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EXTRAPOLATION FROM A_∞ WEIGHTS AND APPLICATIONS

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ABSTRACT. We generalize the A_p extrapolation theorem of Rubio de Francia to A_∞ weights in the context of Muckenhoupt bases. Our result has several important features. First, it can be used to prove weak endpoint inequalities starting from strong-type inequalities, something which is impossible using the classical result. Second, it provides an alternative to the technique of good- λ inequalities for proving L^p norm inequalities relating operators. Third, it yields vector-valued inequalities without having to use the theory of Banach space valued operators. We give a number of applications to maximal functions, singular integrals, potential operators, commutators, multilinear Calderón-Zygmund operators, and multiparameter fractional integrals. In particular, we give new proofs, which completely avoid the good- λ inequalities, of Coifman's inequality relating singular integrals and the maximal operator, of the Fefferman-Stein inequality relating the maximal operator and the sharp maximal operator, and the Muckenhoupt-Wheeden inequality relating the fractional integral operator and the fractional maximal operator.

1. INTRODUCTION

In harmonic analysis, there are a number of important inequalities of the form

$$(1.1) \quad \int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |Sf(x)|^p w(x) dx,$$

$$(1.2) \quad \|Tf\|_{L^{p,\infty}(w)} \leq C \|Sf\|_{L^{p,\infty}(w)}$$

where, typically, T is an operator with some degree of singularity (e.g., a singular integral operator), S is an operator which is, in principle, easier to handle (e.g., a maximal operator), and w is in some class of weights. The standard technique for proving such results is the so-called good- λ inequality, which was introduced by Burkholder and Gundy [BG]. These inequalities compare the measure of the level sets of S and T : for every $\lambda > 0$ and $\epsilon > 0$ small,

$$(1.3) \quad w(\{y \in \mathbb{R}^n : |Tf(y)| > 2\lambda, |Sf(y)| \leq \lambda\epsilon\}) \leq C\epsilon w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}),$$

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where the weight w is assumed to be in the Muckenhoupt class A_∞ . Given inequality (1.3), it is straightforward to prove (1.1) and (1.2) for any p , $0 < p < \infty$. A number of good- λ inequalities are known for specific pairs of operators: singular integrals and the Hardy-Littlewood maximal operator [Coi, CF]; fractional integrals and the fractional maximal operator [MW]; square functions and the maximal operator [GW, CWW, Wil]; the maximal operator and the sharp maximal operator [FS2].

In this paper we describe a different method for proving inequalities of the form (1.1) and (1.2). We show that if either holds for a fixed value $p = p_0$, then it holds for all values of p , $0 < p < \infty$. To put our results in context, recall the A_p extrapolation theorem: If the operator T is bounded on $L^{p_0}(w)$ for some p_0 , $1 < p_0 < \infty$, and every $w \in A_{p_0}$, then for every p , $1 < p < \infty$, T is bounded on $L^p(w)$, $w \in A_p$. This was first proved by Rubio de Francia [Rub]. See also [Gar, Jaw, Duo].

Our basic result extends this theorem from A_p weights to A_∞ weights, to pairs of operators, and to the range $0 < p < \infty$. No assumptions on the operators are needed (e.g., linearity, sublinearity, etc.): we consider any operators that are defined on some class of nice functions. Indeed, as we will show, we can formulate our results in terms of arbitrary pairs of functions and omit any reference to operators. As a consequence we get, for example, the vector-valued inequalities below almost automatically.

Theorem 1.1. *Given two operators S and T , suppose that for some $p = p_0$, $0 < p_0 < \infty$, inequality (1.1) holds for all f in the common domain of S and T such that the lefthand side is finite, and for all weights $w \in A_\infty$ with constant C depending only on the A_∞ constant of w . Then for all p , $0 < p < \infty$, and $w \in A_\infty$, (1.1), (1.2) hold. Further, for $0 < p, q < \infty$,*

$$(1.4) \quad \left\| \left(\sum_j |Tf_j|^q \right)^{1/q} \right\|_{L^p(w)} \leq C \left\| \left(\sum_j |Sf_j|^q \right)^{1/q} \right\|_{L^p(w)},$$

$$(1.5) \quad \left\| \left(\sum_j |Tf_j|^q \right)^{1/q} \right\|_{L^{p,\infty}(w)} \leq C \left\| \left(\sum_j |Sf_j|^q \right)^{1/q} \right\|_{L^{p,\infty}(w)}.$$

The proof of Theorem 1.1 has a corollary which is very useful in some applications.

Corollary 1.2. *The conclusions of Theorem 1.1 hold if the initial hypothesis is replaced by the following: there exists p_0 , $0 < p_0 < \infty$, such that for every $0 < q < p_0$ and every $w \in A_1$,*

$$(1.6) \quad \int_{\mathbb{R}^n} |Tf(x)|^q w(x) dx \leq C \int_{\mathbb{R}^n} |Sf(x)|^q w(x) dx.$$

Theorem 1.1 has two important features. First, since the standard A_p extrapolation theorem is restricted to the range $p > 1$, it cannot be used to prove weak endpoint estimates (e.g., weak $(1, 1)$ inequalities), and other techniques must be used. Indeed, such results may be false. For example, if M is the Hardy-Littlewood maximal

operator, then $T = M \circ M = M^2$ is bounded on $L^p(w)$ for all $p > 1$ and $w \in A_p$, but it is not of weak type $(1, 1)$. On the other hand, since Theorem 1.1 allows us to extrapolate to $p = 1$ in, say, (1.2), we can use a strong-type inequality relating T and S and an endpoint estimate for S to prove a weak endpoint inequality for T .

Second, Theorem 1.1 yields the vector-valued inequalities (1.4) and (1.5). This is not surprising since there is a close connection between such inequalities and extrapolation. However, it is **not** clear how to derive vector-valued estimates from the good- λ inequality (1.3). Our result allows us to prove such inequalities without using the theory of Banach space valued operators developed in [BCP, RRT].

As a consequence of our extrapolation techniques we can prove a number of new theorems and give new proofs of known results. We describe most of these in Section 3 below. Here we give three applications which illustrate the power of our results.

Theorem 1.3. *Let T be a Calderón-Zygmund operator with a standard kernel, I_α , $0 < \alpha < n$, the fractional integral operator, M the Hardy-Littlewood maximal operator, $M^\#$ the sharp maximal operator, and M_α the fractional maximal operator. (See [GR, Duo] for precise definitions.) Then for all p , $0 < p < \infty$, and all $w \in A_\infty$,*

$$(1.7) \quad \int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} Mf(x)^p w(x) dx,$$

$$(1.8) \quad \int_{\mathbb{R}^n} Mf(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} M^\#f(x)^p w(x) dx$$

$$(1.9) \quad \int_{\mathbb{R}^n} |I_\alpha f(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} M_\alpha f(x)^p w(x) dx.$$

Further, each inequality holds with the $L^p(w)$ norm replaced by the $L^{p,\infty}(w)$ norm on each side.

These estimates play a fundamental role in the study of weighted norm inequalities. Inequality (1.7) is due to Coifman [Coi, CF], inequality (1.8) to Fefferman and Stein [FS2], and (1.9) to Muckenhoupt and Wheeden [MW]; all three were proved using a good- λ inequality. Here we will give a proof which avoids good- λ inequalities and instead uses Corollary 1.2 and the following two results.

Theorem 1.4. *Let $f \geq 0$ be such that its level sets $\{x : f(x) > \lambda\}$ have finite measure for all $\lambda > 0$. Then for all weights w (i.e., $w \geq 0$ and locally integrable),*

$$\int_{\mathbb{R}^n} f(x) w(x) dx \leq C \int_{\mathbb{R}^n} M^\#f(x) Mw(x) dx.$$

Theorem 1.4 was recently proved by Lerner [Ler] using a clever Calderón-Zygmund decomposition of w . His result is actually more general: he replaced the sharp maximal operator by the so-called local sharp maximal operator.

Proposition 1.5. *Given T , I_α , M and $M^\#$ as defined above, for all q , $0 < q < 1$, and all $f \in C_0$, there exists a constant $C = C(q, n, \alpha)$*

$$M_q^\#(Tf)(x) \leq C Mf(x), \quad M_q^\#(Mf)(x) \leq C M^\#f(x), \quad M^\#(I_\alpha f)(x) \leq C M_\alpha f(x),$$

where $M_q^\#(g)(x) = M^\#(|g|^q)(x)^{1/q}$.

The first estimate is due to Álvarez and Pérez [AP] and the third to Adams [Ada]; we prove the second in Section 6.1 below. We also note that Lerner [Ler] has obtained similar inequalities with the local sharp maximal function replacing $M_q^\#$.

We can now prove (1.7): given f and $w \in A_1$ (i.e., $Mw(x) \leq Cw(x)$ a.e.), for $0 < q < 1$, by Theorem 1.4 and Proposition 1.5 we have

$$\int_{\mathbb{R}^n} |Tf(x)|^q w(x) dx \leq C \int_{\mathbb{R}^n} M_q^\#(Tf)(x)^q w(x) dx \leq C \int_{\mathbb{R}^n} Mf(x)^q w(x) dx.$$

By Corollary 1.2 we get the desired result. The proofs of (1.8) and (1.9) are identical.

Note that (1.4) and (1.5) hold for T and M , for M and $M^\#$, and for I_α and M_α , and these vector-valued estimates cannot be deduced directly from the good- λ inequalities. Also, with the techniques discussed in Section 3.2 we can give a different proof of (1.9) which does not use the sharp maximal operator.

2. MAIN RESULTS

Theorem 1.1 is a special case of one of our main results. In order to state them, we first introduce some notation. Throughout, w will denote a weight, i.e., a non-negative, locally integrable function. Given a measurable set E , let $w(E) = \int_E w(x) dx$. Given a weight w , for $0 < p < \infty$, let $L^p(w)$ be the weighted Lebesgue space with respect to the measure $w(x) dx$. Similarly, we define the Lorentz spaces $L^{p,q}(w)$, $0 < p, q \leq \infty$.

