WEIGHTED NORM INEQUALITIES FOR SINGULAR INTEGRAL OPERATORS

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Abstract

For a Calderón–Zygmund singular integral operator T, we show that the following weighted inequality holds

$$\int_{\mathbb{R}^n} |Tf(y)|^p\, w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p\, M^{[p]+1}w(y)dy,$$

where M^k is the Hardy–Littlewood maximal operator M iterated k times, and $[p]$ is the integer part of p. Moreover, the result is sharp since it does not hold for $M^{[p]}$.

We also give the following endpoint result:

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, M^2 w(y)dy.$$

1 Introduction and statements of the results

A classical result due to C. Fefferman and E. Stein [4] states that the Hardy–Littlewood maximal operator M satisfies the following inequality for arbitrary $1 < p < \infty$, and weight w

$$\int_{\mathbb{R}^n} |Mf(y)|^p \, w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p \, Mw(y)dy, \tag{1}$$

where C is independent of f. A weight w in $\mathbb{R}^n$ will always be a nonnegative locally integrable function.

The study of weighted inequalities like the above, for other operators has played a central rôle in modern of Harmonic Analysis since they appear in duality arguments. We refer the reader to [5] Chapters 5 and 6 for a very nice exposition.

Although we could work with any Calderón–Zygmund operator (cf. §3), we shall only consider singular integral operators of convolution type defined by:

$$Tf(x) = p.v. \int_{\mathbb{R}^n} k(x-y)f(y)\, dy,$$

where the kernel k is C^1 away from the origin, has mean value on the unit sphere centered at the origin and satisfies for $y \neq 0$

$$|k(y)| \leq \frac{C}{|y|^n} \quad \textit{and} \quad |\nabla k(y)| \leq \frac{C}{|y|^{n+1}}.$$

It is well known that the analogous version of inequality (1) fails for the Hilbert transform for all p. In [3] A. Córdoba and C. Fefferman have shown that there is a similar inequality for any T, but with Mw replaced by the pointwise larger operator $M_r w = M(w^r)^{1/r}$, $r > 1$, that is, for $1 < p < \infty$

$$\int_{\mathbb{R}^n} |Tf(y)|^p \, w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p \, M_r w(y)dy, \tag{2}$$

with C independent of f.

The purpose of this paper is to prove weighted norm inequalities of the form (2), where $M_r w$, $r > 1$, will be replaced by appropiate smaller maximal–type operators $w \to Nw$ satisfying

$$Mw(x) \leq Nw(x) \leq C\, M_r w(x), \tag{3}$$

for each $x \in \mathbb{R}^n$. We shall also be concern with corresponding endpoints results such as weak type $(1,1)$ and H^1–L^1 estimates.

Before stating our main results, we shall make the following observation. Let M^k be the Hardy–Littlewood maximal operator M iterated k times, where $k = 1, 2, \cdots$. We claim that for $k = 2, \cdots$, and $r > 1$, there exists a positive constant C independent of w such that

$$Mw(x) \leq M^k w(x) \leq C\, M_r w(x), \tag{4}$$

for each $x \in \mathbb{R}^n$. The left inequality follows from the Lebesgue differentiation theorem; for the other, we let B be the best constant in Coifman's estimate $M(M_r w) \leq B\, M_r w$, where B is independent of w. Then, it follows easily that $M^k w \leq B^{k-1} M_r w$, $k = 1, 2, \cdots$.

In view of this observation, it is natural to consider whether or not (2) holds for some M^k, with $k = 2, 3, \cdots$. In a very interesting paper [8], M. Wilson has recently obtained the following partial answer to this question: Let $1 < p < 2$, then

$$\int_{\mathbb{R}^n} |Tf(y)|^p\, w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p\, M^2 w(y) dy. \tag{5}$$

Moreover, he shows that this estimate does not hold for $p \geq 2$, and also that when $p = 2$, $M^2 w$ can be replaced by $M^3 w$. However, his method does not yield corresponding estimates for $p > 2$ (cf. §3 of that paper), and $M^2 w$ must be replaced by a much more complicated expression.

