DUAL GARSIDE STRUCTURE AND REDUCIBILITY OF BRAIDS

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Abstract. Benardete, Gutierrez and Nitecki showed an important result which relates the geometrical properties of a braid, as a homeomorphism of the punctured disk, to its algebraic Garside-theoretical properties. Namely, they showed that if a braid sends a *standard* curve to another *standard* curve, then the image of this curve after each factor of the left normal form of the braid (with the classical Garside structure) is also standard. We provide a new simple, geometric proof of the result by Benardete-Gutierrez-Nitecki, which can be easily adapted to the case of the dual Garside structure of braid groups, with the appropriate definition of standard curves in the dual setting. This yields a new algorithm for determining the Nielsen-Thurston type of braids.

1. Introduction

Braid groups are both mapping class groups and Garside groups. The links between these two features of braid groups seem to be very deep and their investigation is currently the goal of several works. In particular the Garside-theoretic approach to the problem of deciding algorithmically the Nielsen-Thurston type of a given braid turns out to be very fruitful [1],[19],[18],[17],[5].

The n-strand braid group is naturally identified with the mapping class group of the n-times punctured disk D_n. Braids induce a (right) action on the set of isotopy classes of simple closed curves in D_n: considering the isotopy class $[\mathcal{C}]$ of a simple closed curve $\mathcal{C}$ and an n-braid x, the isotopy class of simple closed curves resulting from the action of x on $[\mathcal{C}]$ will be denoted by $[\mathcal{C}]^x$. The simple closed curves we shall consider in the present paper will be *nondegenerate*, that is surrounding more than one and less than n punctures.

Let us assume for the moment that D_n is parametrized as the disk with diameter $[0, n+1]$ in $\mathbb{C}$ and points $1, 2, \ldots, n$ removed. In this setting, a curve $\mathcal{C}$ is said to be *standard*, or *round*, if it is isotopic to a geometric circle in D_n. That is, if the punctures enclosed by $\mathcal{C}$ are consecutive. An isotopy class $[\mathcal{C}]$ is said to be standard, or round, if some (hence every) representative is round. Round curves are particularly useful for decomposing a reducible braid into its corresponding components [17]. As every reducible braid has a conjugate which preserves a family of round curves [1], searching for such a conjugate becomes a possible strategy both for determining whether a braid is reducible and for finding its geometric components. Benardete, Gutierrez and Nitecki [1] explain how to determine whether such a conjugate exists, and also how to find it, thanks to the classical Garside structure of braid groups.

Braid groups are the main examples of Garside groups [6]. This means that they admit a lattice structure, and a special element denoted Δ, satisfying some properties first discovered by Garside in [12]. We will refer to this as the *classical Garside structure* of the braid group. Using this structure, one can define the *left normal form* of a braid x, which is a unique decomposition of the form $x = \Delta^p x_1 \cdots x_r$ (see [8],[7]) in which the factors belong to the set of the so-called *simple elements*.

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The result by Benardete, Gutierrez and Nitecki, which relates round curves and left normal forms, is the following:

Theorem 1 ([1],[19])**.** *Let $\mathcal{C}$ be a standard curve in D_n. Let $x = \Delta^p x_1 \cdots x_r$ be a braid in (classical) left normal form. If $[\mathcal{C}]^x$ is standard, then $[\mathcal{C}]^{\Delta^p x_1 \cdots x_m}$ is standard for $m = 1, \ldots, r$.*

Thanks to this result, and applying special conjugations called *cyclings* and *decyclings*, it is possible to determine whether a braid has a conjugate which preserves a family of round curves, hence it is possible to know whether a braid is reducible, with the aid of the Garside structure of the braid group [1].

The method mentioned in the previous paragraph needs to compute a big subset of the conjugacy class of a braid. This has been improved in [18], avoiding the computation of such a big subset, at the cost of enlarging the set of standard curves to include *round* and also *almost-round* curves. This raises the question whether the notion of round curves, and the use of the classical Garside structure, are the best choices for this kind of techniques.

There is another well-known Garside structure of the braid group, discovered by Birman, Ko and Lee [3], which is known as the *dual Garside structure*. With respect to this structure, the left normal form of a braid is a unique decomposition $x = \delta^p x_1 \cdots x_r$, where δ is the special (Garside) element, and the factors are simple, with respect to the dual structure.

In this paper we prove the analogue of Theorem 1 in the dual setting. We remark that the proofs of Theorem 1 given in [1] and [19] cannot be adapted in a natural way to the dual Garside structure of the braid group. For that reason we give a new proof of Theorem 1, in the classical setting, which can be naturally adapted to the dual setting. To this purpose we also introduce a natural notion of standard curve related to the dual Garside structure of the braid group. Namely, as round curves determine standard parabolic subgroups of the braid group, with the Artin strucure, the standard curves in the dual setting will be those determining standard parabolic subgroups, with the dual structure.

It is important to mention that a generalization of Theorem 1 to Artin-Tits groups of spherical type, with the classical Garside structure, is given in [15]. The powerful algebraic methods used in [15] are based in the theory of Garside categories and seem to allow further generalization of Theorem 1 to dual Garside structures [16], although this does not appear in the literature. Here we present simple, geometric proofs, for the particular but important cases of the two well known structures of the braid groups.

Our proof of Theorem 1 can be sketched as follows. We show that if $\mathcal{C}$ is round and $[\mathcal{C}]^{\Delta^p x_1}$ is not round, then $[\mathcal{C}]^{\Delta^p x_1 \cdots x_m}$ is not round for $m = 1, \ldots, r$. Hence $[\mathcal{C}]^{\Delta^p x_1 \cdots x_r}$ cannot be round, contradicting the hypothesis.

We assume the usual Artin generator σ_i, for $i = 1, \ldots, n-1$, to be the counterclockwise half Dehn twist along the segment $[i, i+1]$. We will see that if $[\mathcal{C}]^{\Delta^p x_1}$ is not round, then a portion of (a suitable representative of) $[\mathcal{C}]^{\Delta^p x_1}$ crosses the real line in the way shown in Figure 1, for some $i < j < k$. Moreover, $x_1 \sigma_j$ is simple, meaning that the strands of x_1 ending at j and $j+1$ do not cross.

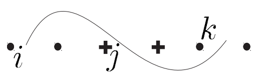

FIGURE 1. Strands which arrive at positions j and $j+1$ (depicted as crosses) have not crossed in x_1.

We then prove by recursion that, as $x_1 \cdots x_r$ is in left normal form, the above properties must hold for $[\mathcal{C}]^{\Delta^p x_1 \cdots x_m}$, for $m = 1, \ldots, r$. That is, some portion of (a suitable representative of) $[\mathcal{C}]^{\Delta^p x_1 \cdots x_m}$ must cross the real line as in Figure 1, for some $i_m < j_m < k_m$, where $x_m \sigma_{j_m}$ is simple. This implies in particular that $[\mathcal{C}]^{\Delta^p x_1 \cdots x_r}$ is not round, showing Theorem 1.

This proof can be adapted to the dual setting as follows. First, in order to work with the dual Garside structure of the braid group, it is more convenient to parametrize D_n as the unit disk in $\mathbb{C}$ with set of punctures $\{\frac{1}{2}e^{\frac{-2ik\pi}{n}},\ k=1,\ldots,n\}$. Throughout the paper the puncture $\frac{1}{2}e^{\frac{-2ik\pi}{n}}$ will be denoted k for brevity, and the disk D_n with this parametrization will be denoted D_n^*. We remark that, if one defines standard curves as isotopy classes of geometric circles, as above, then the analogue of Theorem 1 is not true in the dual setting (see Example 12). Hence we need a different definition for standard curves, adapted to the dual Garside structure:

Definition 2 (See also Definition 13)**.** *A simple closed curve in D_n^* is called* standard *if it is isotopic to a curve which can be expressed, in polar coordinates, as a function $\rho=\rho(\theta)$, for $\theta\in[0,2\pi[$. See Figure 2(a). An isotopy class $[\mathcal{C}]$ is said to be standard if some (hence every) representative is standard.*

The main result of this paper is the following:

Theorem 3. *Let $\mathcal{C}$ be a standard curve in D_n^*. Let $x=\delta^p x_1\cdots x_r$ be a braid in dual left normal form. If $[\mathcal{C}]^x$ is standard, then $[\mathcal{C}]^{\delta^p x_1\cdots x_m}$ is standard for $m=1,\ldots,r$.*

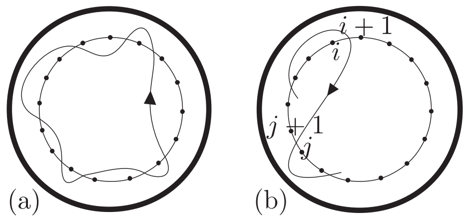

Figure 2. (a) shows a standard curve as in Definition 2. (b) A part of a non-standard curve.

The proof of this result parallels the one of Theorem 1 and its simplicity allows to outline it right here. First recall that the usual generators of the dual Garside structure are the braids $a_{i,j}=a_{j,i}$ corresponding to the counterclockwise half Dehn twist along the chord segment joining the punctures i and j, for each pair $1\leqslant i,j\leqslant n$, with $i\neq j$.