To define the classes of weights we will consider, we first introduce the concept of a basis \mathcal{B} and the maximal operator $M_{\mathcal{B}}$ defined with respect to \mathcal{B} . For complete information, see [Jaw, Pe1]. A basis \mathcal{B} is a collection of open sets $B \subset \mathbb{R}^n$. A weight w is associated with the basis \mathcal{B} , if $w(B) < \infty$ for every $B \in \mathcal{B}$. Given a basis \mathcal{B} , the corresponding maximal operator is defined by

$$M_{\mathcal{B}}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \quad \text{if } x \in \bigcup_{B \in \mathcal{B}} B,$$

and $M_{\mathcal{B}}f(x) = 0$ otherwise. A weight w associated with \mathcal{B} is in the Muckenhoupt class $A_{p,\mathcal{B}}$, $1 < p < \infty$, if there exists a constant C such that for every $B \in \mathcal{B}$,

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} \leq C.$$

When $p = 1$, w belongs to $A_{1,\mathcal{B}}$ if $M_{\mathcal{B}}w(x) \leq Cw(x)$ for almost every $x \in \mathbb{R}^n$. Clearly, if $1 \leq q \leq p < \infty$, then $A_{q,\mathcal{B}} \subset A_{p,\mathcal{B}}$. Further, from the definitions we get the following factorization property: if $w_1, w_2 \in A_{1,\mathcal{B}}$, then $w_1w_2^{1-p} \in A_{p,\mathcal{B}}$. Finally, we let $A_{\infty,\mathcal{B}} = \cup_{p \geq 1} A_{p,\mathcal{B}}$.

We are going to restrict our attention to the following class of bases: A basis \mathcal{B} is a **Muckenhoupt basis** if for each p , $1 < p < \infty$, and for every $w \in A_{p,\mathcal{B}}$, the maximal operator $M_{\mathcal{B}}$ is bounded on $L^p(w)$, that is,

$$(2.1) \quad \int_{\mathbb{R}^n} M_{\mathcal{B}}f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx.$$

These bases were introduced and characterized in [Pe1] (see Theorem 4.1 below). Three immediate examples of Muckenhoupt bases are \mathcal{D} , the set of dyadic cubes in \mathbb{R}^n ; \mathcal{Q} , the set of all cubes in \mathbb{R}^n whose sides are parallel to the coordinate axes, and \mathcal{R} , the set of all rectangles (i.e., parallelepipeds) in \mathbb{R}^n whose sides are parallel to the coordinate axes. (See [Duo].) One advantage of these bases is that by using them we avoid any direct appeal to the underlying geometry: the relevant properties are derived from (2.1), and we do not use covering lemmas of any sort.

Finally, before stating our main results we reconsider the role of the operators S and T in Theorem 1.1. In our proofs, the properties of S and T play no role: all we use is that we have a pair of functions (Tf, Sf) such that (1.1) holds for some value of p with a constant independent of the pair. Therefore, we will eliminate the superfluous operators and concentrate on pairs of functions. Besides simplifying notation, this clarifies the underlying ideas. In particular, this approach is a natural one for considering vector-valued inequalities.

Hereafter, \mathcal{F} will denote a family of ordered pairs of non-negative, measurable functions (f, g) . If we say that for some p , $0 < p < \infty$, and $w \in A_{\infty,\mathcal{B}}$,

$$\int_{\mathbb{R}^n} f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x) dx, \quad (f, g) \in \mathcal{F},$$

we mean that this inequality holds for any $(f, g) \in \mathcal{F}$ such that the lefthand side is finite, and that the constant C depends only upon p and the $A_{\infty,\mathcal{B}}$ constant of w . We will make similar abbreviated statements involving Lorentz spaces. For vector-valued inequalities we will consider sequences $\{(f_j, g_j)\}_j$, where each pair (f_j, g_j) is contained in \mathcal{F} .

To apply our results to the more familiar setting of Theorem 1.1, we will use the following classes: given a pair of operators T and S , let $\mathcal{F}(T, S)$ denote the family of pairs of functions $(|Tf|, |Sf|)$, where f lies in the common domain of T and S , and the lefthand side of the corresponding inequality is finite. To achieve this, the function f may be restricted in some other way, e.g. $f \in C_0^\infty$. In this case we may indicate this by writing $\mathcal{F}(|Tf|, |Sf| : f \in C_0^\infty)$. In Section 3 below, we will give specific examples of such classes.

We can now state our main results.

Theorem 2.1. *Given a family \mathcal{F} , suppose that for some p_0 , $0 < p_0 < \infty$, and for every weight $w \in A_{\infty, \mathcal{B}}$,*

$$(2.2) \quad \int_{\mathbb{R}^n} f(x)^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w(x) dx, \quad (f, g) \in \mathcal{F}.$$

Then:

For all $0 < p < \infty$ and $w \in A_{\infty, \mathcal{B}}$,

$$(2.3) \quad \int_{\mathbb{R}^n} f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x) dx, \quad (f, g) \in \mathcal{F}.$$

For all $0 < p < \infty$, $0 < s \leq \infty$, and $w \in A_{\infty, \mathcal{B}}$,

$$(2.4) \quad \|f\|_{L^{p,s}(w)} \leq C \|g\|_{L^{p,s}(w)}, \quad (f, g) \in \mathcal{F}.$$

For all $0 < p, q < \infty$ and $w \in A_{\infty, \mathcal{B}}$,

$$(2.5) \quad \left\| \left(\sum_j (f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \leq C \left\| \left(\sum_j (g_j)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}.$$

For all $0 < p, q < \infty$, $0 < s \leq \infty$, and $w \in A_{\infty, \mathcal{B}}$,

$$(2.6) \quad \left\| \left(\sum_j (f_j)^q \right)^{\frac{1}{q}} \right\|_{L^{p,s}(w)} \leq C \left\| \left(\sum_j (g_j)^q \right)^{\frac{1}{q}} \right\|_{L^{p,s}(w)}, \quad \{(f_j, g_j)\}_j \subset \mathcal{F}.$$

The proof of Theorem 2.1 is in Section 4 below. Theorem 1.1 is an immediate consequence of Theorem 2.1: replace \mathcal{F} with $\mathcal{F}(T, S)$ and \mathcal{B} with \mathcal{Q} , the basis of cubes in \mathbb{R}^n . The proof of Corollary 1.2 is part of the proof of Theorem 2.1.

Our second main result shows that we can also extrapolate from an initial Lorentz space inequality.

Theorem 2.2. *Given a family \mathcal{F} , suppose that for some p_0 , $0 < p_0 < \infty$, and for every weight $w \in A_{\infty, \mathcal{B}}$,*

$$(2.7) \quad \|f\|_{L^{p_0, \infty}(w)} \leq C \|g\|_{L^{p_0, \infty}(w)}, \quad (f, g) \in \mathcal{F}.$$

Then for all $0 < p < \infty$ and $w \in A_{\infty, \mathcal{B}}$,

$$(2.8) \quad \|f\|_{L^{p, \infty}(w)} \leq C \|g\|_{L^{p, \infty}(w)}, \quad (f, g) \in \mathcal{F}.$$

The proof of Theorem 2.2 is in Section 5 below. We do not know if it is possible to use extrapolation to prove strong (p, p) inequalities beginning with (2.7) and such a result would be of interest. Additionally, it is not clear how to derive vector-valued inequalities from (2.7) (as we did in Theorem 2.1) without passing through the corresponding strong type estimates.

We conclude this section by pointing out that our results extend to spaces of homogeneous type. As is clear from the proofs below, one needs that M or its weighted variant M_w are bounded on weighted Lebesgue and Lorentz spaces. These estimates hold because of the properties of the Muckenhoupt weights and by interpolation. The precise statements of the analogs of Theorems 2.1 and 2.2 are left to the reader. One may apply these results to fractional integrals and their corresponding maximal operators in this setting, see [PW] for more details.

3. APPLICATIONS

In this section we apply our extrapolation theorems to prove weighted norm inequalities and vector-valued estimates. Some of these results are already known, but we believe our approach has advantages over the proofs in the literature. Other results are new; for these we defer the proofs until Section 6. For consistency with the literature, throughout this section we will denote $A_{p,\mathcal{Q}}$ by A_p , and $A_{p,\mathcal{D}}$ by A_p^d .

3.1. Singular integral operators: Let T be a Calderón-Zygmund operator (see [GR, Duo] for a precise definition). From Theorem 1.3 we have that for all $0 < p < \infty$ and $w \in A_\infty$, $\|Tf\|_{L^p(w)} \leq C \|Mf\|_{L^p(w)}$. We stress that our proof, unlike the original, does not use a good- λ inequality. Furthermore, by applying Theorem 2.1 to the family $\mathcal{F}(|Tf|, Mf : f \in C_0^\infty)$ we get the vector-valued inequalities (2.5) and (2.6) which are new. If we combine them with the vector-valued estimates for the maximal function (see [FS1] for the unweighted case, and [AJ] for the weighted case) we obtain a new proof of the following inequalities: if $1 < q < \infty$, then for every $w \in A_1$,

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{L^{1,\infty}(w)} \leq C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^1(w)},$$

and for every $1 < p < \infty$ and every $w \in A_p$,

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \leq C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}.$$

Similar estimates hold for the maximal singular integrals T_* since (1.7) holds with T replaced by T_* . Unlike the proofs of these results in [BCP, RRT], our proof does not involve the theory of Banach space valued operators.

We note two other applications. We can prove analogous results for commutators of Calderón-Zygmund operators with a B.M.O. function. Details are left to the reader; see [Pe4]. Also, Theorem 2.2 can be applied to the problem of the existence of singular integrals with certain properties; see [MPT].

