lie algebra in $\mathcal{P}_{n,q}$ of structure tuple $T \in \mathcal{T}_{n,q}$.
	The evolution algebra in $\mathcal{E}_n(\mathbb{K})$ of structure tuple $T \in \mathcal{T}_n(\mathbb{K})$.
$A^{\cdot}$	A partial-magma algebra based on a magma $([n], \cdot)$.
$d(v)$	The degree of a vertex v in a graph.
$\mathcal{E}_n(\mathbb{K})$	n-dimensional evolution algebras over a field $\mathbb{K}$.
$\mathcal{E}_{n;m}(\mathbb{K})$	Set of evolution algebras in $\mathcal{E}_n(\mathbb{K})$ with an $(n-m)$-dimensional annihilator.
$\mathcal{F}_n(\mathbb{K})$	Set of n-dimensional filiform Lie algebras over a field $\mathbb{K}$.
$\mathcal{L}_{n,q}$	Set of n-dimensional Lie partial-magma algebras over the finite field $\mathbb{F}_q$.
$\mathcal{M}_{n,q}$	Set of n-dimensional Malcev partial-magma algebras over the finite field $\mathbb{F}_q$.
$\mathcal{P}_{n,q}$	Set of n-dimensional pre-filiform Lie algebras over the finite field $\mathbb{F}_q$.
$\mathcal{P}_{n,q;m}$	Set of Lie algebras in $\mathcal{P}_{n,q}$ with an $(n-m-1)$-dimensional center.
S_n	Symmetric group of order n.
$\mathcal{T}_n(\mathbb{K})$	Set of structure tuples in $\mathcal{E}_n(\mathbb{K})$.
$\mathcal{T}_{n,q}$	Set of structure tuples in $\mathcal{P}_{n,q}$.
$\mathcal{T}_{n,q;m}$	Set of structure tuples in $\mathcal{P}_{n,q;m}$.
$\mathcal{V}(I)$	Algebraic set of an ideal of polynomials I.