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Uncertainty principle estimates for vector fields

CARLOS PÉREZ[§] AND RICHARD L. WHEEDEN[¶]

1 Introduction

One form of Hardy's inequality is the estimate

$$\int_{\mathbf{R}^n} |f(x)|^p \frac{dx}{|x|^p} \leq C \int_{\mathbf{R}^n} |\nabla f(x)|^p dx,$$

for $1 \leq p < n$ and any smooth f with compact support, where the constant C is independent of f . This inequality, which can be found in [HLP], has had many important applications. For instance, in Mathematical Physics, it is related (in fact, equivalent) in case $p = 2$ to the Uncertainty Principle of Heisenberg ([RS], vol. II, p. 169).

In this paper, we will derive norm estimates for a wide class of integral operators of potential type. These estimates can be used to obtain inequalities like the one above. In fact, if T is the integral operator defined by

$$Tf(x) = \int_{\mathbf{R}^n} f(y) \frac{1}{|x-y|^{n-1}} dy,$$

then by using the well-known pointwise inequality

$$|f(x)| \leq c_n T(|\nabla f|)(x)$$

for any smooth f with compact support, one can deduce the Hardy estimate above from the corresponding (weighted) L^p norm estimate for T . The same method can be used for more

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general differential operators Xf often called generalized gradients; that is, one can bound norms of f by norms of Xf provided there is an integral operator T which is bounded on appropriate weighted L^p spaces and for which the pointwise estimate $|f| \leq cT(|Xf|)$ is valid. For example, this pointwise estimate is known to hold for vector fields of Hörmander type when T is given by

$$Tf(x) = \int_{\mathbf{R}^n} f(y) \frac{d(x,y)}{|B(x,d(x,y))|} dy,$$

where $d(x,y)$ is the associated Carnot–Carathéodory metric and $B(x,r)$ denotes the metric ball with center x and radius r .

In addition to norm estimates for integral operators, we will also study norm estimates for maximal operators that are closely associated with the integral operators. Our main theorems generalize and sharpen some of the principal results obtained in [SW], [P1] and [P2]. We improve these results in several ways, such as by considering spaces of homogeneous type without any group structure, and by enlarging the classes of weight functions for which some of the results hold. In particular, with regard to weight functions for integral operators, we are able to avoid assuming the doubling conditions that are imposed in some of the results in [SW], as well as improve the results there which deal with generalizations of the Fefferman–Phong “r-bump” condition (see below). We will show that this sort of condition can be replaced by weaker ones like those considered in [P1] and [P2], and which are closely related to estimates derived in [CWW] and [ChW2] for Schrödinger operators.

In order to obtain weighted results, the kinds of conditions that we will impose on the weights are in the spirit of simple sufficient conditions which are close to necessary. For example, in n -dimensional Euclidean space \mathbf{R}^n with the usual metric, the classical Riesz fractional integral operator

$$I_\alpha f(x) = \int_{\mathbf{R}^n} f(y) \frac{1}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n,$$

is known to satisfy the norm inequality

$$\left(\int_{\mathbf{R}^n} \{|I_\alpha f(x)|w(x)\}^q dx \right)^{1/q} \leq c \left(\int_{\mathbf{R}^n} \{|f(x)|v(x)\}^p dx \right)^{1/p}, \quad (1)$$

$1 < p \leq q < \infty$, if for some $r > 1$ and all balls B , the weights satisfy

$$r(B)^{\alpha-n} |B|^{\frac{1}{q} + \frac{1}{p'}} \left(\frac{1}{|B|} \int_B w(x)^{r q} dx \right)^{1/r q} \left(\frac{1}{|B|} \int_B v(x)^{-r p'} dx \right)^{1/r p'} \leq C, \quad (2)$$

where p' denotes the conjugate index of p , that is, $\frac{1}{p} + \frac{1}{p'} = 1$, and $r(B)$ the radius of B . On the other hand, the same condition with $r = 1$, i.e., the condition

$$r(B)^{\alpha-n} \left(\int_B w(x)^q dx \right)^{1/q} \left(\int_B v(x)^{-p'} dx \right)^{1/p'}$$

is necessary but not sufficient for the norm inequality (this is exactly the $A_{p,q}^\alpha$ condition of [SW], but with a different normalization). These facts are proved in [SW]. Extensions to spaces of homogeneous type are also proved there, but with extra restrictions on either the weights or the space, such as doubling conditions on $w^{r q}$ and $v^{-r p'}$, or a group structure for the space. One of our goals is to remove these extra restrictions.

We refer to (2) as a Fefferman–Phong “r–bump” condition. It is simpler in nature than two other kinds of conditions known to be both necessary and sufficient for such norm estimates. To put our results for potential operators in perspective, it may help to briefly recall these other conditions, even though they play no role in the paper. One of them involves “testing” conditions of the type found first in [S] in the usual Euclidean situation, and then generalized and sharpened in [SW], [SWZ], [WZ] and [VW]. Testing conditions are phrased in terms of norm estimates for the integral operator when it is restricted to acting on the weight functions themselves. The second sort of necessary and sufficient condition involves integrals with “tails”, i.e., integrals extended over the entire space of products of weights times suitably truncated powers of the kernel which appears in the integral operator. Such results are known for $1 < p < q < \infty$ but not for $q = p$: see [GK], [SW], [SWZ], [GGK]. Compared with

conditions of these two types, those of the Fefferman–Phong sort have the disadvantage of not being necessary, but they have the advantage of being relatively simple and close to necessary. No simple method is known for proving that Fefferman–Phong conditions imply either of these other two types of conditions.

The r–bump requirement (2) was weakened in [P1] within the Euclidean framework; more general potential operators of convolution form were also studied in [P1]. In the case of the Riesz fractional integral, these weaker assumptions can be described as follows: let Ψ and Φ be doubling Young functions such that both

$$\int_c^\infty \left(\frac{t^q}{\Psi(t)} \right)^{q'-1} \frac{dt}{t} < \infty \text{ and } \int_c^\infty \left(\frac{t^{p'}}{\Phi(t)} \right)^{p'-1} \frac{dt}{t} < \infty \quad (3)$$

for some positive constant c . Some examples of such $\Phi(t)$ are, for large t , $\Phi(t) = t^{p'}(\log t)^{p'-1+\beta}$ and $\Phi(t) = t^{p'}(\log t)^{p'-1}(\log \log t)^{p'-1+\beta}$, where $\beta > 0$. Then (1) holds if, for all balls B , the weights satisfy the condition

$$r(B)^{\alpha-n} |B|^{\frac{1}{q} + \frac{1}{p'}} \|w\|_{\Psi, B} \|v^{-1}\|_{\Phi, B} \leq K, \quad (4)$$

where $\|f\|_{\Psi, B}$ (similarly $\|f\|_{\Phi, B}$) denotes the localized Luxemburg norm

$$\|f\|_{\Psi, B} = \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B \Psi \left(\frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}.$$

See section 4 for more information.

There is a similar situation for maximal functions. Indeed, it was also shown in [P1] that a sufficient condition for the analogue of (1) with $I_\alpha f$ replaced by the fractional maximal function

$$M_\alpha(f)(x) = \sup r(B)^{\alpha-n} \int_B |f(y)| dy, \quad 0 \leq \alpha < n,$$

where the supremum is taken over all balls B containing x , is just the condition (4) with no “bump” on the weight w , namely with $\Psi(t) = t^q$. In this case, the condition on the weights

becomes simply

$$r(B)^{\alpha-n}|B|^{1/p'} \left(\int_B w^q dx \right)^{1/q} \|v^{-1}\|_{\Phi,B} \leq K$$

for some Φ as in (3) and all balls B . An antecedent of the results in [P1], [P3] was given by Neugebauer in [N] for the case $\alpha = 0$. Another goal of this paper is to extend these results for M_α to more general maximal operators and to spaces of homogeneous type.

In case w^q and $v^{-p'}$ are A_∞ weights (in the sense of C. Fefferman and B. Muckenhoupt), as is well-known, condition (2) holds for some $r > 1$ if and only if it holds for $r = 1$. Thus, in this case, no bump is needed in the condition imposed on the weights in order to obtain (1). In a sequel [PW] to this paper, we will show that no bump is needed for classes which are larger than A_∞ , and in spaces which are more general than \mathbf{R}^n . This extends earlier results of the same kind in [SW] in the usual Euclidean case as well as in spaces of homogeneous type. See also [BSa] for results about A_∞ weights in spaces of homogeneous type. For these more general spaces, and for larger classes of weight functions than A_∞ , we also study in [PW] extensions of results in [MW] relating norms of integral operators of potential type to norms of maximal functions.

2 Statements of the main results

Following [SW], we consider potential operators $T = T_K$ of the form

$$Tf(x) = T(fd\mu)(x) = \int_{\mathcal{S}} f(y)K(x,y)d\mu(y), \tag{5}$$

where \mathcal{S} is a space of homogeneous type with underlying doubling measure μ . See §3 for the exact definition of a space of homogeneous type; by a doubling measure, we mean a Borel measure μ with the property that there is a constant C such that for every “ball” $B \subset \mathcal{S}$,

$$\mu(2B) \leq C\mu(B),$$

where $2B$ denotes the ball with the same center as B but twice the radius. If $d(x, y)$ denotes the corresponding quasimetric in \mathcal{S} , we will always assume that the kernel $K(x, y)$ is nonnegative and satisfies the following growth conditions: there exist constants $C_1, C_2 > 1$ such that

$$K(x, y) \leq C_1 K(x', y) \quad \text{if} \quad d(x', y) \leq C_2 d(x, y), \quad (6)$$

$$K(x, y) \leq C_1 K(x, y') \quad \text{if} \quad d(x, y') \leq C_2 d(x, y).$$

The main classical examples of such operators are the Riesz fractional integrals $I_\alpha f$ mentioned in the introduction. An important class of examples for metrics other than the usual Euclidean metric consists of potential operators related to the regularity of subelliptic differential equations. In particular, vector fields of Hörmander type ([H]) as well as the classes of nonsmooth vector fields studied in [FL] lead to integral operators of the type we will study. In addition, the differential operators of Grushin type considered in [FGuW] (at least in the simplest case of Lebesgue measure) are related to integrals of type (5). In fact, for all these examples, the associated potential operator has the form

$$Tf(x) = \int_{\mathcal{S}} f(y) \frac{d(x, y)}{\mu(B(x, d(x, y)))} d\mu(y), \quad (7)$$

where $d(x, y)$ is a distance function that is naturally related to the vector fields and $B(x, r)$ denotes the corresponding ball with center x and radius r .

Associated with the kernel K is a functional $\varphi = \varphi_K$ which acts on balls B and is defined by

$$\varphi(B) = \sup_{\substack{x, y \in B \\ d(x, y) \geq cr(B)}} K(x, y) \quad (8)$$

for a sufficiently small positive geometric constant c (see [SW]), where $r(B)$ denotes the radius of B . For example, in the case of the Riesz potential, we have $K(x, y) = |x - y|^{\alpha - n}$, $0 < \alpha < n$, so that $\varphi(B) \approx r(B)^{\alpha - n}$. In the subelliptic case (7), note that $\varphi(B) \approx r(B)/\mu(B)$.

The conditions (6) on K lead to useful growth properties of φ . If B is a ball and $\theta > 0$, let θB denote the ball concentric with B whose radius is $\theta r(B)$. It is shown in [SWZ, (4.2) and (4.3)] that if $\theta > 1$, there is a constant C depending only on θ, C_1, C_2 , the constant c in (8), and geometric properties of \mathcal{S} so that

$$\varphi(B) \leq C\varphi(\theta B) \text{ for all balls } B \subset \mathcal{S}. \quad (9)$$

Also, for such a constant C (but now C is independent of θ),

$$\varphi(B) \leq C\varphi(B') \text{ for all pairs of balls } B' \subset B. \quad (10)$$

We shall also assume in some of our results that φ satisfies the following condition for some $\epsilon > 0$:

$$\varphi(B_1)\mu(B_1) \leq C \left(\frac{r(B_1)}{r(B_2)} \right)^\epsilon \varphi(B_2)\mu(B_2) \text{ if } B_1 \subset B_2. \quad (11)$$

Observe that in the case of the fractional integrals I_α , we can pick $\epsilon = \alpha$ in (11); for the operator in (7), we can choose $\epsilon = 1$.

Next we define a class of Young functions that plays a key role in our results. Some further facts about Young functions and Orlicz spaces are listed in §4.

Definition 2.1 *Let $1 \leq p < \infty$. A nonnegative function $\Phi(t), t > 0$, satisfies the B_p condition if there is a constant $c > 0$ such that*

$$\int_c^\infty \frac{\Phi(t) dt}{t^p t} < \infty. \quad (12)$$

Simple examples of functions which satisfy B_p are $t^{p-\beta}$ and $t^p(\log(1+t))^{-1-\beta}$, both when $\beta > 0$.

The relevance of condition B_p stems from its relationship to the boundedness of a maximal function that is defined in terms of Φ . In fact, given a Young function Φ , let

$$\|f\|_{\Phi, B} = \inf \left\{ \lambda > 0 : \frac{1}{\mu(B)} \int_B \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1 \right\},$$

and define the corresponding maximal function

$$\mathcal{M}_\Phi f(x) = \sup_{B: x \in B} \|f\|_{\Phi, B}. \quad (13)$$

We will prove the following characterization of B_p , $1 < p < \infty$, in Theorem 5.1 below:

$$\Phi \in B_p \text{ if and only if } \mathcal{M}_\Phi : L^p(\mathcal{S}, \mu) \rightarrow L^p(\mathcal{S}, \mu).$$

For example, in the standard case when $\Phi(t) = t^r$ with $r \geq 1$, so that

$\|f\|_{\Phi, B} = (\mu(B)^{-1} \int_B |f|^r d\mu)^{1/r}$, this statement reduces to the well-known fact that the mapping

$$f \rightarrow \sup_{B: x \in B} \left(\frac{1}{\mu(B)} \int_B |f|^r d\mu \right)^{1/r}$$

is bounded on $L^p(\mathcal{S}, \mu)$ if and only if $p > r$. The characterization of B_p mentioned above was proved in the Euclidean context in [P3] and used to derive sharp two weight estimates for the classical Hardy–Littlewood maximal function. In the general case, the characterization of B_p will play a main role in the proof of the boundedness of T as stated in Theorem 2.2 below.

