

A posteriori error analysis for two non-overlapping domain decomposition techniques

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Abstract

This paper is devoted to the construction of fast solvers for penalty domain decomposition techniques, based upon a posteriori error analysis. We introduce a penalty non-overlapping domain decomposition method (ddm) motivated by the a posteriori error analysis of the method proposed by Chacón and Chacón in [6]. In the new method a $H_{00}^{1/2}(\Gamma)$ penalty term replaces the $L^2(\Gamma)$ one in the original method. The number of iterations needed by the new ddm to yield a solution with an error of the same order as the discretization error is remarkably reduced. We develop an a posteriori error analysis that we use to determine an optimal value of the penalty parameter for a given grid, and also to jointly determine an optimal grid and a penalty parameter to reduce the error below a targeted value. Several numerical tests for model problems exhibit the good performances of our approach and provide to a numerical comparison of the two penalty methods.

Résumé

Cet article a pour but la construction de solveurs rapides pour les techniques de décomposition de domaine avec pénalisation et repose sur une analyse a posteriori. Nous introduisons une méthode de décomposition de domaine sans recouvrement, issue de l'analyse a posteriori de la méthode proposée par Chacón et Chacón [6], où une pénalisation de type $H_{00}^{1/2}(\Gamma)$ remplace celle de type $L^2(\Gamma)$ dans la première méthode. Le nombre d'itérations pour une erreur du même ordre que l'erreur de discrétisation est considérablement réduit. Nous prouvons des estimations d'erreur a posteriori qui permettent d'optimiser le choix du paramètre de pénalisation pour une grille donnée, et aussi lors de l'adaptation de maillage. Plusieurs expériences numériques sur des problèmes académiques montrent les bonnes performances de notre approche et permettent une comparaison numérique des deux méthodes.

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1 Introduction

This paper is devoted to the construction of fast solvers for penalty domain decomposition techniques, based upon a posteriori error analysis. We introduce a penalty non-overlapping domain decomposition method motivated by the a posteriori error analysis of the method studied by Chacón and Chacón in [6] and [7].

The domain decomposition method (ddm) proposed in [7] enforces the continuity of the variables across the interface between adjacent subdomains through a penalty technique. If we denote this interface by Γ , a $L^2(\Gamma)$ penalty term is added to a convenient variational formulation, with a structure that ensures the continuity of the fluxes across the interface. This method provides fair accurate solutions. In fact, it is proved to be equivalent to a penalty ddm with arbitrarily large overlapping region, analyzed by Lions [11] and Lions and Pironneau [13]. However, it has a rather slow convergence rate. In [7] some solutions to this problem were proposed, particularly the use of general acceleration techniques for sequences, such as Aitken or the Minimal Polynomial Extrapolation methods.

In the present paper we address the problem of finding more specific techniques to improve the convergence rate of the method. At first, we introduce a version of the method in which a $H_{00}^{1/2}(\Gamma)$ penalty term replaces the $L^2(\Gamma)$ one in the original method of [7]. In this way we force the jump of the unknown to vanish along the interface Γ in a stronger sense.

The choice of this new penalty term is suggested by the a posteriori error analysis of the ddm introduced in [7] that we perform in the paper at hand. An improvement is deduced from the a priori error analysis that we develop: If the $L^2(\Gamma)$ penalty is replaced by the $H_{00}^{1/2}(\Gamma)$ penalty, the number of iterations needed by the ddm to yield a solution with an error of the same order as the discretization error is remarkably reduced. If the discretization error is of order $\mathcal{O}(h^k)$, the $L^2(\Gamma)$ penalty requires $\mathcal{O}(|\log h|h^{-2k})$ iterations in order to attain this error, while the $H_{00}^{1/2}(\Gamma)$ penalty requires $\mathcal{O}(|\log h|h^{-k})$ iterations. This improvement may be explained in an abstract framework: When $H_{00}^{1/2}(\Gamma)$ penalty is used, the new ddm may be cast as the penalty of a stable mixed formulation of the Poisson problem. This penalty procedure fits into a general framework studied by Girault and Raviart in [9]. When $L^2(\Gamma)$ penalty is used, this mixed formulation is not uniformly stable, due to the lack of an inf-sup condition.

In addition, we develop an a posteriori error analysis that provides independent error indicators for both penalty and discretization errors, and allows to develop strategies to provide further error reductions:

- On one hand, we determine an optimal value for the penalty parameter for a fixed grid that yields a penalty error of the same size as the discretization error, with a minimal computational effort.
- On another hand, we determine both optimal values for the penalty parameter and

optimal grids that allow to reduce the error below a targeted value, also with a minimal computational effort.

Our results show at first that, indeed, the $H_{00}^{1/2}(\Gamma)$ penalty is faster than the $L^2(\Gamma)$ one for a fixed grid and a fixed penalty parameter, in good agreement with the a priori error analysis mentioned above. However, surprisingly, a large computational gain is obtained with $L^2(\Gamma)$ penalty when an adaptive computation strategy is used. This technique turns out to be more flexible and to need much less computational time per iteration, as it does not require the computation of $H_{00}^{1/2}(\Gamma)$ (or even a discrete equivalent) norm.

We refer to [14] for a similar approach (see also [8]). Other types of ddm and the corresponding algorithms are described in [15] and the references therein.

The paper is organized as follows: Section 2 introduces the continuous version of the new ddm. Section 3 performs a joint error analysis of this and the method introduced in [6]. Section 4 characterizes the new ddm as the penalty of a stable mixed method when $H_{00}^{1/2}(\Gamma)$ penalty is used. Section 5 is devoted to the discretization by Lagrange finite element spaces of the two methods considered. In Section 6 the a posteriori error analysis of both methods is performed. In Section 7 the improvement of the convergence rate due to the use of $H_{00}^{1/2}(\Gamma)$ penalty is analyzed, as an a priori error analysis. Section 8 describes practical strategies for the computation of optimal penalty parameters and grids. In Section 9 we report some numerical tests for the Poisson problem as a model problem in significative geometries, that exhibit the good performances of our approaches. Finally, an Appendix contains the proof of some technical results.

2 The penalty problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a simply connected and bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. We consider a simple decomposition of Ω into two non-overlapping subdomains Ω_1 and Ω_2 and set $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, $\Gamma_i = \partial\Omega_i \cap \partial\Omega$. We assume that all of these boundaries are Lipschitz-continuous $(d - 1)$ -dimensional manifolds with positive $(d - 1)$ -dimensional measure. Observe that under these hypotheses, Γ may or may not intersect $\partial\Omega$. However, in this case Ω cannot have a connected boundary, see Figures 1 and 2.

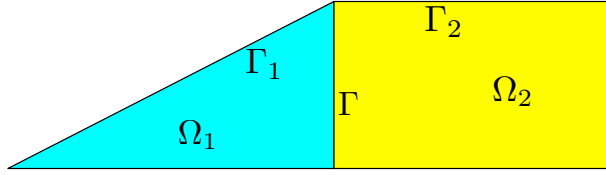


Figure 1: Subdivision of domain Ω when Γ intersects $\partial\Omega$.

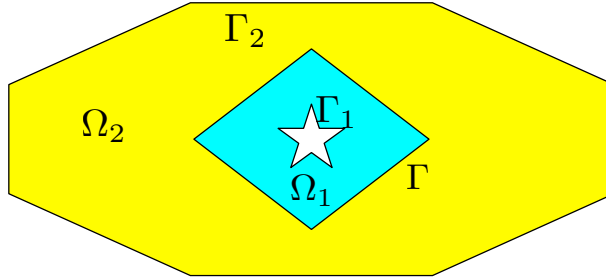


Figure 2: Subdivision of domain Ω when Γ does not intersect $\partial\Omega$.

We denote by \mathbf{n}_{ij} the outward normal vector on Γ pointing from Ω_i into Ω_j , $\partial_{\mathbf{n}_{ij}}$ the partial derivative with respect to \mathbf{n}_{ij} and set $\mathbf{n} = \mathbf{n}_{12}$. For a measurable subset D of \mathbb{R}^n and two functions f, g such that $fg \in L^1(D)$, we denote

$$(f, g)_D = \int_D f(x) g(x) dx.$$

We consider the Sobolev spaces

$$X_i = H^1(\Omega_i; \Gamma_i) = \{v \in H^1(\Omega_i) \text{ such that } v|_{\Gamma_i} = 0\}, \quad i = 1, 2; \quad \mathbf{X} = X_1 \times X_2.$$

We define for $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathbf{X}$ the scalar product and norm on \mathbf{X}

$$((\mathbf{u}, \mathbf{v}))_X = \sum_{i=1}^2 (\nabla u_i, \nabla v_i)_{\Omega_i}, \quad \|\mathbf{u}\|_X^2 = ((\mathbf{u}, \mathbf{u}))_X,$$

Let us consider the Poisson problem in Ω :
Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad (1)$$

for all $v \in H_0^1(\Omega)$. This is a model problem that in despite of its simplicity contains the main difficulties set by the analysis of the ddms that we consider in this paper.

We recall that $H_{00}^{1/2}(\Gamma)$ is the subspace of $H^{1/2}(\Gamma)$ whose extension by zero to $\partial\Omega_1$ (for instance, it could be also to $\partial\Omega_2$) belongs to $H^{1/2}(\partial\Omega_1)$. An intrinsic scalar product on $H_{00}^{1/2}(\Gamma)$ is defined as

$$\begin{aligned} [[w, v]]_\Gamma &= \int_\Gamma w(x) v(x) dx + \int_\Gamma \int_\Gamma \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^d} dx dy \\ &+ \int_\Gamma \frac{w(x) v(x)}{d(x, \partial\Gamma)} dx, \end{aligned} \quad (2)$$

where the first two summands define the $H^{1/2}(\Gamma)$ scalar product (see Adams [1], Theorem 7.48). Its expression involves the distance $d(x, \partial\Gamma)$ to the boundary of Γ . It comes from the restriction to $H_{00}^{1/2}(\Gamma)$ of the scalar product in $H^{1/2}(\partial\Omega_1)$ for instance. It is given by Lions and Magenes in [12], Chapitre 1, Théorème 11.7.

For brevity, we also denote by $[[\cdot, \cdot]]_\Gamma$ the $L^2(\Gamma)$ scalar product, and study both the $L^2(\Gamma)$ and the $H_{00}^{1/2}(\Gamma)$ penalties at the same time with the same notation. We shall distinguish the two cases when this is necessary.

We may now introduce our penalty problem as

$$(P_\varepsilon) \quad \begin{cases} \text{Find } \mathbf{u}^\varepsilon \in \mathbf{X} \text{ such that} \\ ((\mathbf{u}^\varepsilon, \mathbf{v}))_X + \frac{1}{\varepsilon} [[u_1^\varepsilon - u_2^\varepsilon, v_1 - v_2]]_\Gamma = F(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathbf{X}. \end{cases}$$

where F is the $\|\cdot\|_X$ -continuous linear form on \mathbf{X} given by

$$F(\mathbf{v}) = \sum_{i=1}^2 (f, v_i)_{\Omega_i},$$

and $\varepsilon > 0$ is a parameter destined to tend to zero.

This problem has a unique solution for any $\varepsilon > 0$ due to Lax–Milgram Lemma. This is also true when the $(d - 1)$ -dimensional measure of some of the Γ_i is zero, but we have excluded this case for simplicity.

When Γ does not intersect $\partial\Omega$, then the $H_{00}^{1/2}(\Gamma)$ penalty coincides with the $H^{1/2}(\Gamma)$ penalty: If Γ is a manifold with no boundary, then $H_{00}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$. However, for brevity we assume that Γ intersects $\partial\Omega$ and use $H_{00}^{1/2}(\Gamma)$ penalty (or $L^2(\Gamma)$ penalty). All our results are also true if Γ does not intersect $\partial\Omega$ and $H^{1/2}(\Gamma)$ penalty is used.

Remark 1 *The original method introduced in [7] only considers a $L^2(\Gamma)$ penalty term to enhance the continuity of $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ across Γ . We also consider here a $H_{00}^{1/2}(\Gamma)$ penalty term to enhance this continuity in a stronger sense. In both cases, the method may be interpreted as the variational formulation of a coupled system of PDEs with the structure*

$$\left\{ \begin{array}{ll} -\Delta u_1 = f & \text{in } \Omega_1, \\ u_1 = 0 & \text{on } \Gamma_1, \\ \partial_{\mathbf{n}_{12}} u_1 = \frac{1}{\varepsilon} b(u_1 - u_2) & \text{on } \Gamma, \end{array} \right. \quad \left\{ \begin{array}{ll} -\Delta u_2 = f & \text{in } \Omega_2, \\ u_2 = 0 & \text{on } \Gamma_2, \\ \partial_{\mathbf{n}_{21}} u_2 = \frac{1}{\varepsilon} b(u_2 - u_1) & \text{on } \Gamma, \end{array} \right.$$

where b is an injective linear bounded boundary operator on Γ , which reduces to the identity for $L^2(\Gamma)$ penalty. So, the method ensures the continuity of the normal fluxes through Γ and forces by penalty the continuity of \mathbf{u}^ε . Note that, in the case of $H_{00}^{1/2}(\Gamma)$ penalty, this is rather different from a Dirichlet to Neumann or Neumann to Neumann algorithm.

