

TWO WEIGHT EXTRAPOLATION VIA THE MAXIMAL OPERATOR

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ABSTRACT. We give several extrapolation theorems for pairs of weights of the form $(w, M^k w)$ and $(w, (Mw/w)^r w)$, where w is any non-negative function, $r > 1$ and M^k is the k -th iterate of the Hardy-Littlewood maximal operator. As an application we show that our results can be used to extend and sharpen results for square functions and singular integral operators by Chang, Wilson and Wolff [4], Chanillo and Wheeden [5], Wilson [24, 25, 26] and Uchiyama [22]. In the process we prove a conjecture due to Wilson.

1. INTRODUCTION

An extrapolation theorem is a result for deducing the boundedness of an operator on a family of weighted L^p spaces from the fact that the operator is bounded on $L^{p_0}(w)$ for some fixed p_0 (often $p_0 = 2$) and some family of weights. The classical extrapolation theorem is due to Rubio de Francia [19] (see also [11]), who showed that if T is a sublinear operator such that for some p_0 , $1 \leq p_0 < \infty$, T is bounded on $L^{p_0}(w)$ for every $w \in A_{p_0}$ then for every p , $1 < p < \infty$, T is bounded on $L^p(w)$ for every $w \in A_p$. This theorem and its variants have proved to be key in solving many problems in harmonic analysis.

The purpose of this paper is to derive extrapolation results for pairs of weights which do not belong to the class A_∞ . More precisely, we prove two types of extrapolation theorems. The first is for pairs of weights of the form $(w, M^k w)$, where M is the Hardy-Littlewood maximal operator, $M^k = M \cdot M \cdots M$ is the k -th iterate of the maximal operator, and w is any non-negative function. Such pairs of weights arise from attempts to generalize to other operators a result of C. Fefferman and Stein [10] for the maximal operator: for every p , $1 < p < \infty$, every non-negative function w and every function f ,

$$(1.1) \quad \int_{\mathbb{R}^n} (Mf)^p w \, dx \leq C \int_{\mathbb{R}^n} |f|^p M w \, dx.$$

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Our first result is the following. Here and below, by weights we mean non-negative, locally integrable functions.

Theorem 1.1. *Let S and T be operators (not necessarily linear) and let f be a function in a suitable test class for both S and T .*

- (1) *Suppose that there exist positive constants p_0 and C_0 and a positive integer k such that for all weights w*

$$(1.2) \quad \int_{\mathbb{R}^n} |Tf|^{p_0} w \, dx \leq C_0 \int_{\mathbb{R}^n} |Sf|^{p_0} M^k w \, dx.$$

Then for all p , $p_0 < p < \infty$, there exists a constant C_p depending only on C_0 , p_0 , p , k , and n , such that for all weights w ,

$$(1.3) \quad \int_{\mathbb{R}^n} |Tf|^p w \, dx \leq C_p \int_{\mathbb{R}^n} |Sf|^p M^{[kp/p_0]+1} w \, dx,$$

where $[kp/p_0]$ is the largest integer less than or equal to kp/p_0 .

- (2) *Similarly, if for a fixed t and for all weights w ,*

$$(1.4) \quad w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C_0}{t^{p_0}} \int_{\mathbb{R}^n} |Sf|^{p_0} M^k w \, dx,$$

then

$$(1.5) \quad w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C_p}{t^p} \int_{\mathbb{R}^n} |Sf|^p M^{[pk/p_0]+1} w \, dx.$$

If the operator T is sublinear and S is the identity operator then inequalities similar to (1.5) have turned out to be very useful in the study of the two-weight problem for singular integral operators—see [7]. In this case the second half of Theorem 1.1 can be strengthened to the following.

Corollary 1.2. *Let T be a sublinear operator such that there exist positive constants p_0 and C_0 and a positive integer k such that for all weights w and $t > 0$*

$$(1.6) \quad w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C_0}{t^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} M^k w \, dx.$$

Then for all p , $p_0 < p < \infty$, there exists a constant C_p depending only on C_0 , p , k , and n , such that for all weights w ,

$$(1.7) \quad \int_{\mathbb{R}^n} |Tf|^p w \, dx \leq C_p \int_{\mathbb{R}^n} |f|^p M^{[kp/p_0]+1} w \, dx,$$

where $[kp/p_0]$ is the largest integer less than or equal to kp/p_0 .