For other applications to different operators from harmonic analysis, see [P1], [P5], [P6], [CP1] and [CP2].

A Young function Φ has a conjugate function $\bar{\Phi}$ satisfying

$$t \leq \bar{\Phi}(t)\bar{\Phi}^{-1}(t) \leq 2t$$

for all $t > 0$ (cf. §4). For example, if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, the conjugate of t^p is $t^{p'}$, and the conjugate of $t^p(\log(1+t))^{-1-\beta}$, $\beta > 0$, is $t^{p'}(\log(1+t))^{(p'-1)(1+\beta)}$ (cf. [O], p. 275).

We can now state our main result about the boundedness of the potential operators (5).

Theorem 2.2 *Let $1 < p \leq q < \infty$ and T be an integral operator of type (5) with a kernel K such that (6) holds and φ satisfies (11). Let (w, v) be a pair of weights for which*

$$\varphi(B)\mu(B)^{\frac{1}{q} + \frac{1}{p'}} \|w\|_{\Psi, B} \|v^{-1}\|_{\Phi, B} \leq C \quad (14)$$

for all balls B in \mathcal{S} , where Ψ and Φ are Young functions whose corresponding conjugate functions $\bar{\Psi}$ and $\bar{\Phi}$ satisfy $\bar{\Psi} \in B_{q'}$ and $\bar{\Phi} \in B_p$. Then

$$\left(\int_{\mathcal{S}} (|Tf|w)^q d\mu \right)^{1/q} \leq C \left(\int_{\mathcal{S}} (|f|v)^p d\mu \right)^{1/p} \quad (15)$$

with C independent of f .

In the Euclidean setting, condition (14) is a generalization of the sort of condition first considered in [CWW] (see also [P1], [ChW2]).

For example, given p, q with $1 < p \leq q < \infty$, if we choose $\Psi(t) = t^{rq}$ and $\Phi(t) = t^{rp'}$ for any $r > 1$, then condition (14) becomes

$$\phi(B) \mu(B)^{\frac{1}{q} + \frac{1}{p'}} \left(\frac{1}{\mu(B)} \int_B w^{rq} d\mu \right)^{\frac{1}{rq}} \left(\frac{1}{\mu(B)} \int_B v^{-rp'} d\mu \right)^{\frac{1}{rp'}} \leq C.$$

Assuming this r -bump condition, the conclusion (15) was proved in [SW] in the usual Euclidean situation (with $d\mu = dx$); it was also proved there for spaces of homogeneous type but with one of the additional assumptions that both $w^{rq}d\mu, v^{-rp'}d\mu$ are doubling measures or that there is an appropriate group structure for \mathcal{S} . In case $v = 1$ and if T is an integral operator of type (7) with $d\mu = dx$, (15) was proved in [D] without either of these additional assumptions.

It is easy to see that when $p > 1$ and Φ satisfies the doubling property $\Phi(2t) \leq c\Phi(t)$, then

$$\int_c^\infty \frac{\Phi(t) dt}{t^p} \approx \int_c^\infty \left(\frac{t^{p'}}{\bar{\Phi}(t)} \right)^{p-1} \frac{dt}{t}$$

for $c > 0$. Hence, if both $\bar{\Phi}$ and $\bar{\Psi}$ satisfy this doubling condition, then the assumption in Theorem 2.2 that $\bar{\Psi} \in B_{q'}$ and $\bar{\Phi} \in B_p$ is equivalent to assuming both

$$\int_c^\infty \left(\frac{t^q}{\bar{\Psi}(t)} \right)^{q'-1} \frac{dt}{t} < \infty \quad \text{and} \quad \int_c^\infty \left(\frac{t^{p'}}{\bar{\Phi}(t)} \right)^{p-1} \frac{dt}{t} < \infty$$

for some $c > 0$.

The proof of Theorem 2.2 will be given in section 7 and is based on a procedure for discretizing potential operators which appeared independently in [SW] and [JPW], combined with the characterization of B_p given in Theorem 5.1.

The case $p = q$ is important in applications, such as to Schrödinger operators, and in this case by choosing $\Psi(t) = t^p(\log(1+t))^{p-1+\beta}$ and $\Phi(t) = t^{p'}(\log(1+t))^{p'-1+\beta}$, we have the following special case of Theorem 2.2.

Corollary 2.3 *Let $1 < p < \infty$ and T, K and φ be as in Theorem 2.2. Let (w, v) be a pair of weights such that for some $\beta > 0$ and all balls B in \mathcal{S} ,*

$$\varphi(B) \left(\int_B w^p [\log(1 + \frac{w}{w(B)})]^{p-1+\beta} d\mu \right)^{1/p} \left(\int_B v^{-p'} [\log(1 + \frac{v^{-1}}{v^{-1}(B)})]^{p'-1+\beta} d\mu \right)^{1/p'} \leq C.$$

Then

$$\int_{\mathcal{S}} (|Tf|w)^p d\mu \leq C \int_{\mathcal{S}} (|f|v)^p d\mu. \tag{16}$$

Furthermore, this result is sharp in the sense that it does not hold when $\beta = 0$.

Remark 2.4 *It would be interesting to derive an analogue of (16) for Calderón–Zygmund singular integral operators, assuming that the weights satisfy*

$$\left(\int_B w^p [\log(1 + \frac{w}{w(B)})]^{p-1+\beta} d\mu \right)^{1/p} \left(\int_B v^{-p'} [\log(1 + \frac{v^{-1}}{v^{-1}(B)})]^{p'-1+\beta} d\mu \right)^{1/p'} \leq C\mu(B)$$

for all balls B and some $\beta > 0$. This conjecture has been partially confirmed in [TVZ] by means of complex analysis when T is the Hilbert transform in the unit circle. There are corresponding estimates for vector-valued maximal operators in [P6]. Also, in [CP2], some sharp two-weight weak-type inequalities for Calderón–Zygmund operators have been derived assuming that

$$\left(\frac{1}{\mu(B)} \int_B w^p [\log(1 + \frac{w}{w(B)})]^{p-1+\beta} d\mu \right)^{1/p} \left(\frac{1}{\mu(B)} \int_B v^{-rp'} d\mu \right)^{1/rp'} \leq C.$$

Observe that here, we just need a bump on the left-hand weight.

A natural maximal operator of “fractional” type associated with T is defined by

$$M_\varphi f(x) = \sup_{B:x \in B} \varphi(B) \int_B |f| d\mu, \quad (17)$$

where φ is as in (8). For example, in the case of the classical Riesz fractional integral I_α , M_φ is just the fractional maximal operator M_α defined in R^n by

$$M_\alpha f(x) = \sup_{B:x \in B} r(B)^{\alpha-n} \int_B |f(y)| dy.$$

In any case, the pointwise inequality

$$M_\varphi f(x) \leq c T f(x)$$

holds for all $x \in \mathcal{S}$. This follows easily from the fact that $\varphi(B) \leq CK(x, y)$ for all $x, y \in B$ (even if $x, y \in \theta B$ for any fixed $\theta > 1$), as shown in [SWZ, (4.1)]. On the other hand, Tf is often controlled in norm by $M_\varphi f$: in the classical situation, see [MW] and [R], and in more general situations, see [PW].

We can extend the definition of M_φ by considering functionals other than φ . Thus, let

$$M_\psi f(x) = \sup_{B:x \in B} \psi(B) \int_B |f| d\mu$$

where ψ is a nonnegative functional defined on balls. A way in which such a maximal function is related to T is given in the next result, which is proved in §8 using Corollary 2.3. We use the notation Mf for the Hardy–Littlewood maximal function of f defined by

$$Mf(x) = \sup_{B:x \in B} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y), \quad (18)$$

and if k is a positive integer, $M^k f$ denotes the k -fold iterate $M(M(\dots(Mf)\dots))$. Also, $[p]$ denotes the integral part of p .

Theorem 2.5 *Let $1 < p < \infty$ and T, K and φ be as in Theorem 2.2. Then there is a constant C such that for any weight w and all f ,*

$$\int_{\mathcal{S}} |Tf(x)|^p w \, d\mu \leq C \int_{\mathcal{S}} |f(x)|^p M_{\tilde{\varphi}}(M^{[p]}w) \, d\mu, \quad (19)$$

where $\tilde{\varphi}$ is defined by $\tilde{\varphi}(B) = (\varphi(B)\mu(B))^p \mu(B)^{-1}$.

Remark 2.6 *Inequalities in the spirit of (19) but for Calderón–Zygmund operators have been derived in [P4], but the method there is completely different from the one developed here.*

To understand the interest of (19), we note that in the classical situation it can be restated as

$$\int_{\mathbf{R}^n} |I_{\alpha}f(x)|^p w(x) \, dx \leq C \int_{\mathbf{R}^n} |f(x)|^p M_{\alpha p}(M^{[p]}w)(x) \, dx \quad (20)$$

since then $\tilde{\varphi}(B) = r(B)^{\alpha p - n}$ and consequently $M_{\tilde{\varphi}} = M_{\alpha p}$. First we observe that the exponent $[p]$ is sharp in the sense that $M^{[p]}w$ cannot be replaced by $M^{[p-1]}w$ (see [P2]). Second we observe that (20) is sharper than the inequality

$$\int_{\mathbf{R}^n} |I_{\alpha}f(x)|^p w(x) \, dx \leq C \int_{\mathbf{R}^n} |f(x)|^p M_{\alpha p r}(w^r)(x)^{1/r} \, dx \quad (21)$$

proved in [A] since by standard arguments, for any $k = 1, 2, \dots, r > 1$ and $\alpha > 0$, there is a constant C such that for all f ,

$$M_{\alpha}(M^k f) \leq C (M_{\alpha r}(f^r))^{1/r}.$$

The expression on the right here is clearly related to the Fefferman–Phong r -bump condition, while the one on the left is related to the condition considered by Chang, Wilson and Wolff in [CWW]; see [P2], remark 1.5.

We also note that in the case of the operator defined in (7), the functional $\tilde{\varphi}$ in (19) satisfies $\tilde{\varphi}(B) \approx r(B)^p / \mu(B)$ since $\varphi(B) \approx r(B) / \mu(B)$ in this case.

We will also derive an analogue of Theorem 2.2 for the fractional maximal operator M_ψ . For this operator, the condition that we need to impose on the weights is weaker than the one used in Theorem 2.2 for potential operators. We consider any nonnegative function $\psi(B)$ of balls $B \subset \mathcal{S}$ which satisfies the following conditions:

- a) $\psi(B_1) \leq c\psi(B_2)$ if $B_1 \subset B_2 \subset cB_1$
- b) $\psi(B_1)\mu(B_1) \leq c\psi(B_2)\mu(B_2)$ if $B_1 \subset B_2$ (22)
- c) if \mathcal{S} is unbounded, then $\lim_{r(B) \rightarrow \infty} \psi(B) = 0$, in the sense that given $\epsilon > 0$, there exists $N > 0$ such that $\psi(B) < \epsilon$ if $r(B) > N$.

Note that condition b) corresponds to the case $\epsilon = 0$ in (11), and hence b) is weaker than (11). The main example of such a functional is $\psi(B) = r(B)^\alpha/\mu(B)$ with $\alpha > 0$, and in this case, condition c) is true if μ satisfies a reverse doubling condition of order strictly larger than α . Given a functional ψ which satisfies (22), we define the maximal function M_ψ as before:

$$M_\psi f(x) = \sup_{B: x \in B} \psi(B) \int_B |f(y)| d\mu(y). \quad (23)$$

Theorem 2.7 *Let ψ satisfy (22), and let M_ψ be defined by (23). Let $1 < p \leq q < \infty$, ω be a Borel measure, and v be a weight such that*

$$\psi(B)\omega(B)^{1/q}\mu(B)^{\frac{1}{p'}}\|v^{-1}\|_{\Phi, B} \leq C \quad (24)$$

for all balls B in S , where Φ is any Young function whose conjugate function $\bar{\Phi} \in B_p$, i.e., where

$$\int_c^\infty \frac{\bar{\Phi}(t)}{t^p} \frac{dt}{t} < \infty$$

for some $c > 0$. Then

$$\left(\int_{\mathcal{S}} (M_\psi f)^q d\omega \right)^{1/q} \leq C \left(\int_{\mathcal{S}} (|f|v)^p d\mu \right)^{1/p}. \quad (25)$$

In particular, if w, v is a pair of weights which satisfy

$$\psi(B) \left(\int_B w^q d\mu \right)^{1/q} \mu(B)^{\frac{1}{p'}} \|v^{-1}\|_{\Phi, B} \leq C,$$

then

$$\left(\int_{\mathcal{S}} \{(M_\psi f) w\}^q d\mu \right)^{1/q} \leq C \left(\int_{\mathcal{S}} (|f| v)^p d\mu \right)^{1/p}.$$

Observe that in the last condition above, no Orlicz type bump is required on w^q , and so the condition is weaker than the one considered in (14).

The second statement of Theorem 2.7 clearly follows from the first one by choosing

$$d\omega = w^q d\mu.$$

In the next two sections, before proving the results stated above, we give some background facts about spaces of homogeneous type and Orlicz classes. Our main theorems are proved after these sections. When we come to the proofs, we first prove the results about maximal functions and then those for integral operators.

3 Spaces of Homogeneous type

In this section, we briefly recall some basic definitions and facts about spaces of homogeneous type.

A quasimetric d on a set \mathcal{S} is a function $d : \mathcal{S} \times \mathcal{S} \rightarrow [0, \infty)$ which satisfies

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all x, y ;
- (iii) there exists a finite constant $\kappa \geq 1$ such that

$$d(x, y) \leq \kappa(d(x, z) + d(z, y))$$

for all $x, y, z \in \mathcal{S}$.