We will see that if $[\mathcal{C}]^{\delta^p x_1}$ is not standard, then a portion of (a suitable representative of) $[\mathcal{C}]^{\delta^p x_1}$ crosses the circle of radius $\frac{1}{2}$ in the way shown in Figure 2(b), for some $1\leqslant i,j\leqslant n$. Moreover, $x_1 a_{i,j}$ is simple. We then prove by recursion that, as $x_1\cdots x_r$ is in left normal form, the above properties must hold for $[\mathcal{C}]^{\delta^p x_1\cdots x_m}$, for $m=1,\ldots,r$. That is, some portion of (a suitable representative of) $[\mathcal{C}]^{\delta^p x_1\cdots x_m}$ must cross the circle of radius $\frac{1}{2}$ as in Figure 2(b), for some $1\leqslant i_m,j_m\leqslant n$, where $x_m a_{i_m,j_m}$ is simple. This implies in particular that $[\mathcal{C}]^{\delta^p x_1\cdots x_r}$ is not standard, showing Theorem 3.

The following two sections contain the detailed proofs of Theorems 1 and 3, respectively. The last section contains a new algorithm for solving the reducibility problem in braid groups, based on Theorem 3.

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2. The classical Artin-Garside case

This section deals with the classical case. Throughout the section, we thus assume D_n to be parametrized as the disk with diameter $[0,n+1]$ in $\mathbb{C}$ and with the points $1,2,\ldots,n$ removed.

In order to make rigorous the idea expressed in Figure 1, we associate to each isotopy class of simple closed curves $[\mathcal{C}]$ in D_n a unique *reduced* word $W([\mathcal{C}])$ which we shall define in Subsection 2.1. The word $W([\mathcal{C}])$ allows to describe carefully the action of positive braids on $[\mathcal{C}]$; this will be achieved in Subsection 2.2. Subsection 2.3 will be devoted to the proof of Theorem 1.

2.1. **From curves to words.** We will always assume that the curves under consideration are nondegenerate, simple and closed, so unless otherwise stated, the word "curve" alone will mean "nondegenerate simple closed curve".
Let $\mathcal{C}$ be a curve in D_n, and suppose that it has a finite number of crossings with the real line, which are all transverse. We shall associate to $\mathcal{C}$ a word $W(\mathcal{C})$ that we now define.

Our strategy to define the word $W(\mathcal{C})$ already mostly appears in [10], Appendix A. We proceed as follows. Choose a point $*$ of $\mathcal{C}$ which lies on the real line as well as an orientation for $\mathcal{C}$. Running along $\mathcal{C}$ following the chosen orientation, starting and ending at $*$, determines a word in the alphabet $X = \{\smile, \frown, 0, \dots, n\}$ as follows: each arc through the upper half plane contributes a letter $\frown$ to the word, each arc through the lower half plane, a letter $\smile$, and each intersection with the segment $]i, i+1[$ yields the letter i.

The number corresponding to $*$ can be chosen to be either at the beginning or at the end of the word. The word obtained in this way will be denoted $W(\mathcal{C})$ and we call it the *word associated to* $\mathcal{C}$. Choosing another point $*$ or putting the letter determined by the intersection point $*$ at the beginning or at the end of $W(\mathcal{C})$ corresponds to a cyclic permutation of the letters in $W(\mathcal{C})$, whereas choosing the reverse orientation of $\mathcal{C}$ yields the reverse of $W(\mathcal{C})$. Hence, words associated to curves are to be considered up to cyclic permutation of their letters and up to reverse.

Example 4. Let $\mathcal{C}$ be the curve depicted in Figure 3. If we fix $*$ in the interval $]1, 2[$ and the

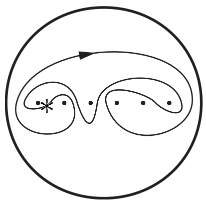

FIGURE 3. The curve $\mathcal{C}$ of Example 4.

clockwise orientation, then the word associated to $\mathcal{C}$ is

$$W(\mathcal{C}) = 1 \frown 2 \smile 0 \frown 6 \smile 3 \frown 5 \smile 6 \frown 3 \smile 2 \frown 0 \smile .$$

Notice that two curves related by an isotopy of D_n fixing the real diameter setwise have the same associated word.

We say that the word associated to $\mathcal{C}$ is *reduced* if it does not contain any subword of the form $i \smile i$ or $i \frown i$. We say that a curve $\mathcal{C}$ is *reduced* if its associated word $W(\mathcal{C})$ is reduced.

Notice that reduced curves are exactly those which do not bound any bigon (see [10]) together with the real line. According to [10], every curve $\mathcal{C}$ is isotopic to a reduced one $\mathcal{C}^{red}$, which is unique up to isotopy of D_n fixing the real diameter setwise. We thus may finally define, for each isotopy class of curves $[\mathcal{C}]$ in D_n, its associated reduced word as $W([\mathcal{C}]) = W(\mathcal{C}^{red})$.

2.2. **The action of positive braids.** Let B_n^+ be the submonoid of B_n generated by $\sigma_1, \dots, \sigma_{n-1}$, called the monoid of positive braids. Instead of the usual Artin generators, it will be convenient to work with the following bigger generating set of B_n^+:

$$\{\Sigma_{p,k} = \sigma_{p-1} \dots \sigma_{p-k}, \quad 1 \leqslant k < p \leqslant n\}.$$

According to our previous convention that σ_i is a counterclockwise half-Dehn twist along the segment $[i, i+1]$, the braid $\Sigma_{p,k}$ corresponds to a move of the puncture numbered p through the upper half plane up to the position $p-k$ while the punctures $p-1, p-2, \ldots, p-k$ are shifted one position to the right.

Given the isotopy class of a curve $\mathcal{C}$ and a positive braid x we are going to describe which transformations have to be performed on the word $W = W([\mathcal{C}])$ in order to obtain the word $W([\mathcal{C}]^x)$. To this purpose we first focus on the case $x = \Sigma_{p,k}$.

Since the generators we are considering are mainly moves of punctures through the upper half plane, we assume that the action of the braids $\Sigma_{p,k}$ mainly modifies the upper arcs occuring in $\mathcal{C}^{red}$ (from which new lower arcs can arise) while the lower arcs are only modified by translating their endpoints. This will be described by the following formulae, which are also depicted in figure 4.

We define, for $i < j$:

$$(i \frown j)^{\Sigma_{p,k}} = \begin{cases} i \frown j & \text{if } [p-k, p[\, \cap \{i,j\} = \emptyset & (F1) \\ (i+1) \frown (j+1) & \text{if } [p-k, p[\, \cap \{i+1,j\} = \{i+1,j\} & (F2) \\ (i+1) \frown (p-k) \smile (p-k-1) \frown j & \text{if } [p-k, p[\, \cap \{i,j\} = \{i\} & (F3) \\ i \frown (p-k-1) \smile (p-k) \frown (j+1) & \text{if } [p-k, p[\, \cap \{i+1,j\} = \{j\} & (F4) \end{cases}$$

We can define in the same way $(j \frown i)^{\Sigma_{p,k}}$. It suffices to take the reverse of the above formulae (the picture is exactly the same than in Figure 4).

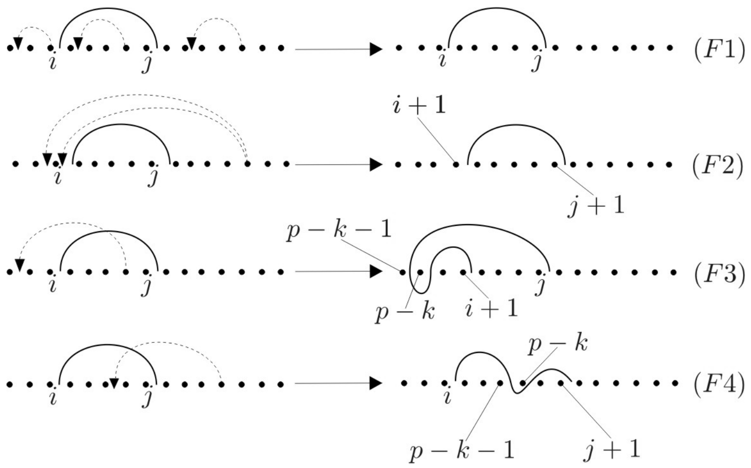

FIGURE 4. How the action of the braid $\Sigma_{p,k}$ does transform upper arcs? In dashed lines is represented the trace of the move of the puncture initially numbered p (up to the position $p-k$). In continuous lines the arc $i \frown j$ on the left hand side and its image $(i \frown j)^{\Sigma_{p,k}}$ on the right hand side.

Now, let $\widetilde{W}$ be the word obtained by replacing each subword $i \frown j$ in W by the corresponding subword $(i \frown j)^{\Sigma_{p,k}}$. This transforms the lower arcs $(i \smile j)$ in W, just by translating their endpoints.