3.2. Potential operators: Let $\Phi \geq 0$ be a locally integrable function for which there exist constants $\delta, c > 0$, and $0 \leq \epsilon < 1$, such that for every $k \in \mathbb{Z}$,

$$(3.1) \quad \sup_{2^k < |x| \leq 2^{k+1}} \Phi(x) \leq \frac{c}{2^{kn}} \int_{\delta(1-\epsilon)2^k < |y| \leq \delta(1+\epsilon)2^{k+1}} \Phi(y) dy.$$

Define the potential operator T_Φ and the maximal operator $M_{\tilde{\Phi}}$, introduced by Kerman and Sawyer [KS], by

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy, \quad M_{\tilde{\Phi}} f(x) = \sup_{x \in Q} \frac{\tilde{\Phi}(\ell(Q))}{|Q|} \int_Q f(y) dy,$$

where $\tilde{\Phi}(t) = \int_{|z| \leq t} \Phi(z) dz$. Functions which satisfy (3.1) include Φ which are radial and monotonic; more generally we can take Φ which satisfy $\Phi(y) \leq c \Phi(x)$ for $|y|/2 \leq |x| \leq 2|y|$. If $\Phi(x) = |x|^{\alpha-n}$, then the operators T_Φ and $M_{\tilde{\Phi}}$ are I_α and M_α , the classical fractional operators.

The discretization method for T_Φ developed in [Pe3] and the ideas in [Pe2] employed for the fractional integrals can be combined to prove the following result. In the special case of fractional integrals, it is a special case of (1.9). The proof, which is similar to that of Proposition 3.2 below, is left to the reader.

Proposition 3.1. *Let Φ satisfy condition (3.1). Then for every weight $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_\Phi f(x)| w(x) dx \leq C \int_{\mathbb{R}^n} M_{\tilde{\Phi}} f(x) w(x) dx.$$

Given Proposition 3.1 we can apply Theorem 2.1 to the family of pairs of functions $\mathcal{F}(|T_\Phi f|, M_{\tilde{\Phi}} f : f \in C_0^\infty)$; the resulting inequalities are new.

3.3. Commutators of fractional integrals: Given $0 < \alpha < n$ and $b \in \text{B.M.O.}$, define the commutator $[I_\alpha, b]$ by

$$[b, I_\alpha] f(x) = b(x) I_\alpha f(x) - I_\alpha (b f)(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} f(y) dy.$$

These commutators were introduced by Chanillo [Cha], who proved that if $1/p - 1/q = \alpha/n$ then $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. A weighted version of this result was first proved in [ST] using a variant of the A_p extrapolation theorem and Banach space valued operators. Another proof was given in [CUF] which used the good- λ inequality relating the maximal function and the sharp maximal function.

We give another proof by showing that these commutators are controlled by fractional Orlicz maximal operators. Then the weighted norm inequalities for such maximal operators in [CUF] yield weighted estimates for the commutator.

Let $\Phi(t) = t \log(e+t)$, and define the fractional Orlicz maximal operator

$$M_{\Phi, \alpha} f(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \|f\|_{\Phi, Q}.$$

For notation and basic facts about Orlicz spaces see Section 6.2 below.

Proposition 3.2. *Given $0 < \alpha < n$, $b \in \text{B.M.O.}$ and $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |[b, I_\alpha]f(x)| w(x) dx \leq C \int_{\mathbb{R}^n} M_{\Phi, \alpha} f(x) w(x) dx.$$

The proof of Proposition 3.2 is in Section 6.2 below and uses a discretization of the commutator. The fact that the exponent is 1 plays an important role in the proof. Given this result, we can apply Theorem 2.1 to the family $\mathcal{F}([b, I_\alpha]f, M_{\Phi, \alpha})$; the resulting vector-valued inequalities are new.

Our proof can be extended to commutators of generalized fractional integrals. Let L be a linear operator on $L^2(\mathbb{R}^n)$ such that $(-L)$ generates an analytic semigroup e^{-tL} . We suppose that this semigroup has a kernel $p_t(x, y)$ which satisfies

$$(3.2) \quad |p_t(x, y)| \leq \frac{C}{t^{\frac{n}{2}}} e^{-c \frac{|x-y|^2}{t}}, \quad \text{for all } x, y \in \mathbb{R}^n; t > 0.$$

For $0 < \alpha < n$, use the subordination formula to define generalized fractional integrals, $L^{-\frac{\alpha}{2}}f(x)$. If $L = -\Delta$, then $L^{-\frac{\alpha}{2}}$ is the classical fractional integral I_α . It follows from (3.2) that the kernel K_α of $L^{-\frac{\alpha}{2}}$ satisfies $|K_\alpha(x, y)| \leq C|x-y|^{\alpha-n}$. In particular, $|L^{-\frac{\alpha}{2}}f(x)| \leq C I_\alpha(|f|)(x)$, so estimates for I_α yield similar results for $L^{-\frac{\alpha}{2}}$. Further,

$$|[b, L^{-\frac{\alpha}{2}}]f(x)| = \left| \int_{\mathbb{R}^n} (b(x) - b(y)) K_\alpha(x, y) f(y) dy \right| \leq C \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y) dy,$$

and the proof of Proposition 3.2 (see (6.2) below) shows that in Proposition 3.2 we can replace $[b, I_\alpha]$ by the operator defined by the righthand side of this inequality. As a consequence we get an analog of Proposition 3.2 and by extrapolation we get weighted norm and vector-valued inequalities for $[b, L^{-\frac{\alpha}{2}}]$.

These commutators were previously studied in [Yan]. There, only unweighted estimates were obtained by using a new sharp maximal function, $M_L^\#$, adapted to the semigroup, which was introduced in [Mar].

3.4. Multilinear Calderón-Zygmund operators: Let T be a multilinear Calderón-Zygmund operator, that is, T is an m -linear operator such that $T : L^{q_1} \times \cdots \times L^{q_m} \rightarrow L^q$, where $1 < q_1, \dots, q_m < \infty$, $0 < q < \infty$ and

$$(3.3) \quad \frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}.$$

The operator T is associated with a Calderón-Zygmund kernel K in the usual way:

$$T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,$$

whenever f_1, \dots, f_m are in C_0^∞ and $x \notin \bigcap_{j=1}^m \text{supp } f_j$. We assume that K satisfies the appropriate decay and smoothness conditions (see [GT1, GT2] for complete details). Such an operator T is bounded on any other product of Lebesgue spaces with exponents $1 < q_1, \dots, q_m < \infty$, $0 < q < \infty$ satisfying (3.3). Further, it also satisfies weak endpoint estimates when some of the q_i 's are equal to one. There are also weighted norm inequalities for multilinear Calderón-Zygmund operators; these were first proved in [GT2] using a good- λ inequality, and later in [PT] using the sharp maximal function. They showed that for $0 < p < \infty$ and for all $w \in A_\infty$,

$$(3.4) \quad \|T(f_1, \dots, f_m)\|_{L^p(w)} \leq C \left\| \prod_{j=1}^m Mf_j \right\|_{L^p(w)}.$$

The same inequality also holds with T replaced by T_* . Beginning with these inequalities, we can apply Theorem 2.1 to the families

$$\mathcal{F}\left(T(f_1, \dots, f_m), \prod_{j=1}^m Mf_j\right), \quad \mathcal{F}\left(T_*(f_1, \dots, f_m), \prod_{j=1}^m Mf_j\right),$$

where $f_1, \dots, f_m \in C_0^\infty$. The scalar estimate (2.4) just (3.4), But the vector-valued inequalities (2.5) and (2.6) are new and immediately yield the following result by applying Hölder's inequality and the norm inequalities for the maximal operator.

Corollary 3.3. *Let T be a multilinear Calderón-Zygmund operator, $1 \leq p_1, \dots, p_m < \infty$, $1 < q_1, \dots, q_m < \infty$ and $0 < p, q < \infty$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}, \quad \frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}.$$

If $1 < p_1, \dots, p_m < \infty$ and $w \in A_{p_1} \cap \dots \cap A_{p_m}$, then

$$(3.5) \quad \left\| \left(\sum_k |T(f_1^k, \dots, f_m^k)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \leq C \prod_{j=1}^m \left\| \left(\sum_k |f_j^k|^{q_j} \right)^{\frac{1}{q_j}} \right\|_{L^{p_j}(w)}.$$

If at least one $p_j = 1$ and $w \in A_1$, then

$$(3.6) \quad \left\| \left(\sum_k |T(f_1^k, \dots, f_m^k)|^q \right)^{\frac{1}{q}} \right\|_{L^{p,\infty}(w)} \leq C \prod_{j=1}^m \left\| \left(\sum_k |f_j^k|^{q_j} \right)^{\frac{1}{q_j}} \right\|_{L^{p_j}(w)}.$$

Moreover, inequalities (3.5) and (3.6) hold with T_ in place T .*

We stress that Corollary 3.3 is proved without using a theory of Banach space valued, multi-linear operators. It is possible that such a theory, analogous to that in [BCP, RRT], could be developed, and such a theory would yield these results. But by extrapolation we avoid this (much longer) route. The strong (p, p) inequality (3.5) was proved independently in [GM]. Their proof used a different extrapolation

technique, one closer in spirit to the original result of Rubio de Francia. That method does not provide end-point estimates. However, in [GM] a multilinear version of the Marcinkiewicz-Zygmund theorem is proved, and this yields a weaker version of (3.6).

3.5. Multiparameter fractional integral operators: We define a multiparameter version of the fractional integral operator of order 1: For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, let

$$Tf(x, y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \frac{f(\bar{x}, \bar{y})}{|x - \bar{x}|^{n-1} |y - \bar{y}|^{m-1}} d\bar{y} d\bar{x}.$$

To motivate this definition, recall that if $f \in C^1$ with compact support, then $|f(x)| \leq C I_1(|\nabla f|)(x)$, where I_1 is the classical fractional integral operator of order 1. An analog of this result holds for T , but now with the crossed second-order derivatives.