Given $x \in \mathcal{S}$ and $r > 0$, let $B(x, r) = \{y \in \mathcal{S} : d(x, y) < r\}$ be the ball with center x and radius r . If $B = B(x, r)$ is a ball, we denote its radius r by $r(B)$ and its center x by x_B . If ν is a measure and E is a measurable set, $\nu(E)$ denotes the ν -measure of E . We sometimes write $|E|_\nu$ instead of $\nu(E)$.

Definition 3.1 *A space of homogeneous type (\mathcal{S}, d, μ) is a set \mathcal{S} together with a quasimetric d and a nonnegative Borel measure μ on \mathcal{S} such that the doubling condition*

$$\mu(B(x, 2r)) \leq C \mu(B(x, r)) \tag{26}$$

holds for all $x \in \mathcal{S}$ and $r > 0$.

The balls $B(x, r)$ are not necessarily open, but by a theorem of Macias and Segovia [MS], there is a continuous quasimetric d' which is equivalent to d (i.e., there are positive constants c_1 and c_2 such that $c_1 d'(x, y) \leq d(x, y) \leq c_2 d'(x, y)$ for all $x, y \in \mathcal{S}$) for which every ball is open. We always assume that the quasimetric d is continuous and that balls are open.

If C is the smallest constant for which (26) holds, then the number $D = \log C$ is called the doubling order of μ . By iterating (26), we have

$$\frac{\mu(B)}{\mu(\tilde{B})} \leq C_\mu \left(\frac{r(B)}{r(\tilde{B})} \right)^D \text{ for all balls } \tilde{B} \subset B. \tag{27}$$

We also assume that all annuli in \mathcal{S} are not empty, i.e., that $B(x, R) \setminus B(x, r)$ is not empty for all $x \in \mathcal{S}$ and $0 < r < R < \infty$. By [W, p.269], any doubling measure μ then satisfies the reverse doubling property: there exist $\delta > 0$ and $c_\mu > 0$ such that

$$\frac{\mu(B)}{\mu(\tilde{B})} \geq c_\mu \left(\frac{r(B)}{r(\tilde{B})} \right)^\delta \text{ for all balls } \tilde{B} \subset B. \tag{28}$$

We shall often use the following observation: if P and B are balls with $P \cap B \neq \emptyset$ and $r(P) \leq \beta r(B)$ for some $\beta > 0$, then

$$P \subset c_\beta B \tag{29}$$

with $c_\beta = \kappa\beta + \kappa^2\beta + \kappa^2$. To verify (29), note that if $z \in B \cap P$ and $y \in P$, then

$$\begin{aligned} d(y, x_B) &\leq \kappa[d(y, x_P) + d(x_P, x_B)] \leq \kappa[r(P) + \kappa(d(x_P, z) + d(z, x_B))] \\ &\leq \kappa[r(P) + \kappa(r(P) + r(B))] \leq \kappa[\beta r(B) + \kappa(\beta r(B) + r(B))] = c_\beta r(B), \end{aligned}$$

which implies (29).

We will use a grid of dyadic sets in \mathcal{S} which are ‘‘almost balls’’, as constructed in [SW]. In fact, the following has been proved there:

If $\rho = 8K^5$, then for any (large negative) integer m , there are points $\{x_j^k\}$ and a family $\mathcal{D}_m = \{E_j^k\}$ of sets for $k = m, m+1, \dots$ and $j = 1, 2, \dots$ such that

- $B(x_j^k, \rho^k) \subset E_j^k \subset B(x_j^k, \rho^{k+1})$
- For each $k = m, m+1, \dots$, the family $\{E_j^k\}$ is pairwise disjoint in j , and $\mathcal{S} = \cup_j E_j^k$.
- If $m \leq k < l$, then either $E_j^k \cap E_i^l = \emptyset$ or $E_j^k \subset E_i^l$.

We call the family $\mathcal{D} = \cup_{m \in \mathbf{Z}} \mathcal{D}_m$ a dyadic cube decomposition of \mathcal{S} and refer to the sets in \mathcal{D} as dyadic cubes. A dyadic cube will usually be denoted by Q , and Q^* will denote the containing ball described above with $\frac{1}{\rho}Q^* \subset Q \subset Q^*$; thus, if $Q = E_j^k$ then $Q^* = B(x_j^k, \rho^{k+1})$. We set $\ell(Q) = r(Q^*)/\rho$ and call $\ell(Q)$ the ‘‘sidelength’’ of Q . We note that while the cubes in each \mathcal{D}_m have the dyadic properties listed above, there may be no nestedness properties of the cubes in \mathcal{D}_{m_1} relative to the cubes in \mathcal{D}_{m_2} if m_1, m_2 are different.

As usual, we say that w is a weight if $w(x)$ is a nonnegative locally integrable function with respect to μ , and for a measurable set E , we write $w(E) = \int_E w(x) d\mu(x)$. Thus, $w(E) = |E|_{w d\mu}$.

4 Orlicz spaces

We next recall some basic definitions and facts about Orlicz spaces, referring to [RR] and [BS] for a complete account.

A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$. It follows that $\Phi(t)/t$ is increasing, and in particular, that

$$\Phi(\gamma t) \geq \gamma \Phi(t) \quad \text{if } \gamma \geq 1 \text{ and } t \geq 0.$$

For Orlicz spaces, we are usually only concerned about the behavior of Young functions for t large. If A, B are two Young functions, we write $A(t) \approx B(t)$ if there are constants $c, c_1, c_2 > 0$ with $c_1 A(t) \leq B(t) \leq c_2 A(t)$ for $t > c$. By definition, the Orlicz space L_Φ consists of all measurable functions f such that

$$\int_S \Phi\left(\frac{|f|}{\lambda}\right) d\mu < \infty$$

for some positive λ . Note that if $0 < \lambda_1 < \lambda_2$, then

$$\Phi\left(\frac{|f|}{\lambda_2}\right) \leq \frac{\lambda_1}{\lambda_2} \Phi\left(\frac{|f|}{\lambda_1}\right),$$

so that

$$\lim_{\lambda \rightarrow \infty} \int_S \Phi\left(\frac{|f|}{\lambda}\right) d\mu = 0 \quad \text{if } f \in L_\Phi.$$

The space L_Φ is a Banach function space with the Luxemburg norm

$$\|f\|_\Phi = \|f\|_{\Phi, \mu} = \inf\{\lambda > 0 : \int_S \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\}.$$

Each Young function Φ has an associated complementary Young function $\bar{\Phi}$ satisfying

$$t \leq \Phi^{-1}(t) \bar{\Phi}^{-1}(t) \leq 2t \tag{30}$$

for all $t > 0$. The function $\bar{\Phi}$ is called the conjugate of Φ , and the space $L_{\bar{\Phi}}$ is called the conjugate space of L_{Φ} . For example, if $\Phi(t) = t^p$ for $1 < p < \infty$, then $\bar{\Phi}(t) = t^{p'}$, $p' = p/(p-1)$, and the conjugate space of $L^p(\mu)$ is $L^{p'}(\mu)$. Another example that will be used frequently is $\Phi(t) \approx t^p(\log t)^{-1-\epsilon}$ for large t , $1 < p < \infty$, $\epsilon > 0$, with complementary function $\bar{\Phi}(t) \approx t^{p'}(\log t)^{(p'-1)(1+\epsilon)}$ (cf. [O]).

A very important property of Orlicz spaces is the generalized Hölder inequality

$$\int_{\mathcal{S}} |fg| d\mu \leq \|f\|_{\Phi} \|g\|_{\bar{\Phi}}. \quad (31)$$

We will sometimes assume that Φ satisfies the doubling condition $\Phi(2t) \leq C\Phi(t)$. If Φ is doubling then $\Phi'(t) \approx \Phi(t)/t$ almost everywhere.

Recall that if X is a rearrangement-invariant function space with respect to the measure μ , then the fundamental function of X , $\varphi_X(t)$, is defined so that if $t > 0$ and E is any measurable set with $\mu(E) = t$, then

$$\varphi_X(t) = \|\chi_E\|_X.$$

See [BS] for more information. In particular, it is shown there that for any Young function Φ , L_{Φ} is a rearrangement-invariant space with fundamental function given by

$$\varphi_{L_{\Phi}}(t) = \frac{1}{\Phi^{-1}(\frac{1}{t})}. \quad (32)$$

In particular, if E is a measurable subset of X , then

$$\|\chi_E\|_{\Phi, \mu} = \frac{1}{\Phi^{-1}(\frac{1}{\mu(E)})}. \quad (33)$$

5 Auxiliary exotic maximal functions

In order to define another maximal function which will play a key role, we need to introduce local versions of Orlicz norms. If Φ is a Young function, let

$$\|f\|_{\Phi,B} = \|f\|_{\Phi,B,\mu} = \inf\{\lambda > 0 : \frac{1}{\mu(B)} \int_B \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\}.$$

For this norm, we will use the fact that if $\lambda > 0$, then $\|f\|_{\Phi,B} > \lambda$ if and only if

$\frac{1}{\mu(B)} \int_B \Phi(|f|/\lambda) d\mu > 1$. Furthermore, the local version of the generalized Hölder inequality (31) is

$$\frac{1}{\mu(B)} \int_B fg d\mu \leq \|f\|_{\Phi,B} \|g\|_{\bar{\Phi},B}. \quad (34)$$

As in (13), there is a corresponding maximal function defined by

$$\mathcal{M}_\Phi f(x) = \sup_{B:x \in B} \|f\|_{\Phi,B}. \quad (35)$$

This maximal function has been used in the usual Euclidean context in [P3] as a tool to derive sharp weighted estimates for the Hardy-Littlewood maximal function. Also, it was considered in the work of T. Iwaniec and Greco [GI] and in [WW] in case $\Phi(t) \approx t \log t$. Note that from 32 we have with the corresponding normalization that

$$\|\chi_K\|_{\Phi,B} = \frac{1}{\Phi^{-1}\left(\frac{\mu(B)}{\mu(B \cap K)}\right)},$$

and therefore

$$\mathcal{M}_\Phi(\chi_K)(x) = \sup_{B:x \in B} \frac{1}{\Phi^{-1}\left(\frac{\mu(B)}{\mu(B \cap K)}\right)}. \quad (36)$$

The main results that we will prove about \mathcal{M}_Φ are summarized in the next theorem.

Theorem 5.1 *Let $1 < p < \infty$ and Φ be a doubling Young function. Then the following statements are equivalent.*

i) $\Phi \in B_p$, i.e., there is a constant $c > 0$ such that

$$\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty. \quad (37)$$

ii) There is a constant $C > 0$ such that

$$\int_S \mathcal{M}_\Phi f(x)^p d\mu(x) \leq C \int_S f(x)^p d\mu(x) \quad (38)$$

for all nonnegative f .

iii) There is a constant $C > 0$ such that

$$\int_S \mathcal{M}_\Phi f(x)^p w(x) d\mu(x) \leq C \int_S f(x)^p Mw(x) d\mu(x) \quad (39)$$

for all nonnegative f and w , where Mw is the Hardy–Littlewood maximal function defined in (18).

iv) There is a constant $C > 0$ such that

$$\int_S Mf(x)^p \frac{w(x)}{[\mathcal{M}_\Phi(u^{1/p})(x)]^p} d\mu(x) \leq C \int_S f(x)^p \frac{Mw(x)}{u(x)} d\mu(x) \quad (40)$$

for all nonnegative f , w and u , where M again denotes the operator defined in (18).

The proof of Theorem 5.1 is based on the following lemma.

Lemma 5.2 *Let Φ be a Young function and f be a bounded nonnegative function with bounded support. For $\lambda > 0$, let $\Omega_\lambda = \{x \in \mathcal{S} : \mathcal{M}_\Phi f(x) > \lambda\}$. If Ω_λ is not empty, then given $\sigma > 1$, there exists a countable family $\{B_i\}$ of pairwise disjoint balls such that*

i) $\cup_i B_i \subset \Omega_\lambda \subset \cup_i B_i^*$, where $B^* = \kappa(4\kappa + 1)B$ (κ is the quasimetric constant),

ii) $\|f\|_{\Phi, B_i} > \lambda$ for all i ,

iii) $\|f\|_{\Phi, B} \leq \lambda$ if B is any ball with $B_i \subset B$ and $r(B) \geq \sigma r(B_i)$ for some i .

Consequently,

$$\mu(\Omega_\lambda) \leq C \int_S \Phi\left(\frac{f}{\lambda}\right) d\mu. \quad (41)$$

Proof: The proof uses a sort of Calderón–Zygmund decomposition combined with Vitali’s lemma. A similar method occurs in [MP]. Fix f and λ . If $x \in \Omega_\lambda$, there is a ball B such that $x \in B$ and $\|f\|_{\Phi, B} > \lambda$. Define $R = R(f, \lambda)$ by

$$R = \sup_{B: \|f\|_{\Phi, B} > \lambda} r(B).$$

We claim that R is finite. Indeed, suppose that the support of f is contained in a ball B_0 , and let B satisfy $\|f\|_{\Phi, B} > \lambda$. Then by definition of the Luxemburg norm, B satisfies

$$1 < \frac{1}{\mu(B)} \int_B \Phi\left(\frac{f}{\lambda}\right) d\mu = \frac{1}{\mu(B)} \int_{B \cap B_0} \Phi\left(\frac{f}{\lambda}\right) d\mu \leq \Phi\left(\frac{\|f\|_{L^\infty}}{\lambda}\right) \frac{\mu(B \cap B_0)}{\mu(B)}.$$

Therefore, B and B_0 must intersect. Assuming as we may that $r(B) \geq r(B_0)$, we easily obtain from (29) that $B_0 \subset \kappa(2\kappa + 1)B$, and then

$$\begin{aligned} 1 < \Phi\left(\frac{\|f\|_{L^\infty}}{\lambda}\right) \frac{\mu(B \cap B_0)}{\mu(B)} &\leq \Phi\left(\frac{\|f\|_{L^\infty}}{\lambda}\right) \frac{\mu(B_0)}{\mu(B)} \\ &\leq \Phi\left(\frac{\|f\|_{L^\infty}}{\lambda}\right) c_{\kappa, \mu} \left(\frac{r(B_0)}{r(B)}\right)^\delta \end{aligned}$$

for some fixed $\delta > 0$ by the reverse doubling property of μ (see (28)). In particular,

$$r(B) \leq c \Phi\left(\frac{\|f\|_{L^\infty}}{\lambda}\right)^{1/\delta} r(B_0). \quad (42)$$

This shows that R is finite. It also shows that each ball B for which $\|f\|_{\Phi, B} > \lambda$ is contained in cB_0 with c depending only on λ and f , since then B intersects B_0 and (42) holds.