Notice that $\widetilde{W}$ is not necessarily reduced, so that $W([\mathcal{C}]^{\Sigma_{p,k}})$ and $\widetilde{W}$ are possibly not the same. The following explains how to turn $\widetilde{W}$ into the reduced word $W([\mathcal{C}]^{\Sigma_{p,k}})$:

Lemma 5. *Let $[\mathcal{C}]$ be an isotopy class of curves and $W = W([\mathcal{C}])$. Let $1 \leqslant k < p \leqslant n$. Let $\widetilde{W}$ be as above, and let $W^{\Sigma_{p,k}}$ be the word obtained from $\widetilde{W}$ by removing all instances of subwords of the form $p \smile p \frown$. Then $W^{\Sigma_{p,k}} = W([\mathcal{C}]^{\Sigma_{p,k}})$.*

Proof. We observe that the formulae defining $(i \frown j)^{\Sigma_{p,k}}$ do not contain any subword of the form $c \smile c$ nor $c \frown c$. Thus the possible instance of such a subword in $\widetilde{W}$ necessarily arises from the

transformation of a lower arc. Notice that a lower arc $c \smile d$ can only be transformed into an arc $k \smile l$, with $k \in \{c, c+1\}$ and $l \in \{d, d+1\}$. Hence the latter arc forms a bigon with the horizontal axis if and only if $c+1 = d$, $k = c+1$ and $l = d$ (up to reverse we may suppose that $c < d$), in which case $\widetilde{W}$ contains the subword $d \smile d$. By the formulae defining $(i \frown j)^{\Sigma_{p,k}}$, this happens if and only if $d = p$, that is $c \smile d = (p-1) \smile p$. In particular we have shown that no subword of the form $c \frown c$ can arise in $\widetilde{W}$.
We now claim that removing the subwords $p \smile p \frown$ is sufficient in order to turn $\widetilde{W}$ into a reduced word; that is, every sequence $p \smile p \frown$ in $\widetilde{W}$ is a subsequence of a larger one, of the form $a \frown p \smile p \frown b$, with $a \neq b$.
Let $h \frown (p-1) \smile p \frown l$ be a subword of W. As $\mathcal{C}$ is a simple curve, h, l must be in one of the following three cases (see Figure 5):

1) $h < p-1 < p < l$,
2) $l \leqslant h < p-1 < p$,
3) $p-1 < p < l \leqslant h$.

We shall show that in the three cases, the subword $a \frown p \smile p \frown b$ of $\widetilde{W}$ yielded by the action of $\Sigma_{p,k}$ satisfies

$$a \in \{p-k, \ldots, p-1\}$$

and

$$b \in \{0, \ldots, p-k-1\} \cup \{p+1, \ldots, n\},$$

and thus $a \neq b$, as claimed.

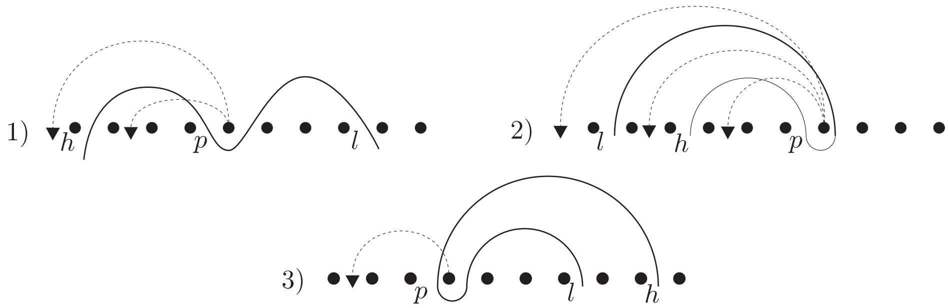

FIGURE 5. The possible cases of subwords $h \frown (p-1) \smile p \frown l$ appearing in W, with the distinct possible moves of the point p.

In the first two cases the arc $(h \frown p)$ will yield $a = h+1$ or $a = p-k$, satisfying $a \in \{p-k, \ldots, p-1\}$. In the third case we will also obtain $a = p-k$. On the other hand, in the first and third cases we have $b = l$ (hence $b > p$); in the second case, either $b = l$ if $p-k \geqslant l+1$ or $b = p-k-1$ otherwise (and $b < p-k$).

This achieves the proof of the lemma. □

We thus can associate to each isotopy class of curves $[\mathcal{C}]$ and each braid $\Sigma_{p,k}$ the word $W([\mathcal{C}])^{\Sigma_{p,k}}$ defined thanks to the above construction. We are now able to define, for each isotopy class of curves $[\mathcal{C}]$, the image of its associated reduced word $W = W([\mathcal{C}])$ under the action of some positive braid x. Indeed, if x is expressed as a product $x = \prod_{m=1}^{r} \Sigma_{p_m,k_m}$, then by Lemma 5, the inductive formula

$$W^x = (W^{\prod_{m=1}^{r-1} \Sigma_{p_m,k_m}})^{\Sigma_{p_r,k_r}}$$

defines a word on X which is the reduced word associated to $[\mathcal{C}]^x$ (hence it does not depend on the chosen decomposition of x in terms of braids $\Sigma_{p,k}$). This can be written $W([\mathcal{C}]^x) = W([\mathcal{C}])^x$.

2.3. **Proof of Theorem 1.** By abuse of notation, instead of speaking about isotopy classes of simple closed curves, we will simply speak about curves, meaning that we are considering the reduced representatives. Consequently, the letter $\mathcal{C}$ will denote the reduced representative of the isotopy class of the curve $\mathcal{C}$ and its associated reduced word will be denoted $W(\mathcal{C})$.

We introduce a class of curves which is larger than the class of round curves:

Definition 6. *A curve $\mathcal{C}$ will be called* almost-round *if the word $W(\mathcal{C})$ (up to cyclic permutation of its letters and reversing) can be written as $W(\mathcal{C}) = w_1w_2$, where the arcs in w_1 are oriented from left to right whereas those in w_2 are oriented from right to left. (See Figure 6).*

FIGURE 6. The curve in Part (a) is almost-round since its associated reduced word can be written as

$$W = w_1w_2 = (1 \frown 4 \smile 6 \frown 9 \smile 10 \frown 11)(\smile 8 \frown 6 \smile 4 \frown 3 \smile).$$

The curve in Part (b) is not almost-round.

The reduced words associated to almost-round curves satisfy the following necessary condition:

Lemma 7. *Let $\mathcal{C}$ be an almost-round, not round curve. Then $W(\mathcal{C})$ (up to cyclic permutation and reversing) must contain a subword of the form $i \frown j \smile l$ for some $0 \leqslant i < j < l \leqslant n$.*

Proof. The curve $\mathcal{C}$ has a unique local minimum (and a unique local maximum) in the horizontal direction. Let $*$ be this local minimum. Choose the clockwise orientation for $\mathcal{C}$, and notice that the arcs oriented from left to right starting at $*$ form a subword $i_1 \frown i_2 \ldots \frown i_k$ (ending with an upper arc, as $\mathcal{C}$ is simple). If $k > 2$, then $i_1 \frown i_2 \smile i_3$ satisfies the required hypothesis. If $k = 2$, choose the counterclockwise orientation. As $\mathcal{C}$ is not round, $W(\mathcal{C})$ must start with a subword $i'_1 \smile i'_2 \ldots \smile i'_k$ with $k \geqslant 4$ and $i'_2 \frown i'_3 \smile i'_4$ does the job. $\square$

Before proving Theorem 1, we introduce some more notation. We will say that some $0 < j < n$ is a *bending point for (or bends)* a curve $\mathcal{C}$ if the reduced word $W(\mathcal{C})$ admits (up to cyclic permutation of its letters and up to reverse) a subword of the form $i \frown j \smile l$, for some $0 \leqslant i < j < l \leqslant n$. Lemma 7 thus asserts that an almost-round, not round curve admits at least one bending point. Given a simple braid s, and a bending point j for some curve $\mathcal{C}$, j will be said to be *compatible* with s if the strands j and $j+1$ of s do not cross in s.

The following is the key for Theorem 1:

Lemma 8. *Let s_1, s_2 be two simple braids such that $s_1 \cdot s_2$ is in left normal form, and let $\mathcal{C}$ be a curve. Let j be a bending point for $\mathcal{C}$, compatible with s_1. Then there exist some bending point j' for $\mathcal{C}^{s_1}$, which is compatible with s_2.*

In order to prove this lemma, we recall that simple braids are those positive braids in which any pair of strands crosses at most once (see [7]). We also recall the following well-known fact (see [11]): any simple braid s can be decomposed in a unique manner as a product

$$s = \prod_{p=2}^{n} \Sigma_{p,k_p},$$

where $0 \leqslant k_p < p$ and $\Sigma_{p,0} = 1$. This allows us to see s as a sequence of moves of the strands numbered from 2 to n (in this order), each of them getting k_p positions to the left (the number k_p depending on s). Notice that the pth strand of s does not cross (in s) any of the strands being to the left of the position $p - k_p$.

Proof of Lemma 8. As $s_1 \cdot s_2$ is in left normal form, it is sufficient to show that there exist some bending point j' for $\mathcal{C}^{s_1}$ such that the strands numbered j' and $j'+1$ at the end of s_1 do not cross in s_1 ([7]).