M. Wilson's approach to this problem is based on certain (difficult) estimates for square functions that he obtained in the same paper, together with a couple of related estimates for the area function, obtained essentially by S. Chanillo and R. Wheeden in [1].

In this paper we give a complete answer to Wilson's problem by means of a different method. Our main result is the following.

Theorem 1.1: Let $1 < p < \infty$, and let T be a singular integral operator. Then, there exists a constant C such that for each weight w

$$\int_{\mathbb{R}^n} |Tf(y)|^p\, w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p\, M^{[p]+1} w(y) dy, \tag{6}$$

where $[p]$ is the integer part of p. Furthermore, the result is sharp since it does not hold for $M^{[p]}$.

The corresponding weak–type $(1,1)$ version of this result is the following.

Theorem 1.2: Let T be a singular integral operator. Then, there exists a constant C such that for each weight w and for all $\lambda > 0$

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, M^2 w(y) dy. \tag{7}$$

Remark 1.3: Let $1 < p < \infty$, a natural question is whether (7) can be extended to the case (p, p), that is whether

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p \, M^{[p]} w(y) dy,$$

holds for some constant C and for all $\lambda > 0$. At the end of section 2 we give an example showing that this inequality is false when p is not an integer; however, we do not know what happens when p is an integer.

Although we do not know whether (7) holds for Mw (cf. remark 1.7) we can give the following estimate. For a measure μ we shall denote by $H^1(\mu)$ the subspace of $L^1(\mu)$ of functions f which can be written as $f = \sum_j \lambda_j a_j$, where a_j are μ–atoms and λ_j are complex numbers with $\sum_j |\lambda_j| < \infty$. A function a is a μ–atom if there is a cube Q for which $supp(a) \subset Q$, so that

$$|a(x)| \leq \frac{1}{\mu(Q)},$$

and

$$\int_Q a(y) \, dy = 0.$$

Theorem 1.4: Let T be a singular integral operator. Then, there exists a constant C such that for each weight w

$$\int_{\mathbb{R}^n} |Tf(y)| \, w(y) dy \leq C \, \|f\|_{H^1(Mw)}. \tag{8}$$

Theorem 1.1 is in fact a consequence of a more precise estimate than (6). The idea is to replace the operator $M^{[p]+1}$ by an optimal class of maximal operators. We explain now what "optimal" means.

We want to define a scale of maximal–type operators $w \to M_A w$ such that

$$Mw(x) \leq M_A w(x) \leq M_r w(x)$$

for each $x \in \mathbb{R}^n$, where $r > 1$. A stands for a Young function; i.e. $A : [0, \infty) \to [0, \infty)$ is continuous, convex and increasing satisfying $A(0) = 0$. To define M_A we introduce for each cube Q the A–average of a function f over Q by means of the following Luxemburg norm

$$\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\}.$$

We define the maximal operator M_A by

$$M_A f(x) = \sup_{x \in Q} \|f\|_{A,Q},$$

where f is a locally integrable functions, and where the supremum is taken over all the cubes containing x. When $A(t) = t^r$ we get $M_A = M_r$, but more interesting examples are provided by Young functions like $A(t) = t \log^{\epsilon}(1+t)$, $\epsilon > 0$.

The optimal class of Young functions A is characterized by the following theorem.

Theorem 1.5: Let $1 < p < \infty$, and let T be a singular integral operator. Suppose that A is a Young function satisfying the condition

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty, \tag{9}$$

for some $c > 0$. Then, there exists a constant C such that for each weight w

$$\int_{\mathbb{R}^n} |Tf(y)|^p \, w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p \, M_A w(y) dy. \tag{10}$$

Furthermore, condition (9) is also necessary for (10) to hold for all the Riesz transforms: $T = R_1, R_2, \cdots, R_n$.

We recall that the j–th Riesz transform R_j, $j = 1, 2, \cdots, n$, is the singular integral operator defined by

$$R_j f(x) = p.v. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) \, dy.$$

The proof of this theorem is given in §2, and it is based on the following inequality of E.M. Stein [7]

$$\int_Q w(y) \log^k(1 + w(y)) \, dy \leq C \int_Q Mw(y) \log^{k-1}(1 + Mw(y)) \, dy, \tag{11}$$

with $k = 1, 2, 3, \cdots$.