3 Penalty error analysis

Our first result is a generalization of the penalty error estimates of [7]. To state it we denote by $\|\cdot\|_\Gamma$ the norm associated with the $[[\cdot, \cdot]]_\Gamma$ scalar product, and by $\|\cdot\|_{*,\Gamma}$ its dual norm.

We use the notation

$$((\mathbf{u}, \mathbf{v}))_\varepsilon = ((\mathbf{u}, \mathbf{v}))_X + \frac{1}{\varepsilon} [[u_1 - u_2, v_1 - v_2]]_\Gamma, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}.$$

Theorem 1 *Assume that one of the following assumptions hold:*

- *Either the solution u of (1) satisfies $\partial_{\mathbf{n}}u \in L^2(\Gamma)$ when $L^2(\Gamma)$ penalty is used,*
- *Or f belongs to $L^2(\Omega)$ when $H_{00}^{1/2}(\Gamma)$ penalty is used.*

Then, for all $\varepsilon > 0$ the following estimates hold

$$\|u_1^\varepsilon - u_2^\varepsilon\|_\Gamma \leq \|\partial_{\mathbf{n}}u\|_{*,\Gamma} \varepsilon, \quad \sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i} \leq \|\partial_{\mathbf{n}}u\|_{*,\Gamma} \sqrt{\varepsilon}. \quad (3)$$

Proof: The case of $L^2(\Gamma)$ penalty has been proved in [7] (Theorem 1), we include it in the statement of the present theorem for completeness. We prove (3) in the case of $H_{00}^{1/2}(\Gamma)$ penalty.

For the solution u of problem (1) we write $\mathbf{u} = (u|_{\Omega_1}, u|_{\Omega_2}) \in \mathbf{X}$. Consider the consistency error $\mathcal{C} \in \mathbf{X}'$ defined by

$$\langle \mathcal{C}, \mathbf{v} \rangle = ((\mathbf{u}, \mathbf{v}))_\varepsilon - F(\mathbf{v}), \quad \forall \mathbf{v} = (v_1, v_2) \in \mathbf{X}. \quad (4)$$

Then the following property holds

$$((\mathbf{u} - \mathbf{u}_\varepsilon, \mathbf{v}))_\varepsilon = \langle \mathcal{C}, \mathbf{v} \rangle, \quad \forall \mathbf{v} = (v_1, v_2) \in \mathbf{X}. \quad (5)$$

To estimate the consistency error, observe that as $\Delta u \in L^2(\Omega_i)$, then $\partial_{\mathbf{n}_2}u \in (H_{00}^{1/2}(\Gamma))'$ due to the relation

$$\langle \partial_{\mathbf{n}_i}u, v_i \rangle_\Gamma = (\nabla u, \nabla v_i)_{\Omega_i} + (\Delta u, v_i)_{\Omega_i}, \quad \forall v_i \in X_i.$$

Thus, we may integrate by parts in the expression of \mathcal{C} to obtain

$$\langle \mathcal{C}, \mathbf{v} \rangle = \langle \partial_{\mathbf{n}}u, v_1 - v_2 \rangle_\Gamma.$$

We may now use the error equation (5) with $\mathbf{v} = \mathbf{u} - \mathbf{u}_\varepsilon$ to obtain

$$\sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i}^2 + \frac{1}{\varepsilon} \|u_1^\varepsilon - u_2^\varepsilon\|_{H_{00}^{1/2}(\Gamma)}^2 = \langle \partial_{\mathbf{n}}u, u_2^\varepsilon - u_1^\varepsilon \rangle_\Gamma \leq \|\partial_{\mathbf{n}}u\|_{(H_{00}^{1/2}(\Gamma))'} \|u_1^\varepsilon - u_2^\varepsilon\|_{H_{00}^{1/2}(\Gamma)}. \quad (6)$$

By Young's inequality,

$$\sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i}^2 + \frac{1}{\varepsilon} \|u_1^\varepsilon - u_2^\varepsilon\|_{H_0^{1/2}(\Gamma)}^2 \leq \frac{\varepsilon}{2} \|\partial_{\mathbf{n}} u\|_{(H_0^{1/2}(\Gamma))'}^2 + \frac{1}{2\varepsilon} \|u_1^\varepsilon - u_2^\varepsilon\|_{H_0^{1/2}(\Gamma)}^2.$$

This yields estimates (3).
Q.E.D.

We next introduce a continuous penalty error indicator (and refer to [14], Thm 3.1, for a similar indicator):

$$\eta^P = \|u_1^\varepsilon - u_2^\varepsilon\|_{H_0^{1/2}(\Gamma)}. \quad (7)$$

This is an optimal indicator, as it satisfies

Theorem 2 *There exist two constants $C > 0$ and $C' > 0$ independent of ε such that*

$$C \eta^P \leq \sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i} \leq C' \eta^P, \quad i = 1, 2. \quad (8)$$

Proof: The first estimate in (8) is obtained by

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{H_0^{1/2}(\Gamma)} \leq \sum_{i=1}^2 \|u - u_i^\varepsilon\|_{H_0^{1/2}(\Gamma)} \leq C_1 \sum_{i=1}^2 \|u - u_i^\varepsilon\|_{H^{1/2}(\partial\Omega_i)},$$

whence

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{H_0^{1/2}(\Gamma)} \leq C_2 \sum_{i=1}^2 \|u - u_i^\varepsilon\|_{1,\Omega_i}.$$

This inequality stems from the usual characterization of the $H_0^{1/2}(\Gamma)$ norm (equivalent to the intrinsic one given by (2), Cf. Adams [1], Theorem 7.53) defined as

$$\inf_{w \in X_i, w|_\Gamma = \varphi} \|w\|_{1,\Omega_i}.$$

To prove the second estimate in (8), let us introduce the subspace of \mathbf{X} ,

$$\tilde{\mathbf{X}} = \{ \tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2) \in \mathbf{X}, \text{ such that } \tilde{v}_1|_\Gamma = \tilde{v}_2|_\Gamma \};$$

which is algebraically isomorphic to $H_0^1(\Omega)$. Observe that

$$((\mathbf{u} - \mathbf{u}^\varepsilon, \tilde{\mathbf{v}}))_X = 0, \quad \forall \tilde{\mathbf{v}} \in \tilde{\mathbf{X}}. \quad (9)$$

There exists a lifting $w_1^\varepsilon \in X_1$ of $(u_1^\varepsilon - u_2^\varepsilon)|_\Gamma$ satisfying

$$|w_1^\varepsilon|_{1,\Omega_1} \leq C_3 \|u_1^\varepsilon - u_2^\varepsilon\|_{H_{00}^{1/2}(\Gamma)}.$$

for some constant C_3 independent of ε . For instance, this lifting is obtained as the solution of the boundary value problem

$$-\Delta w_1^\varepsilon = 0 \text{ in } \Omega_1, \quad w_1^\varepsilon = 0 \text{ on } \Gamma_1, \quad w_1^\varepsilon = u_1^\varepsilon - u_2^\varepsilon \text{ on } \Gamma.$$

Define the function $\mathbf{w}^\varepsilon = (w_1^\varepsilon, 0) \in \mathbf{X}$. Then, taking $\tilde{\mathbf{v}} = \mathbf{u} - \mathbf{u}^\varepsilon + \mathbf{w}^\varepsilon \in \tilde{\mathbf{X}}$ in (9), we have

$$\|\mathbf{u} - \mathbf{u}^\varepsilon\|_X \leq \|\mathbf{w}^\varepsilon\|_X, \text{ and so } \sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i} \leq \sqrt{2} |w_1^\varepsilon|_{1,\Omega_1} \leq C_4 \eta^P.$$

Q.E.D.

Observe that the a posteriori error estimator η^P coincides with the $H_{00}^{1/2}(\Gamma)$ penalty term. In this case, the error estimate for $|u - u_i^\varepsilon|_{1,\Omega_i}$ stated in Theorem 1 is improved by combining the second estimate in (8) with the first estimate in (3) :

Corollary 3 *Assume that $H_{00}^{1/2}(\Gamma)$ penalty is used and that f belongs to $L^2(\Omega)$.*

Then, for all $\varepsilon > 0$ the following estimate holds

$$\sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i} \leq C \|\partial_{\mathbf{n}} u\|_{H_{00}^{1/2}(\Gamma)}, \quad \varepsilon, \quad (10)$$

for some constant C independent of ε .

This suggests to replace the $L^2(\Gamma)$ penalty term by the $H_{00}^{1/2}(\Gamma)$ penalty term to improve the accuracy of our ddm. From the practical point of view this sets the problem of the computation of the non-local $H_{00}^{1/2}(\Gamma)$ norm and scalar products. However, these can be approximated numerically with low cost. We show the good performances of such an approach in our numerical results.

4 Casting as a penalty mixed problem

In the case of $H_{00}^{1/2}(\Gamma)$ penalty, Problem (P_ε) may be cast as a penalty mixed problem, that fits into the general framework of approximation of mixed problems by penalty analysed in Chapter I, Section 4.3 of Girault and Raviart [9]. We also refer to [2] for the introduction of a Lagrangian multiplier to handle the matching conditions in the ddm. To do this, let us denote by M the space $H_{00}^{1/2}(\Gamma)$, and define the bilinear bounded form

$$b(\cdot, \cdot) : \mathbf{X} \times M \mapsto \mathbb{R} \quad \text{by} \quad b(\mathbf{v}, \lambda) = [[v_1 - v_2, \lambda]]_\Gamma.$$

We re-formulate the Poisson problem (1) as the following mixed problem:

Find a pair $(\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2), \lambda) \in \mathbf{X} \times M$ such that

$$(P_M) \quad \begin{cases} ((\tilde{\mathbf{u}}, \mathbf{v}))_X + b(\mathbf{v}, \lambda) = F(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{X}, \\ b(\tilde{\mathbf{u}}, \mu) = 0, & \forall \mu \in M. \end{cases} \quad (11)$$

It can be readily checked that the form $b(\cdot, \cdot)$ satisfies an inf-sup condition: There exists a constant $\beta > 0$ such that

$$\beta \|\mu\|_M \leq \sup_{\mathbf{v} \in \mathbf{X}} \frac{b(\mathbf{v}, \mu)}{\|\mathbf{v}\|_X}, \quad \forall \mu \in M. \quad (12)$$

Thus, problem (P_M) admits a unique solution $(\tilde{\mathbf{u}}, \lambda)$. From the second equation in (11), $\tilde{u}_1 = \tilde{u}_2$ on Γ . Then, the solution of the Poisson problem is recovered from that of (P_M) by

$$u(x) = \begin{cases} \tilde{u}_1(x) & \text{if } x \in \Omega_1, \\ \tilde{u}_2(x) & \text{if } x \in \Omega_2. \end{cases}$$

Conversely, if we define $\tilde{u}_i = u|_{\Omega_i}$, then the second equation in (11) holds and the multiplier λ is obtained from the first equation in (11) thanks to the inf-sup condition (12).

Let us now consider the scalar product in $H_{00}^{1/2}(\Gamma)$,

$$c(\lambda, \mu) = [[\lambda, \mu]]_\Gamma,$$

and the penalty problem

Find a pair $(\tilde{\mathbf{u}}^\varepsilon, \lambda^\varepsilon) \in \mathbf{X} \times M$ such that

$$(P_M^\varepsilon) \quad \begin{cases} ((\tilde{\mathbf{u}}^\varepsilon, \mathbf{v}))_X + b(\mathbf{v}, \lambda^\varepsilon) = F(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{X}. \\ -\varepsilon c(\lambda^\varepsilon, \mu) + b(\tilde{\mathbf{u}}^\varepsilon, \mu) = 0, & \forall \mu \in M. \end{cases} \quad (13)$$

This problem is equivalent to (P_ε) . Indeed, from the second equation in (13) we obtain

$$\lambda^\varepsilon = \frac{1}{\varepsilon} (\tilde{u}_1^\varepsilon - \tilde{u}_2^\varepsilon).$$

Inserting this equality in the first equation of (13), $\tilde{\mathbf{u}}^\varepsilon$ turns out to verify exactly (P_ε) .

Problem (13) is studied in an abstract framework in Chapter I, Section 4.3 of [9]. Its well-posedness is proved, so as the error estimate

$$\|\tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}\|_X + \|\lambda^\varepsilon - \lambda\|_M \leq K \varepsilon \|F\|_{X'},$$

where K is a constant depending on Ω , Γ , d and β . This estimate is similar to the ones that we have obtained in Theorem 1 and Corollary 3.