Theorem 3.4. *Let $f \in C^2(\mathbb{R}^n \times \mathbb{R}^m)$ be a compactly supported function. Then for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,*

$$|f(x, y)| \leq CT(|\nabla_x \nabla_y f|)(x, y),$$

$$\text{where } \nabla_x \nabla_y f = \left(\frac{\partial^2 f}{\partial x_i \partial y_j} \right)_{i,j} \text{ and } |\nabla_x \nabla_y f| = \left(\sum_{i,j} \left| \frac{\partial^2 f}{\partial x_i \partial y_j} \right|^2 \right)^{\frac{1}{2}}.$$

We can prove an analog of Proposition 3.1. Given $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and a function $f \in L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^m)$, define the multi-parameter fractional maximal operators

$$M_1^{(1)} f(x, y) = \sup_{Q_n \ni x} \frac{1}{|Q_n|^{1-\frac{1}{n}}} \int_{Q_n} |f(\bar{x}, y)| d\bar{x}$$

$$M_1^{(2)} f(x, y) = \sup_{Q_m \ni y} \frac{1}{|Q_m|^{1-\frac{1}{m}}} \int_{Q_m} |f(x, \bar{y})| d\bar{y}.$$

A simple estimate shows that $M_1^{(1)} \circ M_1^{(2)} f(x, y) \leq CT(|f|)(x, y)$, and similarly with the order of composition reversed. As in the one-variable case, the reverse inequality does not hold pointwise, but does hold in the sense of weighted L^p norms.

We define a basis in $\mathbb{R}^n \times \mathbb{R}^m$: $\mathcal{B} = \{Q_n \times Q_m : Q_n \subset \mathbb{R}^n, Q_m \subset \mathbb{R}^m\}$, where Q_n and Q_m are cubes with their sides parallel to the coordinate axes. For $1 \leq p \leq \infty$, we will denote $A_{p,\mathcal{B}}$ by $A_p(\mathbb{R}^n \times \mathbb{R}^m)$. For $1 < p < \infty$, $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$ if

$$\left(\frac{1}{|Q_n||Q_m|} \int_{Q_n} \int_{Q_m} w(x, y) dy dx \right) \left(\frac{1}{|Q_n||Q_m|} \int_{Q_n} \int_{Q_m} w(x, y)^{1-p'} dy dx \right)^{p-1} \leq C.$$

When $p = 1$, $w \in A_1(\mathbb{R}^n \times \mathbb{R}^m)$ if $M_{\mathcal{B}} w(x, y) \leq C w(x, y)$, for a.e. $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, where $M_{\mathcal{B}}$ is the strong maximal operator,

$$M_{\mathcal{B}} f(x, y) = \sup_{Q_n \times Q_m \ni (x,y)} \frac{1}{|Q_n||Q_m|} \int_{Q_n} \int_{Q_m} |f(x, y)| dy dx.$$

A key property of these weights is that if $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$, then for almost every $y \in \mathbb{R}^m$ the weight $w^y = w(\cdot, y)$ is an A_p weight in \mathbb{R}^n , and its A_p constant is independent of y . The same is also true for the other variable. (See [GR, Duo].) It then follows by Fubini's theorem that \mathcal{B} is a Muckenhoupt basis.

Proposition 3.5. *For every weight $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m)$,*

$$(3.7) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |Tf(x, y)| w(x, y) dy dx \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} M_1^{(1)} \circ M_1^{(2)} f(x, y) w(x, y) dy dx.$$

and the same holds if the order of $M_1^{(1)}$ and $M_1^{(2)}$ is reversed on the righthand side.

Given inequality (3.7), we can apply Theorem 2.1 to the family of functions $\mathcal{F}(|Tf|, M_1^{(1)} \circ M_1^{(2)} f : f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m))$. Then inequality (2.3), combined with the weighted norm inequalities for the fractional maximal operator (see [MW]), yield weighted norm inequalities for T . Details are left to the reader.

4. PROOF OF THEOREM 2.1

At the heart of our proof is the algorithm of Rubio de Francia for generating A_1 weights with certain properties. (See [GR].) In the special case of the basis \mathcal{Q} of cubes in \mathbb{R}^n , it is possible to avoid this algorithm and actually give a much simpler proof modeled on the proof of the A_p extrapolation theorem in [Duo]. Details are left to the reader. (We want to thank the referee for reminding us of this fact.)

The proof has been broken up into sections corresponding to the four enumerated equations in the statement. We begin with a lemma. Given a weight w , define the weighted maximal function with respect to the basis \mathcal{B} by

$$M_{\mathcal{B}, w} f(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f(y)| w(y) dy \quad \text{if } x \in \bigcup_{B \in \mathcal{B}} B,$$

and $M_{\mathcal{B}, w} f(x) = 0$ otherwise. The follow result is proved in [Pe1].

Theorem 4.1. *Let \mathcal{B} be a Muckenhoupt basis. For every $1 < p < \infty$ and for every $w \in A_{\infty, \mathcal{B}}$, the operator $M_{\mathcal{B}, w}$ is bounded on $L^p(w)$.*

4.1. Proof of Inequality (2.3). We prove this inequality in two steps.

Step 1: We first show that the hypothesis (2.2) is equivalent to a family of weighted inequalities with $A_{1, \mathcal{B}}$ weights.

Theorem 4.2. *Hypothesis (2.2) of Theorem 2.1 is equivalent to the following: for all $0 < q < p_0$, $w \in A_{1, \mathcal{B}}$, and $(f, g) \in \mathcal{F}$,*

$$(4.1) \quad \int_{\mathbb{R}^n} f(x)^q w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q w(x) dx.$$

Note that Corollary 1.2 follows immediately from Theorem 4.2 applied to the family $\mathcal{F}(|Tf|, |Sf|)$ with \mathcal{B} equal to \mathcal{Q} .

Proof of Theorem 4.2. We will prove that (2.2) implies (4.1). This will suffice to complete the proof of inequality (2.3), which in turn immediately implies the converse. Fix $(f, g) \in \mathcal{F}$. We can assume that $g \in L^q(w)$ and $\|f\|_{L^q(w)} > 0$, for otherwise there is nothing to prove. Let $s = p_0/q > 1$. Since $w \in A_{1,\mathcal{B}} \subset A_{s',\mathcal{B}}$, $M_{\mathcal{B}}$ is bounded on $L^{s'}(w)$. Denote the operator norm of $M_{\mathcal{B}}$ on $L^{s'}(w)$ by $\|M_{\mathcal{B}}\|_{L^{s'}(w)}$. For $h \in L^{s'}(w)$, $h \geq 0$, we use the algorithm of Rubio de Francia to define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M_{\mathcal{B}}^k h(x)}{2^k \|M_{\mathcal{B}}\|_{L^{s'}(w)}^k},$$

where $M_{\mathcal{B}}^k$ is the operator $M_{\mathcal{B}}$ iterated k times if $k \geq 1$, and for $k = 0$ is just the identity. From the definition of \mathcal{R} it is immediate that:

- (a) $h(x) \leq \mathcal{R}h(x)$.
- (b) $\|\mathcal{R}h\|_{L^{s'}(w)} \leq 2 \|h\|_{L^{s'}(w)}$.
- (c) $M_{\mathcal{B}}(\mathcal{R}h)(x) \leq 2 \|M_{\mathcal{B}}\|_{L^{s'}(w)} \mathcal{R}h(x)$, so $\mathcal{R}h \in A_{1,\mathcal{B}}$ with constant independent of h .

Since f and g belong to $L^q(w)$ and have positive norms, by (b) we have that

$$H(x) = \mathcal{R} \left(\left(\frac{f}{\|f\|_{L^q(w)}} \right)^{\frac{q}{s'}} + \left(\frac{g}{\|g\|_{L^q(w)}} \right)^{\frac{q}{s'}} \right) (x) \in L^{s'}(w).$$

By (a),

$$(4.2) \quad \left(\frac{f(x)}{\|f\|_{L^q(w)}} \right)^{\frac{q}{s'}} \leq H(x), \quad \left(\frac{g(x)}{\|g\|_{L^q(w)}} \right)^{\frac{q}{s'}} \leq H(x),$$

so $H(x) > 0$ whenever $f(x) > 0$. Further, H is finite a.e. on the set where $w > 0$ because $H \in L^{s'}(w)$. Therefore, by Hölder's inequality,

$$\int_{\mathbb{R}^n} f(x)^q w(x) dx \leq \left(\int_{\mathbb{R}^n} f(x)^{p_0} H(x)^{-s} w(x) dx \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^n} H(x)^{s'} w(x) dx \right)^{\frac{1}{s'}} = I \cdot II.$$

We first estimate I . Since $w, H \in A_{1,\mathcal{B}}$ (by (c)) and $1 + s > 1$, by the factorization property of $A_{p,\mathcal{B}}$ weights, $w H^{-s} = w H^{1-(1+s)} \in A_{1+s,\mathcal{B}} \subset A_{\infty,\mathcal{B}}$. We want to apply (2.2); to do so we must check that I is finite. But by (4.2),

$$\int_{\mathbb{R}^n} f(x)^{p_0} H(x)^{-s} w(x) dx \leq \|f\|_{L^q(w)}^{\frac{q s}{s'}} \int_{\mathbb{R}^n} f(x)^{p_0 - \frac{q s}{s'}} w(x) dx = \|f\|_{L^q(w)}^{q s} < \infty.$$

We can now use (2.2); if we apply (4.2) as before we get

$$I \leq C \left(\int_{\mathbb{R}^n} g(x)^{p_0} H(x)^{-s} w(x) dx \right)^{\frac{1}{s}} \leq C \int_{\mathbb{R}^n} g(x)^q w(x) dx.$$

To estimate II , a straight-forward computation with (b) yields $II \leq 4$. Combining the estimates for I and II gives us the desired inequality. \square