Now, for each x , let

$$R_x = R_x(f, \lambda) = \sup_{B: x \in B, \|f\|_{\Phi, B} > \lambda} r(B),$$

and note that R_x is finite since $R_x \leq R$. Fix $\sigma > 1$. If $x \in \Omega_\lambda$, there is a ball B_x which contains x , whose radius $r(B_x)$ satisfies $R_x/\sigma < r(B_x) \leq R_x$, and for which $\|f\|_{\Phi, B_x} > \lambda$. If B

is any ball with $B_x \subset B$ and $r(B) \geq \sigma r(B_x)$, then $r(B) > R_x$, and consequently $\|f\|_{\Phi, B} \leq \lambda$ since $x \in B$. Thus the ball B_x satisfies ii) and iii). Also note that $\Omega_\lambda = \bigcup_{x \in \Omega_\lambda} B_x$. Picking a Vitali type subcover of $\{B_x\}_{x \in \Omega_\lambda}$ as in [SW], Lemma 3.3, then provides a family of pairwise disjoint balls $\{B_i\} \subset \{B_x\}_{x \in \Omega_\lambda}$ satisfying i). Therefore $\{B_i\}$ satisfies i), ii) and iii). Finally, (41) follows in a standard way from i) and ii) by the doubling property of μ and the disjointness of the B_i :

$$\begin{aligned} \mu(\Omega_\lambda) &\leq \sum \mu(B_i^*) \leq \sum C\mu(B_i) \\ &\leq C \sum \int_{B_i} \Phi\left(\frac{f}{\lambda}\right) d\mu \leq C \int_S \Phi\left(\frac{f}{\lambda}\right) d\mu. \end{aligned}$$

□

Proof of Theorem 5.1

We may assume that f is bounded with bounded support and that $f \geq 0$. We start by proving that i) implies ii). Let $\Omega_\lambda = \{x \in \mathcal{S} : \mathcal{M}_\Phi f(x) > \lambda\}$. For each $\lambda > 0$, we split f as usual: $f = f_1 + f_2$ where $f_1(x) = f(x)$ if $f(x) > \lambda/2$ and $f_1(x) = 0$ otherwise. We may assume without loss of generality that Φ is normalized so that $\Phi(1) = 1$. Since $f_2 \leq \lambda/2$, it then follows that $\mathcal{M}_\Phi(f_2) \leq \lambda/2$, and consequently that $\mathcal{M}_\Phi(f) \leq \mathcal{M}_\Phi(f_1) + \mathcal{M}_\Phi(f_2) \leq \mathcal{M}_\Phi(f_1) + \lambda/2$. Using this combined with (41), we get

$$\mu(\Omega_\lambda) \leq C \int_{x \in \mathcal{S}: f(x) > \lambda/2} \Phi\left(\frac{2f(x)}{\lambda}\right) d\mu(x).$$

Then

$$\begin{aligned} \int_S \mathcal{M}_\Phi(f)^p d\mu &= p \int_0^\infty \lambda^p \mu(\Omega_\lambda) \frac{d\lambda}{\lambda} \\ &\leq C \int_0^\infty \lambda^p \int_{x \in \mathcal{S}: f(x) > \lambda/2} \Phi\left(\frac{2f(x)}{\lambda}\right) d\mu(x) \frac{d\lambda}{\lambda} = C \int_S \int_0^{2f(x)} \lambda^p \Phi\left(\frac{2f(x)}{\lambda}\right) \frac{d\lambda}{\lambda} d\mu(x) \\ &= C \int_S f(x)^p \int_1^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} d\mu(x) = C \int_S f(x)^p d\mu(x) \end{aligned}$$

since $\Phi \in B_p$. This proves that i) implies ii).

We now show that ii) implies iii). For $k \in \mathbf{Z}$ and $\gamma > 1$, γ to be chosen, let

$$\Omega_k = \{x \in \mathcal{S} : \mathcal{M}_\Phi(f)(x) > \gamma^k\}.$$

For $\sigma > 1$ to be chosen, by applying Lemma 5.2 to each Ω_k , we obtain balls $\{B_j^k\}_j$ with $\bigcup_j B_j^k \subset \Omega_k \subset \bigcup_j cB_j^k$, and for each k the balls $\{B_j^k\}$ are disjoint in j . Furthermore, for all k and j , $\|f\|_{\Phi, B_j^k} > \gamma^k$ and $\|f\|_{\Phi, B} \leq \gamma^k$ if B is any ball with $B_j^k \subset B$ and $r(B) \geq \sigma r(B_j^k)$. Then

$$\begin{aligned} \int_{\mathcal{S}} \mathcal{M}_\Phi(f)^p w \, d\mu &\leq \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}_\Phi(f)^p w \, d\mu \leq \sum_k \gamma^{(k+1)p} w(\Omega_k) \\ &\leq c \sum_{k,j} \|f\|_{\Phi, B_j^k}^p w(cB_j^k) = c \sum_{k,j} \|f\|_{\Phi, B_j^k}^p \frac{w(cB_j^k)}{\mu(cB_j^k)} \mu(cB_j^k) \\ &\leq c \sum_{k,j} \left\| \left(\frac{w(cB_j^k)}{\mu(cB_j^k)} \right)^{1/p} f \right\|_{\Phi, B_j^k}^p \mu(B_j^k). \end{aligned} \quad (43)$$

Now consider the family of sets $\{E_j^k\}_{k,j}$ defined by $E_j^k = B_j^k \setminus \Omega_{k+1}$. Observe that the E_j^k are disjoint in both k, j . We will show that for sufficiently large γ there exists a constant c such that for all k, j ,

$$\mu(B_j^k) \leq c \mu(E_j^k)$$

Assuming this for the moment, we obtain from (43) that

$$\begin{aligned} \int_{\mathcal{S}} \mathcal{M}_\Phi(f)^p w \, d\mu &\leq c \sum_{k,j} \left\| (\inf_{B_j^k} Mw)^{1/p} f \right\|_{\Phi, B_j^k}^p \mu(E_j^k) \leq c \sum_{k,j} \int_{E_j^k} \mathcal{M}_\Phi(f(Mw)^{1/p})^p \, d\mu \\ &\leq c \int_{\mathcal{S}} \mathcal{M}_\Phi(f(Mw)^{1/p})^p \, d\mu \leq C \int_{\mathcal{S}} f^p Mw \, d\mu \text{ by hypothesis,} \end{aligned}$$

and thus iii) would be proved.

To prove that $\mu(B_j^k) \leq c\mu(E_j^k)$ if γ is large, it is enough to show that

$$\mu(B_j^k \cap \Omega_{k+1}) < \frac{1}{2} \mu(B_j^k) \text{ if } \gamma \text{ is large.} \quad (44)$$

Recall that by Lemma 5.2, the sets $\{B_m^{k+1}\}$ are disjoint in m for each k , and that $\Omega_{k+1} \subset \bigcup_m cB_m^{k+1}$ with $c = \kappa(4\kappa + 1)$. Moreover, $\|f\|_{\Phi, B_m^{k+1}} > \gamma^{k+1}$, and $\|f\|_{\Phi, B} \leq \gamma^k$ if B satisfies $B_j^k \subset B$ and $r(B) \geq \sigma r(B_j^k)$ for some j, k (with $\sigma > 1$ still to be chosen). Thus

$$\mu(B_j^k \cap \Omega_{k+1}) \leq \sum_m \mu(B_j^k \cap cB_m^{k+1}). \quad (45)$$

We claim that if $B_j^k \cap cB_m^{k+1} \neq \emptyset$ then $r(B_j^k) > r(B_m^{k+1})$. To see this, first note that if B_j^k and cB_m^{k+1} intersect and $r(B_j^k) \leq r(B_m^{k+1})$, then $B_j^k \subset c_1 B_m^{k+1}$ for a geometric constant $c_1 > 1$.

Since Φ is a Young function, $\Phi(t)/t$ is increasing, so that

$$\Phi\left(\frac{f}{\gamma^k}\right) = \Phi\left(\frac{\gamma f}{\gamma^{k+1}}\right) \geq \gamma \Phi\left(\frac{f}{\gamma^{k+1}}\right), \quad \gamma > 1,$$

and therefore by the doubling property of μ , there is a geometric constant $C > 1$ such that

$$\begin{aligned} \frac{1}{\mu(c_1 B_m^{k+1})} \int_{c_1 B_m^{k+1}} \Phi\left(\frac{f}{\gamma^k}\right) d\mu &\geq \frac{\gamma}{C \mu(B_m^{k+1})} \int_{B_m^{k+1}} \Phi\left(\frac{f}{\gamma^{k+1}}\right) d\mu \\ &> \frac{\gamma}{C} > 1, \end{aligned}$$

if we choose $\gamma > C$. (Recall that for any $\lambda > 1$, the inequality $\|f\|_{\Phi, B} > \lambda$ is the same as $\frac{1}{\mu(B)} \int_B \Phi(f/\lambda) d\mu > 1$.) This implies that $\|f\|_{\Phi, c_1 B_m^{k+1}} > \gamma^k$. However, if we now choose σ with $1 < \sigma \leq c_1$, we obtain from the inequality $r(B_j^k) \leq r(B_m^{k+1})$ that the ball $c_1 B_m^{k+1}$ has radius at least $\sigma r(B_j^k)$, and therefore since $B_j^k \subset c_1 B_m^{k+1}$, we must have $\|f\|_{\Phi, c_1 B_m^{k+1}} \leq \gamma^k$ by construction of B_j^k . This contradiction shows our claim.

Thus if $B_j^k \cap cB_m^{k+1} \neq \emptyset$, then $r(B_j^k) > r(B_m^{k+1})$, and consequently $B_m^{k+1} \subset c_1 B_j^k$. Hence by (45) and the doubling property of μ ,

$$\begin{aligned} \mu(B_j^k \cap \Omega_{k+1}) &\leq \sum_{m: B_m^{k+1} \subset c_1 B_j^k} \mu(B_j^k \cap cB_m^{k+1}) \leq c \sum_{m: B_m^{k+1} \subset c_1 B_j^k} \mu(B_m^{k+1}) \\ &\leq c \sum_{m: B_m^{k+1} \subset c_1 B_j^k} \int_{B_m^{k+1}} \Phi\left(\frac{f}{\gamma^{k+1}}\right) d\mu \leq c \int_{c_1 B_j^k} \Phi\left(\frac{f}{\gamma^{k+1}}\right) d\mu \end{aligned}$$

since the sets $\{B_m^{k+1}\}_m$ are disjoint. Again using the fact that $\Phi(t)/t$ is increasing together with property iii) of Lemma 5.2 for B_j^k , we can continue the last chain of inequalities with

$$\leq \frac{c}{\gamma} \int_{c_1 B_j^k} \Phi\left(\frac{f}{\gamma^k}\right) d\mu \leq \frac{c}{\gamma} \mu(c_1 B_j^k) \leq \frac{C}{\gamma} \mu(B_j^k) < \frac{1}{2} \mu(B_j^k)$$

if γ is large. This completes the proof of (44) and also shows that part ii) of the theorem implies part iii).

We prove now that iii) implies iv). Assume then that iii) holds. Since (40) is equivalent to

$$\int_{\mathcal{S}} M(fg)(x)^p \frac{w(x)}{[\mathcal{M}_{\bar{\Phi}}(g)(x)]^p} d\mu(x) \leq c \int_{\mathcal{S}} f(x)^p M w(x) d\mu(x),$$

for all nonnegative functions f, g , and w , iv) follows immediately from (39) after an application of the inequality

$$M(fg)(x) \leq \mathcal{M}_{\Phi} f(x) \mathcal{M}_{\bar{\Phi}} g(x), \quad x \in \mathcal{S},$$

which is a consequence of the local version (34) of the generalized Hölder's inequality.

To prove that iv) implies i), we let $w = 1$ in (40), obtaining

$$\int_{\mathcal{S}} M f(x)^p \frac{1}{[\mathcal{M}_{\bar{\Phi}}(u^{1/p})(x)]^p} d\mu(x) \leq C \int_{\mathcal{S}} f(x)^p \frac{1}{u(x)} d\mu(x) \quad (46)$$

for all nonnegative functions f and u . Fix any $z \in \mathcal{S}$ and $r > 0$, and let $K = B(z, r)$. WE USE HERE THAT THE SPACE HAS MORE THAN ONE POINT BY THE ANNULI CONDITION. Choosing $f = u = \chi_K$ in (46) gives

$$\int_{\mathcal{S}} M(\chi_K)(x)^p \frac{1}{[\mathcal{M}_{\bar{\Phi}}(\chi_K)(x)]^p} d\mu(x) \leq C$$

(where C depends on $\mu(K)$). On the other hand, by (36),

$$\mathcal{M}_{\bar{\Phi}}(f)(x) = \sup_{B: x \in B} \frac{1}{\bar{\Phi}^{-1}\left(\frac{\mu(B)}{\mu(B \cap K)}\right)}.$$

Now, since $t \rightarrow \frac{1}{\bar{\Phi}^{-1}(\frac{1}{t})}$ is increasing, it is easy to see that there is a positive constant b depending on $\mu(K)$ such that if $d(x, z) > \eta r$ for a large geometric constant $\eta > 1$ to be chosen, then

$$\mathcal{M}_{\bar{\Phi}}(\chi_K)(x) = \frac{1}{\bar{\Phi}^{-1}(b \mu(B(x, d(x, z))))}.$$

Similarly, the Hardy-Littlewood maximal function satisfies $M(\chi_K)(x) \geq c/\mu(B(x, d(x, z)))$.