Observe that the $L^2(\Gamma)$ penalty does not fit into this framework, as an inf-sup condition (12) does not hold. Indeed, in this case, the quantity

$$\sup_{\mathbf{v} \in \mathbf{X}} \frac{b(\mathbf{v}, \mu)}{\|\mathbf{v}\|_X},$$

is a norm equivalent to the $(H_{00}^{1/2}(\Gamma))'$ norm, and not to the $L^2(\Gamma)$ norm.

5 Discretization of the penalty problem

To discretize (P_ε) let us assume that the domains Ω , Ω_1 and Ω_2 are polygonal or polyhedral. Consider a regular family of triangulations $\{\mathcal{T}_{ih}\}_{\{h>0\}}$ of each $\bar{\Omega}_i$ formed by polygons ($d = 2$) or polyhedra ($d = 3$) such that Γ is the union of whole faces or sides of elements K in each \mathcal{T}_{ih} . As usual, $h > 0$ denotes the largest diameter of the elements of $\mathcal{T}_h = \mathcal{T}_{1h} \cup \mathcal{T}_{2h}$.

We assume that \mathcal{T}_{1h} and \mathcal{T}_{2h} have the same trace sets on Γ ; i.e., the trace sets

$$\partial_\Gamma \mathcal{T}_{ih} = \{e \subset \mathbb{R}^d \text{ such that } e = \partial K \cap \bar{\Gamma} \text{ for some } K \in \mathcal{T}_{ih}\}; \quad i = 1, 2$$

of \mathcal{T}_{1h} and \mathcal{T}_{2h} are equal to a same set, that we denote by \mathcal{E}_h^Γ . Then, \mathcal{T}_h is a triangulation of Ω , and the family $\{\mathcal{T}_h\}_{\{h>0\}}$ is regular.

Next we consider a family of finite element subspaces X_{ih} of X_i ($i = 1, 2$), built on the grids \mathcal{T}_{ih} . Note that the two spaces X_{1h} and X_{2h} may be constructed with piecewise polynomials of different degrees, or different finite elements.

We set $\mathbf{X}_h = X_{1h} \times X_{2h}$. We define the spaces

$$\tilde{\mathbf{X}}_h = \{\tilde{\mathbf{v}}_h = (\tilde{v}_{1h}, \tilde{v}_{2h}) \in \mathbf{X}_h, \text{ such that } \tilde{v}_{1h|_\Gamma} = \tilde{v}_{2h|_\Gamma}\}, \text{ and}$$

$$V_h = \{v_h : \bar{\Omega} \mapsto \mathbb{R}^d \text{ such that } v_h|_{\Omega_1} = \tilde{v}_{1h} \text{ and } v_h|_{\Omega_2} = \tilde{v}_{2h} \text{ for some } (\tilde{v}_{1h}, \tilde{v}_{2h}) \in \tilde{\mathbf{X}}_h\}.$$

Then, V_h is a finite element subspace of $H_0^1(\Omega)$ built on the triangulation \mathcal{T}_h . It is not reduced to $\{0\}$ in the situations that we take into consideration.

The structure of these finite element spaces allows compatible discretizations of our penalty ddm, in the sense that a discrete orthogonality property (9) holds. We consider the standard finite element Galerkin approximation $u_h \in V_h$ of u , satisfying

$$(P_h) \quad \begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ (\nabla u_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \text{ for all } v_h \in V_h. \end{cases}$$

Our **penalty discrete problem** is the standard finite element Galerkin approximation $\mathbf{u}_h^\varepsilon = (u_{1h}^\varepsilon, u_{2h}^\varepsilon) \in \mathbf{X}_h$ of \mathbf{u}^ε , solution of

$$(P_{\varepsilon,h}) \quad \begin{cases} \text{Find } \mathbf{u}_h^\varepsilon \in \mathbf{X}_h \text{ such that} \\ ((\mathbf{u}_h^\varepsilon, \mathbf{v}_h))_\varepsilon = F(\mathbf{v}_h) \text{ for all } \mathbf{v}_h \in \mathbf{X}_h. \end{cases}$$

This problem admits a unique solution for any $\varepsilon > 0$.

We consider the **parallel technique** by Schwarz successive approximations introduced in [7] to solve $(P_{\varepsilon,h})$:

For $n = 0, 1, 2, \dots$, $u_1^{n+1} = u_{1h}^{n+1,\varepsilon} \in X_{1h}$ and $u_2^{n+1} = u_{2h}^{n+1,\varepsilon} \in X_{2h}$ are computed from u_{1h}^n and u_{2h}^n by solving (we drop the exponent ε and index h for simplicity),

$$(P_{\varepsilon,h}^n) \quad \begin{cases} (\nabla u_1^{n+1}, \nabla v_{1h})_{\Omega_1} + \frac{1}{\varepsilon} [[u_1^{n+1} - u_2^n, v_{1h}]]_\Gamma = (f, v_{1h})_{\Omega_1}, \quad \forall v_{1h} \in X_{1h}, \\ (\nabla u_2^{n+1}, \nabla v_{2h})_{\Omega_2} + \frac{1}{\varepsilon} [[u_2^{n+1} - u_1^n, v_{2h}]]_\Gamma = (f, v_{2h})_{\Omega_2}, \quad \forall v_{2h} \in X_{2h}. \end{cases} \quad (14)$$

For $k \geq 1$ we assume that u_h is a finite element approximation of degree k of u , in the sense that when $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ the error committed with u_h belongs to V_h is $\mathcal{O}(h^k)$; specifically

$$|u_h - u|_{1,\Omega} \leq C_0 h^k |u|_{k+1,\Omega} \quad (15)$$

for some constant $C_0 > 0$.

We also assume that the spaces X_{ih} verify the approximation estimates

$$\forall v_i \in X_i, \exists v_{ih} \in X_{ih} \text{ s. t. } \|v_i - v_{ih}\|_{\alpha,\Omega_i} \leq C_0 h^{1-\alpha} |v_i|_{1,\Omega_i}, \quad 0 \leq \alpha \leq 1, \quad (16)$$

$$\text{and} \quad \|v_i - v_{ih}\|_{\alpha,\Gamma} \leq C_0 h^{1/2-\alpha} |v_i|_{1,\Omega_i}, \quad 0 \leq \alpha \leq 1/2. \quad (17)$$

Sufficient conditions that guarantee that rather general families of affine equivalent finite element spaces satisfy these properties are given in Bernardi et al. [4], Chapitre IX (Cf. Corollaire IX.3.9 and Théorème IX.3.10 for the proofs). In particular, these conditions are verified by the Lagrange finite element spaces constructed with triangles ($d = 2$) or tetrahedra ($d = 3$), as

$$X_{ih}^* = \{w_h \in H^1(\Omega_i); w_{h|_K} \in P_{k_i}(K), \forall K \in \mathcal{T}_{ih}, w_{h|_{\Gamma_i}} = 0 \}, \quad (18)$$

where $P_{k_i}(K)$ denotes the space of restrictions to K of polynomials with d variables and total degree $\leq k_i$ for positive integers k_i .

Note that if $k_1 \neq k_2$, say $k_1 > k_2$, then the first component $v_{1h} \in X_{1h}^*$ of a pair $(v_{1h}, v_{2h}) \in \tilde{\mathbf{X}}_h$ must degenerate to polynomials of $P_{k_2}(e)$ on the sides or faces $e \in \mathcal{E}_h^\Gamma$ when the X_{ih} are equal to the X_{ih}^* .

6 A posteriori error analysis

From now on, we assume that each of the spaces X_{ih} contains the space X_{ih}^* defined in (18), and that its trace space on Γ is

$$W_{ih} = \{v_h \in H_0^1(\Gamma) : v_h|_e \in P_{k_i}(e), \forall e \in \mathcal{E}_h^\Gamma\}. \quad (19)$$

To introduce our error indicators, for each $K \in \mathcal{T}_{ih}$, $i = 1, 2$, let us denote by \mathcal{E}_K the set of edges ($d = 2$) or faces ($d = 3$) of K that are not contained in Γ_i . Let us also consider a piecewise polynomial approximation f_h of f on the triangulation \mathcal{T}_h . We introduce

- a family of **discretization error indicators**: For $i = 1, 2$ and $K \in \mathcal{T}_{ih}$,

$$\eta_i^K = h_K \|f_h + \Delta u_{ih}^\varepsilon\|_{L^2(K)} + \sum_{e \in \mathcal{E}_K} h_e^{1/2} \|[\partial_{n_K} u_{ih}^\varepsilon]\|_{L^2(e)},$$

where

- h_K and h_e stand for the diameters of K and e , respectively,
- n_K stands for the unit outward normal to ∂K , and
- $[\partial_{n_K} u_{ih}^\varepsilon]$ stands for the jump of $\partial_{n_K} u_{ih}^\varepsilon$ across e if e is not included in Γ , and for $\partial_{n_K} u_{1h}^\varepsilon - \partial_{n_K} u_{2h}^\varepsilon$ if e is included in Γ .

- a **penalty error indicator**:

$$\eta_h^P = \|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_{00}^{1/2}(\Gamma)}.$$

The penalty error is estimated by the following discrete version of Theorem 2:

Theorem 4 *There exist two constants $C > 0$ and $C' > 0$ independent of ε and h such that*

$$C \eta_h^P \leq \sum_{i=1}^2 |u_h - u_{ih}^\varepsilon|_{1, \Omega_i} \leq C' \eta_h^P, \quad i = 1, 2. \quad (20)$$

Proof: It is parallel to that of Theorem 2. The first inequality is proved in the same way.

To prove the second inequality, we start from the discrete orthogonality property

$$((\mathbf{u}_h - \mathbf{u}_h^\varepsilon, \tilde{\mathbf{v}}_h))_X = 0, \quad \forall \tilde{\mathbf{v}}_h \in \tilde{\mathbf{X}}_h, \quad \text{with } \mathbf{u}_h = (u_{1h}, u_{2h}), \quad \mathbf{u}_h^\varepsilon = (u_{1h}^\varepsilon, u_{2h}^\varepsilon). \quad (21)$$

Denote by φ_h the extension by zero of $(u_{1h}^\varepsilon - u_{2h}^\varepsilon)|_\Gamma$ to $\partial\Omega_1$. Since $(u_{1h}^\varepsilon - u_{2h}^\varepsilon)|_\Gamma$ vanishes on $\partial\Gamma$, assuming for instance $k_1 \geq k_2$, then φ_h belongs to the trace space on $\partial\Omega_1$ of X_{1h} . Thus, following Bernardi et al. [4] (Théorème IX.4.1), there exists a lifting $w_{1h}^\varepsilon \in X_{1h}$ of φ_h that satisfies

$$|w_{1h}^\varepsilon|_{1, \Omega_1} \leq C_1 \|\varphi_h\|_{H^{1/2}(\partial\Omega_1)},$$

for some constant C_1 independent of ε . As

$$\|\varphi_h\|_{H^{1/2}(\partial\Omega_1)} \leq C_2 \|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_0^{1/2}(\Gamma)}, \quad \text{then} \quad |w_{1h}^\varepsilon|_{1,\Omega_1} \leq C_3 \|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_0^{1/2}(\Gamma)}.$$

Define the function $\mathbf{w}_h^\varepsilon = (w_{1h}^\varepsilon, 0) \in \mathbf{X}_h$. Then, it is enough to take $\tilde{\mathbf{v}}_h = \mathbf{u}_h - \mathbf{u}_h^\varepsilon + \mathbf{w}_h^\varepsilon \in \tilde{\mathbf{X}}_h$ in (21) and repeat the remaining arguments of the end of proof of Theorem 2.

Q.E.D.

We may now state our first a posteriori error analysis result:

Theorem 5 *The solution of the discrete penalty problem $(P_{h,\varepsilon})$ satisfies the following a posteriori error estimate*

$$\begin{aligned} \sum_{i=1}^2 |u - u_{ih}^\varepsilon|_{1,\Omega_i} &\leq C \left\{ \left(1 + \frac{h^\gamma}{\varepsilon}\right) \eta_h^P \right. \\ &\quad \left. + \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} (\eta_i^K)^2 + h_K^2 \|f - f_h\|_{L^2(K)}^2 \right)^{1/2} \right\}, \end{aligned} \quad (22)$$

where $\gamma = \begin{cases} 1/2 & \text{if } L^2(\Gamma) \text{ penalty is used,} \\ 0 & \text{if } H_0^{1/2}(\Gamma) \text{ penalty is used.} \end{cases}$

Proof. We start from the triangle inequality

$$\sum_{i=1}^2 |u - u_{ih}^\varepsilon|_{1,\Omega_i} \leq \sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i} + \sum_{i=1}^2 |u_i^\varepsilon - u_{ih}^\varepsilon|_{1,\Omega_i} \quad (23)$$

To estimate the first term in the right-hand side of (23), we use (8):

$$\sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i} \leq C_1 \|u_1^\varepsilon - u_2^\varepsilon\|_{H_0^{1/2}(\Gamma)} \leq C_1 \left(\|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_0^{1/2}(\Gamma)} + \sum_{i=1}^2 \|u_i^\varepsilon - u_{ih}^\varepsilon\|_{H_0^{1/2}(\Gamma)} \right).$$

Combining this estimate with (23) we obtain

$$\sum_{i=1}^2 |u - u_{ih}^\varepsilon|_{1,\Omega_i} \leq C_2 \left(\eta_h^P + \sum_{i=1}^2 |u_i^\varepsilon - u_{ih}^\varepsilon|_{1,\Omega_i} \right), \quad (24)$$

for some constant $C_2 \geq 0$ independent of ε and h .