Step 2: We now show that for all $0 < p < \infty$ and for every $w \in A_{\infty, \mathcal{B}}$, (2.3) holds. Fix $0 < p < \infty$ and $w \in A_{\infty, \mathcal{B}}$. Assume that $(f, g) \in \mathcal{F}$ with $f \in L^p(w)$, $g \in L^p(w)$. Since $A_{p_1, \mathcal{B}} \subset A_{p_2, \mathcal{B}}$ if $1 \leq p_1 \leq p_2$, there exists $0 < q < \min\{p, p_0\}$ such that $w \in A_{p/q, \mathcal{B}}$. Let $r = p/q > 1$. Since $w \in A_{r, \mathcal{B}}$, $w^{1-r'} \in A_{r', \mathcal{B}}$. Given $h \in L^{r'}(w^{1-r'})$, $h \geq 0$, we apply the Rubio de Francia algorithm to define

$$(4.3) \quad \mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M_{\mathcal{B}}^k h(x)}{2^k \|M_{\mathcal{B}}\|_{L^{r'}(w^{1-r'})}^k},$$

where $\|M_{\mathcal{B}}\|_{L^{r'}(w^{1-r'})}$ is the operator norm of $M_{\mathcal{B}}$ on $L^{r'}(w^{1-r'})$; this is finite since $w^{1-r'} \in A_{r', \mathcal{B}}$. Again, we have

- (a) $h(x) \leq \mathcal{R}h(x)$.
- (b) $\|\mathcal{R}h\|_{L^{r'}(w^{1-r'})} \leq 2 \|h\|_{L^{r'}(w^{1-r'})}$.
- (c) $M_{\mathcal{B}}(\mathcal{R}h)(x) \leq 2 \|M_{\mathcal{B}}\|_{L^{r'}(w^{1-r'})} \mathcal{R}h(x)$, so $\mathcal{R}h \in A_{1, \mathcal{B}}$ with constant independent of h .

We now argue as follows: by duality,

$$\|f\|_{L^p(w)}^q = \|f^q\|_{L^r(w)} = \sup_h \int_{\mathbb{R}^n} f(x)^q h(x) w(x) dx,$$

where the supremum is taken over all $h \in L^{r'}(w)$ with $h \geq 0$ and $\|h\|_{L^{r'}(w)} = 1$. Fix such a function h . Then $hw \in L^{r'}(w^{1-r'})$ and $\|hw\|_{L^{r'}(w^{1-r'})} = \|h\|_{L^{r'}(w)} = 1$. By (c), $\mathcal{R}(hw) \in A_{1, \mathcal{B}}$. Hence, by (a) and Theorem 4.2,

$$\int_{\mathbb{R}^n} f(x)^q h(x) w(x) dx \leq \int_{\mathbb{R}^n} f(x)^q \mathcal{R}(hw)(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}(hw)(x) dx,$$

provided the middle term is finite. But, since $f \in L^p(w)$ and $\mathcal{R}(hw) \in L^{r'}(w^{1-r'})$, $0 < w < \infty$ almost everywhere in the set where $f^q \mathcal{R}(hw) > 0$; thus,

$$\int_{\mathbb{R}^n} f(x)^q \mathcal{R}(hw)(x) dx \leq \|f\|_{L^p(w)}^q \|\mathcal{R}(hw)\|_{L^{r'}(w^{1-r'})} \leq 2 \|f\|_{L^p(w)}^q < \infty.$$

The same argument also holds for g instead of f . Therefore,

$$\int_{\mathbb{R}^n} f(x)^q h(x) w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}(hw)(x) dx \leq C \|g\|_{L^p(w)}^q.$$

The desired inequality follows at once. \square

4.2. Proof of Inequality (2.4). We first prove a lemma. Given two weights u and v , we say that $u \in A_{1,\mathcal{B}}(v)$ if for almost every x , $M_{\mathcal{B},v}u(x) \leq C u(x)$.

Lemma 4.3. *If $w_1 \in A_{p,\mathcal{B}}$, $1 \leq p \leq \infty$, and $w_2 \in A_{1,\mathcal{B}}(w_1)$, then $w_1 w_2 \in A_{p,\mathcal{B}}$.*

Proof. First observe that if $w_2 \in A_{1,\mathcal{B}}(w_1)$, then for $B \in \mathcal{B}$,

$$\frac{1}{|B|} \int_B w_1(x) w_2(x) dx = \frac{w_1(B)}{|B|} \frac{1}{w_1(B)} \int_B w_2(x) w_1(x) dx \leq C \frac{w_1(B)}{|B|} \operatorname{ess\,inf}_B w_2.$$

The desired conclusion follows if we substitute this into the definition of $A_{p,\mathcal{B}}$. \square

Proof of (2.4). Fix $p, s, w \in A_{\infty,\mathcal{B}}$ and $(f, g) \in \mathcal{F}$ with $f, g \in L^{p,s}(w)$. Fix $0 < q < \min\{p, s\}$ and set $r = p/q > 1$, $\tilde{r} = s/q > 1$. (If $s = \infty$, take $0 < q < p$ and $\tilde{r} = \infty$.) Then

$$\|f\|_{L^{p,s}(w)}^q = \|f^q\|_{L^{r,\tilde{r}}(w)} = \sup_h \int_{\mathbb{R}^n} f(x)^q h(x) w(x) dx,$$

where the supremum is taken over all $h \in L^{r',\tilde{r}'}(w)$ with $h \geq 0$ and $\|h\|_{L^{r',\tilde{r}'}(w)} = 1$. Fix such a function h . Apply the Rubio de Francia algorithm to define

$$\mathcal{R}_w h(x) = \sum_{k=0}^{\infty} \frac{M_{\mathcal{B},w}^k h(x)}{2^k \|M_{\mathcal{B},w}\|_\star},$$

where $\|M_{\mathcal{B},w}\|_\star$ is the operator norm of $M_{\mathcal{B},w}$ on $L^{r',\tilde{r}'}(w)$ endowed with a norm equivalent to $\|\cdot\|_{L^{r',\tilde{r}'}(w)}$. Since $M_{\mathcal{B},w}$ is bounded on $L^p(w)$, by Marcinkiewicz interpolation in the scale of Lorentz spaces (see [BS, p. 225]) it is bounded on $L^{r',\tilde{r}'}(w)$. Therefore,

- (a) $h(x) \leq \mathcal{R}_w h(x)$.
- (b) $\|\mathcal{R}_w h\|_{L^{r',\tilde{r}'}(w)} \leq C \|h\|_{L^{r',\tilde{r}'}(w)} = C$.
- (c) $M_{\mathcal{B},w}(\mathcal{R}_w h)(x) \leq 2 \|M_{\mathcal{B},w}\|_\star \mathcal{R}_w h(x)$, so $\mathcal{R}_w h \in A_{1,\mathcal{B}}(w)$ with constant independent of h .

By Lemma 4.3, $\mathcal{R}_w h w \in A_{\infty,\mathcal{B}}$. As we showed above, (2.3) holds with exponent q and the $A_{\infty,\mathcal{B}}$ weight $\mathcal{R}_w h w$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)^q h(x) w(x) dx &\leq \int_{\mathbb{R}^n} f(x)^q \mathcal{R}_w h(x) w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^q \mathcal{R}_w h(x) w(x) dx \\ &\leq C \|g^q\|_{L^{r,\tilde{r}}(w)} \|\mathcal{R}_w h\|_{L^{r',\tilde{r}'}(w)} \leq C \|g\|_{L^{p,s}(w)}^q, \end{aligned}$$

provided the second integral is finite. But, this is the case since

$$\int_{\mathbb{R}^n} f(x)^q \mathcal{R}_w h(x) w(x) dx \leq \|f^q\|_{L^{r,\tilde{r}}(w)} \|\mathcal{R}_w h\|_{L^{r',\tilde{r}'}(w)} \leq C \|f\|_{L^{p,s}(w)}^q < \infty.$$

The desired inequality now follows at once. \square

4.3. Proof of Inequalities (2.5) and (2.6). Fix $0 < q < \infty$. By the monotone convergence theorem it is enough to prove the vector-valued inequalities only for finite sums. Fix $N \geq 1$ and define

$$f_q(x) = \left(\sum_{j=1}^N f_j(x)^q \right)^{\frac{1}{q}}, \quad g_q(x) = \left(\sum_{j=1}^N g_j(x)^q \right)^{\frac{1}{q}},$$

where $\{(f_j, g_j)\}_{j=1}^N \subset \mathcal{F}$. Now form a new family \mathcal{F}_q consisting of the pairs (f_q, g_q) . Then, for every $w \in A_{\infty, \mathcal{B}}$ and $(f_q, g_q) \in \mathcal{F}_q$, by (2.3) we have that

$$\|f_q\|_{L^q(w)}^q = \sum_{j=1}^N \int_{\mathbb{R}^n} f_j(x)^q w(x) dx \leq C \sum_{j=1}^N \int_{\mathbb{R}^n} g_j(x)^q w(x) dx = C \|g_q\|_{L^q(w)}^q.$$

But this inequality implies that the hypotheses of Theorem 2.1 are fulfilled by \mathcal{F}_q with $p_0 = q$. Therefore, by inequalities (2.3) and (2.4), for all $0 < p < \infty$, $0 < s \leq \infty$, $w \in A_{\infty, \mathcal{B}}$, and $(f_q, g_q) \in \mathcal{F}_q$, $\|f_q\|_{L^p(w)} \leq C \|g_q\|_{L^p(w)}$ and $\|f_q\|_{L^{p,s}(w)} \leq C \|g_q\|_{L^{p,s}(w)}$.

5. PROOF OF THEOREM 2.2

The proof is similar to the proof of Theorem 2.1, so it is organized in the same way and minor details which are the same in both proofs have been omitted.

Step 1: Prove the analog of Theorem 4.2.