The second extrapolation theorem we prove is for pairs of the form $(w, (Mw/w)^r w)$, where again w is a weight, M is the maximal operator and $r > 1$. We were led to consider such pairs of weights by a result of Chanillo and Wheeden [5] for the square function: if f is in the Schwartz class and $2 < p < \infty$ then for every non-negative w ,

$$(1.8) \quad \int_{\mathbb{R}^n} S(f)^p w \, dx \leq C_p \int_{\mathbb{R}^n} |f|^p (Mw/w)^{p/2} w \, dx.$$

(For further details, see Section 2 below.)

Our results for such pairs of weights are analogous to the results above, and we summarize them compactly as follows.

Theorem 1.3. *Theorem 1.1 and Corollary 1.2 remain true if in inequalities (1.2), (1.4) and (1.6) $M^k w$ is replaced Mw and in inequalities (1.3), (1.5) and (1.7) $M^{[kp/p_0]+1} w$ is replaced by $(Mw/w)^{p/p_0} w$. The constant C_p in each case depends only on C_0 , p_0 , p and n .*

By slightly modifying the proof of Theorem 1.1, we can prove sharp weighted norm inequalities for the vector-valued maximal operator. Given a vector-valued function $f = \{f_i\}$, define the vector-valued maximal operator $\mathcal{M}f = \{Mf_i\}$, and for $1 < q < \infty$ define the real-valued operator \overline{M}_q by

$$\overline{M}_q f(x) = \|\mathcal{M}f\|_q = \left(\sum_{i=1}^{\infty} Mf_i(x)^q \right)^{1/q}.$$

This operator was introduced by C. Fefferman and Stein [10] as a generalization of both the Hardy-Littlewood maximal function and the Marcinkiewicz integral. It follows from the Fefferman-Stein inequality (1.1) that if $p = q$ then

$$(1.9) \quad \int_{\mathbb{R}^n} (\overline{M}_q f)^p w \, dx \leq C \int_{\mathbb{R}^n} \|f\|_q^p Mw \, dx,$$

where $\|\cdot\|_q$ denotes the ℓ^q norm. It can be easily shown using vector-valued interpolation between the endpoints $q = \infty$ and $q = p$ that this inequality also holds for $1 < p < q$. The case $p > q$, however, is more interesting since it reflects the higher ‘‘singularity’’ of \overline{M}_q for small values of q .

Theorem 1.4. *Let $1 < q < p < \infty$.*

- (1) *There exists a constant C , depending on p , q and n , such that for all locally integrable f and weights w ,*

$$(1.10) \quad \int_{\mathbb{R}^n} (\overline{M}_q f)^p w \, dx \leq C \int_{\mathbb{R}^n} \|f\|_q^p M^{[p/q]+1} w \, dx.$$

(2) Inequality (1.10) is sharp since there is no finite constant C such that

$$(1.11) \quad \int_{\mathbb{R}^n} (\overline{M}_q f)^p w \, dx \leq C \int_{\mathbb{R}^n} \|f\|_q^p M^{[p/q]} w \, dx$$

holds for all locally integrable f and weights w . The analogous weak-type (p, p) inequality is also false.

This result was first proved in [18] by different means.

We now make a number of observations about our results.

Remark 1.5. Theorems 1.1, 1.3 and 1.4, and Corollary 1.2 remain true if the maximal operator is everywhere replaced by the dyadic maximal operator M_d . The proofs below go through with only minor alterations; details are left to the reader.

Remark 1.6. In inequalities (1.2) and (1.10) the number of iterates, k , can be thought of as measuring the “singularity” of the operator. For example, the Fefferman-Stein inequality (1.1) shows that if $T = M$ and S is the identity, then for any $p_0 > 1$, (1.2) holds with $k = 1$. But for singular integral operators, the sharp exponent is $k = [p_0] + 1$. (See (2.2) below.) Further, for higher order commutators or for nonlinear commutators the sharp exponent k is larger than $[p_0] + 1$, reflecting a worse singularity. (See [17].)

Remark 1.7. Unlike the extrapolation theorem of Rubio de Francia, we can only extrapolate “up” and cannot go “down”. A simple counter-example is given by $M_r f = M(|f|^r)^{1/r}$, $r > 1$. For by the Fefferman-Stein inequality,

$$\int_{\mathbb{R}^n} M_r(f)^p w \, dx \leq C \int_{\mathbb{R}^n} |f|^p M w \, dx$$

holds for all f and w when $p > r$ but fails for $p = r$.

For an example of a two-weight extrapolation theorem which goes down, see Neugebauer [12].