Let

$$s_1 = (\prod_{p=2}^{j-1} \Sigma_{p,k_p}) \cdot \Sigma_{j,k_j} \cdot \Sigma_{j+1,k_{j+1}} \cdot (\prod_{p=j+2}^{n} \Sigma_{p,k_p})$$

be the decomposition of s_1 in terms of the braids $\Sigma_{p,k}$. Acting on $\mathcal{C}$ by s_1 following the above factorization, we will be able to find at each step a bending point ι for the resulting curve such that the strands at respective positions ι and $\iota+1$ at the end of this step have not yet crossed in s_1.

The first factor $\prod_{p=2}^{j-1} \Sigma_{p,k_p}$ involves moves of punctures which are to the left of j. Hence, due to Formulae $(F1)$ and $(F3)$ and Lemma 5, j is still a bending point for the resulting curve. Moreover strands ending at j and $j+1$ have not crossed in this first factor.

The factor Σ_{j,k_j} moves the jth puncture to the left; once again due to Formulae $(F1)$ and $(F3)$ and Lemma 5, the resulting reduced word admits a subword $i' \frown j \smile$ (with $i' < j$); j is still a bending point for the corresponding curve. Notice that the former jth puncture lies (now at position $j - k_j$) under the arc $i' \frown j$ and that strands ending at positions j and $j+1$ have not crossed.

Then, the factor $\Sigma_{j+1,k_{j+1}}$ moves the $j+1$st puncture to the left. As j was a bending point for $\mathcal{C}$, compatible with s_1, the strands which started at positions j and $j+1$ do not cross in s_1, hence this movement of the $j+1$st puncture cannot exceed the position $j - k_j$: it ends at the diameter of the upper arc $i' \frown j$ and by Formula $(F4)$ and Lemma 5, the position $\iota_{j+1} := j - k_{j+1}$ is now a bending point for the resulting curve (see Figure 7). Moreover by construction the strands ending at ι_{j+1} and $\iota_{j+1}+1$ have not crossed in s_1.

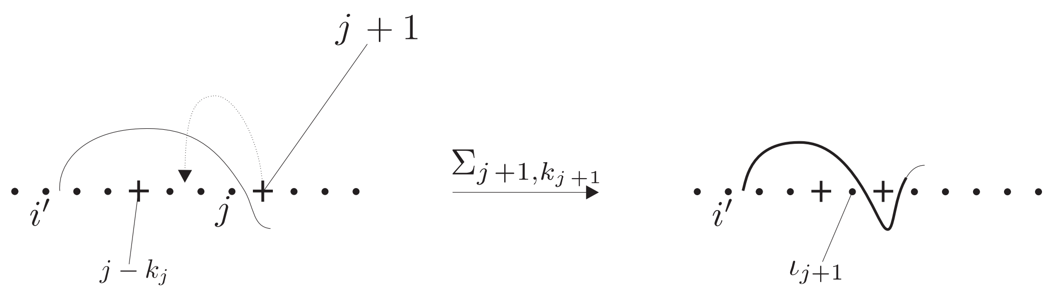

FIGURE 7. The action of the braid $\Sigma_{j+1,k_{j+1}}$; as the strands represented as bold crosses cannot cross, the point $j+1$ is "blocked" in its move to the left by the position $j - k_j$.

Finally, observe that for $j+2 \leqslant q \leqslant n$, the movement of the qth puncture yields some bending point ι_q for the resulting curve: this is $\iota_q = \iota_{q-1}+1$ if this movement ends to the left of ι_{q-1} and $\iota_q = \iota_{q-1}$ otherwise. Thus $j' := \iota_n$ is a bending point of $\mathcal{C}^{s_1}$ and strands numbered j' and $j'+1$ at the end of s_1 do not cross in s_1 by construction. This shows that the bending point j' of $\mathcal{C}^{s_1}$ is compatible with s_2. $\square$

We can now achieve the proof of Theorem 1. Recall that we consider a braid $x = \Delta^p x_1 \dots x_r$ in left normal form which sends a round curve $\mathcal{C}$ to another round curve $\mathcal{C}' = \mathcal{C}^x$. Since the braid Δ corresponds to a rotation of D_n, the curve $\mathcal{C}^{\Delta^t}$ is round for all integers t and we may suppose, up to multiplication by Δ^{-p}, that the left normal form of x is just $x_1 \dots x_r$. In order to prove Theorem 1, it is sufficient to prove that the curve $\mathcal{C}^{x_1}$ is round, since once this fact is proven, we are given a braid $x_2 \dots x_r$ in left normal form whose action transforms the round curve $\mathcal{C}^{x_1}$ into the round curve $\mathcal{C}'$, and the result follows by induction on the number of factors in the left normal form of x. We shall give a proof by contradiction, assuming that the curve $\mathcal{C}^{x_1}$ is not round. However it is almost round, as x_1 is a simple braid, see [18].

According to Lemma 7, there must exist a bending point j_1 for $\mathcal{C}^{x_1}$. Notice that punctures which lie above the curve $\mathcal{C}^{x_1}$ come from strands which started to the right of $\mathcal{C}$ at the beginning of x_1 as well as punctures which lie below $\mathcal{C}^{x_1}$ come from strands which started to the left of $\mathcal{C}$. By Lemma 7, the puncture j_1 lies either below the curve $\mathcal{C}^{x_1}$ or is enclosed by it whereas the puncture j_1+1 lies either above, or is enclosed by $\mathcal{C}^{x_1}$. Moreover, at most one of them is enclosed by $\mathcal{C}^{x_1}$. This implies that the strands in positions j_1 and j_1+1 at the end of x_1 do not cross in x_1. In other words, j_1 is a bending point for $\mathcal{C}^{x_1}$ compatible with x_2 (as $x_1 \cdot x_2$ is in left normal form).

Now, by induction on $m = 1, \ldots, r-1$ and by Lemma 8, there exist $j_2, \ldots, j_{r-1}$ such that j_m bends the curve $\mathcal{C}^{x_1 \cdots x_m}$ and is compatible with x_{m+1}, for all $m = 1, \ldots, r-1$. The proof of Lemma 8 also ensures that $\mathcal{C}^{x_1 \cdots x_r}$ has a bending point j_r.

In particular we have shown that the curve $\mathcal{C}^{x_1 \cdots x_r} = \mathcal{C}^x$ is not round, provided that $\mathcal{C}^{x_1}$ is not round. This is a contradiction; therefore $\mathcal{C}^{x_1}$ is round and Theorem 1 is shown.

3. The dual case, proof of Theorem 3

In this section we shall be interested in the dual (or BKL) Garside structure of the braid group [3]. We thus adopt the parametrization D_n^* of the n-times punctured disk defined in the Introduction. We shall associate to each isotopy class of simple closed curves $[\mathcal{C}]$ in D_n^*, a unique reduced word $W([\mathcal{C}])$ to be defined in Subsection 3.1. Reduced words will be a tool for describing the action of dual positive braids on simple closed curves in D_n^* in Subsection 3.2. Finally, Theorem 3 will be proved in Subsection 3.3.

3.1. **From curves in D_n^* to reduced words.**

As in the previous section, all the curves we are considering are nondegenerate, simple and closed. We shall speak about "curves" when we really mean "nondegenerate simple closed curves".

Notation : the circle of radius $\frac{1}{2}$ in D_n^* centered at the origin will be denoted by Γ. Given $i, j \in \{1, \ldots, n\}$, the move of the puncture i clockwise along Γ up to the position j describes an arc of Γ which we will denote by (i, j). The arc (i, i) is just the puncture i.

Let $\mathcal{C}$ be a curve in D_n^*, and suppose that it has a finite number of crossings with the circle Γ, which are all transverse. We shall associate to $\mathcal{C}$ a word $W(\mathcal{C})$ that we now define.

Choose a point $*$ of $\mathcal{C}$ which lies on the circle Γ and choose an orientation for $\mathcal{C}$. Running along $\mathcal{C}$ following the chosen orientation, starting and ending at $*$, determines a word in the alphabet $Y = \{\smile, \curvearrowleft, \curvearrowright, \circlearrowleft, \circlearrowright, 1, \ldots, n\}$ as follows.
Each arc through the inner component of $D_n^* - \Gamma$ contributes a letter $\smile$ to the word. Each intersection of $\mathcal{C}$ with the arc $(i, i+1)$ of Γ contributes a letter i. Finally arcs of $\mathcal{C}$ through the outer component of $D_n^* - \Gamma$ contribute a letter $\curvearrowright$, ($\curvearrowleft$ respectively) if they are oriented clockwise (counterclockwise, respectively); except those as in Figure 8 (a) (thus having their endpoints in the same arc $(i, i+1)$ of Γ) which contribute a letter $\circlearrowright$ or $\circlearrowleft$, in a a natural way according to their orientation.