To estimate the last term in the right-hand side of this inequality, we use the problems (P_ε) and $(P_{\varepsilon,h})$ to deduce the residual equation:

$$((\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon = F(\mathbf{v}) - ((\mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon = F(\mathbf{v} - \mathbf{v}_h) - ((\mathbf{u}_h^\varepsilon, \mathbf{v} - \mathbf{v}_h))_\varepsilon, \quad \forall \mathbf{v} \in \mathbf{X}, \quad \forall \mathbf{v}_h \in \mathbf{X}_h.$$

Then,

$$((\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon = \sum_{i=1}^2 (f_h, v_i - v_{ih})_{\Omega_i} - ((\mathbf{u}_h^\varepsilon, \mathbf{v} - \mathbf{v}_h))_\varepsilon + \sum_{i=1}^2 (f - f_h, v_i - v_{ih})_{\Omega_i}.$$

Whence,

$$\begin{aligned} ((\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon &= \sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} \left(\int_K (f_h + \Delta u_{ih}^\varepsilon) (v_i - v_{ih}) - \sum_{e \in \mathcal{E}_K} \int_e \partial_{n_K} u_{ih}^\varepsilon (v_i - v_{ih}) \right) \\ &+ \sum_{i=1}^2 (f - f_h, v_i - v_{ih}) - \frac{1}{\varepsilon} [[u_{1h}^\varepsilon - u_{2h}^\varepsilon, v_1 - v_{1h} - (v_2 - v_{2h})]]_\Gamma. \end{aligned} \quad (25)$$

The first three summands in (25) are estimated in a standard way from the approximation properties (16) and (17) to give (see [16], Proposition 1.5)

$$\begin{aligned} ((\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon &\leq \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} (\eta_i^K)^2 + h_K^2 \|f - f_h\|_{L^2(K)}^2 \right)^{1/2} \sum_{i=1}^2 |v_i|_{1, \Omega_i} \\ &+ \frac{1}{\varepsilon} [[u_{1h}^\varepsilon - u_{2h}^\varepsilon, (v_1 - v_{1h}) - (v_2 - v_{2h})]]_\Gamma. \end{aligned} \quad (26)$$

To estimate the last term in (26), consider first the $L^2(\Gamma)$ penalty case. Using (17),

$$[[u_{1h}^\varepsilon - u_{2h}^\varepsilon, v_1 - v_{1h} - (v_2 - v_{2h})]]_\Gamma \leq \|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{0, \Gamma} \sum_{i=1}^2 \|v_i - v_{ih}\|_{0, \Gamma},$$

whence

$$[[u_{1h}^\varepsilon - u_{2h}^\varepsilon, v_1 - v_{1h} - (v_2 - v_{2h})]]_\Gamma \leq C_5 h^{1/2} \|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_{00}^{1/2}(\Gamma)} \sum_{i=1}^2 |v_i|_{1, \Omega_i} \quad (27)$$

Consider now the $H_{00}^{1/2}(\Gamma)$ penalty case. Then,

$$[[u_{1h}^\varepsilon - u_{2h}^\varepsilon, v_1 - v_{1h} - (v_2 - v_{2h})]]_\Gamma \leq \|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_{00}^{1/2}(\Gamma)} \sum_{i=1}^2 \|v_i - v_{ih}\|_{H_{00}^{1/2}(\Gamma)},$$

whence

$$[[u_{1h}^\varepsilon - u_{2h}^\varepsilon, v_1 - v_{1h} - (v_2 - v_{2h})]]_\Gamma \leq C_6 \|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_{00}^{1/2}(\Gamma)} \sum_{i=1}^2 |v_i|_{1, \Omega_i} \quad (28)$$

To conclude the proof, we need to use

$$\sum_{i=1}^2 |w_i|_{1,\Omega_i} \leq \sup_{\mathbf{v} \in \mathbf{X}} \frac{((\mathbf{w}, \mathbf{v}))_\varepsilon}{((\mathbf{v}, \mathbf{v}))_\varepsilon}, \quad \forall \mathbf{w} = (w_1, w_2) \in \mathbf{X},$$

and next to combine (26) with either (27) or (28).

Q.E.D.

We next analyze the optimality of the error indicators that we have introduced. To do it, we need some further notation. For a given element $K \in \mathcal{T}_{ih}$, $i = 1, 2$, we define the union ω_K of the elements $K' \in \mathcal{T}_{1h} \cup \mathcal{T}_{2h}$ that share an edge ($d = 2$) or a face ($d = 3$) with K . We denote by \mathcal{E}_K^Γ (respectively, \mathcal{E}_K^0) the set of edges or faces of \mathcal{E}_K that are (resp., are not) included in Γ . We also define the seminorm

$$|\mathbf{v}|_{1,\omega_K,h} = \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}, K \subset \omega_K} |u_i^\varepsilon - u_{ih}^\varepsilon|_{1,K}^2 \right)^{1/2}.$$

We set the following result.

Theorem 6 *The error indicators η_i^K and η_h^P verify the following estimates*

$$\eta_i^K \leq C (|\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon|_{1,\omega_K,h} + h_K \|f - f_h\|_{L^2(\omega_K)}), \quad (29)$$

$$\eta_h^P \leq C' \sum_{i=1}^2 |u - u_{ih}^\varepsilon|_{1,\Omega_i}, \quad (30)$$

for some constants $C > 0$ and $C' > 0$ that are independent on ε and h .

Proof.

- **Discretization error indicator.** We start from the residual equation (25), where we set $v_{ih} = 0$:

$$\begin{aligned} ((\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon &= \sum_{i=1}^2 \left(\sum_{K \in \mathcal{T}_{ih}} \int_K (f_h + \Delta u_{ih}^\varepsilon) v_i + (f - f_h, v_i)_{\Omega_i} \right) \\ &- \sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} \left(\frac{1}{2} \sum_{e \in \mathcal{E}_K^0} \int_e [\partial_{n_K} u_{ih}^\varepsilon] v_i - \sum_{e \in \mathcal{E}_K^\Gamma} \int_e \partial_{n_K} u_{ih}^\varepsilon v_i \right) \\ &- \frac{1}{\varepsilon} [[u_{1h}^\varepsilon - u_{2h}^\varepsilon, v_1 - v_2]]_\Gamma. \end{aligned} \quad (31)$$

For each $K \in \mathcal{T}_{ih}$, the estimate of $\|f_h + \Delta u_{ih}^\varepsilon\|_{L^2(K)}$ is standard. To do it, v_i must be chosen as the product of $(f_h + \Delta u_{ih}^\varepsilon)|_K$ times the bubble function (the product of

the barycentric coordinates of K) on element K and to zero elsewhere. The use of inverse inequalities allows to prove that (Cf. Verfürth [16], Proposition 1.5)

$$h_K \|f_h + \Delta u_{ih}^\varepsilon\|_{L^2(K)} \leq C_1 \left(|\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon|_{H^1(K)} + h_K \|f - f_h\|_{L^2(K)} \right). \quad (32)$$

Next, let us estimate $\|[\partial_n u_{ih}^\varepsilon]\|_{L^2(e)}$, for some $e \in \mathcal{E}_K^\Gamma$. There exist two elements $K_1 \in \mathcal{T}_{1h}$ and $K_2 \in \mathcal{T}_{2h}$ that share the edge or face e . We choose $v_i \in X_i$ with support in K_i such that $v_{1|\Gamma} = v_{2|\Gamma} = \varphi$, where φ is a suitable polynomial function that we choose later. Then the interface product vanishes in the residual equation (31), that reduces to

$$((\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon = \sum_{i=1}^2 \left(\int_{\partial K_i} [\partial_{n_K} u_{ih}^\varepsilon] v_i + \int_{K_i} (f_h + \Delta u_{ih}^\varepsilon) v_i + (f - f_h, v_i) \right). \quad (33)$$

The choice of the v_i is standard: If ψ_e denotes the bubble function on e , defined as the product of the barycentric coordinates of e , then

$$\varphi = [\partial_{n_K} u_{ih}^\varepsilon] \psi_e.$$

This is a polynomial function of degree, say r , vanishing on ∂e . Then, the v_i are defined by

$$v_i = \begin{cases} \mathcal{R}_{K_i, e}(\varphi) & \text{in } K_i \\ 0 & \text{in } \bar{\Omega} \setminus (K_1 \cup K_2) \end{cases},$$

where $\mathcal{R}_{K_i, e}$ is a lifting operator from the space of polynomials $P_r(e)$ vanishing on ∂e onto the space of polynomials $P_r(K_i)$ vanishing on $\partial K_i \setminus e$. This operator is constructed from a fixed lifting operator on a reference triangle by affine transformation. It verifies the estimate (Cf. Verfürth [16], Lemma 1.3 and Proposition 1.5)

$$|\mathcal{R}_{K_i, e}(\varphi)|_{H^1(K)} + h_K^{-1} \|\mathcal{R}_{K_i, e}(\varphi)\|_{L^2(K)} \leq C_2 h_e^{-1/2} \|\varphi\|_{L^2(e)}.$$

From this estimate, again standard arguments allow to deduce from (33) that

$$\begin{aligned} h_e^{1/2} \|[\partial_n u_{ih}^\varepsilon]\|_{L^2(e)} &\leq C_3 \left(|\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon|_{H^1(K_1 \cup K_2)} + h_K \|f_h + \Delta u_{ih}^\varepsilon\|_{L^2(K_1 \cup K_2)} \right. \\ &\quad \left. + h_K \|f - f_h\|_{L^2(K_1 \cup K_2)} \right). \end{aligned}$$

The same choice of the v_i allows to estimate $\|[\partial_n u_{ih}^\varepsilon]\|_{L^2(e)}$ for edges faces $e \in \mathcal{E}_K$ not situated on Γ . In this case the support of the v_i does not intersect Γ , so the scalar product interface term in the expression of the residual directly vanishes.

Estimate (29) follows.

- **Penalty error estimator.** The estimate for η_h^P is also obtained by a triangle inequality, as in Theorem 2,

$$\eta_h^P \leq \|u_{1h}^\varepsilon - u\|_{H_{00}^{1/2}(\Gamma)} + \|u_{2h}^\varepsilon - u\|_{H_{00}^{1/2}(\Gamma)} \leq C_4 \sum_{i=1}^2 \|u - u_{ih}^\varepsilon\|_{1, \Omega_i}.$$

Q.E.D.

Note that the estimator η_i^K is not directly bounded in (29) in terms of the total error $|\mathbf{u} - \mathbf{u}_h^\varepsilon|_{1,\omega_K,h}$, but in terms of the discretization error $|\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon|_{1,\omega_K,h}$. This should not be surprising, as η_i^K is the error arising in the discretization step of the penalized problem (P_ε) to obtain $(P_{\varepsilon,h})$.

Let us consider the penalty error E_ε and the discretization error E_h , respectively defined by

$$E_\varepsilon = \sum_{i=1}^2 |u - u_i^\varepsilon|_{1,\Omega_i}, \quad E_h = \sum_{i=1}^2 |u_i^\varepsilon - u_{ih}^\varepsilon|_{1,\Omega_i}.$$

Theorems 5 and 6 set the following bounds (assuming for simplicity that there is no error stemming from the source terms),

$$\begin{aligned} C_1 \left\{ \eta_h^P + \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} (\eta_i^K)^2 \right)^{1/2} \right\} &\leq E_\varepsilon + E_h \\ &\leq C_2 \left\{ \eta_h^P + \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} (\eta_i^K)^2 \right)^{1/2} + \frac{h^\gamma}{\varepsilon} \eta_h^P \right\}. \end{aligned} \quad (34)$$

where C_1 and C_2 are constants independent of h and ε . From these bounds we conclude that our error indicators are quasi-optimal. There is only a lack of optimality due to the h^γ/ε term in the upper bound.

However, under some mild additional conditions we may considerably improve the estimate on the right-hand side of (34).