As for the strong case, there is an estimate sharper than (7).

Theorem 1.6: Let T be a singular integral operator. For arbitrary $\epsilon > 0$, consider the Young function

$$A_\epsilon(t) = t \log^\epsilon(1+t). \tag{12}$$

Then, there exists a constant C such that for each weight w and for all $\lambda > 0$

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, M_{A_\epsilon} w(y) dy. \tag{13}$$

Remark 1.7: For $1 < p < \infty$ let us denote by B_p the callection of all Young functions A satisfying condition (9):

$$\int_c^\infty \left(\frac{t}{A(t)} \right)^{p'-1} \frac{dt}{t} < \infty,$$

for some $c > 0$. Observe that $B_p \subset B_q$, $1 < p < q < \infty$. Then it follows easily from the proof of last theorem that we may replace A_ϵ by any Young function belonging to the smallest class $\cap_{p>1} B_p$. We could consider for instance

$$A_\epsilon(t) = t \log(1+t)[\log\log(1+t)]^\epsilon. \tag{14}$$

If we let $\epsilon = 0$ in (12) $M_{A_0} = M$ is the Hardy–Littlewood maximal operator. Since A_0 does not belong to $\cap_{p>1} B_p$ we think that the estimate:

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, Mw(y) dy, \tag{15}$$

for some constant C, and for all $\lambda > 0$, does not hold.

2 Proof of the Theorems

Proof of Theorem 1.5:

We prove first that condition (9) is sufficient for (10) to hold for any singular integral operator T.

We may assume that $M_A w$ is finite almost everywhere, and we let T^* be the adjoint operator of T. T^* is also a singular integral operator with kernel $k^*(x) = k(-x)$. Then, by duality (10) is equivalent to

$$\int_{\mathbb{R}^n} |T^*f(y)|^{p'} M_A w(y)^{1-p'} dy \le C \int_{\mathbb{R}^n} |f(y)|^{p'} w(y)^{1-p'} dy. \tag{16}$$

We shall be using some well known facts about the A_p theory of weights for which we remit the reader to [5] Chapter 4.

To prove (16) we shall use the following fundamental estimate due to Coifman ([2]):
Let T be any singular integral operator; then for each $0 < p < \infty$, and each $u \in A_\infty$, there exists $C = C_{u,p} > 0$ such that for each $f \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |Tf(y)|^p u(y) dy \le C \int_{\mathbb{R}^n} Mf(y)^p u(y) dy. \tag{17}$$

Therefore, to apply this estimate to T^* we need to show that $(M_A w)^{1-p'}$ satisfies the A_∞ condition.

To check this, we claim first that $(M_A w)^\delta$ satisfies the A_1 condition for $0 < \delta < 1$. However, this is an straightforward generalization of the well known fact that $(Mw)^\delta \in A_1$, $0 < \delta < 1$, also due to Coifman (cf. [5] p. 158), and we shall omit its proof.

Now, since $w^{1-r} \in A_r$, for any $w \in A_1$ and $r > 1$, we have that

$$(M_A w)^{1-p'} = \left[(M_A w)^{\frac{p'-1}{r-1}} \right]^{1-r} \in \cap_{r>p'} A_r \subset A_\infty.$$

After these observations, we have reduced the problem to showing that

$$\int_{\mathbb{R}^n} Mf(y)^{p'} M_A w(y)^{1-p'} dy \le C \int_{\mathbb{R}^n} |f(y)|^{p'} w(y)^{1-p'} dy. \tag{18}$$

But this is a particular instance of the following characterization which can be found in [6] Theorem 4.4.