The number corresponding to the intersection point $*$ can be chosen to be either at the beginning or at the end of the word. The word obtained in this way will be denoted by $W(\mathcal{C})$ and we call it the *word associated to $\mathcal{C}$*. Choosing another point $*$ or putting the letter determined by $*$ at the beginning or at the end of $W(\mathcal{C})$ corresponds to a cyclic permutation of the letters in $W(\mathcal{C})$, whereas choosing the reverse orientation of $\mathcal{C}$ yields the reverse of $W(\mathcal{C})$, exchanging with each other the letters $\curvearrowright$ and $\curvearrowleft$ ($\circlearrowright$ and $\circlearrowleft$, respectively). Hence, words associated to curves are to be considered up to cyclic permutation of their letters and up to reverse, exchanging the orientation of outer arcs.

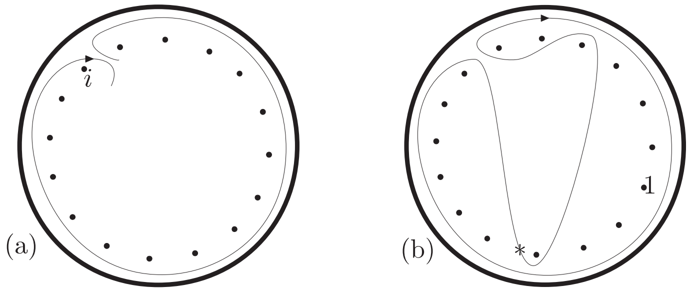

FIGURE 8. (a) An arc $i \circlearrowright i$. (b) The curve of Example 9

Example 9. Let $\mathcal{C}$ be the curve depicted in Figure 8 (b); here we have $n = 16$. The point $*$ and the orientation are also indicated in the figure. This curve yields the word

$$W(\mathcal{C}) = 4 \curvearrowleft 3 \smile 13 \curvearrowleft 12 \smile 10 \circlearrowright 10 \smile .$$

Notice that two curves related by an isotopy of D_n^* fixing the circle Γ setwise have the same associated word.

We say that the word associated to $\mathcal{C}$ is *reduced* if it does not contain any subword of the form $i \smile i$, $i \curvearrowright i$ or $i \curvearrowleft i$. We say that a curve $\mathcal{C}$ is *reduced* if its associated word $W(\mathcal{C})$ is reduced. Notice that reduced curves are exactly those which do not bound any bigon (see [10]) together with the circle Γ. According to [10], every curve $\mathcal{C}$ is isotopic to a reduced one $\mathcal{C}^{red}$, which is unique up to isotopy of D_n^* fixing the circle Γ setwise. We finally define, for each isotopy class of curves $[\mathcal{C}]$ in D_n^*, its associated reduced word as $W([\mathcal{C}]) = W(\mathcal{C}^{red})$.

3.2. **The action of dual positive braids.** Let B_n^{+*} be the submonoid of B_n generated by the braids $a_{i,j}$, $1 \leqslant i, j \leqslant n$, $i \neq j$, called the monoid of dual positive braids. We shall use a bigger generating set, namely

$$\mathcal{P} = \left\{ a_{i_1,i_2}a_{i_2,i_3}\dots a_{i_{r-2},i_{r-1}}a_{i_{r-1},i_r}, \;\middle|\; \begin{array}{c} 2 \leqslant r \leqslant n, \\ i_1, \dots, i_r \text{ all distinct and placed in this order,} \\ \text{following the circle } \Gamma \text{ clockwise from } i_1 \text{ to } i_r \end{array} \right\}.$$

The elements of $\mathcal{P}$ are naturally called *polygons* according to their geometric representation in D_n^*; and they correspond to a counterclockwise rotation of a neighborhood of the convex polygon in D_n^* whose vertices are $i_1, i_2, \dots, i_r$, following the cyclic permutation $[i_r, i_{r-1}, \dots, i_2, i_1]$.

The following are well-known (see [3]):

- Let $P_1, P_2 \in \mathcal{P}$, seen as polygons in B_n^*. If their respective convex hulls are disjoint, then $P_1P_2 = P_2P_1$. In this situation, we will say that P_1 and P_2 are *disjoint*.
- Let $i_1, \dots, i_r$ be r $(2 \leqslant r \leqslant n)$ punctures placed in this order following Γ clockwise from i_1 up to i_r. Then all the braid words obtained as the concatenation of $r-1$ consecutive letters taken from the sequence $(a_{i_1,i_2}, a_{i_2,i_3}, \dots, a_{i_{r-2},i_{r-1}}, a_{i_{r-1},i_r}, a_{i_r,i_1})$ in this order, up to cyclic permutation, are representatives of the same braid P, which is an element of $\mathcal{P}$. Moreover, for each pair $1 \leqslant d < e \leqslant r$, the letter a_{i_d,i_e} is a prefix of P.

Let us now consider the isotopy class of a curve $\mathcal{C}$ and its associated reduced word $W = W([\mathcal{C}])$. Given a dual positive braid x, we are going to describe which transformations have to be performed on the word W in order to obtain the word $W([\mathcal{C}]^x)$. To this purpose we first focus on the case $x = P \in \mathcal{P}$.

We assume that the action of P will mainly modify the inner arcs whereas the outer arcs are only modified by shifting their endpoints along the circle Γ.

We first observe that any inner arc $i \smile j$ separates the punctures into two disjoint subsets: one containing the punctures $i+1$ and j, and the other containing the punctures $j+1$ and i. We say that P is *disjoint* from the arc $i \smile j$ if all the vertices of P lie in only one of these sets. If it is so, we set $(i \smile j)^P = i \smile j$.

Otherwise we say that P is *transverse* to the arc $i \smile j$. In that case, let $p_{i,j}$ be the rightmost vertex of P lying in $(j+1, i)$; similarly let $q_{i,j}$ be the rightmost vertex of P lying in $(i+1, j)$. By abuse of notation, we will write $i \in P$ to mean that i is a vertex of P. Observe that $p_{i,j} = i \Leftrightarrow i \in P$ and $q_{i,j} = j \Leftrightarrow j \in P$. A priori, looking at the pictures, we would define:

$$(i \smile j)^P \quad = \quad i \smile p_{i,j} \curvearrowleft (p_{i,j}-1) \smile (q_{i,j}-1) \curvearrowright q_{i,j} \smile j \qquad (F')$$

Formula (F') is depicted in Figure 9.

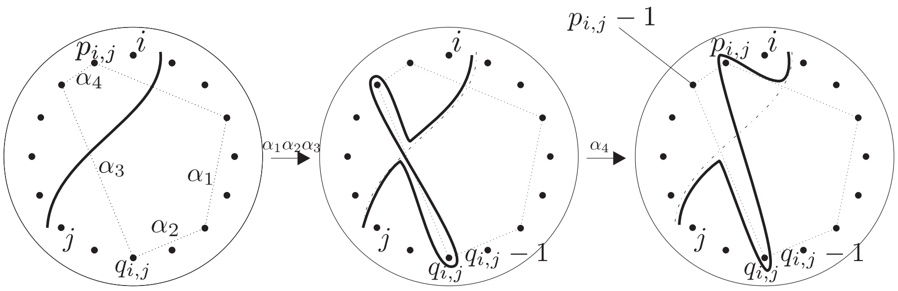

FIGURE 9. In this example P (in dotted lines) can be expressed as the product $\alpha_1 \ldots \alpha_4$; $\alpha_1\alpha_2$ acts trivially on the arc $i \smile j$, α_3 yields the arc in the middle part, and the action of α_4 on the latter yields the arc in the right-hand side, which is $(i \smile j)^P$, the image under the action of P of the arc $i \smile j$.

But notice that application of Formula (F') produces bigons with the circle Γ if either $i \in P$ or $j \in P$. That is why we set:

$$(i \smile j)^P = \begin{cases} i \smile p_{i,j} \curvearrowleft (p_{i,j}-1) \smile (q_{i,j}-1) \curvearrowright q_{i,j} \smile j & \text{if } i \notin P \text{ and } j \notin P \quad (F'0) \\ i \smile p_{i,j} \curvearrowleft (p_{i,j}-1) \smile (j-1) & \text{if } i \notin P \text{ and } j \in P \quad (F'1) \\ (i-1) \smile (q_{i,j}-1) \curvearrowright q_{i,j} \smile j & \text{if } i \in P \text{ and } j \notin P \quad (F'2) \\ (i-1) \smile (j-1) & \text{if } i \in P \text{ and } j \in P \quad (F'3) \end{cases}$$

Later, we shall need the following:

Remark 10. The image of an inner arc $i \smile j$ under the action of a polygon P lies (up to deformation) in a neighborhood of the union of $i \smile j$ with P.