Remark 1.8. For the applications which we consider in Section 2 below, it follows from well-known results that inequalities of the form

$$(1.12) \quad \int_{\mathbb{R}^n} |Tf|^p w \, dx \leq C \int_{\mathbb{R}^n} |Sf|^p M_r w \, dx$$

hold for $r > 1$. However, since $M_r w$ is an A_1 weight (see [11]), $Mw \leq M(M_r w) \leq CM_r w$, so by iteration, $M^k w \leq C^{k-1} M_r w$. Hence inequality (1.3) is sharper than inequality (1.12). Further, we note that the weights $M^k w$, $k \geq 1$, are not necessarily A_∞ weights. (It is an open question to characterize the weights w such that Mw is in A_∞ . For partial results see [6].)

Remark 1.9. Because of the generality of these results the restrictions on f must be vague. In the proof, the only requirement is that f be such that the left-hand side of inequality (1.3) is finite. In practice (e.g. in the examples considered in Section 2 below) it usually suffices to assume f is in C_0^∞ , in the Schwartz class, or in L^p .

Remark 1.10. In the cases when Theorems 1.1 and 1.3 overlap (i.e. when $k = 1$ in inequality (1.2)), neither result is necessarily stronger than the other. To see this, consider the following two examples. First, let

$$w(x) = \chi_{[0,1]}(x) + x^{-1}\chi_{(1,\infty)}(x).$$

Then for $x \geq e$, $M^k w(x) \approx x^{-1}(\log x)^k$, so if we let $p_0 = 2$ and $p = 4$ then

$$\frac{M^{[p/p_0]+1}w}{(Mw/w)^{p/p_0}w} = \frac{M^3w}{(Mw/w)^2w} \approx \log x.$$

Hence, for large x , $M^3w \gg (Mw/w)^2w$.

Second, let

$$w(x) = x\chi_{[0,1]}(x) + \chi_{(1,\infty)}(x).$$

Then $M^3w(x) = Mw(x) = 1$, so for $0 < x < 1$,

$$\frac{M^3w}{(Mw/w)^2w} = x.$$

Hence, for x close to 0, $(Mw/w)^2w \gg M^3w$.

Remark 1.11. Theorem 1.1 and the corresponding part of Theorem 1.3 remain true if we replace Tf and Sf , where f is a fixed function, by arbitrary but fixed functions f_1 and f_2 respectively. This is not the case for Corollary 1.2.

Remark 1.12. Theorems 1.1 and 1.3 can be extended to give extrapolation results for mixed norm inequalities. For example, given p_0, q_0 such that

$$\left(\int_{\mathbb{R}^n} |Tf|^{p_0} w \, dx \right)^{1/p_0} \leq C_0 \left(\int_{\mathbb{R}^n} |Sf|^{q_0} M^k w \, dx \right)^{1/q_0},$$

then for $p > p_0, q > q_0$ such that $p/q = p_0/q_0$,

$$\left(\int_{\mathbb{R}^n} |Tf|^p w \, dx \right)^{1/p} \leq C_p \left(\int_{\mathbb{R}^n} |Sf|^q M^{[kq/q_0]+1} w \, dx \right)^{1/q}.$$

Further details are left to the reader.

Finally, we make some observations about the proofs of our results. The proofs of both Theorems 1.1 and 1.3 depend on duality, “separating” $M^k(gw)$ into M_0gM_1w (where here M_0 and M_1 denote certain appropriate maximal operators), and the norm inequalities for M_0 . Thus in the proof of Theorem 1.3, we use the relatively simple

observation that $M(gw)$ can be factored into $M_w g \cdot Mw$, where M_w is the weighted, centered Hardy-Littlewood maximal operator, and the well known fact that M_w is a bounded operator on $L^p(w)$, $1 < p < \infty$.

In the proof of Theorem 1.1 the “separation” involves Orlicz spaces, and leads to sharper versions of this theorem and Corollary 1.2. Given a Young function A , we define the maximal operator M_A by

$$M_A f(x) = \sup_{Q \ni x} \|f\|_{A,Q},$$

where $\|f\|_{A,Q}$ denotes the localized Luxemburg norm

$$\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A \left(\frac{|f|}{\lambda} \right) dy \leq 1 \right\}.$$

We say that a Δ_2 Young function A satisfies the B_p condition if there exists $c > 0$ such that

$$(1.13) \quad \int_c^\infty \frac{A(t) dt}{t^p t} < \infty.$$

An obvious example of such a function is $t^{p+\delta}$, with $\delta > 0$. More interesting are examples of the form

$$A(t) \approx t^p (\log t)^{p-1+\delta}, \quad \delta > 0.$$

A key step in our approach is following result from [16]: $A \in B_p$ if and only if $M_A : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$. Given $k \geq 1$ we show that there exist Young functions A and C such that $M^k(gw) \leq 2M_A w \cdot M_C g$. Further, we can choose A and C so that $M_A w \approx M^{\lfloor kp/p_0 \rfloor + 1} w$ and C satisfies the $B_{(p/p_0)'}$ condition, giving us the desired norm inequality.