Letting $B(z, \eta^k r) = B_k$ and $A_k = B_{k+1} \setminus B_k$, we obtain from these estimates and doubling that

$$\begin{aligned} \int_S M(\chi_K)(x)^p \frac{1}{[\mathcal{M}_{\bar{\Phi}}(\chi_K)(x)]^p} d\mu(x) &\geq C \int_{d(x,z) > \eta r} \bar{\Phi}^{-1}(b \mu(B(x, d(x, z))))^p \frac{d\mu(x)}{\mu(B(x, d(x, z)))^p} \\ &= C \sum_{k=1}^{\infty} \int_{A_k} \bar{\Phi}^{-1}(b \mu(B(x, d(x, z))))^p \frac{d\mu(x)}{\mu(B(x, d(x, z)))^p} \\ &\approx \sum_{k=1}^{\infty} \int_{A_k} \bar{\Phi}^{-1}(b' \mu(B(z, d(x, z))))^p \frac{d\mu(x)}{\mu(B(z, d(x, z)))^p} \\ &\approx \sum_{k=1}^{\infty} \frac{\mu(A_k)}{\mu(B_k)^p} \bar{\Phi}^{-1}(b' \mu(B_k))^p. \end{aligned}$$

Recall that since annuli are not empty, the reverse doubling property (28) of μ implies that

$$\frac{\mu(B_{k+1})}{\mu(B_k)} \geq c_\mu \eta^\delta.$$

If we choose η so large that $c_\mu \eta^\delta > \frac{3}{2}$, then $\mu(A_k) = \mu(B_{k+1} \setminus B_k) > \frac{1}{2} \mu(B_k)$. Combining this with (30), it follows that the last sum is larger than a multiple of

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\mu(B_k)}{\bar{\Phi}^{-1}(b' \mu(B_k))^p} &\geq C \sum_{k=1}^{\infty} \int_{b' \mu(B_k)}^{b' \mu(B_{k+1})} \frac{t}{\bar{\Phi}^{-1}(t)^p} \frac{dt}{t} \\ &= C \int_{b' \mu(B_1)}^{\infty} \frac{t}{\bar{\Phi}^{-1}(t)^p} \frac{dt}{t} \approx \int_c^{\infty} \frac{\Phi(t)}{t^p} \frac{dt}{t}. \end{aligned}$$

The last formula follows from the change of variables $s = \Phi(t)$ and from the fact that $\Phi'(t) \approx \Phi(t)/t$ since Φ is doubling. The constants depend on z and r . This gives condition (37) and concludes the proof of Theorem 5.1.

□

6 Proof of Theorem 2.7

The proof uses arguments similar to ones in the proof of Theorem 5.1, particularly those showing that ii) implies iii). Recall from (23) that the maximal function $M_\psi f$ is defined by

$$M_\psi f(x) = \sup_{B:x \in B} \psi(B) \int_B |f| d\mu,$$

where $\psi(B)$ is assumed to be nonnegative and to satisfy (22), i.e.,

- a) if $B_1 \subset B_2 \subset cB_1$, then $\psi(B_1) \leq c\psi(B_2)$;
- b) if $B_1 \subset B_2$, then $\psi(B_1)\mu(B_1) \leq c\psi(B_2)\mu(B_2)$;
- c) if \mathcal{S} is unbounded, then $\lim_{r(B) \rightarrow \infty} \psi(B) = 0$.

We need a version of Lemma 5.2 adapted to M_ψ .

Lemma 6.1 *Let f be a bounded nonnegative function with bounded support, and let ψ and $M_\psi f$ be as above for any measure μ (μ need not be a doubling measure here). For $\lambda > 0$, let $\Omega_\lambda = \{x \in \mathcal{S} : M_\psi f(x) > \lambda\}$. Then given $x \in \Omega_\lambda$ and $\sigma > 1$, there is a ball $B_x \subset \Omega_\lambda$ containing x with*

$$\psi(B_x) \int_{B_x} f d\mu > \lambda$$

and such that if B is any ball with $B_x \subset B$ and $r(B) > \sigma r(B_x)$, then

$$\psi(B) \int_B f d\mu \leq \lambda.$$

Moreover, if Ω_λ is not empty, then given $\sigma > 1$, there is a countable family $\{B_i\}$ of pairwise disjoint balls such that

$$i) \cup_i B_i \subset \Omega_\lambda \subset \cup_i B_i^*, \text{ where } B^* = \kappa(4\kappa + 1)B;$$

$$ii) \psi(B_i) \int_{B_i} f d\mu > \lambda \text{ for all } i;$$

iii) if B is any ball such that $B_i \subset B$ and $r(B) > \sigma r(B_i)$ for some i , then $\psi(B) \int_B f d\mu \leq \lambda$.

Proof: If $x \in \Omega_\lambda$, there is a ball B with $x \in B$ and $\psi(B) \int_B f d\mu > \lambda$. For $x \in \Omega_\lambda$, let R_x be defined by

$$R_x = \sup\{r(B) : x \in B \text{ and } \psi(B) \int_B f d\mu > \lambda\}.$$

We claim that R_x is finite. If \mathcal{S} is bounded this is obvious. If \mathcal{S} is unbounded and the support of f is contained in a ball B_0 , then any ball B for which $\psi(B) \int_B f d\mu > \lambda$ must intersect B_0 and satisfy

$$\lambda < \psi(B) \int_B f d\mu \leq \|f\|_{L^\infty} \mu(B_0) \psi(B).$$

Since λ , f and B_0 are fixed, the last inequality means that there is a constant $c > 0$ so that $\psi(B) > c$ for any such B , and consequently, by property c) of ψ , that $r(B)$ is bounded for such B . This shows that R_x is finite and in fact bounded in x for $x \in \Omega_\lambda$. Moreover, since every such B intersects B_0 , it now follows that any B which satisfies $\psi(B) \int_B f d\mu > \lambda$ lies in a fixed enlargement (depending on f, λ) of B_0 .

Thus, if $\sigma > 1$ and $x \in \Omega_\lambda$, there is a ball B_x containing x whose radius satisfies

$R_x/\sigma < r(B_x) \leq R_x$ and for which $\psi(B_x) \int_{B_x} f d\mu > \lambda$. This ball satisfies ii), and if B is any ball containing B_x with $r(B) > \sigma r(B_x)$, then $r(B) > R_x$ and hence $\psi(B) \int_B f d\mu \leq \lambda$. Also observe that $\Omega_\lambda = \bigcup_{x \in \Omega_\lambda} B_x$. Picking a Vitali type subcover of $\{B_x\}_{x \in \Omega_\lambda}$ gives us a family of pairwise disjoint balls $\{B_i\} \subset \{B_x\}_{x \in \Omega_\lambda}$ satisfying all the desired properties. □

For $\gamma > 1$ to be chosen and $k \in \mathbf{Z}$, let $\Omega_k = \Omega_{\gamma^k}$. Then

$$\int_{\mathcal{S}} (M_\psi f)^q d\omega = \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} (M_\psi f)^q d\omega \leq \sum_k \gamma^{(k+1)q} \omega(\Omega_k). \quad (47)$$

Assuming as we may that f is nonnegative, bounded and has bounded support, and given $\sigma > 1$, we can use Lemma 6.1 for each k to find a family $\{B_j^k\}_j$ of pairwise disjoint balls with

$\bigcup_j B_j^k \subset \Omega_k \subset \bigcup_j cB_j^k$ and $\psi(B_j^k) \int_{B_j^k} f d\mu > \gamma^k$. Moreover, if B is any ball with $B_j^k \subset B$ and $r(B) > \sigma r(B_j^k)$ for some k, j , then $\psi(B) \int_B f d\mu \leq \gamma^k$. Then for the last sum in (47), we have

$$\sum_k \gamma^{(k+1)q} \omega(\Omega_k) \leq \gamma^q \sum_{k,j} \left(\psi(B_j^k) \int_{B_j^k} f d\mu \right)^q \omega(cB_j^k).$$

By the local generalized Hölder inequality (34),

$$\frac{1}{\mu(B_j^k)} \int_{B_j^k} f d\mu = \frac{1}{\mu(B_j^k)} \int_{B_j^k} f v v^{-1} d\mu \leq \|f v\|_{\bar{\Phi}, B_j^k} \|v^{-1}\|_{\Phi, B_j^k}.$$

Collecting estimates, we obtain

$$\begin{aligned} \int_{\mathcal{S}} (M_\psi f)^q d\omega &\leq \gamma^q \sum_{k,j} \psi(B_j^k)^q \left(\mu(B_j^k) \|f v\|_{\bar{\Phi}, B_j^k} \|v^{-1}\|_{\Phi, B_j^k} \right)^q \omega(cB_j^k) \\ &\leq \gamma^q \sum_{k,j} \left[\psi(cB_j^k) \mu(cB_j^k)^{\frac{1}{p'}} \omega(cB_j^k)^{\frac{1}{q}} \|v^{-1}\|_{\Phi, cB_j^k} \right]^q \|f v\|_{\bar{\Phi}, B_j^k}^q \mu(B_j^k)^{q/p}. \end{aligned}$$

Since we are assuming (see (24)) that for all balls B ,

$$\psi(B) \mu(B)^{\frac{1}{p'}} \omega(B)^{\frac{1}{q}} \|v^{-1}\|_{\Phi, B} \leq C,$$

it follows that the last expression is bounded by

$$C \sum_{k,j} \|f v\|_{\bar{\Phi}, B_j^k}^q \mu(B_j^k)^{q/p} \leq C \left[\sum_{k,j} \|f v\|_{\bar{\Phi}, B_j^k}^p \mu(B_j^k) \right]^{q/p} \quad (48)$$

since $q \geq p$.

Consider the family of sets $\{E_j^k\}_{k,j}$ defined by $E_j^k = B_j^k \setminus \Omega_{k+1}$ and observe that the E_j^k are disjoint in both k and j . We claim that if γ is sufficiently large, there is a constant c such that $\mu(B_j^k) \leq c \mu(E_j^k)$ for all k, j . To prove this, it is enough to show that if γ is large, then

$$\mu(B_j^k \cap \Omega_{k+1}) < \frac{1}{2} \mu(B_j^k).$$

Since $\Omega_{k+1} \subset \bigcup_m cB_m^{k+1}$,

$$\mu(B_j^k \cap \Omega_{k+1}) \leq \sum_m \mu(B_j^k \cap cB_m^{k+1}). \quad (49)$$

Let B_m^{k+1} satisfy $B_j^k \cap cB_m^{k+1} \neq \emptyset$, and suppose that $r(B_j^k) \leq r(B_m^{k+1})$. Then $B_j^k \subset c_1 B_m^{k+1}$ for some geometric constant $c_1 > 1$, and therefore, using properties of B_m^{k+1} and property a) of ψ , we obtain

$$\psi(c_1 B_m^{k+1}) \int_{c_1 B_m^{k+1}} f d\mu \geq c \psi(B_m^{k+1}) \int_{B_m^{k+1}} f d\mu \geq c \gamma^{k+1} > \gamma^k$$

if γ is large enough. Now pick $\sigma = c_1$ and let $B = c_1 B_m^{k+1}$. Then $B_j^k \subset B$, $r(B) \geq \sigma r(B_j^k)$ and, by the last estimate, $\psi(B) \int_B f d\mu > \gamma^k$, in contradiction to the properties of B_j^k . Thus $r(B_j^k) > r(B_m^{k+1})$ if $B_j^k \cap cB_m^{k+1} \neq \emptyset$, and consequently, $B_m^{k+1} \subset c_1 B_j^k$ in this case. Hence, by (49) and the doubling property of μ ,

$$\begin{aligned} \mu(B_j^k \cap \Omega_{k+1}) &\leq \sum_{m: B_m^{k+1} \subset c_1 B_j^k} \mu(B_j^k \cap cB_m^{k+1}) \leq C \sum_{m: B_m^{k+1} \subset c_1 B_j^k} \mu(B_m^{k+1}) \\ &\leq \frac{C}{\gamma^{k+1}} \sum_{m: B_m^{k+1} \subset c_1 B_j^k} \psi(B_m^{k+1}) \mu(B_m^{k+1}) \int_{B_m^{k+1}} f d\mu. \end{aligned}$$

Using property b) of ψ together with the fact that the sets $\{B_m^{k+1}\}_m$ are pairwise disjoint, we can continue the chain of estimates above with

$$\begin{aligned} &\leq \frac{C}{\gamma^{k+1}} \mu(c_1 B_j^k) \psi(c_1 B_j^k) \sum_{m: B_m^{k+1} \subset c_1 B_j^k} \int_{B_m^{k+1}} f d\mu \leq \frac{C}{\gamma^{k+1}} \mu(B_j^k) \psi(c_1 B_j^k) \int_{c_1 B_j^k} f d\mu \\ &\leq \frac{C}{\gamma^{k+1}} \mu(B_j^k) \gamma^k = \frac{C}{\gamma} \mu(B_j^k), \end{aligned}$$

because of the properties of B_j^k and the fact that $\sigma = c_1$. To conclude the proof of the claim, we just choose γ so large that $C/\gamma < 1/2$.

It follows from (48) and the claim that

$$\begin{aligned} \int_{\mathcal{S}} (M_\psi f)^q d\omega &\leq C \left[\sum_{k,j} \|f v\|_{\Phi, B_j^k}^p \mu(E_j^k) \right]^{q/p} \\ &\leq C \left[\sum_{k,j} \int_{E_j^k} \mathcal{M}_{\Phi}(f v)^p d\mu \right]^{q/p} \leq C \left(\int_{\mathcal{S}} \mathcal{M}_{\Phi}(f v)^p d\mu \right)^{q/p} \end{aligned}$$

$$\leq C \left(\int_{\mathcal{S}} (fv)^p d\mu \right)^{q/p}.$$

This completes the proof of Theorem 2.7. □

7 Proof of Theorem 2.2

Recall that the potential operator T is defined by

$$Tf(x) = \int_{\mathcal{S}} f(y) K(x, y) d\mu(y) \quad (50)$$

where the kernel $K(x, y)$ satisfies (6) and μ is the underlying (doubling) measure on the space \mathcal{S} of homogeneous type. Associated with K is the functional $\varphi = \varphi_K$ acting on balls defined by

$$\varphi(B) = \sup_{\substack{x, y \in B \\ d(x, y) \geq cr(B)}} K(x, y) \quad (51)$$

for a sufficiently small positive constant c . We recall that we are assuming that for some $\epsilon > 0$, φ satisfies

$$\varphi(B_1)\mu(B_1) \leq c \left(\frac{r(B_1)}{r(B_2)} \right)^\epsilon \varphi(B_2)\mu(B_2) \quad \text{if } B_1 \subset B_2. \quad (52)$$

We divide the proof of Theorem 2.2 into several steps.