Let us replace each subword $(i \smile j)$ in W by the corresponding subword $(i \smile j)^P$ as defined above. This transforms the outer arcs in W by shifting their endpoints along the circle Γ. Moreover, letters $\curvearrowright$ ($\curvearrowleft$, respectively) need to be transformed into $\circlearrowright$ ($\circlearrowleft$, respectively) if they correspond in W to an arc $(c+1) \curvearrowright c$, where $c+1$ is shifted up to the position c (or if they correspond to an arc $(c-1) \curvearrowleft c$, where c is shifted up to the position $c-1$, respectively). See Figure 10 (a). Similarly, letters $\circlearrowright$ ($\circlearrowleft$, respectively) need to be transformed into $\curvearrowright$ ($\curvearrowleft$, respectively) if the endpoints of the arc do not coincide any more after the suitable translation. Let us denote by $\widetilde{W}$ the word on Y obtained in this way. Notice that $\widetilde{W}$ is not necessarily reduced, so that $W([\mathcal{C}]^P)$ needs not to be the same as $\widetilde{W}$. The following is the analogue of Lemma 5 in the dual setting:

Lemma 11. *Let $[\mathcal{C}]$ be an isotopy class of curves, and $W = W([\mathcal{C}])$. Let $P \in \mathcal{P}$ be a polygon. Let $\widetilde{W}$ be as above, and let W^P be the word obtained from $\widetilde{W}$ by removing all instances of subwords of the form $(p-1) \curvearrowright (p-1) \smile$ (or $(p-1) \curvearrowleft (p-1) \smile$) where p is a vertex of P whereas $p-1$ is not. Then we have $W^P = W([\mathcal{C}]^P)$.*

Proof. We observe that formulae defining $(i \smile j)^P$ do not contain any subword of the form $c \smile c$ nor $c \frown c$ (by $\frown$ we mean either $\curvearrowright$ or $\curvearrowleft$). Thus the possible instance of such a subword in $\widetilde{W}$ necessarily arises from the transformation of an outer arc of W. According to the formulae above, the translations of punctures involved in such a transformation turn the extremities c and d of an outer arc into $k \in \{c, c-1\}$ and $l \in \{d, d-1\}$ respectively.

First notice that transformations of outer arcs of the form $c \circlearrowright c$ or $c \circlearrowleft c$ cannot yield subwords of the form $c \frown c$. Now, an arc $c \frown d$ of W (thus with $c \neq d$) will produce a bigon with the circle Γ only if the above k and l are the same and $|c-d| = 1$. There are two possibilities (up to reverse), shown in Figure 10.

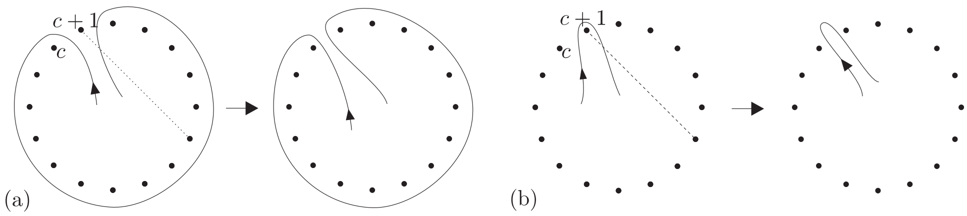

FIGURE 10. Transformation of an outer arc $c \smile d$ of W, with $|c-d| = 1$, into an outer arc of $\widetilde{W}$ having the same extremities (in dashed line is depicted an edge of P). (a) The transformation $c \curvearrowleft (c+1) \rightsquigarrow c \circlearrowleft c$ was already mentioned: no bigon is formed. (b) The only way (up to reverse) to get a bigon: $c \curvearrowright (c+1) \rightsquigarrow c \curvearrowright c$.

Finally, by the formulae defining $(i \smile j)^P$, a necessary condition for a transformation as in Figure 10 (b) to happen is that $c+1$ is a vertex of P whereas c is not. We have shown in particular that no subword of the form $c \smile c$ can arise in $\widetilde{W}$, and that the only subwords of the form $c \frown c$ which possibly arise are $(p-1) \frown (p-1)$, where p is a vertex of P whereas $p-1$ is not.

We now claim that removing all the instances of these subwords is sufficient in order to turn $\widetilde{W}$ into a reduced word; that is, every sequence $(p-1) \frown (p-1)$ in $\widetilde{W}$ is a subsequence of a larger one of the form

$$a \smile (p-1) \frown (p-1) \smile b,$$

with $a \neq b$. Let $r \smile (p-1) \curvearrowright p \smile v$ be a subword of W to be transformed into the above one under the action of P (hence P is transverse to the arc $p \smile v$). See Figure 11.

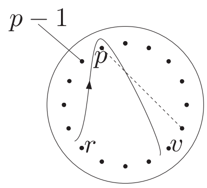

FIGURE 11. The arc $r \smile (p-1) \curvearrowright p \smile v$ and an edge of P in dashed line.

We have (since $p-1$ is not a vertex of P)

$$(r \smile (p-1))^P = \begin{cases} r \smile (p-1) & \text{if } P \text{ disjoint from } r \smile (p-1), \\ \dots (q_{r,p-1} - 1) \curvearrowright q_{r,p-1} \smile (p-1) & \text{otherwise,} \end{cases}$$

and

$$(p \smile v)^P = \begin{cases} (p-1) \smile (q_{p,v} - 1) & \text{if } v \in P, \\ (p-1) \smile (q_{p,v} - 1) \curvearrowright q_{p,v} \smile v & \text{if } v \notin P. \end{cases}$$

Thus it remains to be proved that the values $a \in \{r, q_{r,p-1}\}$ and $b = q_{p,v} - 1$ are distinct. Notice that by definition, $q_{p,v}$ lies in $(p+1, v)$ so that $b = q_{p,v} - 1$ lies in $(p, v-1)$. On the other hand, as $\mathcal{C}^{red}$ is simple and W is its associated reduced word, $r \in (v, p-2)$ (see Figure 11). Finally by definition, $q_{r,p-1}$ lies in $(r+1, p-1) \subset (v+1, p-1)$. Hence in any case, $a \in (v, p-1)$; this shows that $a \neq b$ and achieves the proof of Lemma 11. $\square$

We thus can associate to each isotopy class of curves $[\mathcal{C}]$ and each braid $P \in \mathcal{P}$ the word $W([\mathcal{C}])^P$ defined thanks to the above construction. We are now able to define, for each isotopy class of curves $[\mathcal{C}]$, the image of its associated reduced word $W = W([\mathcal{C}])$ under the action of some dual positive braid x. Indeed, if x is expressed as a product $x = \prod_{m=1}^{r} P_m$ (where each factor lies in $\mathcal{P}$), then by Lemma 11, the inductive formula

$$W^x = (W^{\prod_{m=1}^{r-1} P_m})^{P_r}$$

defines a word on Y which is the reduced word associated to $[\mathcal{C}]^x$ (hence it does not depend on the chosen decomposition of x in terms of braids in $\mathcal{P}$). This can be written $W([\mathcal{C}]^x) = W([\mathcal{C}])^x$.

3.3. **Proof of Theorem 3.** By abuse of notation, we will speak about curves instead of isotopy classes of curves, meaning that we are always considering the reduced representatives. Consequently the letter $\mathcal{C}$ will denote the reduced representative of the isotopy class of the curve $\mathcal{C}$ and its associated reduced word will be denoted by $W(\mathcal{C})$.
We shall now prove the analogue of Theorem 1 in the dual setting. As mentioned in the introduction, the statement of Theorem 1, with round curves defined as circles surrounding a set of consecutive punctures is false in this setting, as shows the following example:

Example 12. Let $n = 4$. Consider the braid $x = a_{1,2}.a_{1,4}$ which is in left normal form as written. Figure 12 shows that roundness is not preserved after each factor of the left normal form if we define it as the property of being homotopic to a geometric circle surrounding a set of consecutive punctures.

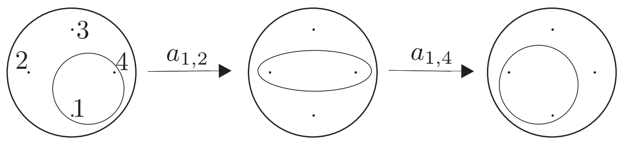

FIGURE 12. The curve in the middle part fails to be homotopic to a circle surrounding a set of consecutive punctures, although it is the image, under the first factor of the left normal form of x, of a "round" curve which is sent by x to a "round" curve.

We then need to define a suitable class of curves, which will play the role played by round curves in the classical setting. It is the following:

Definition 13. *A curve $\mathcal{C}$ will be called* standard *if $W(\mathcal{C})$ only admits letters in $\{1, \dots, n, \curvearrowleft, \smile\}$ (or in $\{1, \dots, n, \curvearrowright, \smile\}$, up to reverse).*

Notice that this is equivalent to Definition 2, in the Introduction. It turns out that Theorem 1 holds in the dual setting, if we replace round curves by *standard* curves. This is the statement of Theorem 3, which we will now prove.

First, we recall from [3] the following facts:

Claim 14. *Every dual simple braid can be written in a unique manner (up to permutation of the factors) as a product of pairwise disjoint elements of $\mathcal{P}$ (hence a commutative product).*

Therefore, by Remark 10, studying the action of dual simple braids on the arc $i \smile j$ of the curve $\mathcal{C}$ through the decomposition $s = P_1 \dots P_g$ can be done quite easily, since all inner arcs obtained by application of Formulae (F') for some P_l are invariant under the action of $P_k, k \neq l$. See Figure 13.