Theorem 2.1: Let $1 < p < \infty$. Let A be a Young function, and denote $B = \overline{A(t^{p'})}$. Then the following are equivalent.
i)

$$\int_c^\infty \left(\frac{t}{A(t)} \right)^{p-1} \frac{dt}{t} < \infty; \tag{19}$$

ii) there is a constant c such that

$$\int_{\mathbb{R}^n} M_B f(y)^p dy \le c \int_{\mathbb{R}^n} f(y)^p dy \tag{20}$$

for all nonnegative, locally integrable functions f;
iii) there is a constant c such that

$$\int_{\mathbb{R}^n} M_{B} f(y)^p \, u(y) dy \leq c \int_{\mathbb{R}^n} f(y)^p \, Mu(y) dy \tag{21}$$

for all nonnegative, locally integrable functions f and u;
iv) there is a constant c such that

$$\int_{\mathbb{R}^n} Mf(y)^p \, \frac{u(y)}{[M_{A}(w)(y)]^{p-1}} dy \leq c \int_{\mathbb{R}^n} f(y)^p \, \frac{Mu(y)}{w(y)^{p-1}} dy, \tag{22}$$

for all nonnegative, locally integrable functions f, w and u.

Observe that (18) follows from (22) by taking $u = 1$, and by replacing p by p'.

Now we shall prove that condition (9) is also necessary for (10) to hold for all the Riesz transforms. That is, suppose that the Young function A is fixed, and that the inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \, M_A w(x) dx, \tag{23}$$

is verified for each Riesz transform $T = R_j$, $j = 1, 2, \cdots, n$.

Fix one of these j. As above, by duality (23) is equivalent to

$$\int_{\mathbb{R}^n} |R_j f(x)|^{p'} \, M_A w(x)^{1-p'} dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p'} \, w(x)^{1-p'} dx, \tag{24}$$

We shall adapt an argument from [5] p. 561. We define the cone

$$E_j = \{x \in \mathbb{R}^n : \max\{|x_1|, |x_2|, \cdots, |x_n|\} = x_j\},$$

so that $\mathbb{R}^n = \cup_{j=1}^n (E_j \cup (-E_j))$. Let B be the unit ball, and consider the function $f = w = \chi_{B \cap (-E_j)}$. Then, (24) implies

$$\infty > C \int_{\mathbb{R}^n} |f(x)|^{p'} \, w(x)^{1-p'} dx = C \, |B \cap (-E_j)| \geq$$

$$\geq \int_{E_j \cap \{|x| > 2\}} |R_j f(x)|^{p'} \, M_A f(x)^{1-p'} dx.$$

Observe that for $|x|>2$, $M_A f(x) \approx A^{-1}(|x|^n)^{-1}$. Also, for every $x \in E_j$

$$R_j f(x) = C\int_{B\cap(-E_j)} \frac{x_j - y_j}{|x-y|^{n+1}}\,dy \geq C\int_{B\cap(-E_j)} \frac{1}{|x-y|^n}\,dy \geq \frac{C}{|x|^n}.$$

Therefore

$$\int_{E_j\cap\{|x|>2\}} \frac{1}{|x|^{np'}} A^{-1}(|x|^n)^{p'-1}\,dx \leq C\,|B\cap(-E_j)|.$$

A corresponding estimate can be proved for E_j, and for each $j=1,2,\cdots,n$, by using in each case the corresponding Riesz transform. Since the family of cones $\{\overset{+}{-}E_j\}_{j=1,2,\cdots,n}$ is disjoint, we finally have that

$$\int_{|x|>2} \frac{1}{|x|^{np'}} A^{-1}(|x|^n)^{p'-1}\,dx \approx \int_c^\infty \frac{1}{t^{p'}} A^{-1}(t)^{p'-1} t\,\frac{dt}{t} \approx$$

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty,$$

since $tA'(t) \approx A(t)$. This concludes the proof of the theorem.

□

Proof of Theorem 1.6:

We shall assume that $M_{A_\epsilon} w$ is finite almost everywhere, since otherwise there is nothing to be proved.

For $f \in C_0^\infty(\mathbb{R}^n)$ we consider the standard Calderón–Zygmund decomposition of f at level λ (cf. [5] p. 414).

Let $\{Q_j\}$ be the Calderón–Zygmund nonoverlapping dyadic cubes satisfying

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\,dx \leq 2^n\,\lambda. \tag{25}$$

If we let $\Omega = \cup_j Q_j$, we also have that $|f(x)| \leq \lambda$ a.e. $x \in \mathbb{R}^n \setminus \Omega$.