7.1 Step 1: Discretization of the potential operator

Let D_m be the grid of dyadic cubes associated with $\rho = 8\kappa^5 > 1$ and a fixed $m \in \mathbf{Z}$ as in section 3. For $f \geq 0$, let

$$T_m f(x) = \int_{d(x, y) > \rho^m} K(x, y) f(y) d\mu(y).$$

Momentarily fix x, y with $d(x, y) > \rho^m$ and pick the integer $\ell \geq m$ for which

$\rho^\ell < d(x, y) \leq \rho^{\ell+1}$. Select $Q \in D_m$ with $l(Q) = \rho^\ell$ and $x \in Q$. Let $B(Q)$ denote the

containing ball of Q , and let x_Q denote its center $x_{B(Q)}$. Thus, $\frac{1}{\rho}B(Q) \subset Q \subset B(Q)$ and $r(B(Q)) = \rho^{\ell+1}$. We then have

$$d(y, x_Q) \leq \kappa(d(y, x) + d(x, x_Q)) \leq \kappa(\rho^{\ell+1} + \rho^{\ell+1}) = 2\kappa r(B(Q)),$$

so that $y \in 2\kappa B(Q)$. Since $d(x, y) > \rho^\ell = r(2\kappa B(Q))/2\kappa\rho$, then by definition and property (10) of φ ,

$$K(x, y) \leq \varphi(2\kappa B(Q)) \leq C\varphi(B(Q)).$$

Hence,

$$K(x, y) \leq c\varphi(B(Q))\chi_Q(x)\chi_{2\kappa B(Q)}(y) \leq c \sum_{Q \in D_m} \varphi(B(Q))\chi_Q(x)\chi_{2\kappa B(Q)}(y),$$

where the last estimate holds for all x, y with $d(x, y) > \rho^m$. Therefore,

$$T_m f(x) \leq c \sum_{Q \in D_m} \varphi(B(Q))\chi_Q(x) \int_{2\kappa B(Q)} f(y) d\mu(y), \quad (53)$$

and then if $g \geq 0$, we obtain

$$\int_S (T_m f) g w d\mu \leq c \sum_{Q \in D_m} \varphi(B(Q)) \int_{2\kappa B(Q)} f d\mu \int_Q g w d\mu. \quad (54)$$

For $k \in \mathbf{Z}$ and $\gamma > 1$ to be chosen, let

$$\mathcal{C}^k = \{Q \in D_m : \gamma^k < \frac{1}{\mu(Q)} \int_Q g w d\mu \leq \gamma^{k+1}\}.$$

Assuming as we may that g is bounded and has bounded support, we can choose maximal cubes $\{Q_j^k\}_j$ in D_m with

$$\gamma^k < \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} g w d\mu.$$

If I_j^k is the next largest dyadic cube containing Q_j^k , then

$$\gamma^k < \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} g w d\mu \leq c_{\mu, \rho} \frac{1}{\mu(I_j^k)} \int_{I_j^k} g w d\mu \leq c_{\mu, \rho} \gamma^k \leq \gamma^{k+1} \quad (55)$$

by choosing $\gamma \geq c_{\mu, \rho}$. Thus $Q_j^k \in \mathcal{C}^k$. Since each cube $Q \in D_m$ must lie in some \mathcal{C}^k , it must be contained in some Q_j^k . Of course, the sets $\{Q_j^k\}_j$ are pairwise disjoint for fixed k . Then

$$\begin{aligned} \int_S (T_m f) g w d\mu &\leq c \sum_k \sum_{Q \in \mathcal{C}^k} \varphi(B(Q)) \mu(Q) \int_{2\kappa B(Q)} f d\mu \frac{1}{\mu(Q)} \int_Q g w d\mu \\ &\leq c \sum_k \gamma^{k+1} \sum_j \sum_{Q \in \Delta_m(Q_j^k)} \varphi(B(Q)) \mu(Q) \int_{2\kappa B(Q)} f d\mu, \end{aligned} \quad (56)$$

where we have used the notation $\Delta_m(Q_0) = \{Q \in D_m : Q \subset Q_0\}$.

Lemma 7.1 *Let $f \geq 0$ and φ satisfy (52), and let $\Delta_m(Q_0) = \{Q \in D_m : Q \subset Q_0\}$ if $Q_0 \in D_m$. There exists a geometric constant C such that for each $Q_0 \in D_m$,*

$$\sum_{Q \in \Delta_m(Q_0)} \varphi(B(Q)) \mu(Q) \int_{2\kappa B(Q)} f d\mu \leq C \varphi(B(Q_0)) \mu(Q_0) \int_{\kappa(2\kappa+1)B(Q_0)} f d\mu. \quad (57)$$

Proof: The left side of (57) equals

$$\sum_{\ell=0}^{\infty} \sum_{\substack{Q \in \Delta_m(Q_0) \\ \ell(Q) = \rho^{-\ell} \ell(Q_0)}} \varphi(B(Q)) \mu(Q) \int_{2\kappa B(Q)} f d\mu,$$

which by (52) is at most

$$\begin{aligned} &c \sum_{\ell=0}^{\infty} \sum_{\substack{Q \in \Delta_m(Q_0) \\ \ell(Q) = \rho^{-\ell} \ell(Q_0)}} \rho^{-\ell \epsilon} \varphi(B(Q_0)) \mu(Q_0) \int_{2\kappa B(Q)} f d\mu \\ &= c \varphi(B(Q_0)) \mu(Q_0) \sum_{\ell=0}^{\infty} \rho^{-\ell \epsilon} \sum_{\substack{Q \in \Delta_m(Q_0) \\ \ell(Q) = \rho^{-\ell} \ell(Q_0)}} \int_{2\kappa B(Q)} f d\mu. \end{aligned} \quad (58)$$

To estimate the last expression, first observe that if $Q \subset Q_0$ and $\ell(Q) \leq \ell(Q_0)$, then $2\kappa B(Q) \subset \kappa(2\kappa+1)B(Q_0)$, since if $y \in 2\kappa B(Q)$ then

$$\begin{aligned} d(y, x_{Q_0}) &\leq \kappa[d(y, x_Q) + d(x_Q, x_{Q_0})] \leq \kappa[2\kappa r(B(Q)) + r(B(Q_0))] \\ &= \kappa[2\kappa \rho \ell(Q) + r(B(Q_0))] \leq \kappa[2\kappa \rho \ell(Q_0) + r(B(Q_0))] = \kappa(2\kappa+1) r(B(Q_0)). \end{aligned}$$

Thus (58) is at most

$$c \varphi(B(Q_0)) \mu(Q_0) \sum_{\ell=0}^{\infty} \rho^{-\epsilon \ell} \int_{\kappa(2\kappa+1)B(Q_0)} \sum_{\substack{Q \in D_m \\ \ell(Q)=\rho^{-\ell}\ell(Q_0)}} \chi_{2\kappa B(Q)}(x) f(x) d\mu(x),$$

and therefore (57) will follow if we show that

$$\sum_{\substack{Q \in D_m \\ \ell(Q)=\rho^{-\ell}\ell(Q_0)}} \chi_{2\kappa B(Q)}(x) \leq C \tag{59}$$

uniformly in x, j, k, l, m . To prove this, fix x, j, k, l, m and write $r = \rho^{-\ell}\ell(Q_0)$. If $Q \in D_m$, $\ell(Q) = r$ and $x \in 2\kappa B(Q)$, then for any $y \in Q$ we have

$$\begin{aligned} d(x, y) &\leq \kappa[d(x, x_Q) + d(x_Q, y)] \leq \kappa[2\kappa r(B(Q)) + r(B(Q))] \\ &\leq \kappa(2\kappa + 1)\rho\ell(Q) = c_1 r, \end{aligned}$$

so that $Q \subset B(x, c_1 r)$. But those $Q \in D_m$ with $\ell(Q) = r$ are disjoint, and consequently by doubling, since each Q has sidelength comparable to the radius of $B(x, c_1 r)$, the number of such $Q \subset B(x, c_1 r)$ is bounded uniformly in x and r . This proves (59) and so also (57).

We will apply Lemma 7.1 to each Q_j^k . In fact, by combining (56), (57) and (55), we obtain

$$\begin{aligned} \int_S (T_m f) g w d\mu &\leq c \sum_{k,j} \gamma^{k+1} \varphi(B(Q_j^k)) \mu(Q_j^k) \int_{\kappa(2\kappa+1)B(Q_j^k)} f d\mu \\ &\leq c\gamma \sum_{k,j} \varphi(B(Q_j^k)) \int_{\kappa(2\kappa+1)B(Q_j^k)} f d\mu \int_{Q_j^k} g w d\mu. \end{aligned} \tag{60}$$

This completes the process of discretizing T_m .

7.2 Applying the condition on the weights

For simplicity, let $\tilde{Q}_j^k = \kappa(2\kappa + 1)B(Q_j^k)$. We estimate (60) by using the generalized Hölder inequality (34) and the growth condition (9):

$$\int_S (T_m f) g w d\mu$$

$$\begin{aligned}
&\leq c \sum_{k,j} \varphi(B(Q_j^k)) \mu(\tilde{Q}_j^k) \left(\frac{1}{\mu(\tilde{Q}_j^k)} \int_{\tilde{Q}_j^k} f d\mu \right) \left(\frac{1}{\mu(Q_j^k)} \int_{Q_j^k} g w d\mu \right) \mu(Q_j^k) \\
&\leq c \sum_{k,j} \varphi(\tilde{Q}_j^k) \mu(\tilde{Q}_j^k) \|fv\|_{\Phi, \tilde{Q}_j^k} \|v^{-1}\|_{\Phi, \tilde{Q}_j^k} \|g\|_{\Psi, Q_j^k} \|w\|_{\Psi, Q_j^k} \mu(Q_j^k),
\end{aligned}$$

where we have used (10) to majorize $\varphi(B(Q_j^k))$ by a multiple of $\varphi(\tilde{Q}_j^k)$. By the doubling of μ , we can also majorize $\|w\|_{\Psi, Q_j^k}$ by a fixed multiple of $\|w\|_{\Psi, \tilde{Q}_j^k}$. Then using condition (14) on the weights and Hölder inequality, we can continue the estimates above with

$$\begin{aligned}
&\leq c \sum_{k,j} \left(\varphi(\tilde{Q}_j^k) \mu(\tilde{Q}_j^k)^{\frac{1}{q} + \frac{1}{p'}} \|w\|_{\Psi, \tilde{Q}_j^k} \|v^{-1}\|_{\Phi, \tilde{Q}_j^k} \right) \|fv\|_{\Phi, \tilde{Q}_j^k} \mu(\tilde{Q}_j^k)^{\frac{1}{p}} \|g\|_{\Psi, Q_j^k} \mu(Q_j^k)^{\frac{1}{q'}} \\
&\leq c \sum_{k,j} \|fv\|_{\Phi, \tilde{Q}_j^k} \mu(\tilde{Q}_j^k)^{\frac{1}{p}} \|g\|_{\Psi, Q_j^k} \mu(Q_j^k)^{\frac{1}{q'}} \\
&\leq c \left(\sum_{k,j} \|fv\|_{\Phi, \tilde{Q}_j^k}^q \mu(\tilde{Q}_j^k)^{\frac{q}{p}} \right)^{1/q} \left(\sum_{k,j} \|g\|_{\Psi, Q_j^k}^{q'} \mu(Q_j^k) \right)^{1/q'} \\
&\leq c \left(\sum_{k,j} \|fv\|_{\Phi, \tilde{Q}_j^k}^p \mu(Q_j^k) \right)^{1/p} \left(\sum_{k,j} \|g\|_{\Psi, Q_j^k}^{q'} \mu(Q_j^k) \right)^{1/q'} \tag{61}
\end{aligned}$$

since $q \geq p$ and μ is doubling.