It follows immediately that we can strengthen Theorem 1.1 and Corollary 1.2 by replacing $M^{\lfloor kp/p_0 \rfloor + 1} w$ in the conclusions by $M_A w$ for suitably chosen A . We will show, for example, that we can take $A(t) \approx t(\log t)^{(kp/p_0)-1+\epsilon}$ for any $\epsilon > 0$. Frequently these estimates are sharp in that we cannot take $\epsilon = 0$. (See, for instance, part (2) of Theorem 1.4 or the main counterexample in [7].)

The remainder of this paper is organized as follows: in Section 2 we give a number of applications of our results and in Section 3 we give the proofs.

Throughout this paper all notation is standard or will be defined as needed. Given a positive real number x , $[x]$ will denote the largest integer less than or equal to x . All cubes are assumed to have their sides parallel to the coordinate axes. By weights we will always mean non-negative, locally integrable functions. Given a measurable set E and a weight v , $|E|$ will denote the Lebesgue measure of E , and $v(E) = \int_E v dx$. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent of p . Finally, C will denote a positive constant whose value may change at each appearance.

2. APPLICATIONS

In this section we give several applications of Theorems 1.1 and 1.3. In each case we show how existing results for which inequality (1.2) is known for $0 < p_0 \leq 2$ can be extended to the range $p > 2$.

2.1. Square Functions. Our principal application is to square functions and area integrals. Let $\phi \in C^\infty$ be a radial function such that ϕ has compact support and $\int \phi dx = 0$, and let $\phi_t(x) = t^{-n}\phi(x/t)$. Define the square function S_ϕ by

$$S_\phi(f)(x) = \left(\int_{|x-y|<t} |(f * \phi_t)(y)|^2 \frac{dtdy}{t^{n+1}} \right)^{1/2}.$$

Chang, Wilson and Wolff [4] showed that inequality (1.2) holds for $p_0 = 2$, $T = S_\phi$, S equal to the identity, and $k = 1$. Chanillo and Wheeden generalized their result as follows: let ψ be a Schwartz function such that $\int \psi dx = 0$, let $\psi_t(x) = t^{-n}\psi(x/t)$, and define the area function

$$S_\psi(f)(x) = \left(\int_{|x-y|<t} |\nabla_{y,t}(f * \psi_t)(y)|^2 \frac{dtdy}{t^{n-1}} \right)^{1/2}.$$

They showed that inequality (1.2) holds for $1 < p_0 \leq 2$, $T = S_\psi$, S equal to the identity and $k = 1$. It follows immediately from these results and from Theorem 1.1 that for all $p > 2$

$$\int_{\mathbb{R}^n} |S_\psi(f)|^p w dx \leq C \int_{\mathbb{R}^n} |f|^p M^{[p/2]+1} w dx.$$

An example given by Chanillo and Wheeden shows that when $p > 2$ the exponent $[p/2] + 1$ is sharp. Alternatively, their inequality (1.8) above follows immediately from Theorem 1.3.

In the dyadic case, Uchiyama [22] noted that the arguments of Chang, Wilson and Wolff, and Chanillo and Wheeden showed that inequality (1.2) holds for $1 < p_0 \leq 2$, S equal to the identity, T equal to the dyadic square function

$$S_d(f)(x) = \left(\sum_{x \in Q \in D} (f_Q - f_{\tilde{Q}})^2 \right)^{1/2},$$

(where D is the collection of all dyadic cubes in \mathbb{R}^n , \tilde{Q} is the smallest dyadic cube properly containing Q and f_Q is the average of f on Q) and with M replaced by M_d . He extended these results to the range $p > 2$ by showing that

$$\int_{\mathbb{R}^n} S_d(f)^p w dx \leq C \int_{\mathbb{R}^n} |f|^p M_d^{[p/2]+2} w dx.$$

By combining his observation in the case $p = 2$ with Theorem 1.1 we can improve his result to the following:

$$\int_{\mathbb{R}^n} S_d(f)^p w \, dx \leq C \int_{\mathbb{R}^n} |f|^p M_d^{[p/2]+1} w \, dx.$$

If p is not an even integer, $[p/2] + 1 = -[-p/2]$, so in this case we have answered in the affirmative a question posed by Wilson [25]. (See also Derrick [8].)