Theorem 5.1. *The hypothesis (2.7) of Theorem 2.2 is equivalent to the following: for all $0 < q < p_0$, $w \in A_{1, \mathcal{B}}$, and $(f, g) \in \mathcal{F}$, $\|f\|_{L^q(w)} \leq C \|g\|_{L^q(w)}$.*

Proof. Fix $(f, g) \in \mathcal{F}$; we may assume that both $\|f\|_{L^q(w)}$ and $\|g\|_{L^q(w)}$ are finite and strictly positive. Let $s = p_0/q > 1$. Since $w \in A_{1, \mathcal{B}}$, $M_{\mathcal{B}}$ is bounded from $L^{s', \infty}(w)$ to $L^{s', \infty}(w)$; denote its norm by $\|M_{\mathcal{B}}\|_{L^{s', \infty}(w)}$. (As in the proof of Theorem 2.1, this follows via interpolation.) For $h \in L^{s', \infty}(w)$, $h \geq 0$, define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M_{\mathcal{B}}^k h(x)}{2^k \|M_{\mathcal{B}}\|_{L^{s', \infty}(w)}^k}.$$

As before, from the definition of \mathcal{R} we have that $h(x) \leq \mathcal{R}h(x)$, that \mathcal{R} is bounded on $L^{s', \infty}(w)$ with norm at most 2, and that $\mathcal{R}h \in A_{1, \mathcal{B}}$. Next, define

$$H(x) = \mathcal{R} \left(\left(\frac{f}{\|f\|_{L^q(w)}} \right)^{\frac{q}{s'}} + \left(\frac{g}{\|g\|_{L^q(w)}} \right)^{\frac{q}{s'}} \right) (x) \in L^{s', \infty}(w).$$

Note that $H > 0$ on the set where $f > 0$ and that H is finite for almost every x such that $w(x) > 0$. Further, $W = H^{-s} w \in A_{1+s, \mathcal{B}} \subset A_{\infty, \mathcal{B}}$ because w and H are in $A_{1, \mathcal{B}}$. By the duality of $L^{s,1}(W)$ and $L^{s', \infty}(W)$, for $\lambda > 0$ we have that

$$w(\{x : f(x) > \lambda\}) = w(E_\lambda) \leq \|\chi_{E_\lambda}\|_{L^{s,1}(W)} \|H^s\|_{L^{s', \infty}(W)}.$$

We analyze each term separately. For the first, we use that $W \in A_{\infty, \mathcal{B}}$ and (2.7):

$$\|\chi_{E_\lambda}\|_{L^{s,1}(W)} = \lambda^{-q} \left(\lambda W(E_\lambda)^{\frac{1}{p_0}} \right)^q \leq \lambda^{-q} \|f\|_{L^{p_0, \infty}(W)}^q \leq C \lambda^{-q} \|g\|_{L^{p_0, \infty}(W)}^q,$$

where the last inequality holds provided that $\|f\|_{L^{p_0, \infty}(W)}$ is finite. For $x \in E_\lambda$, by (a), we have

$$\lambda^{\frac{q}{s'}} \|f\|_{L^{q, \infty}(w)}^{-\frac{q}{s'}} \leq \left(\frac{f(x)}{\|f\|_{L^{q, \infty}(w)}} \right)^{\frac{q}{s'}} \leq H(x),$$

and so

$$\lambda^{p_0} W(E_\lambda) = \lambda^{p_0} \int_{\mathbb{R}^n} \chi_{E_\lambda}(x) H(x)^{-s} w(x) dx \leq \|f\|_{L^{q, \infty}(w)}^{p_0 - q} \lambda^q w(E_\lambda) \leq \|f\|_{L^{q, \infty}(w)}^{p_0}.$$

Thus, we have proved that $\|f\|_{L^{p_0, \infty}(W)} \leq \|f\|_{L^{q, \infty}(w)} < \infty$. The same computation also holds with g place of f , and so

$$\|\chi_{E_\lambda}\|_{L^{s,1}(W)} \leq C \lambda^{-q} \|g\|_{L^{p_0, \infty}(W)}^q \leq C \lambda^{-q} \|g\|_{L^{q, \infty}(w)}^q.$$

We now estimate the second term: since \mathcal{R} is bounded on $L^{s', \infty}(w)$,

$$\begin{aligned} \|H^s\|_{L^{s', \infty}(W)} &= \sup_{\alpha > 0} \alpha \left(\int_{\mathbb{R}^n} \chi_{\{x: H(x)^s > \alpha\}}(x) H(x)^{-s} w(x) dx \right)^{\frac{1}{s'}} \\ &\leq \sup_{\alpha > 0} \alpha^{\frac{1}{s}} w(\{x : H(x)^s > \alpha\})^{\frac{1}{s'}} = \|H\|_{L^{s', \infty}(w)} \leq 4. \end{aligned}$$

Combining these two estimates we get the desired result. \square

Step 2: We now show that for all $0 < p < \infty$ and for every $w \in A_{\infty, \mathcal{B}}$, (2.8) holds. Fix $0 < p < \infty$, $w \in A_{\infty, \mathcal{B}}$, and $(f, g) \in \mathcal{F}$; we may assume that f, g are in $L^{p, \infty}(w)$. Take $0 < q < \min\{p, p_0\}$ such that $w \in A_{r, \mathcal{B}}$, where $r = p/q > 1$. We use the Rubio de Francia algorithm exactly as we did in Step 2 of the proof of Theorem 2.1. Hence, given \mathcal{R} as defined by (4.3), we have that

$$\begin{aligned} \|f\|_{L^{p, \infty}(w)}^q &= \|f^q\|_{L^{r, \infty}(w)} = \sup_{\lambda > 0} \left(\int_{\mathbb{R}^n} (\lambda \chi_{\{x: f(x)^q > \lambda\}}(x))^r w(x) dx \right)^{\frac{1}{r}} \\ &= \sup_{\lambda > 0} \sup_h \int_{\mathbb{R}^n} \lambda \chi_{\{x: f(x)^q > \lambda\}} h(x) w(x) dx, \end{aligned}$$

where the second supremum is taken over all $h \in L^{r'}(w)$ with $h \geq 0$ and $\|h\|_{L^{r'}(w)} = 1$. Fix $\lambda > 0$ and such a function h . Then $\|h w\|_{L^{r'}(w^{1-r'})} = \|h\|_{L^{r'}(w)} = 1$. Further, by the properties of \mathcal{R} , $W = \mathcal{R}(h w) \in A_{1, \mathcal{B}}$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} \lambda \chi_{\{x: f(x)^q > \lambda\}} h(x) w(x) dx &\leq \int_{\mathbb{R}^n} \lambda \chi_{\{x: f(x)^q > \lambda\}} \mathcal{R}(h w)(x) dx \\ &= \lambda W(\{x : f(x)^q > \lambda\}) \leq \|f\|_{L^{q, \infty}(W)}^q \leq C \|g\|_{L^{q, \infty}(W)}^q, \end{aligned}$$

where in the last inequality we used Theorem 5.1. In order to do so we must have $\|f\|_{L^{q,\infty}(W)} < \infty$. But, by Hölder's inequality and the boundedness of \mathcal{R} ,

$$W\{x : f(x) > \alpha\} \leq w\{x : f(x) > \alpha\}^{\frac{1}{r}} \|\mathcal{R}(hw)\|_{L^{r'(w^{1-r'})}} \leq 2w\{x : f(x) > \alpha\}^{\frac{1}{r}}.$$

Therefore, $\|f\|_{L^{q,\infty}(W)} \leq C \|f\|_{L^{p,\infty}(w)} < \infty$. The same computation also hold for g , so we have that $\|g\|_{L^{q,\infty}(W)} \leq C \|g\|_{L^{p,\infty}(w)}$. Thus

$$\int_{\mathbb{R}^n} \lambda \chi_{\{x:f(x)^q>\lambda\}} h(x) w(x) dx \leq C \|g\|_{L^{q,\infty}(W)}^q \leq C \|g\|_{L^{p,\infty}(w)}^q,$$

which yields the desired estimate. \square

6. PROOFS RELATED TO THE APPLICATIONS

In this section we prove the results stated in Sections 1 and 3.

6.1. Maximal functions: Proof of Proposition 1.5. As we noted above, we will show the second estimate in (1.5). Our proof is based on the proof by Torchinsky [Tor] that the maximal operator is bounded on B.M.O. Fix $x_0 \in \mathbb{R}^n$ and fix a cube Q containing x_0 . To get the desired estimate it will suffice to show that there exists a positive constant C depending only on n and q such that

$$\frac{1}{|Q|} \int_Q |Mf(x)^q - ((Mf)^q)_Q| dx \leq CM^\# f(x_0)^q.$$

Let $Q^+ = \{x \in Q : Mf(x)^q > ((Mf)^q)_Q\}$; then

$$\frac{1}{|Q|} \int_Q |Mf(x)^q - ((Mf)^q)_Q| dx = \frac{2}{|Q|} \int_{Q^+} Mf(x)^q - ((Mf)^q)_Q dx.$$

We now introduce two auxiliary operators. For $x \in Q$ let

$$M_Q f(x) = \sup \left\{ \frac{1}{|P|} \int_P |f(y)| dy : x \in P \subset 3Q \right\},$$

$$M^Q f(x) = \sup \left\{ \frac{1}{|P|} \int_P |f(y)| dy : x \in P, P \cap (\mathbb{R}^n \setminus 3Q) \neq \emptyset \right\}.$$

It follows immediately from this that for $x \in Q$,

$$Mf(x) = \max(M_Q f(x), M^Q f(x)) \quad \text{and} \quad M_Q f(x) \leq M(f\chi_{3Q})(x).$$