We introduce the following notation:

- π_{ih}^Γ denotes the orthogonal projection operator from either $L^2(\Gamma)$ or $H_{00}^{1/2}(\Gamma)$ onto W_{ih} with respect to the $[[\cdot, \cdot]]_\Gamma$ scalar product.
- For some $K \in \mathcal{T}_{ih}$, Δ_K denotes the set of all elements K' of \mathcal{T}_{ih} such that $K \cap K' \neq \emptyset$.
- For any face or side e of some element $K \in \mathcal{T}_{ih}$, Δ_e denotes the set of all elements K' of \mathcal{T}_{ih} such that $e \cap K' \neq \emptyset$.
- \mathcal{V}_i denotes the set of Lagrange interpolation nodes contained in $\overline{\Omega}_i$ associated with the space X_{ih} .
- \mathcal{V}_i^0 denotes the subset of the nodes of \mathcal{V}_i located in Ω_i .
- \mathcal{V}_i^Γ denotes the subset of the nodes of \mathcal{V}_i located on Γ .
- \mathcal{T}_{ih}^Γ is the set of elements of \mathcal{T}_{ih} that intersect Γ .

- \mathcal{T}_{ih}^0 is the set of elements of \mathcal{T}_{ih} that do not intersect Γ .

With each a in \mathcal{V}_i , we associate the Lagrange function φ_a in X_{ih} which is equal to 1 in a and to 0 in all $a' \neq a$ in \mathcal{V}_i . Thus, we consider the Clément interpolation operators

$$\mathcal{C}_{ih}^* v = \sum_{a \in \mathcal{V}_i} (\pi_a v)(a) \varphi_a,$$

where π_a is a local L^2 projection operator (for instance, on an element K_a of \mathcal{T}_{ih} containing a). We also consider the modified Clément operator

$$\mathcal{C}_{ih} v = \sum_{a \in \mathcal{V}_i^0} (\pi_a v)(a) \varphi_a + \sum_{a \in \mathcal{V}_i^\Gamma} (\pi_{ih}^\Gamma v)(a) \varphi_a.$$

We state the following result.

Lemma 1 *For any function v_i of $H^1(\Omega_i)$, the following estimates hold*

$$\|v_i - \mathcal{C}_{ih} v_i\|_{L^2(K)}^2 \leq C h_K |v_i|_{H^1(\Delta_K)}, \quad \text{if } K \cap \Gamma = \emptyset; \quad (35)$$

$$\|v_i - \mathcal{C}_{ih} v_i\|_{L^2(e)}^2 \leq C h_e^{1/2} |v_i|_{H^1(\Delta_e)}, \quad \text{if } e \cap \Gamma = \emptyset; \quad (36)$$

$$\left(\sum_{K \in \mathcal{T}_{ih}^\Gamma} h_K^{-2} \|v_i - \mathcal{C}_{ih} v_i\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \leq C (1 + \lambda_h) |v_i|_{H^1(\Omega_i)}; \quad (37)$$

$$\left(\sum_{K \in \mathcal{T}_{ih}^\Gamma} \sum_{e \in \mathcal{E}_K} h_e^{-1} \|v_i - \mathcal{C}_{ih} v_i\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \leq C (1 + \lambda_h) |v_i|_{H^1(\Omega_i)}; \quad (38)$$

where

i) In the case of $L^2(\Gamma)$ projection,

$$\lambda_h = (h_M/h_m)^{1/2}, \quad \text{with } h_M = \max_{K \cap \Gamma \neq \emptyset} h_K, \quad h_m = \min_{K \cap \Gamma \neq \emptyset} h_K;$$

ii) In the case of $H_{00}^{1/2}(\Gamma)$ projection,

$$\lambda_h = (h_M/h_m + h_m^{-1})^{1/2}.$$

We report the proof of this Lemma to an appendix to keep on the main line of our argumentation. Note that in the case of $L^2(\Gamma)$ projection, the coefficient λ_h is bounded if only a finite number of grid refinements are performed, as it is usual. However, in the case of $H_{00}^{1/2}(\Gamma)$ projection, there is a $h_m^{-1/2}$ loss of optimality. Nevertheless, this loss is limited to the elements, sides or faces that intersect the interface Γ .

From the technical point of view, this loss of optimality could be avoided by means of a duality estimate of the error $\|v_i - \pi_{ih}^\Gamma v_i\|_{L^2(\Gamma)}$ in terms of the $H_{00}^{1/2}(\Gamma)$ norm of v_i , that is lacking up to our knowledge.

As a consequence of this Lemma, we deduce our main a posteriori error estimate result,

Theorem 7 Assume that the trace spaces W_{1h} and W_{2h} defined in (19) coincide. Then, the solution of the discrete penalty problem $(P_{h,\varepsilon})$ satisfies the following a posteriori error estimate

$$\sum_{i=1}^2 |u - u_{ih}^\varepsilon|_{1,\Omega_i} \leq C \left\{ \eta_h^P + \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} (\mu_K \eta_i^K)^2 + (\mu_K h_K)^2 \|f - f_h\|_{L^2(K)}^2 \right)^{1/2} \right\}, \quad (39)$$

$$\text{where } \mu_K = \begin{cases} (1 + \lambda_h) & \text{if } K \cap \Gamma \neq \emptyset, \\ 1 & \text{if } K \cap \Gamma = \emptyset. \end{cases}$$

Proof. We adapt the proof of Theorem 5, that we only change from the residual equation (25) between problems (P_ε) and $(P_{h,\varepsilon})$.

Consider $v_i \in X_i$, $i = 1, 2$. We put $v_{ih} = \mathcal{C}_{ih}v_i$ in the residual equation (25). By construction of \mathcal{C}_{ih} , the trace on Γ of v_{ih} is $\pi_{ih}^\Gamma v_i|_\Gamma$ that belongs to W_{ih} . As $W_{1h} = W_{2h}$, then $u_{1h}^\varepsilon - u_{2h}^\varepsilon$ belongs to W_{ih} , and it is orthogonal to $\{(v_1 - v_{1h}) - (v_2 - v_{1h})\}|_\Gamma$ with respect to the $[[\cdot, \cdot]]_\Gamma$ scalar product. Then, the boundary term in the residual equation vanishes and this reduces to

$$\begin{aligned} ((\mathbf{u}^\varepsilon - \mathbf{u}_h^\varepsilon, \mathbf{v}))_\varepsilon &= \sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} \left(\int_K (f_h + \Delta u_{ih}^\varepsilon) (v_i - \mathcal{C}_{ih}v_i) - \sum_{e \in \mathcal{E}_K} \int_e \partial_{n_K} u_{ih}^\varepsilon (v_i - \mathcal{C}_{ih}v_i) \right) \\ &+ \sum_{i=1}^2 (f - f_h, v_i - \mathcal{C}_{ih}v_i) = I + II + III. \end{aligned} \quad (40)$$

We bound the first term of the right-hand side of (40) by

$$\begin{aligned} I &\leq C_1 \sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} \|f_h + \Delta u_{ih}^\varepsilon\|_{L^2(K)}^2 \|v_i - \mathcal{C}_{ih}v_i\|_{L^2(K)} \\ &\leq C_2 \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^\Gamma} (\mu_K \eta_K)^2 \right)^{1/2} \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^\Gamma} (\mu_K h_K)^{-2} \|v_i - \mathcal{C}_{ih}v_i\|_{L^2(K)}^2 \right)^{1/2} \\ &+ C_2 \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^0} \eta_K^2 \right)^{1/2} \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^0} h_K^{-2} \|v_i - \mathcal{C}_{ih}v_i\|_{L^2(K)}^2 \right)^{1/2} \\ &\leq C_3 \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} (\mu_K \eta_K)^2 \right)^{1/2} \left(\sum_{i=1}^2 |v_i|^2 \right)^{1/2}, \end{aligned}$$

where the last bound comes from estimates (35) and (37) of Lemma 1. Similar arguments lead to the bound for the quantity III . The second term of the right-hand side of (40) is

bounded by

$$\begin{aligned}
II &\leq \frac{1}{2} \sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} \sum_{e \in \mathcal{E}_K} \|[\partial_{n_K} u_{ih}^\varepsilon]\|_{L^2(e)} \|v_i - \mathcal{C}_{ih} v_i\|_{L^2(e)} \\
&\leq C_4 \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^\Gamma} (\mu_K \eta_K)^2 \right)^{1/2} \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^\Gamma} \sum_{e \in \mathcal{E}_K} \mu_K^{-2} h_e^{-1} \|v_i - \mathcal{C}_{ih} v_i\|_{L^2(e)}^2 \right)^{1/2} \\
&+ C_4 \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^0} \eta_K^2 \right)^{1/2} \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}^0} \sum_{e \in \mathcal{E}_K} h_e^{-1} \|v_i - \mathcal{C}_{ih} v_i\|_{L^2(e)}^2 \right)^{1/2} \\
&\leq C_5 \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} (\mu_K \eta_K)^2 \right)^{1/2} \left(\sum_{i=1}^2 |v_i|^2 \right)^{1/2}.
\end{aligned}$$

where the last bound comes from estimates (36) and (38) of Lemma 1.

The remaining of the proof is similar to that of Theorem 5.

Q.E.D.

7 Improvement of convergence rate

If $H_{00}^{1/2}(\Gamma)$ penalty is used, the convergence rate of the parallel iteration process (14) is remarkably increased. This is proved thanks to Theorem 4 by replacing the $L^2(\Gamma)$ penalty by the $H_{00}^{1/2}(\Gamma)$ one in the proof of Theorem 2 in [7]. This yields

$$\|u_{1h}^\varepsilon - u_{2h}^\varepsilon\|_{H_{00}^{1/2}(\Gamma)} \leq C (\sqrt{\varepsilon} h^k + \varepsilon),$$

where $C = C(u)$ is a positive constant that only depends on u . Combining this estimate with (20), we deduce

$$\sum_{i=1}^2 |u_h^\varepsilon - u_{ih}^\varepsilon|_{1,\Omega_i} \leq C (\sqrt{\varepsilon} h^k + \varepsilon). \quad (41)$$

Note that

$$\sum_{i=1}^2 |u_i - u_{ih}^\varepsilon|_{1,\Omega_i} \leq \sum_{i=1}^2 |u_i - u_i^\varepsilon|_{1,\Omega_i} + \sum_{i=1}^2 |u_i^\varepsilon - u_h^\varepsilon|_{1,\Omega_i} + \sum_{i=1}^2 |u_h^\varepsilon - u_{ih}^\varepsilon|_{1,\Omega_i},$$

where the second term in the right-hand side is of order h^k . Then, the best choice for ε in (41) is $\varepsilon = O(h^k)$, to achieve an overall $O(h^k)$ accuracy. Other choices improving the accuracy between u_h^ε and u_{ih}^ε would require a larger computational effort without diminishing the discretization error which in any case is of order h^k . This choice $\varepsilon = O(h^k)$ gives

$$\sum_{i=1}^2 |u_h - u_{ih}^\varepsilon|_{1,\Omega_i} \leq C h^k, \quad \text{and then} \quad \sum_{i=1}^2 |u - u_{ih}^\varepsilon|_{1,\Omega_i} \leq C h^k. \quad (42)$$

This allows to state the

Theorem 8 *Assume that hypothesis (15) on the solution u of problem (1) hold. Assume that the sequence $\{(u_{1h}^{n+1,\varepsilon}, u_{2h}^{n+1,\varepsilon})\}_{n \geq 1}$ is computed by the full discretized problem $(P_{\varepsilon,h}^n)$, with $H_{00}^{1/2}(\Gamma)$ penalty and a penalty parameter ε of size $\varepsilon = O(h^k)$. Then, if the number of iterations n is of order $O(|\log h| h^{-k})$, the following error estimate holds,*

$$\sum_{i=1}^2 |u - u_{ih}^{n,\varepsilon}|_{1,\Omega_i} \leq C h^k, \quad (43)$$

where C is a constant independent of ε and h .

Proof. An argument similar to that developed in the proof of Theorem 3 of [7] yields:

$$\sum_{i=1}^2 |u_{ih}^\varepsilon - u_{ih}^{\varepsilon,n+1}|_{1,\Omega_i} \leq \frac{E_0}{\sqrt{\varepsilon}} \frac{1}{(1 + 2C_0\varepsilon)^{n/2}}, \quad (44)$$

where $E_0 = \sum_{i=1}^2 \|u_{ih}^{0,\varepsilon} - u_{ih}^\varepsilon\|_{0,\Gamma}$ and C_0 is a positive constant independent of h and ε .

As $\varepsilon = \mathcal{O}(h^k)$, and the number of iterations n is of order $\mathcal{O}(|\log h| h^{-k})$ estimate (44) yields an accuracy of order h^k between u_{ih}^ε and $u_{ih}^{\varepsilon,n+1}$ in $H^1(\Omega_i)$ norm. It is enough to combine it with estimate (42) to obtain (43).

Q.E.D.