Using the notation $f_{Q_j} = \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx$, we write $f = g + b$ where g, the "good part", is given by

$$g(x) = \begin{cases} f(x) & x \in \mathbb{R}^n \setminus \Omega \\ f_{Q_j} & x \in Q_j \end{cases}$$

Observe that $|g(x)| \leq 2^n\,\lambda$ a.e.

The "bad part" can be split as $b = \sum_j b_j$, where $b_j(x) = (f(x) - f_{Q_j})\chi_{Q_j}(x)$. Let $\tilde{Q}_j = 2Q_j$ and $\tilde{\Omega} = \cup_j \tilde{Q}_j$.

We have

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda/2\}) \le$$
$$\le w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tg(y)| > \lambda/2\}) + 2\, w(\tilde{\Omega}) + w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tb(y)| > \lambda/2\}).$$

Pick any $p > 1$ such that $1 < p < 1 + \epsilon$. Then, it follows that $A_\epsilon = t \log^\epsilon(1+t)$ satisfies condition

$$\int_c^\infty \left(\frac{t}{A_\epsilon(t)}\right)^{p'-1} \frac{dt}{t} < \infty,$$

for some $c > 0$. Thus, we can apply Theorem 1.5 with this p to the first term, together with the fact that $|g(x)| \le 2^n \lambda$ a.e. Then, using an idea from [1] p. 282

$$w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tg(y)| > \lambda/2\}) \le \frac{C}{\lambda^p} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tg(y)|^p \, w(y) dy \le$$
$$\le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |g(y)|^p \, M_{A_\epsilon}(w\chi_{\mathbb{R}^n \setminus \tilde{\Omega}})(y) dy \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |g(y)| \, M_{A_\epsilon}(w\chi_{\mathbb{R}^n \setminus \tilde{\Omega}})(y) dy =$$
$$\frac{C}{\lambda} \left(\int_{\mathbb{R}^n \setminus \Omega} |f(y)| \, M_{A_\epsilon} w(y) dy + \int_\Omega |g(y)| \, M_{A_\epsilon}(w\chi_{\mathbb{R}^n \setminus \tilde{\Omega}})(y) dy \right) =$$
$$\frac{C}{\lambda}(I + II)$$

Since $I \le \int_{\mathbb{R}^n} |f(y)| \, M_{A_\epsilon} w(y) dy$ we only need to estimate II:

$$II \le \sum_j \int_{Q_j} |f_{Q_j}| \, M_{A_\epsilon}(w\chi_{\mathbb{R}^n \setminus \tilde{\Omega}})(y) dy \le$$
$$\sum_j \int_{Q_j} |f(x)| \, dx \frac{1}{|Q_j|} \int_{Q_j} M_{A_\epsilon}(w\chi_{\mathbb{R}^n \setminus \tilde{\Omega}})(y) \, dy.$$

We shall make use of the following fact: for arbitrary Young function A, non-negative function w with $M_A w(x) < \infty$ a.e., cube Q, and $R > 1$ we have

$$M_A(\chi_{\mathbb{R}^n \setminus RQ} w)(y) \approx M_A(\chi_{\mathbb{R}^n \setminus RQ} w)(z) \tag{26}$$

for each $y, z \in Q$. This is an observation whose proof follows exactly as for the case of the Hardy–Littlewood maximal operator M, cf. for instance [5] p. 159.

Then,

$$II \leq C \sum_j \int_{Q_j} |f(x)|\, dx \quad \inf_{Q_j} M_{A_\epsilon}(w\chi_{\mathbb{R}^n \setminus 2Q_j}) \leq C \sum_j \int_{Q_j} |f(x)|\, M_{A_\epsilon} w(x) dx$$

$$\leq C \int_{\mathbb{R}^n} |f(x)|\, M_{A_\epsilon} w(x) dx.$$

The second term is estimated as follows:

$$w(\tilde{\Omega}) \leq C \sum_j \frac{w(\tilde{Q}_j)}{\left|\tilde{Q}_j\right|} |Q_j| \leq$$

$$\frac{C}{\lambda} \sum_j \frac{w(\tilde{Q}_j)}{\left|\tilde{Q}_j\right|} \int_{Q_j} |f(x)|\, dx \leq \frac{C}{\lambda} \sum_j \int_{Q_j} |f(x)|\, Mw(x) dx \leq$$