Lemma 15. [3]

1) *If s is a dual simple braid and $s = P_1 \dots P_t$ is its decomposition into pairwise disjoint factors in $\mathcal{P}$, then $sa_{i,j}$ is simple if and only if $P_u a_{i,j}$ is simple for every $u = 1, \dots, t$.*
2) *If $P \in \mathcal{P}$, then $Pa_{i,j}$ is simple if and only if no word representative of P can be written with a letter $a_{k,l}$ such that $k \in (j+1, i)$ and $l \in (i+1, j)$.*

Then, we introduce, as in the previous section, some further notation. Given a curve $\mathcal{C}$ in D_n^*, and an unordered pair i, j of punctures, we say that i, j is a *bending pair* for $\mathcal{C}$ if the reduced word $W(\mathcal{C})$ contains a sequence of the form $\curvearrowright i \smile j \curvearrowleft$ or $\curvearrowright i \smile j \circlearrowleft$ (up to reverse). Also, given a dual simple element s, a bending pair i, j for a curve $\mathcal{C}$ will be said to be *compatible* with s if $a_{i,j}$ is not a prefix of s.

We are now able to state and prove the key-lemma, aiming to Theorem 3; it is the analogue of Lemma 8 in the dual setting:

Lemma 16. *Let s_1, s_2 be two dual simple braids such that $s_1 \cdot s_2$ is in dual left normal form. Let $\mathcal{C}$ be a curve and assume that $\mathcal{C}$ admits some bending pair i, j, which is compatible with s_1. Then there exist a bending pair i', j' for $\mathcal{C}^{s_1}$ which is compatible with s_2.*

Proof. Consider the decomposition $s_1 = P_1 \dots P_g$ of s_1 into pairwise disjoint elements of $\mathcal{P}$. Notice that, as $s_1 \cdot s_2$ is in left normal form, $a_{i',j'}$ is not a prefix of s_2 whenever $s_1 a_{i',j'}$ is simple, thus it is sufficient to find a bending pair i', j' for $\mathcal{C}^{s_1}$ such that $s_1 a_{i',j'}$ is simple.

By hypothesis, $a_{i,j}$ is not a prefix of s. Thus, none of the polygons $P_1, \dots, P_g$ has both i and j as vertices. First, the action of any polygon among $P_1, \dots, P_g$ which is disjoint from the arc $i \smile j$ in $W(\mathcal{C})$ results in a curve for which i, j is still a bending pair (see Lemma 11). Moreover, by Lemma 15, if P is such a polygon, then $Pa_{i,j}$ is simple.

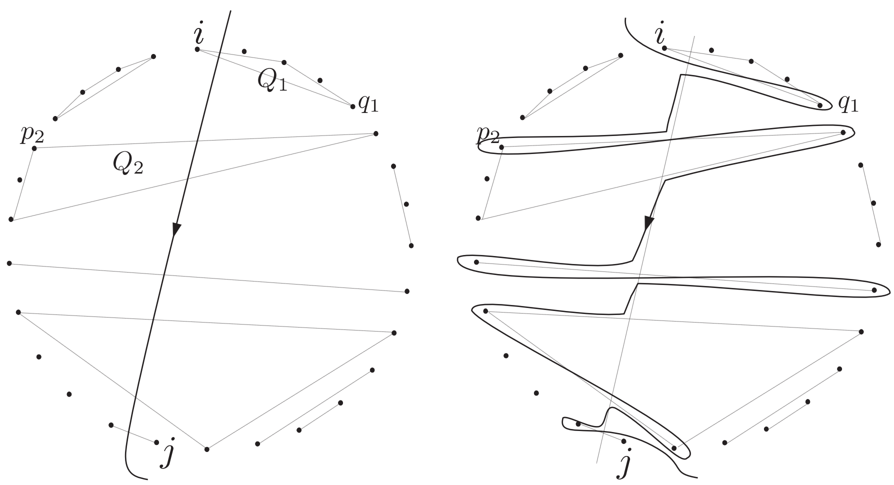

FIGURE 13. The braid s_1, as a product of pairwise disjoint polygons, is depicted in dashed lines. We can see, from left to right, the action of this braid on the arc $i \smile j$.

Then, the action of the polygons among $P_1, \dots, P_g$ which are transverse to the arc $i \smile j$ can be studied as follows (see Figure 13). The involved polygons can be ordered by running along the arc $i \smile j$ starting at $(i, i+1)$: $Q_1, \dots, Q_h$. For $t = 1, \dots, h$, if p_t is the rightmost vertex of Q_t in $(j+1, i)$ and q_t is the rightmost vertex of Q_t in $(i+1, j)$, then by Formulae (F') and Lemma 11, the pair

$$i', j' = \begin{cases} q_1, p_2 & \text{if } h > 1 \\ i, p_1 & \text{if } h = 1 \text{ and } i \text{ is not a vertex of } Q_1 \\ q_1, j & \text{if } h = 1 \text{ and } j \text{ is not a vertex of } Q_1 \end{cases}$$

is a bending pair for the curve $\mathcal{C}^{s_1}$. Moreover, by Lemma 15, in any case the braid $s_1 a_{i',j'}$ is simple. It follows that i', j' is a bending pair for the curve $\mathcal{C}^{s_1}$ compatible with s_2. $\square$

We now achieve the proof of Theorem 3. Recall that we consider a braid $x = \delta^p x_1 \dots x_r$ in dual left normal form which sends some standard curve $\mathcal{C}$ to another standard curve $\mathcal{C}' = \mathcal{C}^x$. Since the braid δ corresponds to a rotation of D_n^*, and thus sends standard curves to standard curves, up to multiplication by a power of δ we may assume that x is a dual positive braid, whose left normal form is $x_1 \dots x_r$. By a direct induction on the number of factors in the left normal form of x, it is sufficient to show that $\mathcal{C}^{x_1}$ is standard. We shall give a proof by contradiction, assuming that $\mathcal{C}^{x_1}$ is nonstandard. We will see (by induction on $m = 1, \dots, r$) that none of the curves $\mathcal{C}^{x_1 \cdots x_m}$ for $1 \leqslant m \leqslant r$ can be standard, contradicting the fact that $\mathcal{C}^x$ is standard.

Let $x_1 = P_1 \dots, P_g$ be the decomposition of x_1 into pairwise disjoints elements of $\mathcal{P}$. On the other hand suppose that $W(\mathcal{C})$ is written only with letters in $\{\smile, \curvearrowleft, 1, \dots, n\}$ (that is choose the counterclockwise orientation for $\mathcal{C}$).

As $\mathcal{C}^{x_1}$ is nonstandard, so must be $\mathcal{C}^{P_t}$ for at least one of the P_t's (according to Remark 10). We may assume that $\mathcal{C}^{P_1}$ is nonstandard. Therefore, there exist an arc $i \smile j$ in $W(\mathcal{C})$ such that P_1 is transverse to $i \smile j$ and j is not a vertex of P_1 (hence $W(\mathcal{C}^{P_1})$ has at least one letter $\curvearrowright$, see Formulae $(F'0)$ and $(F'2)$).

Consider all the polygons among $P_1, \dots, P_g$ which are transverse to the arc $i \smile j$. By Remark 10, only these polygons witness the action of x_1 on $i \smile j$. Running along the arc $i \smile j$ starting at $(i, i+1)$ allows us to order them in a natural way (see the proof of Lemma 16): $Q_1, \dots, Q_h$. Let q_1 be the rightmost vertex of Q_1 in $(i+1, j)$ (so that $q_1 \neq j$) and if $h > 1$ let p_2 be the rightmost vertex of Q_2 in $(j+1, i)$.

Then we set $i_1 = q_1$ and $j_1 = p_2$ if $h > 1$, $j_1 = j$ otherwise. Formulae (F') and Lemma 11 now imply that $W(\mathcal{C}^{x_1})$ contains the subword $\curvearrowright i_1 \smile j_1 \curvearrowleft$ (or $\curvearrowright i_1 \smile j_1 \circlearrowleft$). By construction, the braid $x_1 a_{i_1, j_1}$ is simple according to Lemma 15.

In other words we saw that i_1, j_1 is a bending pair for $\mathcal{C}^{x_1}$, and since $x_1 \cdot x_2$ is in left normal form, this bending pair is compatible with x_2. It follows by induction on m and Lemma 16 that one can find, for each $m = 1, \dots, r-1$, a bending pair i_m, j_m for the curve $\mathcal{C}^{x_1 \cdots x_m}$ which is compatible with x_{m+1}. The existence of a bending pair i_r, j_r for the curve $\mathcal{C}^{x_1 \cdots x_r}$ also follows from the proof of Lemma 16.

We proved in particular that the curve $\mathcal{C}^{x_1 \cdots x_r} = \mathcal{C}^x$ is not standard. This is a contradiction which achieves the proof of Theorem 3.

4. Deciding the dynamical type of braids

The above results give rise to an algorithm for deciding the Nielsen-Thurston type of a given braid, in the spirit of [19], [18], using the dual structure. Thurston's classification Theorem [21] asserts that the elements of Mapping Class Groups of surfaces (and therefore, in particular, braids) split into three mutually exclusive types, according to their dynamical properties: periodic, reducible non-periodic and pseudo-Anosov. The reader is referred to [21], [9] for details and precise definitions. We restrict ourselves to recalling that reducible braids are those preserving a family of pairwise disjoint isotopy classes of nondegenerate simple closed curves in the n-times punctured disk. Periodic braids being easy to detect [2], the main problem to be solved is to decide whether a non-periodic braid is pseudo-Anosov or reducible. In what follows, we will assume that the braids under consideration are not periodic and the curves, simple, closed and nondegenerate, will be considered up to isotopy so that the term *curve* will be applied to the isotopy class of a curve.