7.3 Patching the pieces together

Recall that the family $\{Q_j^k\}_j$ consists of maximal dyadic cubes satisfying

$$\gamma^k < \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} g w d\mu,$$

and that we also have

$$\frac{1}{\mu(Q_j^k)} \int_{Q_j^k} g w d\mu \leq c_{\mu, \rho} \gamma^k. \tag{62}$$

Let $\Omega_k = \{x : M_m^d g(x) > \gamma^k\}$ where $M_m^d g$ is the dyadic maximal function defined by

$$M_m^d g(x) = \sup_{\substack{Q: x \in Q \\ Q \in \mathcal{D}_m}} \frac{1}{\mu(Q)} \int_Q |g| d\mu,$$

and note that $\Omega_k = \cup_j Q_j^k$. As before, consider the sets $E_j^k = Q_j^k \setminus \Omega_{k+1}$. These are pairwise disjoint in both j and k , and we will show that there is a universal constant c such that for each j, k ,

$$\mu(Q_j^k) \leq c\mu(E_j^k).$$

The proof is somewhat easier than before due to the dyadic structure. In fact, since $\mu(Q_j^k) = \mu(Q_j^k \cap \Omega_{k+1}) + \mu(E_j^k)$, it is enough to show that

$$\mu(Q_j^k \cap \Omega_{k+1}) \leq \frac{c_{\mu,\rho}}{\gamma} \mu(Q_j^k)$$

and then pick $\gamma > c_{\mu,\rho}$. We have

$$\mu(Q_j^k \cap \Omega_{k+1}) = \sum_{\ell} \mu(Q_j^k \cap Q_{\ell}^{k+1}).$$

If $Q_j^k \cap Q_{\ell}^{k+1}$ is nonempty, then by the dyadic structure, either $Q_j^k \subset Q_{\ell}^{k+1}$ or $Q_{\ell}^{k+1} \subset Q_j^k$. If Q_j^k were strictly contained in Q_{ℓ}^{k+1} , then by the maximality of Q_j^k we would have

$$\frac{1}{\mu(Q_{\ell}^{k+1})} \int_{Q_{\ell}^{k+1}} g w d\mu \leq \gamma^k,$$

which contradicts the fact that this average exceeds γ^{k+1} . Consequently, $Q_{\ell}^{k+1} \subset Q_j^k$ if these sets intersect, and therefore

$$\begin{aligned} \mu(Q_j^k \cap \Omega_{k+1}) &= \sum_{\ell: Q_{\ell}^{k+1} \subset Q_j^k} \mu(Q_{\ell}^{k+1}) \leq \sum_{\ell: Q_{\ell}^{k+1} \subset Q_j^k} \frac{1}{\gamma^{k+1}} \int_{Q_{\ell}^{k+1}} g w d\mu \\ &\leq \frac{1}{\gamma^{k+1}} \int_{Q_j^k} g w d\mu \leq \frac{c_{\mu,\rho}}{\gamma} \mu(Q_j^k), \end{aligned}$$

which proves the assertion above.

Consequently, by (61),

$$\int_S (T_m f) g w d\mu \leq c \left(\sum_{k,j} \|f v\|_{\bar{\Phi}, \bar{Q}_j^k}^p \mu(Q_j^k) \right)^{1/p} \left(\sum_{k,j} \|g\|_{\bar{\Psi}, \bar{Q}_j^k}^{q'} \mu(Q_j^k) \right)^{1/q'}$$

$$\begin{aligned}
&\leq c \left(\sum_{k,j} \|fv\|_{\bar{\Phi}, \bar{Q}_j^k}^p \mu(E_j^k) \right)^{1/p} \left(\sum_{k,j} \|g\|_{\bar{\Psi}, \bar{Q}_j^k}^{q'} \mu(E_j^k) \right)^{1/q'} \\
&\leq c \left(\sum_{k,j} \int_{E_j^k} \mathcal{M}_{\bar{\Phi}}(fv)^p d\mu \right)^{1/p} \left(\sum_{k,j} \int_{E_j^k} \mathcal{M}_{\bar{\Psi}}(g)^{q'} d\mu \right)^{1/q'} \\
&\leq c \left(\int_S \mathcal{M}_{\bar{\Phi}}(fv)^p d\mu \right)^{1/p} \left(\int_S \mathcal{M}_{\bar{\Psi}}(g)^{q'} d\mu \right)^{1/q'} \leq c \left(\int_S (fv)^p d\mu \right)^{1/p} \left(\int_S g^{q'} d\mu \right)^{1/q'}
\end{aligned}$$

by using Theorem 5.1, since $\bar{\Phi} \in B_p$ and $\bar{\Psi} \in B_{q'}$ by hypothesis. Since the constant c is independent of m , Theorem 2.2 now follows from duality by letting $m \rightarrow \infty$.

□

8 Proof of Theorem 2.5

8.1 A local version of a classical lemma of Wiener

We will derive a local version of a result of N. Wiener which leads to a way to control the $L \log L$ norm of a function by the L^1 norm of its Hardy–Littlewood maximal function. We first need a local version of the classical Calderón–Zygmund decomposition as shown in [MP].

We adapt the arguments there to our context.

Fix $\delta > 0$ and a ball B_0 , and consider the following family of balls adapted to B_0 :

$$\mathcal{B} = \mathcal{B}_{B_0, \delta} = \{B : x_B \in B_0 \text{ and } r(B) \leq \delta r(B_0)\}. \quad (63)$$

This family has the properties listed in the next lemma. We use the notation $\widehat{B}_0 = (1 + \delta)\kappa B_0$, where κ is the quasimetric constant of d , and we also denote $f_B = \frac{1}{\mu(B)} \int_B f d\mu$.

In order to obtain our local version of the Calderón–Zygmund lemma, we begin with the following observations.

Lemma 8.1 *Let B_0 be a ball, \mathcal{B} be defined as in (63), and D be the doubling order of μ relative to \widehat{B}_0 , i.e.,*

$$\frac{\mu(\widehat{B}_0)}{\mu(B)} \leq c_\mu \left(\frac{r(\widehat{B}_0)}{r(B)} \right)^D \text{ if } B \subset \widehat{B}_0.$$

Let f be a nonnegative function which is integrable on \widehat{B}_0 .

a) *If $B \in \mathcal{B}$ then $B \subset \widehat{B}_0$.*

b) *If $B \in \mathcal{B}$ and $f_B > \lambda$, then*

$$r(B) \leq \left(c_\mu \frac{f_{\widehat{B}_0}}{\lambda} \right)^{1/D} r(\widehat{B}_0).$$

If we also assume that $\lambda \geq \gamma f_{\widehat{B}_0}$, where $\gamma = c_\mu \frac{M}{\delta}$ and $M > 0$, then

$$r(B) \leq \left(\frac{\delta}{M} \right)^{1/D} r(\widehat{B}_0).$$

Proof: The first observation follows from the quasimetric inequality, since if $B \in \mathcal{B}$ and $x \in B$, then

$$\begin{aligned} d(x, x_{B_0}) &\leq \kappa[d(x, x_B) + d(x_B, x_{B_0})] < \kappa[r(B) + r(B_0)] \\ &\leq \kappa[\delta r(B_0) + r(B_0)] = \kappa(1 + \delta)r(B_0). \end{aligned}$$

To show b), let $B \in \mathcal{B}$ and $f_B > \lambda$. Then by using a) and the doubling property of μ , we have

$$\lambda < f_B \leq \frac{\mu(\widehat{B}_0)}{\mu(B)} f_{\widehat{B}_0} \leq c_\mu \left(\frac{r(\widehat{B}_0)}{r(B)} \right)^D f_{\widehat{B}_0},$$

and the first part of b) follows. The second part of b) is a simple corollary of the first part. \square

Given an integrable function f on \widehat{B}_0 , the maximal function of f associated to \mathcal{B} is defined by

$$M_{\mathcal{B}}f(x) = \sup_{B: x \in B \in \mathcal{B}} \frac{1}{\mu(B)} \int_B |f| d\mu$$

if x belongs to an element of the basis \mathcal{B} , and $M_{\mathcal{B}}f(x) = 0$ otherwise. Therefore, if $\lambda > 0$ and we define

$$\Omega_\lambda = \{x \in \mathcal{S} : M_{\mathcal{B}}f(x) > \lambda\},$$

then $\Omega_\lambda \subset \widehat{B}_0$.

The version of the Calderón–Zygmund lemma that we will use is given in the next lemma.

Lemma 8.2 *Let f be a nonnegative and integrable function on \widehat{B}_0 , γ be as in part c) of Lemma 8.1, and $\lambda \geq \gamma f_{\widehat{B}_0}$. If Ω_λ is not empty, then given $\sigma > 1$, there exists a countable family $\{B_i\}$ of pairwise disjoint balls such that*

- i) $\cup_i B_i \subset \Omega_\lambda \subset \cup_i B_i^*$, where $B^* = \kappa(4\kappa + 1)B$ and κ is the quasimetric constant
- ii) $r(B_i) \leq \left(\frac{\delta}{M}\right)^{1/D} r(\widehat{B}_0)$ for all i , where D is the doubling order of μ ,
- iii) $\frac{1}{\mu(B_i)} \int_{B_i} f d\mu > \lambda$ for all i ,
- iv) $\frac{1}{\mu(\sigma B_i)} \int_{\sigma B_i} f d\mu \leq \lambda$ if $\sigma B_i \in \mathcal{B}$.

Proof: The proof is similar to that of Lemma 5.2. Fix f and λ . If $x \in \Omega_\lambda$, there exists a ball $B' \in \mathcal{B}$ with $x \in B'$ and $\frac{1}{\mu(B')} \int_{B'} f d\mu > \lambda$. Define $R = R(x, f, \lambda)$ by

$$R = \sup\{r(B) : B \in \mathcal{B}, x \in B \text{ and } f_B > \lambda\}.$$

Lemma 8.1 implies that

$$R \leq \left(\frac{\delta}{M}\right)^{1/D} r(\widehat{B}_0).$$

Then there is a ball B_x with $x \in B \in \mathcal{B}$ whose radius satisfies $\frac{R}{\sigma} < r(B_x) \leq R$ and for which $f_{B_x} > \lambda$. For this ball, ii), iii) and iv) hold with B_x in place of B_i . Part iii) implies that $\Omega_\lambda = \bigcup_{x \in \Omega_\lambda} B_x$. Picking a Vitali type subcover of $\{B_x\}_{x \in \Omega_\lambda}$ as in [SW], Lemma 3.3, we obtain

a family of pairwise disjoint balls $\{B_i\} \subset \{B_x\}_{x \in \Omega_\lambda}$ satisfying i) as well as the rest of the properties. □

In our context, we have the following version of a classical estimate due to Wiener.

Lemma 8.3 *Let f be a nonnegative locally integrable function, $\delta > 0$, B_0 be any ball, and γ be as in Lemma 8.1 with M chosen to satisfy*

$$M \geq [\kappa^2(4\kappa + 1)(1 + \delta)]^D \delta^{1-D},$$

where κ is the quasimetric constant and D is the doubling order of μ . Then there exists a constant $A = A_{\kappa, \mu}$ such that for each $\lambda \geq \gamma f_{\widehat{B}_0}$,

$$\frac{1}{\lambda} \int_{\{x \in \widehat{B}_0 : f(x) > \lambda\}} f d\mu \leq A \mu(\{x \in \widehat{B}_0 : M_{\mathcal{B}}(f)(x) > \lambda\}). \quad (64)$$

Remark 8.4 *In \mathbf{R}^n , if we consider a cube Q instead of a ball, and if M_Q^d denotes the usual dyadic maximal operator with respect to Q , then it is not difficult to see that a corresponding inequality holds with no “blow-up” in the constant. To be more precise, we then have*

$$\frac{1}{\lambda} \int_{\{x \in Q : f(x) > \lambda\}} f d\mu \leq 2^n \mu(\{x \in Q : M_Q^d f(x) > \lambda\})$$

for $\lambda > f_Q$.

Proof of Lemma 8.3: Fix $\lambda \geq \gamma f_{\widehat{B}_0}$ with $\gamma = c_\mu M / \delta$ and M to be chosen. Note that

$\Omega_\lambda = \{x \in \widehat{B}_0 : M_{\mathcal{B}}(f)(x) > \lambda\}$. We may assume without loss of generality that f is bounded.

We may also assume that Ω_λ is not empty since $\{x \in \widehat{B}_0 : f(x) > \lambda\} \subset \Omega_\lambda$ (except possibly for a set of μ -measure zero) by the Lebesgue differentiation theorem. Applying Lemma 8.2 to f and λ with $\sigma = \kappa(4\kappa + 1)$, we obtain a family of disjoint balls $\{B_i\}$ satisfying

$\bigcup_i B_i \subset \Omega_\lambda \subset \bigcup_i \sigma B_i$. Furthermore, for all i , $\frac{1}{\mu(B_i)} \int_{B_i} f d\mu > \lambda$ and $\frac{1}{\mu(\sigma B_i)} \int_{\sigma B_i} f d\mu \leq \lambda$ since

$\sigma B_i \in \mathcal{B}$ if we choose M with $M^{1/D} \geq \kappa^2(4\kappa + 1)(1 + \delta)\delta^{1/D-1}$, because σB_i is centered in B_0 and

$$r(\sigma B_i) = \sigma r(B_i) \leq \sigma \left(\frac{\delta}{M} \right)^{1/D} r(\widehat{B}_0) = \sigma \left(\frac{\delta}{M} \right)^{1/D} (1 + \delta)\kappa r(B_0) \leq \delta r(B_0)$$

for such M .

Then, since $\{x \in \widehat{B}_0 : f(x) > \lambda\} \subset \Omega_\lambda$ a.e., these properties together with the doubling of μ imply that

$$\begin{aligned} \int_{\{x \in \widehat{B}_0 : f(x) > \lambda\}} f d\mu &\leq \int_{\Omega_\lambda} f d\mu \leq \sum_i \int_{\sigma B_i} f d\mu \\ &\leq \lambda \sum_i \mu(\sigma B_i) \leq A \lambda \sum_i \mu(B_i) \leq A \lambda \mu(\Omega_\lambda), \end{aligned}$$

which proves the lemma. □

8.2 Proof of Theorem 2.5

We will show that Theorem 2.5 follows from Corollary 2.3. Thus, we must show that there are positive constants C and β so that the pair of weights $(w^{1/p}, \{M_{\tilde{\varphi}}(M^{[p]}w)\}^{1/p})$ satisfies the condition

$$L_B := \varphi(B)\mu(B) \|w^{1/p}\|_{L^p(\log L)^{p-1+\beta}(B)} \|\{M_{\tilde{\varphi}}(M^{[p]}w)\}^{-1/p}\|_{L^{p'}(\log L)^{p'-1+\beta}(B)} \leq C$$

for all balls B in \mathcal{S} . Recalling that $\tilde{\varphi}(B) = (\varphi(B)\mu(B))^p \mu(B)^{-1}$ and that $\widehat{B} = (1 + \delta)\kappa B$, we have that $M_{\tilde{\varphi}}(M^{[p]}w)(x)^{-1/p} \leq \left(\frac{[\varphi(\widehat{B})\mu(\widehat{B})]^p}{\mu(\widehat{B})} \int_{\widehat{B}} M^{[p]}w d\mu \right)^{-1/p}$ for all $x \in B$. Hence,

$$\begin{aligned} L_B^p &\leq [\varphi(B)\mu(B)]^p \|w^{1/p}\|_{L^p(\log L)^{p-1+\beta}(B)}^p \left(\frac{[\varphi(\widehat{B})\mu(\widehat{B})]^p}{\mu(\widehat{B})} \int_{\widehat{B}} M^{[p]}w d\mu \right)^{-1} \\ &\leq c \|w\|_{L(\log L)^{p-1+\beta}(B)} \left(\frac{1}{\mu(\widehat{B})} \int_{\widehat{B}} M^{[p]}w d\mu \right)^{-1}, \end{aligned}$$

by the reverse doubling properties of $\varphi(B)$ and $\mu(B)$ (see (9) for φ).