We also consider the “converse” inequality for the dyadic square function:

$$(2.1) \quad \int_{\mathbb{R}^n} |f|^p w \, dx \leq C \int_{\mathbb{R}^n} S_d(f)^p M_d^k w \, dx.$$

(There is a similar inequality for the continuous square function S_ϕ .) C. Fefferman [9] asked if inequality (2.1) was true when $p = 2$ and $k = 1$. A counter-example (actually for the continuous square function) was given by Chang, Wilson and Wolff [4]. Wilson [24, 26] then gave a relatively straightforward proof that inequality (2.1) holds for $0 < p < 2$ and $k = 1$ and a more difficult argument showing that it was true for $p \geq 2$ with $k = [p/2] + 1$. (He has similar results for a variant of the continuous square function: see [25].) However, his results for $p \geq 2$ follow immediately from the case $p < 2$ and from Theorem 1.1. Further, Theorem 1.3 implies a new result: if $p \geq 2$ and $0 < p_0 < 2$ then

$$\int_{\mathbb{R}^n} |f|^p w \, dx \leq C \int_{\mathbb{R}^n} S_d(f)^p (Mw/w)^{p/p_0} w \, dx.$$

2.2. Calderón-Zygmund singular integral operators. Another application of our results is to Calderón-Zygmund singular integral operators. Wilson [25] showed that if T is a regular singular integral operator (see [11]) and $1 < p < 2$, then for every function $f \in \bigcup_{q>1} L^q$ and weight w

$$(2.2) \quad \int_{\mathbb{R}^n} |Tf|^p w \, dx \leq C \int_{\mathbb{R}^n} |f|^p M^2 w \, dx.$$

Further, he showed that if $p = 2$ then M^2 must be replaced by M^3 . Using a different method, it was shown in [15] that for $1 < p < \infty$ and for arbitrary Calderón-Zygmund singular integral operators

$$\int_{\mathbb{R}^n} |Tf|^p w \, dx \leq C \int_{\mathbb{R}^n} |f|^p M^{[p]+1} w \, dx,$$

where the exponent $[p] + 1$ is sharp. Using Theorem 1.1 we can now deduce this result for $p \geq 2$ directly from Wilson’s result for $1 < p < 2$.

3. PROOFS OF THEOREMS

3.1. Preliminaries. We begin by recalling a few facts about Orlicz spaces. (For further details see Bennett and Sharpley [1].) A function $B : [0, \infty) \rightarrow [0, \infty)$ is a Young function if it is continuous, convex and increasing, and if $B(0) = 0$ and $B(t) \rightarrow \infty$ as $t \rightarrow \infty$. A Young function satisfies the Δ_2 condition if $B(2t) \leq CB(t)$ for all $t > 0$. Each Young function B has associated to it a complementary Young function \bar{B} such that for all $t > 0$

$$(3.1) \quad t \leq B^{-1}(t)\bar{B}^{-1}(t) \leq 2t.$$

Given a Young function B , we define the B -average of a function f over a cube Q by

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B \left(\frac{|f|}{\lambda} \right) dy \leq 1 \right\}.$$

Given three Young functions A , B and C such that for all $t > 0$

$$(3.2) \quad A^{-1}(t)C^{-1}(t) \leq B^{-1}(t),$$

then we have the following generalized Hölder's inequality due to O'Neil [13]: for any cube Q and all functions f and g ,

$$(3.3) \quad \|fg\|_{B,Q} \leq 2\|f\|_{A,Q}\|g\|_{C,Q}.$$

In particular, given complementary functions A and \bar{A} , inequality (3.1) becomes

$$(3.4) \quad \frac{1}{|Q|} \int_Q |fg| dx \leq 2\|f\|_{A,Q}\|g\|_{\bar{A},Q}.$$

(This particular case is originally due to Weiss [23].)

Finally, define the maximal operator M_B by

$$M_B f(x) = \sup_{Q \ni x} \|f\|_{B,Q}.$$

It follows at once from inequality (3.3) that if A , B and C satisfy (3.2) then for all $x \in \mathbb{R}^n$,

$$(3.5) \quad M_B(fg)(x) \leq 2M_A f(x)M_C g(x);$$

or given complementary functions A and \bar{A} ,

$$(3.6) \quad M(fg)(x) \leq 2M_A f(x)M_{\bar{A}} g(x).$$

3.2. Proof of Theorem 1.1. First suppose that inequality (1.2) holds. Fix $p, p_0 < p < \infty$, and let $r = p/p_0$. Then by duality

$$\left(\int_{\mathbb{R}^n} |Tf|^p w \, dx \right)^{1/r} = \sup_g \int_{\mathbb{R}^n} |Tf|^{p_0} g w \, dx,$$

where the supremum is taken over all $g \in C_0^\infty$ such that $\|g\|_{L^{r'}(w)} = 1$. Therefore, to show inequality (1.3) it will suffice to show that for any such g ,

$$\int_{\mathbb{R}^n} |Tf|^{p_0} g w \, dx \leq C_p \left(\int_{\mathbb{R}^n} |Sf|^p M^{[kr]+1} w \, dx \right)^{1/r}.$$