Remark 2 *The number of iterations $\mathcal{O}(|\log h| h^{-k})$ given by this result to obtain an $\mathcal{O}(h^k)$ accuracy with $H_{00}^{1/2}(\Gamma)$ penalty must be compared with a number of iterations of order $\mathcal{O}(|\log h| h^{-2k})$, which corresponds to $L^2(\Gamma)$ penalty.*

8 Application to penalty parameter optimization and grid adaptation

We describe a strategy to optimize the penalty parameter and to adapt the grid, based upon the error indicators developed in the previous sections. Our purpose is to determine the penalty parameter that yields an error of the same size or slightly smaller than the error due to the discretization, for a fixed grid. Indeed, computing a numerical solution with a penalty parameter that yields a penalty error neatly smaller than that of the discretization would require a useless extra computational effort.

Adaptation algorithm

We use the penalty error and discretization error indicators (for both $H_{00}^{1/2}(\Gamma)$ and $L^2(\Gamma)$ penalties) by

$$\eta_h^{n+1,P} = \|u_{1h}^{n+1,\varepsilon} - u_{2h}^{n+1,\varepsilon}\|_{H_{00}^{1/2}(\Gamma)}, \quad (45)$$

$$\eta_h^{n+1,D} = \left(\sum_{i=1}^2 \sum_{K \in \mathcal{T}_{ih}} \left(\eta_i^{n+1,K} \right)^2 \right)^{1/2}, \quad \text{with} \quad (46)$$

$$\eta_i^{n+1,K} = h_K \|f_h + \Delta u_{ih}^{n+1,\varepsilon}\|_{L^2(K)} + \sum_{e \in \mathcal{E}_K} h_e^{1/2} \|[\partial_n u_{ih}^{n+1,\varepsilon}]\|_{L^2(e)}.$$

Initialization: Set three tolerance levels η_1 , η_2 and η_3 and a parameter ρ in $[0, 1]$.

- Set an initial penalty parameter $\varepsilon_e^0 > 0$.
- Build an initial triangulation \mathcal{T}_h^0 such that the corresponding approximation error $h \|f - f_h\|_{L^2(\Omega)}$ is smaller than η_1 .

Iteration: Once solved the full discrete problem $(P_{\varepsilon,h}^n)$ with a current penalty error ε^n , compute the error indicators by (45) and (46) Then, the adaptation strategy is made in three steps:

★ **Step 1: Adaptation of the penalty parameter.**

- If $\eta_h^{n+1,P} > \eta_h^{n+1,D}$, set $\varepsilon^{n+1} = \theta^n \varepsilon^n$, with $\theta^n = \frac{(1 - \rho) \eta_h^{n+1,D}}{\eta_h^{n+1,P}}$;
- If $\eta_h^{n+1,P} < \eta_h^{n+1,D}$, set $\varepsilon^{n+1} = \varepsilon^n$.

★ **Step 2: Adaptation of the mesh.**

We compute the η_i^K and their mean value $\bar{\eta}_h$. For all K such that η_i^K is larger than $\bar{\eta}_h$, we divide K into smaller triangles such that the diameters of these new elements behave like h_i^K multiplied by the ratio $\bar{\eta}_h/\eta_i^K$.

★ **Step 3: Solution of $(P_{\varepsilon,h}^n)$.**

Iterate the Schwarz procedure to obtain a refined solution of $(P_{\varepsilon,h}^n)$, until

$$\|\mathbf{u}_h^{m+1,\varepsilon} - \mathbf{u}_h^{m,\varepsilon}\|_{\mathbf{X}} / \|\mathbf{u}_h^{m,\varepsilon}\|_{\mathbf{X}} < \eta_2, \quad \text{where } \mathbf{u}_h^{m,\varepsilon} = (u_{1h}^{m,\varepsilon}, u_{2h}^{m,\varepsilon}) \quad (47)$$

★ **Stop test:** *If $\max\{\eta_h^{n+1,P}, \eta_h^{n+1,D}\} \leq \eta_3$, stop.*

We refer to [3], Section 5, for first tests of this strategy in a complete different framework. In [3], Step 2 is iterated 4 of 5 times for each iteration of Step 1. In our case we have observed a faster convergence when Step 2 is iterated just once. Also, Step 3 in practice is needed to guarantee the convergence of the overall process, particularly for large grid sizes.

Optimization of penalty parameter:

By skipping Step 2, we only optimize the value of ε for a fixed mesh.

9 Numerical experiments

This section is aimed to two objectives: Comparing the practical efficiency of $H_{00}^{1/2}(\Gamma)$ versus $L^2(\Gamma)$ penalty, and testing the grid adaptation + penalty parameter optimization strategy. In a first step, we test the theoretical predictions of convergence rate and error estimates for the single Schwarz iteration process (14). Next we check the efficiency of the two types of indicators. In a third step, we propose an adaptive strategy aimed to optimize the choice of ε with and without refining the mesh.

Our results are rather surprising: For a fixed grid the $H_{00}^{1/2}(\Gamma)$ penalty is faster than the $L^2(\Gamma)$ penalty when the penalty parameter ε is small enough, as expected from the theoretical predictions. However, when a combined strategy of grid adaptation + penalty parameter optimization is used, the situation turns to be the opposite: The $L^2(\Gamma)$ penalty is faster, and the gain of computing time increases as h decreases. This is mainly due to the cost required by the computation of the $H_{00}^{1/2}(\Gamma)$ norm. In fact, we use a discrete equivalent norm much less costly, but this does not change our conclusions: The $H_{00}^{1/2}(\Gamma)$ penalty is faster for a fixed grid, but the $L^2(\Gamma)$ penalty is much faster when grid adaptation strategies are used.

In this section, we work in dimension $d = 2$ and in the case where the discrete spaces are made of piecewise affine functions ($k_1 = k_2 = 1$). All experiments are performed on the finite element code FreeFEM++, see [10], using a Toshiba laptop computer with an Intel Pentium processor to 1.80 Ghz.

We recall that the scalar product of $H_{00}^{1/2}(\Gamma)$ as defined in (2) is non local, hence difficult to compute. So, in the numerical experiments that follow, we use instead a discrete analogue: If e_j , $1 \leq j \leq J(h)$, denote the edges of \mathcal{E}_h^Γ such that the endpoints of each e_j are x_j and x_{j+1} , and if h_j stands for the length $|x_{j+1} - x_j|$,

$$[[w, v]]_{h\Gamma} = \sum_{j=1}^{J(h)} h_j w^j v^j + \sum_{j=1}^{J(h)} \sum_{k=1, k \neq j}^{J(h)} h_j h_k \frac{(w^j - w^k)(v^j - v^k)}{(x_j - x_k)^2} + \sum_{j=1}^{J(h)} h_j \frac{w^j v^j}{d(x_j, \partial\Gamma)}, \quad (48)$$

where w^j and v^j denotes the values of w and v at the node x_j . Indeed, it is proved in [5], Lemma 5.1, that the norm associated with the two first terms in this summand is equivalent to the norm of $H^{1/2}(\Gamma)$ on the spaces W_{1h} and W_{2h} , with equivalence constants independent of h , in the case of a uniformly regular family of triangulations. However the extension of this result to a regular family of triangulations and also to the $H_{00}^{1/2}(\Gamma)$ norm seems easy to establish.

We consider two test cases:

L-shaped domain. Here, $\Omega =]0, 1[\setminus]\frac{1}{2}, 1]^2$. It is divided into two sub-domains symmetric with respect to the straight line $x = y$. The boundary Γ is the segment of this line between the points $(0, 0)$ and $(1, 1)$. We consider the solution u given by

$$u(x, y) = \sin(3x + y)xy(1 - x)(1 - y)\left(\frac{1}{2} - x\right)\left(\frac{1}{2} - y\right),$$

which satisfies the homogeneous Dirichlet boundary condition, and is not symmetric with respect to the boundary Γ . Figure 3 displays one of the initial triangulations that we have used for this domain and its division into two sub-domains.

Square domain. Here, Ω is the square $]0, 1[^2$, divided into the rectangles $\Omega_1 =]0, 1[\times]0, \frac{1}{2}[$ and $\Omega_2 =]0, 1[\times]\frac{1}{2}, 1[$. We consider the solution u given by

$$u(x, y) = \sin(xy) x(1-x)y(1-y),$$

which satisfies the desired boundary conditions.

We mainly present the results for the L-shape domain test, but we stress that all results that we report are qualitatively similar for the square domain test.

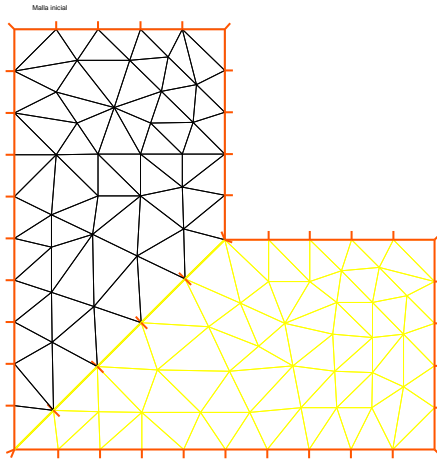


Figure 3: Initial triangulation of L-shaped domain.

9.1 Test of a priori error analysis

We test here the analysis of penalty error and the improvement of convergence rate performed in Sections 3 and 7, respectively.

Theorem 1 and Corollary 3 respectively state penalty error estimates proportional to $\sqrt{\epsilon}$ for $L^2(\Gamma)$ penalty and to ϵ for $H_{00}^{1/2}(\Gamma)$ penalty. We have observed that in practice these estimates are sharp, in the sense that the rate of $H_{00}^{1/2}(\Gamma)$ to $L^2(\Gamma)$ errors is proportional to $\sqrt{\epsilon}$. This probably corresponds to asymptotic behaviors as

$$E_{H^{1/2}}(\epsilon) \simeq C \epsilon, \quad E_{L^2}(\epsilon) \simeq C' \sqrt{\epsilon} \quad \text{as } \epsilon \rightarrow 0. \quad (49)$$

Here we denote

$$E_{H^{1/2}}(\epsilon) = \|\mathbf{u} - \mathbf{u}^\epsilon\|_{\mathbf{X}}, \quad E_{L^2}(\epsilon) = \|\mathbf{w} - \mathbf{w}^\epsilon\|_{\mathbf{X}},$$

where \mathbf{u}^ϵ and \mathbf{w}^ϵ respectively are the solutions of (P_ϵ) for $H_{00}^{1/2}(\Gamma)$ and $L^2(\Gamma)$ penalty.

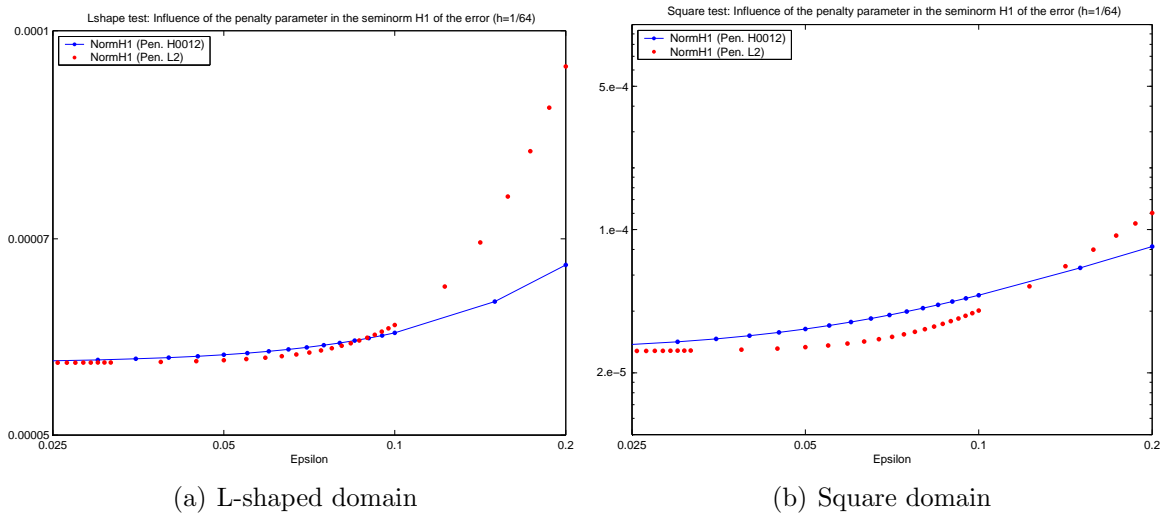


Figure 4: Comparison of penalty errors due to $H_{00}^{1/2}(\Gamma)$ penalty (for penalty parameter ϵ) and to $L^2(\Gamma)$ (for penalty parameter ϵ^2).