From Theorem 3, it follows that if a braid preserves a family of standard curves, then its cyclic sliding [13] also preserves a family of standard curves (see Proposition 4.2. and Corollary 4.3. in [17]). Therefore:

Proposition 17. *Let x be a reducible braid. Then there exist some y in $SC_{BKL}(x)$ (the set of sliding circuits of x [13], with respect to the dual structure) which preserves a family of (pairwise disjoint) standard curves.*

The main result of this section asserts that this last condition, i.e. preserving a (non-empty) family of pairwise disjoint standard curves, is checkable in polynomial time. In fact, we shall prove:

Theorem 18. *There is an algorithm which decides whether a given n-strand braid x in dual left normal form $x = \delta^p x_1 \dots x_\ell$ admits a standard invariant curve. Moreover this algorithm takes time $O(\ell \cdot n^4)$.*

Proof. Given a subset I_0 of $\{1, \dots, n\}$ (of cardinality $2 \leqslant \#(I_0) \leqslant n-1$) and a braid x in dual left normal form $x = \delta^p x_1 \dots x_\ell$, the main task of the algorithm is to construct a bigger set $S(I_0, x)$ of punctures, which must be enclosed by the image under x of any standard curve $\mathcal{C}$ surrounding punctures in I_0, provided $\mathcal{C}^x$ is standard. This is achieved by the following, which is the key-result:

Lemma 19. *Suppose we are given a dual simple braid s, decomposed into pairwise disjoint polygons $s = P_1 \dots P_g$, together with a proper subset I of $\{1, \dots, n\}$ of cardinality at least 2, whose elements are enumerated $p_1, \dots, p_k$ in this order (up to cyclic permutation) running along the circle Γ clockwise (we will put $p_{k+1} = p_1$). For each $i = 1, \dots, k$, consider all the polygons among $P_1, \dots, P_g$ having at least one vertex, but not all, in $(p_i + 1, p_{i+1} - 1)$ and which does not have p_i as a vertex, and take for each of them its leftmost vertex in the arc $(p_i + 1, p_{i+1} - 1)$. Let I' be the union of I with all punctures collected in the above way. If $\mathcal{C}$ is a standard curve surrounding the punctures in I (and possibly other punctures) such that $\mathcal{C}^s$ is standard, then $\mathcal{C}$ surrounds the punctures in I'. Moreover, if $I' = I$, the standard curve whose set of inner punctures is exactly I is sent to a standard curve by s.*

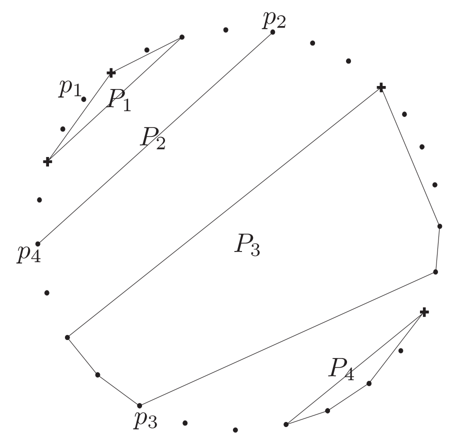

FIGURE 14. An example that illustrates Lemma 19: here the simple braid s is decomposed as $s = P_1 \dots P_4$, and I is made of 4 points p_1, p_2, p_3, p_4. Then running along each arc (p_i, p_{i+1}) allows the construction of I', which consists of adding punctures depicted as crosses to I.

Proof. First observe that the image of a standard curve under s is standard if and only if the image of this curve under each of the P_i's is standard (Remark 10). Now, for a polygon Q and a standard curve $\mathcal{C}$ oriented counterclockwise, according to Formulae (F'), the following are equivalent:

- $\mathcal{C}^Q$ is standard.
- for each inner arc $a \smile b$ of $\mathcal{C}$ which is transverse to Q, the puncture b is a vertex of Q.

Let $\mathcal{C}$ be a standard curve (oriented counterclockwise) surrounding the punctures in I, such that $\mathcal{C}^s$ is standard. The punctures in $I'-I$ (that is the punctures added to I by the process of Lemma 19) cannot lie in the outer component of $D_n^* - \mathcal{C}$. Indeed, the belonging of such a puncture to the outer component of $D_n^* - \mathcal{C}$ would yield some inner arc $a \smile b$ of $\mathcal{C}$, transverse to a polygon in s not having b as a vertex; this would be in contradiction with the above remarks. See Figure 14.

Now let $\mathcal{C}_I$ be the standard curve surrounding exactly the punctures in I (oriented counterclockwise) and suppose that the process of Lemma 19 yields $I' = I$. Then, whenever a polygon in s and an inner arc $a \smile b$ of $\mathcal{C}_I$ are transverse, the puncture b is a vertex of the involved polygon (otherwise some puncture in $(b+1, a)$ would be added to I by the process of Lemma 19). By the remark above, it follows that $\mathcal{C}_I^s$ is standard, as claimed. □

The set I' of the above lemma depends only on I and s, and we will denote by $S(I, s)$ the set $\rho(s)(I')$ (where ρ is the natural morphism $B_n \longrightarrow S_n$).

Lemma 19 says that if $\mathcal{C}$ is a standard curve surrounding the punctures in I and $\mathcal{C}^s$ is standard, then $\mathcal{C}^s$ must surround the punctures in $S(I, s)$. Using Theorem 3 and an induction on the number of non-δ factors in the left normal form of x allows to construct the set $S(I_0, x)$.

First, we set $S(I_0, \delta^p x_1) = S(\rho(\delta^p)(I_0), x_1)$. Then we define, for $i = 1, \dots, \ell - 1$,

$$S(I_0, \delta^p x_1 \dots x_{i+1}) = S(S(I_0, \delta^p x_1 \dots x_i), x_{i+1}).$$

Notice that the set $S(I_0, x)$ can be computed in time $O(\ell \cdot n)$. Notice also that, in virtue of Lemma 19, the equality $S(I, x) = I$ implies that the curve whose set of inner punctures is exactly I is x-invariant.

The last step in the proof of Theorem 18 is the following:

Proposition 20. *Let a, b be any pair of punctures in $\{1, \dots, n\}$. There is an algorithm which decides whether a given n-braid x of length ℓ admits a standard invariant reduction curve surrounding the punctures a and b. Moreover this algorithm runs in time $O(\ell \cdot n^2)$.*

Proof. The algorithm does the following:

- set $I_0 = \{a, b\}$,
- for $m = 1, \dots, n-2$, compute the set $I_m = S(I_{m-1}, x) \cup I_{m-1}$.

Remark that $I_{i-1} \subset I_i$ for all i. If $I_{n-2} = \{1, \dots, n\}$, then the algorithm answers negatively; otherwise, the standard curve which surrounds exactly the punctures in I_{n-2} is x-invariant. Indeed, in the latter case, there must exist some k, $1 \leqslant k < n-2$, such that $\#(I_k) = \#(I_{k+1})$, and therefore $I_k = I_{k+1} = I_{n-2}$. This means that $S(I_k, x) = I_k$ and therefore the standard curve whose set of inner punctures is exactly I_{n-2} (and thus contains a and b) is x-invariant. The complexity of the algorithm is $O(\ell \cdot n^2)$, according to the above estimation about the computation of $S(I_0, x)$. □

Iterating the above algorithm for each pair of points in $\{1, \dots, n\}$ yields the algorithm in the statement of Theorem 18, since the number of pairs of points in $\{1, \dots, n\}$ is $\frac{n \cdot (n-1)}{2}$, so that the complexity of the whole algorithm is $O(l \cdot n^4)$ as claimed. □

We observe that this algorithm is the analogue of Theorem 2.9. in [18]. The result in [18] deals with the classical structure, but needs an additional hypothesis about the triviality of the "inner" braid; we believe that obtaining the same conclusion without this hypothesis could be an indication that the dual structure is more adapted for the kind of problems we are dealing with.

The algorithm deciding the Nielsen-Thurston type of a given braid x is as follows:

1. Test whether x is periodic [2], and if it is so, return "periodic", and stop.
2. Compute the set of sliding circuits (for the dual structure) of x [14].
3. For each element y of $SC_{BKL}(x)$, for each $k = 1, \ldots, \frac{n}{2}$, apply the algorithm of Theorem 18 to the braid y^k (a curve belonging to a family fixed by y must indeed be fixed by some power y^k, with $k = 1, \ldots, \frac{n}{2}$).
4. Stop whenever a positive answer is found, and return "reducible"; otherwise return "pseudo-Anosov".

The complexity of this algorithm is not bounded above by a polynomial in n and ℓ since the size of the set of sliding circuits is known to be exponential in general [13], [20].

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