Therefore, showing that $L_B \leq C$ will follow from proving that

$$\|w\|_{L(\log L)^{p-1+\beta}(B)} \leq \frac{C^p}{\mu(\widehat{B})} \int_{\widehat{B}} M^{[p]}w \, d\mu \quad (65)$$

for an appropriate β . Now, if we choose $\beta = [p] - p + 1 > 0$, we must check that

$$\|w\|_{L(\log L)^{[p]}(B)} \leq \frac{C^p}{\mu(\widehat{B})} \int_{\widehat{B}} M^{[p]}w \, d\mu.$$

Hence, everything is reduced to the following general lemma.

Lemma 8.5 *Let $\delta > 0$, $k = 1, 2, \dots$, B be a ball and $\widehat{B} = (1 + \delta)\kappa B$ where κ is the quasimetric constant. Then there is a positive a constant c such that for any measurable function w ,*

$$\|w\|_{L(\log L)^k(B)} \leq \frac{c}{\mu(\widehat{B})} \int_{\widehat{B}} M^k w \, d\mu. \quad (66)$$

This lemma, in the same form but in the context of \mathbf{R}^n and with balls replaced by cubes (with no “blow-up” $(1 + \delta)\kappa$ in the constant) can be found in [P7]. A similar estimate is also given in both [GI] and [WW]. The idea of deducing $L \log L$ behavior of a function from integrability of its maximal function goes back to E. Stein in [St], although the inequality proved there does not preserve homogeneity as ours.

Proof: By definition of the Luxemburg norm, (66) will follow from showing that for some constant $c > 1$, c independent of w ,

$$\frac{1}{\mu(B)} \int_B \frac{w}{\lambda_B} \log^k \left(1 + \frac{w}{\lambda_B}\right) \, d\mu \leq 1, \quad (67)$$

where we denote $\lambda_B = \frac{c}{\mu(\widehat{B})} \int_{\widehat{B}} M^k w \, d\mu$.

To prove this, we will use induction. We start by proving (67) with $k=1$. Let $f = w/\lambda_B$.

Recall that $f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$ so that $0 \leq f_{\widehat{B}} \leq \frac{1}{c}$ by the Lebesgue differentiation theorem and the definition of λ_B . Using the formula

$$\int_X \Phi(f) \, d\nu = \int_0^\infty \Phi'(\lambda) \nu(\{x \in X : f(x) > \lambda\}) \, d\lambda,$$

which holds for any Young function Φ , we have

$$\begin{aligned} \frac{1}{\mu(B)} \int_B f \log(1+f) d\mu &= \frac{1}{\mu(B)} \int_0^\infty \frac{1}{1+\lambda} f(\{x \in B : f(x) > \lambda\}) d\lambda \\ &= \frac{1}{\mu(B)} \int_0^{\gamma f_{\widehat{B}}} + \frac{1}{\mu(B)} \int_{\gamma f_{\widehat{B}}}^\infty \dots = I + II, \end{aligned}$$

where γ is given in Lemma (8.1) and where we use the notation $f(E) = \int_E f d\mu$ for any measurable set E . Recalling that $f_{\widehat{B}} \leq 1/c$, we have by the doubling property of μ that

$$I \leq C \gamma f_{\widehat{B}}^2 \leq \frac{C\gamma}{c^2}.$$

For II , we use estimate (64):

$$\begin{aligned} II &= \frac{1}{\mu(B)} \int_{\gamma f_{\widehat{B}}}^\infty \frac{1}{1+\lambda} f(\{x \in B : f(x) > \lambda\}) d\lambda \\ &\leq \frac{A}{\mu(B)} \int_{\gamma f_{\widehat{B}}}^\infty \frac{\lambda}{1+\lambda} \mu(\{x \in \widehat{B} : M_B f(x) > \lambda\}) d\lambda \\ &\leq \frac{A}{\mu(B)} \int_0^\infty \mu(\{x \in \widehat{B} : M f(x) > \lambda\}) d\lambda = \frac{A}{\mu(B)} \int_{\widehat{B}} M f d\mu = \frac{A}{\mu(B)} \int_{\widehat{B}} M w d\mu \frac{1}{\lambda_B} \leq \frac{C}{c} \end{aligned}$$

by definition of λ_B and doubling. Therefore,

$$I + II \leq \frac{C\gamma}{c^2} + \frac{C}{c} \leq 1$$

if c is large enough.

We now assume that the estimate holds for a certain k . Then with $f = w/\lambda_B$ and

$$\lambda_B = \frac{c}{\mu(\widehat{B})} \int_{\widehat{B}} M^{k+1} w d\mu,$$

$$\begin{aligned} \frac{1}{\mu(B)} \int_B f \log^{k+1}(1+f) d\mu &= \frac{k+1}{\mu(B)} \int_0^\infty \frac{\log^k(1+\lambda)}{1+\lambda} f(\{x \in B : f(x) > \lambda\}) d\lambda \\ &= \frac{k+1}{\mu(B)} \left(\int_0^{\gamma f_{\widehat{B}}} + \int_{\gamma f_{\widehat{B}}}^\infty \dots \right) = I + II. \end{aligned}$$

Again by the Lebesgue differentiation theorem, we can choose $c \geq 1$ independent of B and w such that $I \leq 1/2$. Let $\Phi(\lambda) = \lambda \log^k(1 + \lambda)$. Then $\Phi'(\lambda) = \log^k(1 + \lambda) + \frac{k\lambda \log^{k-1}(1+\lambda)}{1+\lambda}$, and again for II we use estimate (64):

$$\begin{aligned}
II &= \frac{k+1}{\mu(B)} \int_{\gamma f_{\widehat{B}}}^{\infty} \frac{\log^k(1+\lambda)}{1+\lambda} f(\{x \in B : f(x) > \lambda\}) d\lambda \\
&\leq \frac{A(k+1)}{\mu(B)} \int_{\gamma f_{\widehat{B}}}^{\infty} \frac{\lambda \log^k(1+\lambda)}{1+\lambda} \mu(\{x \in \widehat{B} : M_{\mathcal{B}}f(x) > \lambda\}) d\lambda \\
&\leq \frac{A(k+1)}{\mu(B)} \int_0^{\infty} \Phi'(\lambda) \mu(\{x \in \widehat{B} : Mf(x) > \lambda\}) d\lambda \quad \text{since } \frac{\lambda}{1+\lambda} \leq 1 \\
&\leq \frac{A(k+1)}{\mu(B)} \int_{\widehat{B}} Mf \log^k(1 + Mf) d\mu \leq 1/2.
\end{aligned}$$

In the last inequality, we have used the induction hypothesis applied to Mw since $f = w/\lambda_B$ and $\lambda_B = \frac{c}{\mu(\widehat{B})} \int_{\widehat{B}} M^{k+1}w d\mu \geq \frac{c}{\mu(\widehat{B})} \int_{\widehat{B}} M^k w d\mu$, and we have chosen an appropriately large constant c in order to compensate for the factor $A(k+1)$. This completes the proof of the lemma and hence the proof of Theorem 2.5.

□

References

- [A] D. R. Adams, *Weighted nonlinear potential theory*, Trans. Amer. Math. Soc. **297** (1986), 73–94.
- [BS] C. Bennett, R. C. Sharpley, *Interpolation of Operators*, Pure and Appl. Math. **129**, Academic Press, 1988.
- [BSa] A. Bernardis and O. Salinas, *Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type*, Studia Math. **108** (1994), 201–207.

- [CW] R. R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. **242**, Springer-Verlag, 1971.
- [CWW] A. Chang, M. Wilson, T. Wolff, *Some weighted norm inequalities concerning the Schrödinger operator*, Comm. Math. Helv. **60** (1985), 217–246.
- [ChW2] S. Chanillo, R. L. Wheeden, *L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators*, Comm. P. D. E. **10** (1985), 1077–1116.
- [CP1] D. Cruz-Uribe, SFO, and C. Pérez, *Two weight extrapolation via the maximal operator*, preprint.
- [CP2] D. Cruz-Uribe, C. Pérez, *Two-weight, weak-type norm inequalities for singular integral operators*, to appear in Mathematical Research Letters, **6** 1999, 1-11.
- [D] D. Danielli, *A Fefferman–Phong inequality and applications to quasilinear subelliptic equations*, Potential Anal. **4** 1999, 387-413.
- [FP] C. Fefferman, D. H. Phong, *Subelliptic eigenvalue estimates*, Conference on Harmonic Analysis, Chicago, W. Beckner et al. ed., Wadsworth, 1981, 590–606.
- [FL] B. Franchi, E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), 523–541.
- [FLW] B. Franchi, G. Lu, R. L. Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier **45** (1995), 577–604.
- [FGuW] B. Franchi, C. E. Gutiérrez, R. L. Wheeden, *Weighted Sobolev–Poincaré inequalities for Grushin type operators*, Comm. P. D. E. **19** (1994), 523–604.

- [GGK] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, *Criteria of general weak type inequalities for integral transforms with positive kernels*, Proc. Georgian Acad. Sci. (Math) **1** (1993), 11–34.
- [GI] L. Greco and T. Iwaniec, *New inequalities for the Jacobian*, Ann. Inst. Henri Poincaré, **11** (1994), 17–35.
- [GK] M. Gabidzashvili, V. Kokilashvili, *Two weight weak type inequalities for fractional type integrals*, Ceskoslovenska Akademie Ved., **45** (1989), 1–11.
- [H] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.
- [HLP] G. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, 1934.
- [Je] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander’s condition*, Duke Math. J. **53** (1986), 503–523.
- [JPW] B. Jawerth, C. Pérez, and G. Welland, *The positive cone in Triebel–Lizorkin spaces and the relation among potential and maximal operators*, Contemporary Mathematics (M. Milman editor), Amer. Math. Soc., Providence (1989).
- [MP] P. MacManus, C. Pérez, *Generalized Poincaré inequalities: Sharp Self–Improving Properties*, International Mathematics Research Notices **2** (1998), 101–116.
- [MS] R. Macias, C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), 257–270.
- [MW] B. Muckenhoupt, R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–275.

- [NSW] A. Nagel, E. M. Stein, S. Wainger, *Balls and metrics defined by vector fields, I: Basic properties*, Acta Math. **155** (1985), 103–147.
- [N] C. J. Neugebauer, *Inserting A_p weights*, Proc. Amer. Math. Soc. **87** (1983), 272–276.
- [O] R. O’Neil, *Integral transforms and tensor products on Orlicz spaces and $L_{p,q}$ spaces*, J. D’Anal. Math. 21 (1968), 1–276.
- [P1] C. Pérez, *Two weighted norm inequalities for potential and fractional maximal operators*, Indiana U. Math. J. **43** (1994), 663–683.
- [P2] C. Pérez, *Sharp L^p -weighted Sobolev inequalities*, Ann. Inst. Fourier **45** (1995), 809–824.
- [P3] C. Pérez, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p -spaces with different weights*, Proc. London Math. Soc. 71 (1995), 135–57.
- [P4] C. Pérez, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc. 49 (1994), 296–308.
- [P5] C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, J. Fourier Anal. Appl. 3 (1997), 743–756.
- [P6] C. Pérez, *Sharp weighted inequalities for the vector-valued maximal function*, to appear Trans. Amer. Math. Soc.
- [P7] C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. of Functional Analysis **127** (1995), 163–185.

- [PW] C. Pérez, R. L. Wheeden, *Norm estimates relating fractional integrals and maximal functions*, preprint.
- [R] Y. Rakotondratsimba, *Improved Muckenhoupt–Wheeden inequality and weighted inequalities for potential operators*, *Publicacions Mat.* **39** (1995), 23–41.
- [RR] M. Rao, Z. Ren, *Theory of Orlicz spaces*, Marcel Dekker, Inc. New York, 1991.
- [RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. II, Fourier Analysis, Self-adjointness, Academic Press, 1975.
- [S] E. T. Sawyer, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, *Trans. Amer. Math. Soc.* **308** (1988), 533–545.
- [SW] E. T. Sawyer, R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, *Amer. J. Math.* **114** (1992), 813–874.
- [SWZ] E. T. Sawyer, R. L. Wheeden, S. Zhao, *Weighted inequalities for operators of potential type and fractional maximal functions*, *Potential Analysis* **5** (1996) 523–580.
- [St] E. M. Stein, *Note on the class $L \log L$* , *Studia Math.* **32** (1969), 305–310.
- [TVZ] S. Treil, A. Volberg and D. Zheng, *Hilbert transform, Hankel operators, and invariant A_∞ weights*, preprint (1997).
- [VW] I. E. Verbitsky, R. L. Wheeden, *Weighted norm inequalities for integral operators*, *Trans. Amer. Math. Soc.* **350** (1998), 3371–3391.
- [W] R. L. Wheeden, *A characterization of some weighted norm inequalities for the fractional maximal function*, *Studia Math.* **107** (1993), 251–272.

- [WW] R. L. Wheeden, J. M. Wilson, *Weighted norm estimates for gradients of half-space extensions*, Indiana U. Math. J. **44** (1995), 917–969.
- [WZ] R. L. Wheeden, S. Zhao, *Weak type estimates for operators of potential type*, Studia Math. **119** (1996), 149–160.

C. Pérez: Departamento de Matemáticas

Universidad Autónoma de Madrid

28049 Madrid, Spain

carlos.perez@uam.es

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R. L. Wheeden: Department of Mathematics

Rutgers University

New Brunswick, New Jersey 08903, USA

wheeden@math.rutgers.edu

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