Hence, if we let $W_1 = \{x \in Q^+ : M_Q f(x) > M^Q f(x)\}$ and $W_2 = Q^+ \setminus W_1$, then

$$\begin{aligned} \frac{2}{|Q|} \int_{Q^+} Mf(x)^q - ((Mf)^q)_Q dx &= \frac{2}{|Q|} \int_{W_1} M_Q f(x)^q - ((Mf)^q)_Q dx \\ &\quad + \frac{2}{|Q|} \int_{W_2} M^Q f(x)^q - ((Mf)^q)_Q dx \end{aligned}$$

$$= A + B.$$

We estimate each integral in turn. To estimate A , note that by the triangle inequality, if $x \in Q$, $M_Q f(x) \leq M((f - f_{3Q})\chi_{3Q})(x) + f_{3Q}$. Further, for all $x \in Q$, $Mf(x) \geq f_{3Q}$. Therefore, since $0 < q < 1$, by Kolmogorov's inequality,

$$A \leq \frac{2}{|Q|} \int_{W_1} M((f - f_{3Q})\chi_{3Q})(x)^q \leq C \left(\frac{1}{|3Q|} \int_{3Q} |f(x) - f_{3Q}| dx \right)^q \leq C M^\# f(x_0)^q.$$

To estimate B it will suffice to show that for any $x \in Q$, $M^Q f(x)^q - ((Mf)^q)_Q \leq C M^\# f(x_0)^q$. Fix $x \in Q$ and let P be any cube containing x such that $P \cap (\mathbb{R}^n \setminus 3Q) \neq \emptyset$. Then $Q \subset 3P$, and so for all $y \in Q$, $Mf(y) \geq f_{3P}$. Therefore, since $0 < q < 1$,

$$\begin{aligned} (f_P)^q - ((Mf)^q)_Q &\leq (f_P)^q - (f_{3P})^q \leq |f_P - f_{3P}|^q \leq \left(\frac{1}{|P|} \int_P |f(x) - f_{3P}| dx \right)^q \\ &\leq C \left(\frac{1}{|3P|} \int_{3P} |f(x) - f_{3P}| dx \right)^q \leq C M^\# f(x_0)^q. \end{aligned}$$

If we now take the supremum over all such cubes P we get the desired inequality.

6.2. Commutators of Fractional integrals: Proof of Proposition 3.2. The proof is similar to the arguments given in [Pe2] (see also [SW]) for fractional integral operators, and we will draw upon that proof extensively; we recommend that the reader consult it for complete details.

We first state some definitions and basic facts about Orlicz spaces. For complete information, see [RR, BS]. Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a Young function: i.e., a continuous, convex, increasing function with $\Phi(0) = 0$ and such that $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Each Young function Φ has associated to it a complementary Young function $\bar{\Phi}$. We denote by L_Φ the usual Orlicz space endowed with its Luxemburg norm $\|\cdot\|_\Phi$. For example, if $\Phi(t) = t^p$ for $1 < p < \infty$, then $L_\Phi = L^p(\mu)$ and $\bar{\Phi}(t) = t^{p'}$. More importantly, if $\Phi(t) = t \log(e + t)$ then L_Φ is the Zygmund space $L \log L$ and the complementary function, $\bar{\Phi}(t) \approx e^t - 1$, gives the Zygmund space $\exp L$.

We also need a localized version of the Luxemburg norm: for every Q , define

$$\|f\|_{\Phi, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

There is a generalized Hölder's inequality associated with this norm:

$$(6.1) \quad \frac{1}{|Q|} \int_Q |f(x) g(x)| dx \leq 2 \|f\|_{\Phi, Q} \|g\|_{\bar{\Phi}, Q}.$$

Finally, we define the fractional maximal operator associated to an Orlicz norm by

$$M_{\Phi, \alpha} f(x) = \sup_{Q \ni x} \ell(Q)^\alpha \|f\|_{\Phi, Q}, \quad 0 \leq \alpha < n.$$

Proof of Proposition 3.2. Throughout the proof, let $\Phi(t) = t \log(e+t)$. Fix f ; without loss of generality we may assume that $f \geq 0$. The first step of the proof is to discretize the commutator:

$$\begin{aligned}
(6.2) \quad & \int_{\mathbb{R}^n} |[b, I_\alpha]f(x)| w(x) dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y) dy w(x) dx \\
& = \sum_{Q \in \mathcal{D}} \int_{\mathbb{R}^n} \chi_Q(x) \int_{\frac{\ell(Q)}{2} < |x-y| \leq \ell(Q)} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y) dy w(x) dx \\
& \leq C \sum_{Q \in \mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|} \int_Q |b(x) - b_Q| w(x) dx \int_{3Q} f(y) dy \\
& \quad + C \sum_{Q \in \mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|} \int_{3Q} |b(y) - b_Q| f(y) dy \int_Q w(x) dx = C(A + B).
\end{aligned}$$

We estimate each term separately. For A we use the fact that the weight $w \in A_\infty$ satisfies a reverse Hölder inequality: there exists $\theta > 1$ such that for every cube Q ,

$$\left(\frac{1}{|Q|} \int_Q w(x)^\theta dx \right)^{\frac{1}{\theta}} \leq C \frac{1}{|Q|} \int_Q w(x) dx.$$

In particular for $Q \in \mathcal{D}$, by Hölder's inequality and the John-Nirenberg inequality,

$$\frac{1}{|Q|} \int_Q |b(x) - b_Q| w(x) dx \leq C \|b\|_{\text{B.M.O.}} \frac{1}{|Q|} \int_Q w(x) dx.$$

Therefore,

$$A \leq C \sum_{Q \in \mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|} \int_{3Q} f(y) dy \int_Q w(x) dx.$$

At this point we want to apply an argument from [Pe2]; to do so we need to modify w so that it has bounded support. Given any finite collection \mathcal{D}_0 of dyadic cubes, there exists a cube Q_0 that contains each cube of \mathcal{D}_0 . (Such a cube exists precisely because \mathcal{D}_0 is finite). Let $w_0 = w \chi_{Q_0}$; then we have

$$\sum_{Q \in \mathcal{D}_0} \frac{\ell(Q)^\alpha}{|Q|} \int_{3Q} f(y) dy \int_Q w(x) dx \leq \sum_{Q \in \mathcal{D}_0} \frac{\ell(Q)^\alpha}{|Q|} \int_{3Q} f(y) dy \int_Q w_0(x) dx.$$

Now it was shown in [Pe2] that because w_0 has compact support,

$$\sum_{Q \in \mathcal{D}_0} \frac{\ell(Q)^\alpha}{|Q|} \int_{3Q} f(y) dy \int_Q w_0(x) dx \leq C \int_{\mathbb{R}^n} M_\alpha f(x) w(x) dx,$$

where C does not depend on Q_0 . Thus, if we let $\mathcal{D}_0 \nearrow \mathcal{D}$ we get

$$A \leq C \int_{\mathbb{R}^n} M_\alpha f(x) w(x) dx \leq C \int_{\mathbb{R}^n} M_{\Phi, \alpha} f(x) w(x) dx.$$

To estimate B , note that by the John-Nirenberg inequality, $\|b - b_Q\|_{\exp L, 3Q} \leq C \|b\|_{\text{B.M.O.}}$. As noted above, the complementary function of $e^t - 1$ is $t \log(e + t)$. Therefore, by (6.1), for every cube Q ,

$$\frac{1}{|Q|} \int_{3Q} |b(y) - b_Q| f(y) dy \leq 2 \|b - b_Q\|_{\exp L, 3Q} \|f\|_{L \log L, 3Q} \leq C \|b\|_{\text{B.M.O.}} \|f\|_{L \log L, 3Q}.$$

Consequently, we conclude that

$$(6.3) \quad B \leq C \sum_{Q \in \mathcal{D}} \ell(Q)^\alpha \|f\|_{L \log L, 3Q} \int_Q w(x) dx.$$

We can take the sum over a finite set \mathcal{D}_0 so as to restrict the support of w to a bounded set Q_0 . Let $w_0 = w \chi_{Q_0}$; we will work with (6.3) with w_0 replacing w . We will show that there is a constant C such that for any dyadic cube P ,

$$(6.4) \quad \sum_{\substack{Q \in \mathcal{D} \\ Q \subset P}} \ell(Q)^\alpha |Q| \|f\|_{L \log L, 3Q} \leq C \ell(P)^\alpha |P| \|f\|_{L \log L, 3P}.$$

To do so, we need the following characterization of Orlicz norms (see [RR]):

$$\|f\|_{\Phi, Q} \approx \inf_{\lambda > 0} \left\{ \lambda + \frac{\lambda}{|Q|} \int_Q \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \right\}.$$

Then for any $\lambda > 0$,

$$\begin{aligned} \sum_{\substack{Q \in \mathcal{D} \\ Q \subset P}} \ell(Q)^\alpha |Q| \|f\|_{L \log L, 3Q} &\leq C \lambda \sum_{\substack{Q \in \mathcal{D} \\ Q \subset P}} \ell(Q)^\alpha \int_{3Q} \left(1 + \Phi \left(\frac{|f(x)|}{\lambda} \right) \right) dx \\ &\leq C \lambda \ell(P)^\alpha \int_{3P} \left(1 + \Phi \left(\frac{|f(x)|}{\lambda} \right) \right) dx \\ &= C \ell(P)^\alpha |P| \left(\lambda + \frac{\lambda}{|3P|} \int_{3P} \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \right), \end{aligned}$$

where we have used an inequality in [Pe2, Lemma 3.1]. This estimate holds for every $\lambda > 0$ and so we can take the infimum over all λ to get (6.4).