Figure 4 displays approximations of the computed errors $E_{H^{1/2}}(\epsilon)$ and $E_{L^2}(\epsilon^2)$ for the L-shape and square domains with grids of size $h = 1/64$, as a function of ϵ . We approximate \mathbf{u} by its Lagrange interpolate $\mathcal{I}_h \mathbf{u}$ on space V_h . We also approximate for instance \mathbf{u}^ϵ by a close-to-convergence $\mathbf{u}_h^{m+1,\epsilon}$ in the iterative Schwarz process (14). Specifically, we have taken $\mathbf{u}^\epsilon = \mathbf{u}_h^{m+1,\epsilon}$ when the quantity

$$\|\mathbf{u}_h^{m+1,\epsilon} - \mathbf{u}_h^{m,\epsilon}\|_{\mathbf{X}} / \|\mathbf{u}_h^{m,\epsilon}\|_{\mathbf{X}} < \eta = 10^{-7}. \quad (50)$$

These errors indeed appear to be proportional (in fact, are almost equal) for ϵ small enough, in agreement with relations (49). We have observed the same qualitative behavior for several grids of both test domains. We just present one of these results for brevity. It is striking that these errors appear to be asymptotically equal instead of simply proportional, but we have obtained the same behavior in all our tests. We infer that the constants C and C' in (49) are equal for some reason.

Also, as stated in Remark 2, the $H_{00}^{1/2}(\Gamma)$ penalty provides a better convergence rate than the $L^2(\Gamma)$ penalty in the Schwarz iterative process (14). To test this point, we have run both penalty procedures on a same grid, and compared the number of iterations needed for the iterative processes to converge.

Figure 5 displays the number of iterations required by the $H_{00}^{1/2}(\Gamma)$ and $L^2(\Gamma)$ penalty procedures for the L-shape and square domains test, as a function of ϵ . Here again, we approximate \mathbf{u}^ϵ by the $\mathbf{u}_\epsilon^{m+1}$ verifying the stop test (50). We observe a linear dependence of this number of iterations with respect to ϵ for $L^2(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$ penalty, with a better slope for the $H_{00}^{1/2}(\Gamma)$ one. We have observed exactly the same qualitative behavior for

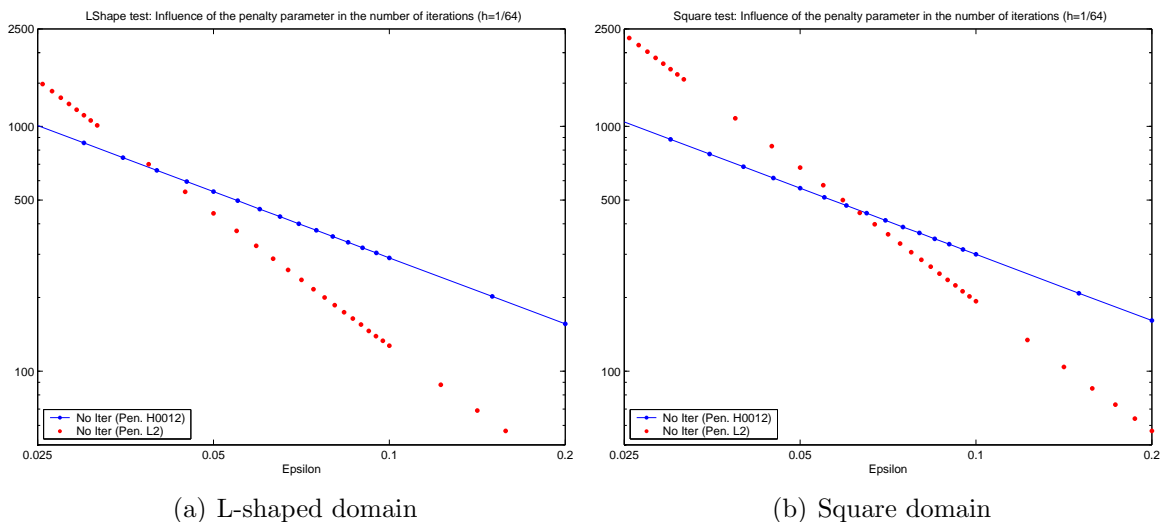


Figure 5: Comparison of number of iterations required for convergence of Schwarz iterative processes.

other grid sizes h . We conclude that for ϵ small enough the $H_{00}^{1/2}(\Gamma)$ penalty procedure is faster than the $L^2(\Gamma)$ one.

9.2 The efficiency of the indicators

To test the efficiency of the indicators, we first work with a uniform structured mesh for the square domain, with h equal to $1/32$ and ϵ decreasing from 5 to 5×10^{-3} . Figure 6(a) presents, in logarithmic scales and as a function of ϵ ,

- the curves of the errors

$$\sum_{i=1}^2 |\mathcal{I}_h \mathbf{u} - \mathbf{u}_{ih}^\epsilon|_{H^1(\Omega_i)} \quad \text{and} \quad \sum_{i=1}^2 \|\mathcal{I}_h \mathbf{u} - \mathbf{u}_{ih}^\epsilon\|_{L^2(\Omega_i)},$$

in dotted dashed line and dotted line, respectively.

- the indicator η_h^P in plain line and the Hilbertian sum η_h^D of all indicators η_i^K in dashed line.

The error decreases with ϵ until the penalization error (which behaves like $c\epsilon$) is of the same order as the finite element discretization error, for a critical value $\epsilon_c \simeq 0.2$. For ϵ sufficiently larger than ϵ_c , the error indicator curve η_h^P is parallel to the curves of the errors. In contrast, the quantity η_h^D turns out to be fully independent of ϵ .

Next, we fix ϵ equal to 0.01 and we still use uniform (unstructured) meshes. Figure 6(b) presents the same curves as previously, now in semi-logarithmic scales and for h decreasing from 0.25 to 1.5×10^{-2} . There also the error decreases with h until it reaches

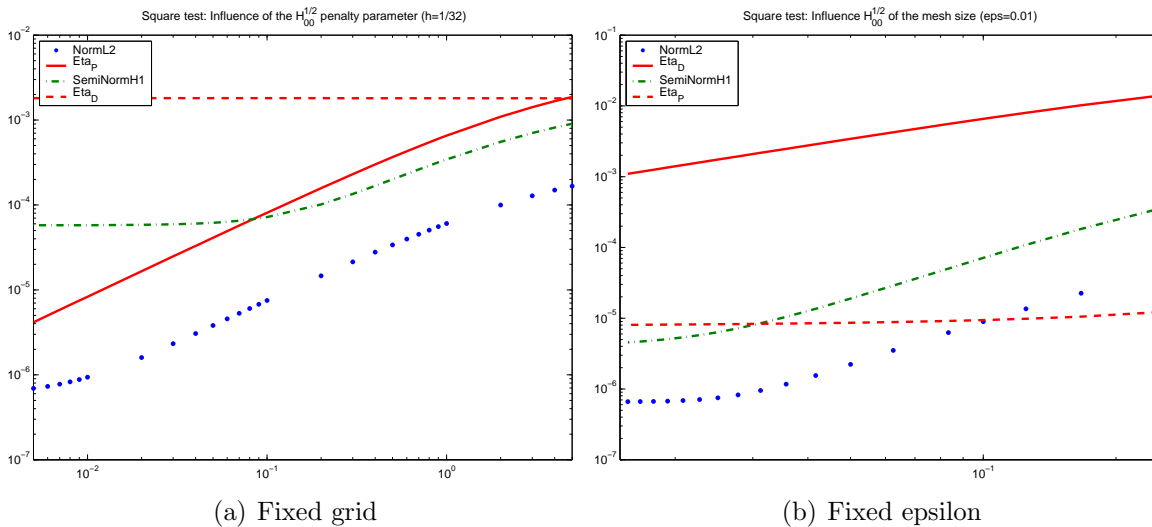


Figure 6: Influence of penalty parameter and grid size in error indicators: Square domain

the penalization error. Also, the curves of the indicators and of the errors are almost parallel for h sufficiently larger than a critical h_c .

Figure 7 represents the same curves for the L-shaped domain, where the behavior of the indicators is qualitatively the same. Let us also remark that the behavior of indicators corresponding to L^2 penalty is also similar, for both the square and the L-shaped domain.

We can infer from these results that the error estimate (39) is probably suboptimal, in the sense that the parameter μ_K appears in practice to be of order one. If this parameter had an asymptotic growth to $+\infty$ as $h \rightarrow 0$, we would not have obtained the behavior of the indicators exhibited in Figures 6(b) and 7(b).

These results resemble in a striking way to those obtained in [3], Section 5, for a completely different type of penalization. In this last paper, the Stokes problem is solved by non-mixed finite elements, and penalty is used to treat the incompressibility restriction. This similar behavior is probably due to the common formulation of both discretizations as mixed problems, in the way indicated in Section 4.

9.3 Testing the adaptivity strategy and penalty parameter optimization

The adaptivity strategy provides a large saving of computational effort for both $H_{00}^{1/2}(\Gamma)$ and $L^2(\Gamma)$ penalties. The most surprising result is that the $L^2(\Gamma)$ penalty is computationally faster than the $H_{00}^{1/2}(\Gamma)$ one.

We present the results obtained with the L-shape domain test. Figures 8(a), 8(b) and 9 respectively display an intermediate solution, the final solution and the grids associated to these solutions. These results are obtained starting from the initial triangulation pre-

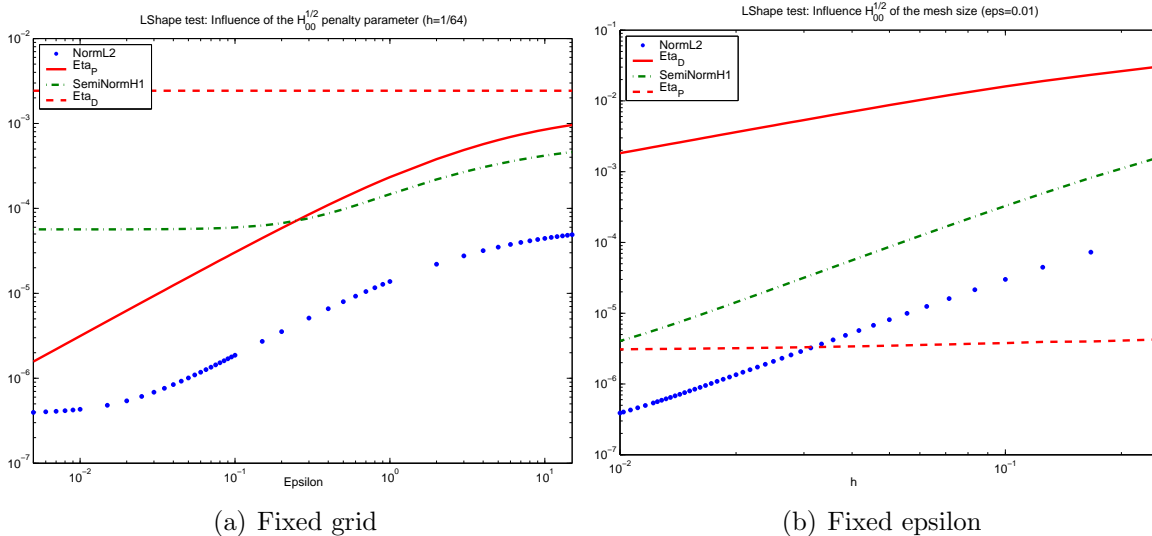


Figure 7: Influence of penalty parameter and grid size in error indicators: L-shaped domain

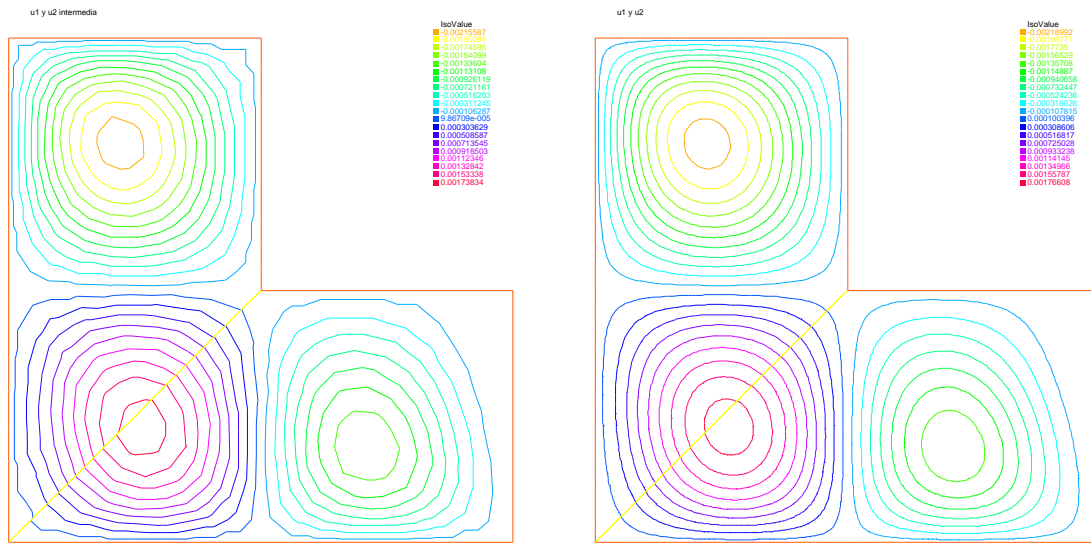
sented in Figure 3, with $H_{00}^{1/2}(\Gamma)$ penalty. We observe that the continuity and even the differentiability of iso-lines of the solution across the interface Γ is progressively improved in the adaptive process. The intermediate solution could be acceptable for some applications not requiring high precision. The high accuracy of the final solution is rather costly, as the density of triangles of the adapted grid is quite large. This density is larger in the zones of strong gradients of the solution, as could be expected, but is not particularly high in the neighborhood of Γ .