By our hypothesis

$$(3.7) \quad \int_{\mathbb{R}^n} |Tf|^{p_0} g w \, dx \leq C_0 \int_{\mathbb{R}^n} |Sf|^{p_0} M^k(gw) \, dx.$$

A result of Stein [20] implies that $M^k(gw) \approx M_B(gw)$, where $B(t) = t \log(1+t)^{k-1}$. (For details see [16, p. 151] or Carozza and Passarelli [3].) Fix $\epsilon > 0$ such that $kr - 1 + \epsilon = [kr]$. Then

$$B^{-1}(t) \approx \frac{t}{(\log t)^{k-1}} = A^{-1}(t)C^{-1}(t),$$

where A and C are Young functions such that

$$A(t) \approx t^r (\log t)^{kr-1+\epsilon} \quad \text{and} \quad C(t) \approx t^{r'} (\log t)^{-1-(r'-1)\epsilon}.$$

(This triple of Young functions is due to O'Neil [14]. For details see [17].) Therefore, by inequality (3.5) and by Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}^n} |Sf|^{p_0} M_B(gw) \, dx &\leq 2 \int_{\mathbb{R}^n} |Sf|^{p_0} M_A(w^{1/r}) M_C(gw^{1/r'}) \, dx \\ &\leq 2 \left(\int_{\mathbb{R}^n} |Sf|^p M_A(w^{1/r})^r \, dx \right)^{1/r} \left(\int_{\mathbb{R}^n} M_C(gw^{1/r'})^{r'} \, dx \right)^{1/r'} \end{aligned}$$

A computation shows that C satisfies the $B_{r'}$ condition (1.13): there exists $c > 0$ such that

$$\int_c^\infty \frac{C(t)}{t^{r'}} \frac{dt}{t} < \infty.$$

As we noted above (again see [16]) this is a necessary and sufficient condition for M_C to be bounded on $L^{r'}(\mathbb{R}^n)$. Therefore

$$\int_{\mathbb{R}^n} M_C(gw^{1/r'})^{r'} \, dx \leq K \int_{\mathbb{R}^n} (gw^{1/r'})^{r'} \, dx = K,$$

where the constant K depends only on r' and n . Furthermore, if we let $\tilde{A}(t) = A(t^{1/r}) \approx t(\log t)^{kr-1+\epsilon}$ then again by the result of Stein used above, $M_A(w^{1/r})^r = M_{\tilde{A}}w \approx M^{[kr]+1}w$. Thus

$$\int_{\mathbb{R}^n} |Sf|^{p_0} M^k(gw) dx \leq C_p \int_{\mathbb{R}^n} |Sf|^p M^{[kr]+1}w dx,$$

where C_p depends only on C_0, p, p_0, k and n . This concludes the proof of inequality (1.3).

Now suppose that inequality (1.4) holds. Again fix $p > p_0$ and let $r = p/p_0$. Fix $t > 0$ and define $E_t = \{x \in \mathbb{R}^n : |Tf(x)| > t\}$. Then $w(E_t)^{1/r} = \|\chi_{E_t}\|_{L^r(w)}$, and by duality

$$\|\chi_{E_t}\|_{L^r(w)} = \sup_g \int_{\mathbb{R}^n} g\chi_{E_t}w dx = \sup_g (gw)(E_t),$$

where the supremum is taken over all $g \in C_0^\infty$ such that $\|g\|_{L^{r'}(w)} = 1$. But by inequality (1.4),

$$(gw)(E_t) \leq \frac{C_0}{t^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} M(gw) dx.$$

To prove inequality (1.5) we now estimate the integral on the right-hand side exactly as we did in the proof of inequality (1.3) above.