Fix $a > 2^n$. Since w_0 has compact support, for each $k \in \mathbb{Z}$ there exists a collection $\{Q_{k,j}\}$ of disjoint maximal dyadic cubes such that

$$D_k = \{x \in \mathbb{R}^n : M^d w_0(x) > a^k\} = \bigcup_j Q_{k,j}, \quad a^k < \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w_0(x) dx \leq 2^n a^k.$$

Further, every cube which satisfies the first inequality is contained in a unique cube $Q_{k,j}$. Finally, if we define $E_{k,j} = Q_{k,j} \setminus D_{k+1}$, then $|Q_{k,j}| \approx |E_{k,j}|$ and so $w(Q_{k,j}) \approx$

$w(E_{k,j})$, since $w \in A_\infty$. (See [Pe3].) For each $k \in \mathbb{Z}$, define

$$\mathcal{C}^k = \left\{ Q \in \mathcal{D} : a^k < \frac{1}{|Q|} \int_Q w_0(x) dx \leq a^{k+1} \right\}.$$

We can now argue as in [Pe2], replacing a sum over all dyadic cubes with a sum over Calderón-Zygmund cubes: by (6.4),

$$\begin{aligned} & \sum_{Q \in \mathcal{D}} \ell(Q)^\alpha \|f\|_{L \log L, Q} \int_Q w_0(x) dx \\ & \leq C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w_0(x) dx \sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_{k,j}}} \ell(Q)^\alpha |Q| \|f\|_{L \log L, 3Q} \\ & \leq C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(x) dx \ell(Q_{k,j})^\alpha |Q_{k,j}| \|f\|_{L \log L, 3Q_{k,j}} \\ & \leq C \sum_{k,j} \int_{E_{k,j}} \ell(Q_{k,j})^\alpha \|f\|_{L \log L, 3Q_{k,j}} w(x) dx \leq C \int_{\mathbb{R}^n} M_{\Phi, \alpha} f(x) w(x) dx, \end{aligned}$$

since the sets $\{E_{j,k}\}_{j,k}$ are pairwise disjoint. Thus we have shown that

$$B \leq C \int_{\mathbb{R}^n} M_{\Phi, \alpha} f(x) w(x) dx.$$

Combining the estimates for A and B we get the desired result. \square

6.3. Multiparameter fractional integral operators: Proof of Theorem 3.4 and Proposition 3.5. Theorem 3.4 is an almost immediate consequence of a local version which is stated below. We use the following notation: Q_n and Q_m denote cubes in \mathbb{R}^n and \mathbb{R}^m respectively whose sides are parallel to the coordinate axes. Further, for any cubes Q_n and Q_m , let

$$f_{Q_n}^y = \frac{1}{|Q_n|} \int_{Q_n} f(\bar{x}, y) d\bar{x}, \quad f_{Q_m}^x = \frac{1}{|Q_m|} \int_{Q_m} f(x, \bar{y}) d\bar{y},$$

and

$$f_{Q_n \times Q_m} = \frac{1}{|Q_n| |Q_m|} \int_{Q_n} \int_{Q_m} f(\bar{x}, \bar{y}) d\bar{y} d\bar{x}.$$

Proposition 6.1. *Given two cubes $Q_n \subset \mathbb{R}^n$ and $Q_m \subset \mathbb{R}^m$, and $f \in C^2(Q_n \times Q_m)$, for every $(x, y) \in Q_n \times Q_m$,*

$$|f(x, y) - f_{Q_n}^y - f_{Q_m}^x + f_{Q_n \times Q_m}| \leq CT(|\nabla_x \nabla_y f| \chi_{Q_n \times Q_m})(x, y).$$

Proof of Theorem 3.4. By Proposition 6.1, for every $(x, y) \in Q_n \times Q_m$,

$$|f(x, y) - f_{Q_n}^y - f_{Q_m}^x + f_{Q_n \times Q_m}| \leq CT(|\nabla_x \nabla_y f|)(x, y).$$

Since f has compact support, $f_{Q_n}^y, f_{Q_m}^x, f_{Q_n \times Q_m}$ tend to 0 as $Q_n \nearrow \mathbb{R}^n$ and $Q_m \nearrow \mathbb{R}^m$, and the desired inequality follows if we take these limits. \square

Proof of Proposition 6.1. Let I denote the lefthand side of the desired inequality. Then,

$$I \leq \frac{1}{|Q_n||Q_m|} \int_{Q_n} \int_{Q_m} |f(x, y) - f(\bar{x}, y) - f(x, \bar{y}) + f(\bar{x}, \bar{y})| d\bar{y} d\bar{x}.$$

For $t \in [0, 1]$ let $g(t) = f(x + t(\bar{x} - x), y) - f(x + t(\bar{x} - x), \bar{y})$. Then,

$$\begin{aligned} |f(x, y) - f(\bar{x}, y) - f(x, \bar{y}) + f(\bar{x}, \bar{y})| &= |g(1) - g(0)| \leq \int_0^1 |g'(t)| dt \\ &= \int_0^1 |h_t(0) - h_t(1)| dt \leq \int_0^1 \int_0^1 |h'_t(s)| ds dt \\ &\leq \int_0^1 \int_0^1 |\nabla_x \nabla_y f(x + t(x - \bar{x}), y + s(y - \bar{y}))| |\bar{x} - x| |\bar{y} - y| ds dt. \end{aligned}$$

where, for $s \in [0, 1]$, the function h is defined by

$$h_t(s) = \langle \nabla_x f(x + t(\bar{x} - x), y + s(\bar{y} - y)), \bar{x} - x \rangle.$$

Therefore,

$$I \leq \frac{1}{|Q_n||Q_m|} \int_0^1 \int_0^1 \int_{Q_n} \int_{Q_m} |\nabla_x \nabla_y f(\dots, \dots)| |\bar{x} - x| |\bar{y} - y| d\bar{y} d\bar{x} ds dt.$$

Now perform the following changes of variables on \bar{x} and \bar{y} : $\tilde{x} = x + t(\bar{x} - x)$, $\tilde{y} = y + s(\bar{y} - y)$. Since $x, \bar{x} \in Q_n$, $\tilde{x} \in Q_n$. On the other hand,

$$|\tilde{x} - x| = t|\bar{x} - x| \leq \sqrt{n} t \ell(Q_n),$$

and consequently $\tilde{x} \in Q_n \cap B(x, \sqrt{n} t \ell(Q_n))$. Denote this set by Q_n^t . In the same way, $\tilde{y} \in Q_m^s = Q_m \cap B(y, \sqrt{m} s \ell(Q_m))$. Hence,

$$\begin{aligned} &\int_0^1 \int_0^1 \int_{Q_n} \int_{Q_m} |\nabla_x \nabla_y f(\dots, \dots)| |\bar{x} - x| |\bar{y} - y| d\bar{y} d\bar{x} ds dt \\ &\leq \int_0^1 \int_0^1 \int_{Q_n^t} \int_{Q_m^s} |\nabla_x \nabla_y f(\tilde{x}, \tilde{y})| \frac{|\tilde{x} - x|}{t} \frac{|\tilde{y} - y|}{s} \frac{d\tilde{y}}{s^m} \frac{d\tilde{x}}{t^n} ds dt \\ &\leq \int_{Q_n} \int_{Q_m} \left(\int_{\frac{|\tilde{x}-x|}{\sqrt{n}\ell(Q_n)}}^\infty \frac{1}{t^n} \frac{dt}{t} \right) \left(\int_{\frac{|\tilde{y}-y|}{\sqrt{m}\ell(Q_m)}}^\infty \frac{1}{s^m} \frac{ds}{s} \right) |\nabla_x \nabla_y f(\tilde{x}, \tilde{y})| |\tilde{x} - x| |\tilde{y} - y| d\tilde{y} d\tilde{x} \\ &\leq C |Q_n| |Q_m| T(|\nabla_x \nabla_y f| \chi_{Q_n \times Q_m})(x, y), \end{aligned}$$

and this yields the desired estimate. \square

Proof of Proposition 3.5. This result will follow from (1.9) if T can be written as the composition of two fractional integrals of order 1, one in each variable. Fix f ; without loss of generality, $f \geq 0$. We will use the following notation: $f^y(x) = f(x, y)$ and $f^x(y) = f(x, y)$. Define

$$I_1^{(1)} f(x, y) = I_1^{(1)} f^y(x) = \int_{\mathbb{R}^n} \frac{f^y(\bar{x})}{|x - \bar{x}|^{n-1}} d\bar{x} = \int_{\mathbb{R}^n} \frac{f(\bar{x}, y)}{|x - \bar{x}|^{n-1}} d\bar{x}$$

and

$$I_1^{(2)} f(x, y) = I_1^{(2)} f^x(y) = \int_{\mathbb{R}^m} \frac{f^x(\bar{y})}{|y - \bar{y}|^{m-1}} d\bar{y} = \int_{\mathbb{R}^m} \frac{f(x, \bar{y})}{|y - \bar{y}|^{m-1}} d\bar{y}.$$

Hence, $Tf(x, y) = I_1^{(1)} \circ I_1^{(2)} f(x, y)$, and so

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} Tf(x, y) w(x, y) dy dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x - \bar{x}|^{n-1}} \int_{\mathbb{R}^m} I_1^{(2)} f^{\bar{x}}(y) w^x(y) dy d\bar{x} dx.$$

For a.e. $x, \bar{x} \in \mathbb{R}^n$, since $w^x \in A_\infty(\mathbb{R}^m)$ with constant independent of x , so by (1.9),

$$\int_{\mathbb{R}^m} I_1^{(2)} f^{\bar{x}}(y) w^x(y) dy \leq C \int_{\mathbb{R}^m} M_1^{(2)} f^{\bar{x}}(y) w^x(y) dy = \int_{\mathbb{R}^m} M_1^{(2)} f(\bar{x}, y) w(x, y) dy.$$

Similarly, for a.e. $y \in \mathbb{R}^n$, the weight w^y is uniformly in $A_\infty(\mathbb{R}^n)$, so again by (1.9),

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} Tf(x, y) w(x, y) dy dx &\leq C \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} I_1^{(1)} ((M_1^{(2)} f)^y)(x) w^y(x) dx dy \\ &\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} M_1^{(1)} \circ M_1^{(2)} f(x, y) w(x, y) dy dx. \end{aligned}$$

\square

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