Combined grid adaptation-penalty parameter optimization

In order to compare the $H_{00}^{1/2}(\Gamma)$ and $L^2(\Gamma)$ penalties, we have run the L-shape domain test with the same set of data for both: The initial grid displayed in Figure 3, $\varepsilon^0 = 0.1$, $\eta_1 = 0.1$, $\eta_2 = 5 \times 10^{-2}$, $\rho = 0.05$. To obtain an accurate solution, we have stopped the process when

$$\|\mathbf{u}_h^{n,\varepsilon} - \mathcal{I}_h \mathbf{u}\|_{\mathbf{X}} / \|\mathcal{I}_h \mathbf{u}\|_{\mathbf{X}} < 3 \times 10^{-3}. \quad (51)$$

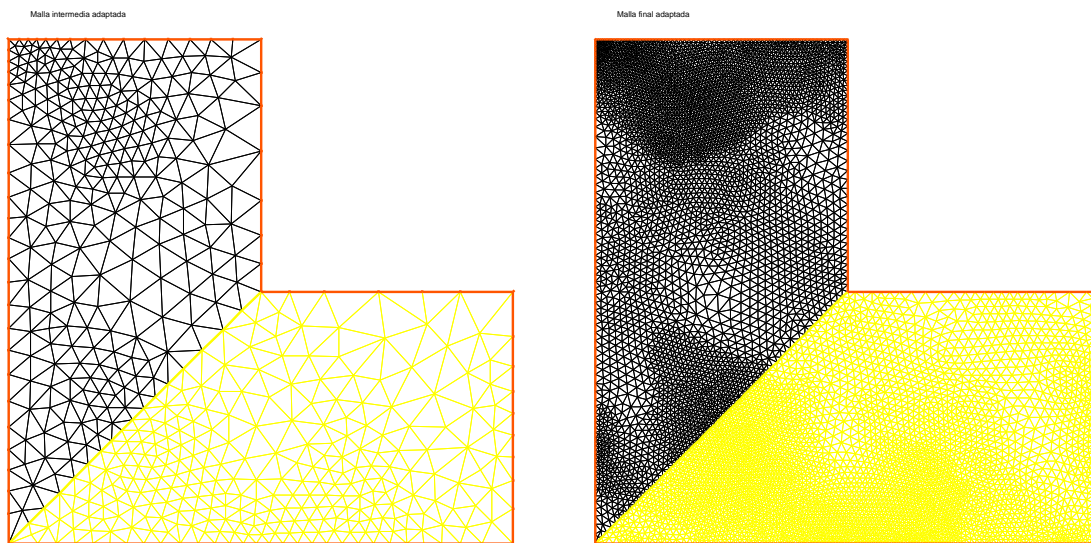
Table 1 compares the results obtained. The number of iterations stands for the total number of applications of the Schwarz iteration procedure along the adaptation process. We observe that the values of the estimators η_h^P and η_h^D and the number of triangles of the final grids are quite close for both penalties. Also, that the optimal ε for $L^2(\Gamma)$ penalty is close to the square of the optimal ε for $H_{00}^{1/2}(\Gamma)$ penalty. We may observe a large saving of CPU time for $L^2(\Gamma)$ penalty. This is due to two effects: The faster convergence of the $L^2(\Gamma)$ penalty, that requires much less iterations, and the cost due to the computation of the (even discrete) $H_{00}^{1/2}(\Gamma)$ norm.



(a) Intermediate solution

(b) Final solution

Figure 8: Snapshots of iterated solutions in adaptivity strategy



(a) Intermediate grid

(b) Final grid

Figure 9: Triangulations of iterated solutions in adaptivity strategy

Comparison of $H_{00}^{1/2}(\Gamma)$ versus $L^2(\Gamma)$ penalties.						
Penalty	ε_{opt}	η_h^P	η_h^D	Triangles	Iterations	CPU time
$H_{00}^{1/2}(\Gamma)$	0.0387	0.00180	0.00136	8148	471	287.7s
$L^2(\Gamma)$	0.0035	0.00180	0.00134	8230	56	41.7s

Table 1.

Also, we have tested the improvement of computational time due to the adaptive strategy with respect to the single iterating Schwarz procedure. To do it we have run the Schwarz procedure, using the optimized grids and the optimal penalty parameters. We have obtained the results presented in Table 2. We remark for both penalties a large gain of CPU time due to the adaptivity strategy, with a ratio larger than 4 in both cases.

Schwarz iteration procedure with optimized data		
Penalty	ε_{opt}	CPU time
$H_{00}^{1/2}(\Gamma)$	0.0387	1158.6s
$L^2(\Gamma)$	0.0035	266.5s

Table 2.

Penalty parameter optimization

We have performed a final test, where only the penalty parameter is optimized for a fixed grid. As stated in Section 8, this is achieved by skipping the grid adaptation step (Step 2) in the adaptation algorithm. We have used a non-structured grid with 3092 triangles, corresponding to $h = 1/64$. The procedure stops when

$$\eta_h^{n+1,P} \leq \eta_h^D,$$

as in this case the discretization error is fixed, and the purpose is to decrease the penalty error below this value.

Table 3 displays the results obtained. Again, the optimal ε for $L^2(\Gamma)$ penalty is close to the square of the optimal one for $H_{00}^{1/2}(\Gamma)$ penalty. Both penalties take a similar number of iterations, but the $H_{00}^{1/2}(\Gamma)$ one is neatly more costly.

All the results presented in this subsection are qualitatively the same for the square test. We do not present them for brevity.

Penalty parameter optimization for fixed grid.			
Penalty	ε_{opt}	Iterations	CPU time
$H_{00}^{1/2}(\Gamma)$	0.0503	165	126.3s
$L^2(\Gamma)$	0.0029	157	99.0s

Table 3.

Appendix

This appendix is devoted to the rather technical proof of Lemma 1.

Proof of (35) and (36). If some element $K \in \mathcal{T}_{ih}$ does not intersect Γ , then $\mathcal{C}_{ih}^* v_i = \mathcal{C}_{ih} v_i$ on K . Also, if some edge e of some element of \mathcal{T}_{ih} does not intersect Γ , then $\mathcal{C}_{ih}^* v_i = \mathcal{C}_{ih} v_i$ on e . Thus, estimates (35) and (36) are standard properties of Clément operator (Cf. [4], Théorème IX.3.7).

Proof of (37) and (38). We write the triangle inequality

$$\|v - \mathcal{C}_{ih} v_i\|_{L^2(K)} \leq \|v_i - \mathcal{C}_{ih}^* v_i\|_{L^2(K)} + \|\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i\|_{L^2(K)}.$$

The estimates (Cf. [4], Corollaire IX.3.10)

$$\|v_i - \mathcal{C}_{ih}^* v_i\|_{L^2(K)} \leq C h_K |v_i|_{H^1(\Delta_K)}, \quad \|v_i - \mathcal{C}_{ih}^* v_i\|_{L^2(e)} \leq C h_e^{1/2} |v_i|_{H^1(\Delta_K)}, \quad \forall e \in \mathcal{E}_K$$

imply

$$\left(\sum_{K \in \mathcal{T}_{ih}^\Gamma} h_K^{-2} \|v_i - \mathcal{C}_{ih}^* v_i\|_{L^2(K)}^2 + \sum_{K \in \mathcal{T}_{ih}^\Gamma} \sum_{e \in \mathcal{E}_K} h_e^{-1} \|v_i - \mathcal{C}_{ih}^* v_i\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \leq C |v_i|_{H^1(\Omega_i)}. \quad (52)$$

We consider now the quantity $\|\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i\|_{L^2(K)}$, and note that the support of $\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i$ is included in the union of the elements K of \mathcal{T}_{ih} that intersect Γ . On such a K , we have

$$\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i = \sum_{a \in \mathcal{V}^\Gamma \cap K} ((\pi_a v_i)(a) - (\pi_{ih}^\Gamma v_i)(a)) \varphi_a.$$

Denote by e_a a side or a face containing a and included in Γ . Then,

$$\|\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i\|_{L^2(K)} \leq c h_K^{\frac{d}{2}} \max_{a \in \mathcal{V}^\Gamma \cap K} \|\pi_a v_i - \pi_{ih}^\Gamma v_i\|_{L^\infty(e_a)}.$$

We next use the inverse inequality

$$\|\pi_a v_i - \pi_{ih}^\Gamma v_i\|_{L^\infty(e_a)} \leq C h_K^{-\frac{d-1}{2}} \|\pi_a v_i - \pi_{ih}^\Gamma v_i\|_{L^2(e_a)}.$$

We deduce

$$\sum_{K \in \mathcal{T}_{ih}^\Gamma} h_K^{-2} \|\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i\|_{L^2(K)}^2 \leq C h_m^{-1} \left(\sum_{a \in \mathcal{V}^\Gamma} \|v_i - \pi_a v_i\|_{L^2(e_a)}^2 + \|v_i - \pi_{ih}^\Gamma v_i\|_{L^2(\Gamma)}^2 \right). \quad (53)$$

Consider now a side or face e of some element K of \mathcal{T}_{ih}^Γ . In this case,

$$\|\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i\|_{L^2(e)} \leq C h_e^{\frac{d-1}{2}} \max_{a \in \mathcal{V}^\Gamma \cap K} \|\pi_a v_i - \pi_{ih}^\Gamma v_i\|_{L^\infty(e_a)}.$$

We now deduce

$$\sum_{K \in \mathcal{T}_{ih}^\Gamma} \sum_{e \in \mathcal{E}_K} h_e^{-1} \|\mathcal{C}_{ih}^* v_i - \mathcal{C}_{ih} v_i\|_{L^2(e)}^2 \leq C h_m^{-1} \left(\sum_{a \in \mathcal{V}^\Gamma} \|v_i - \pi_a v_i\|_{L^2(e_a)}^2 + \|v_i - \pi_{ih}^\Gamma v_i\|_{L^2(\Gamma)}^2 \right) \quad (54)$$

Thus, the proofs of (37) and (38) are reduced to estimate the term

$$\sum_{a \in \mathcal{V}^\Gamma} \|v_i - \pi_a v_i\|_{L^2(e_a)}^2 + \|v_i - \pi_{ih}^\Gamma v_i\|_{L^2(\Gamma)}^2. \quad (55)$$

To estimate the first summand of the right-hand side, we recall [4], Corollaire IX.3.9,

$$\|v_i - \pi_a v_i\|_{L^2(e_a)} \leq C h_{K_a}^{1/2} \|v_i\|_{H^1(\Delta_{K_a})},$$

where K_a is some element of the triangulation \mathcal{T}_{ih} such that a is a side or a face of K_a . Then,

$$\sum_{a \in \mathcal{V}^\Gamma} \|v_i - \pi_a v_i\|_{L^2(e_a)}^2 \leq C h_M |v|_{H^1(\Omega_i)}^2. \quad (56)$$

The estimate of the second summand in the right-hand side of (55) is made as follows.

i) Assume that π_{ih}^Γ is the $L^2(\Gamma)$ projector. As π_{ih}^Γ is a projection,

$$\|v_i - \pi_{ih}^\Gamma v_i\|_{L^2(\Gamma)}^2 \leq \|v_i - \mathcal{C}_{ih}^* v_i\|_{L^2(\Gamma)}^2 \leq C h_M |v_i|_{H^1(\Omega_i)}^2. \quad (57)$$

ii) Assume now that π_{ih}^Γ is the $H_{00}^{1/2}(\Gamma)$ projector. Also, using that π_{ih}^Γ is a projection,

$$\|v_i - \pi_{ih}^\Gamma v_i\|_{L^2(\Gamma)} \leq \|v_i - \pi_{ih}^\Gamma v_i\|_{H_{00}^{1/2}(\Gamma)} \leq \|v_i\|_{H_{00}^{1/2}(\Gamma)} \leq C |v_i|_{H^1(\Omega_i)} \quad (58)$$

Combining (52), (53) and (54) with (56), (57) and (58), we deduce (37) and (38).
Q.E.D.

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