3.3. Proof of Corollary 1.2. The proof of this result depends on the following weighted interpolation theorem.

Lemma 3.1. *Let $0 < p_0 < p_1 < \infty$ and suppose T is a sublinear operator such that, for pairs of weights (w, v_i) , $i = 0, 1$,*

$$w(\{x : |Tf(x)| > t\}) \leq \frac{C_i}{t^{p_i}} \int_{\mathbb{R}^n} |f|^{p_i} v_i dx, \quad i = 0, 1,$$

for every $t > 0$. Fix $p, p_0 < p < p_1$, let θ be such that

$$(3.8) \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

and let $v_\theta = v_0^{(1-\theta)p/p_0} v_1^{\theta p/p_1}$. Then

$$\int_{\mathbb{R}^n} |Tf|^p w dx \leq C_0^{1-\theta} C_1^\theta \int_{\mathbb{R}^n} |f|^p v_\theta dx.$$

Proof. Lemma 3.1 follows at once from the real method of interpolation, since we have the identity

$$(L^{p_0}(v_0), L^{p_1}(v_1))_{\theta, p} = L^p(v_\theta).$$

See Bergh and Löfström [2] for further details. (Also see Stein and Weiss [21].) \square

Now suppose that inequality (1.6) holds for all w . Fix w and $p_0 < p < \infty$. Choose $p_1 > p$ such that $[kp_1/p_0] = [kp/p_0]$. Then by Theorem 1.1 and Lemma 3.1 with $v_0 = M^k w$ and $v_1 = M^{[kp/p_0]+1} w$,

$$(3.9) \quad \int_{\mathbb{R}^n} |Tf|^p w dx \leq C_0^{1-\theta} C_1^\theta \int_{\mathbb{R}^n} |f|^p v_\theta dx,$$

where θ is defined by equation (3.8) and

$$v_\theta = (M^k w)^{(1-\theta)p/p_0} (M^{[kp_1/p_0]+1} w)^{\theta p/p_1} = (M^k w)^{(1-\theta)p/p_0} (M^{[kp/p_0]+1} w)^{\theta p/p_1}.$$

Since $0 < \theta < 1$ and $M^k w \leq M^{[kp/p_0]+1} w$, $v_\theta \leq M^{[kp/p_0]+1} w$, so inequality (1.7) follows at once from inequality (3.9).

3.4. Proof of Theorem 1.3. Suppose first that

$$\int_{\mathbb{R}^n} |Tf|^{p_0} w dx \leq C_0 \int_{\mathbb{R}^n} |Sf|^{p_0} M w dx;$$

we want to show that for all $p > p_0$,

$$(3.10) \quad \int_{\mathbb{R}^n} |Tf|^p w dx \leq C_p \int_{\mathbb{R}^n} |Sf|^p (Mw/w)^{p/p_0} w dx.$$

The proof of this proceeds exactly as the proof of inequality (1.3) in Theorem 1.1, with the following changes. At inequality (3.7), rather than use inequality (3.5) we argue as follows: given functions g and w , we have that $M(gw) \leq C M_c(gw)$, where M_c is the unweighted, centered Hardy-Littlewood maximal operator, and C is a constant depending only on the dimension n . Furthermore,

$$\begin{aligned} M_c(gw)(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} gw \\ &= \sup_{r>0} \frac{w(B_r(x))}{|B_r(x)|} \frac{\int_{B_r(x)} gw}{w(B_r(x))} \\ &\leq M_{w,c} g(x) M_c w(x) \\ &= \left(M_{w,c} g(x) w(x)^{1/r'} \right) \left(w(x)^{-1/r'} M_c w(x) \right), \end{aligned}$$

where $M_{w,c}$ is the weighted, centered maximal operator, and $r = p/p_0 > 1$. We now apply Hölder's inequality as before, and use the well-known fact that $M_{w,c}$ is bounded on $L^p(w)$, $1 < p < \infty$, with a constant that depends only on p and n . Inequality (3.10) now follows with a constant that only depends on C_0 , p_0 , p and n .

The proof of the corresponding weak-type inequality is gotten from the proof of inequality (1.5) above with exactly the same modifications.

Finally, suppose that T is a sublinear operator and that

$$(3.11) \quad w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C_0}{t^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} Mw \, dx,$$

holds for all weights w , $t > 0$ and all f . Fix $p_0 < p < \infty$; we need to show that

$$(3.12) \quad \int_{\mathbb{R}^n} |Tf|^p w \, dx \leq C_p \int_{\mathbb{R}^n} |f|^p (Mw/w)^{p/p_0} w \, dx.$$

To see this, fix $p_1 > p$; then by the weak-type inequality of Theorem 1.3,

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C_0}{t^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0} (Mw/w)^{p_1/p_0} \, dx.$$

Fix θ so that (3.8) holds; then by Lemma 3.1 inequality (3.12) follows immediately since

$$(Mw)^{(1-\theta)p/p_0} [(Mw/w)^{p_1/p_0} w]^{\theta p/p_1} = (Mw)^{p/p_0} w^{\theta p/p_1 - \theta p/p_0} = (Mw/w)^{p/p_0} w.$$

3.5. Proof of Theorem 1.4. Let $r = p/q > 1$. Then by duality there exists a non-negative function g with $\|g\|_{L^{r'}(\mathbb{R}^n)} = 1$ such that

$$\begin{aligned} \left(\int_{\mathbb{R}^n} (\overline{M}_q f)^p w \, dx \right)^{1/r} &= \left\| \sum_{i=1}^{\infty} (Mf_i)^q \right\|_{L^r(w)} \\ &= \left\| \sum_{i=1}^{\infty} w^{1/r} (Mf_i)^q \right\|_{L^r(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} \sum_{i=1}^{\infty} (Mf_i)^q g w^{1/r} \, dx \\ &= \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} (Mf_i)^q g w^{1/r} \, dx. \end{aligned}$$

Given any Young function A , by the Fefferman-Stein inequality (1.1), inequality (3.6) and Hölder's inequality, we get

$$\begin{aligned}
\sum_{i=1}^{\infty} \int_{\mathbb{R}^n} (Mf_i)^q g w^{1/r} dx &\leq C \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} |f_i|^q M(g w^{1/r}) dx \\
&\leq C \sum_i \int_{\mathbb{R}^n} |f_i|^q M_A(w^{1/r}) M_{\bar{A}} g dx \\
&= C \int_{\mathbb{R}^n} \|f\|_q^q M_A(w^{1/r}) M_{\bar{A}} g dx \\
&\leq \left(\int_{\mathbb{R}^n} \|f\|_q^p M_A(w^{1/r})^r dx \right)^{1/r} \left(\int_{\mathbb{R}^n} (M_{\bar{A}} g)^{r'} dx \right)^{1/r'}.
\end{aligned}$$

Now fix $A(t) \approx t^r (\log t)^{r-1+\epsilon}$, with ϵ such that $r-1+\epsilon = [r]$. Then a computation shows that its complementary function $\bar{A}(t) \approx t^{r'} (\log t)^{-1-\epsilon(r'-1)}$ (cf. [14]) satisfies the $B_{r'}$ condition (1.13). As we noted above (again see [16]), this implies that the maximal function $M_{\bar{A}}$ is bounded on $L^{r'}(\mathbb{R}^n)$. Therefore, if we let $\tilde{A}(t) = A(t^{1/r})$, then $M_{\tilde{A}} w = M_A(w^{1/r})^r$, so it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} (\bar{M}_q f)^p w dx &\leq C \left(\int_{\mathbb{R}^n} \|f\|_q^p M_{\tilde{A}} w dx \right) \left(\int_{\mathbb{R}^n} g^{r'} dx \right)^{r/r'} \\
&= C \int_{\mathbb{R}^n} \|f\|_q^p M_A w dx.
\end{aligned}$$

But arguing as we did in the proof of Theorem 1.1, by our choice of A , $M_{\tilde{A}} w \leq C M^{[r]+1} w$, which completes the proof of part (1) of Theorem 1.4.

For the counterexample in part (2) of Theorem 1.4, fix $n = 1$ and let N be a large positive integer. Let $r = p/q > 1$, $w = \chi_{(0,1)}$, and define $f_i = (\log x)^{-1/q} \chi_{(e^i, e^{i+1})}(x)$ for each i , $1 \leq i \leq N-1$, and $f_i = 0$ for $i \geq N$. Since for $x \geq e$, $M^{[r]} w \approx x^{-1} (\log x)^{[r]-1}$, a computation shows that for any $r > 1$,

$$\left\| \left(\sum_{i=1}^{\infty} (f_i)^q \right)^{1/q} \right\|_{L^p(M^{[r]} w)}^p \approx \int_e^{e^N} (\log x)^{-r} (\log x)^{[r]-1} \frac{dx}{x} \leq \log N.$$

On the other hand,

$$\| \bar{M}_q f \|_{L^p(w)}^p = \int_0^1 \left(\sum_{i=1}^{N-1} Mf_i(x)^q \right)^r dx \geq \int_0^1 \left(\sum_{i=1}^{N-1} \frac{1}{i} \right)^r dx \geq (\log N)^r,$$

since for $0 < x < 1$ and $1 \leq i \leq N - 1$,

$$Mf_i(x) \geq \frac{1}{e^{i+1}} \int_e^{e^{i+1}} (\log(y))^{-1/q} \chi_{(e^i, e^{i+1})}(y) dy \approx \frac{1}{i^{1/q}}.$$

Thus if inequality (1.11) holds, there exists a constant C such that $(\log N)^r \leq C \log N$ holds for all large values of N , which is a contradiction. A similar calculation using the same example shows that the analogous weak-type inequality does not hold.

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