$$\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, Mw(x) dx.$$

To estimate the last term we use the inequality

$$\int_{\mathbb{R}^n \setminus \tilde{Q}_j} |Tb_j(y)|\, w(y) dy \leq C \int_{\mathbb{R}^n} b_j(y)\, Mw(y) dy,$$

with C independent of b_j, which can be found in Lemma 3.3, p. 413, of [5]. Now, using this estimate with w replaced by $w\chi_{\mathbb{R}^n \setminus \tilde{Q}_j}$ we have

$$w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tb(y)| > \lambda/2\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tb(y)|\, w(y) dy \leq$$

$$\frac{C}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |Tb_j(y)|\, w(y) dy \leq \frac{C}{\lambda} \sum_j \int_{\mathbb{R}^n} |b_j(y)|\, M(w\chi_{\mathbb{R}^n \setminus \tilde{Q}_j})(y) dy \leq$$

$$\frac{C}{\lambda} \sum_j \int_{Q_j} |b(y)|\, M(w\chi_{\mathbb{R}^n \setminus \tilde{Q}_j})(y) dy.$$

Since $b = f - g$ this is at most

$$\frac{C}{\lambda} \sum_j \left(\int_{Q_j} |f(y)|\, Mw(y) dy + \int_{Q_j} |g(y)|\, M(w\chi_{\mathbb{R}^n \setminus \tilde{Q}_j})(y) dy \right) = \frac{C}{\lambda}(A + B)$$

To conclude the proof of the theorem is clear that we only need to estimate B. However

$$B = \sum_j \int_{Q_j} |f_{Q_j}|\, M(w\chi_{\mathbb{R}^n\setminus\tilde{Q}_j})(y)dy \leq$$

$$\sum_j \int_{Q_j} |f(x)|\, dx \frac{1}{|Q_j|} \int_{Q_j} M(w\chi_{\mathbb{R}^n\setminus\tilde{Q}_j})(x)\, dx \leq$$

$$\sum_j \int_{Q_j} |f(x)|\, dx \quad \inf_{Q_j} M(w\chi_{\mathbb{R}^n\setminus 2Q_j}) \leq \sum_j \int_{Q_j} |f(x)|\, M(w\chi_{\mathbb{R}^n\setminus 2Q_j})(x)\, dx \leq$$

$$C \int_{\mathbb{R}^n} |f(y)|\, Mw(y)dy$$

Here we have used again that $M(\chi_{\mathbb{R}^n\setminus 2Q}\mu)(y) \approx M(\chi_{\mathbb{R}^n\setminus 2Q}\mu)(z)$ for each $y, z \in Q$.

This concludes the proof of the theorem since we always have that $Mw(x) \leq M_A w(x)$ for each Young function A and for each x.

□

Proof of Theorem 1.1:

Let us assume that $M^{[p]+1}w$ is finite almost everywhere, since otherwise (6) is trivial. Let A be the Young function

$$A(t) = t \log^{[p]}(1+t).$$

A simple computation shows that A satisfies condition (9), which is the hypothesis of Theorem 1.5. Then, Theorem 1.1 will follow if we prove the pointwise inequality

$$M_A w(x) \leq C\, M^{[p]+1}w(x). \tag{27}$$

Recall that M_A is defined by $M_A f(x) = \sup_{x\in Q} \|f\|_{A,Q}$, where

$$\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\}.$$

Then, it is enough to prove that there is constant C such that for each cube Q

$$\|f\|_{A,Q} \leq \frac{C}{|Q|} \int_Q M^{[p]}w(x)\, dx.$$

By assumption, the right hand side average is finite, and by homogeneity we can assume that is equal to one. Then, by the definition of Luxemburg norm we need to prove

$$\frac{1}{|Q|}\int_Q A(w(y))\,dy = \frac{1}{|Q|}\int_Q w(y)\log^{[p]}(1+w(y))\,dy \le C.$$

But this is a consequence of iterating the following inequality of E.M. Stein [7]

$$\int_Q w(y)\log^k(1+w(y))\,dy \le C\int_Q Mw(y)\log^{k-1}(1+Mw(y))\,dy, \tag{28}$$

with $k = 1, 2, 3, \cdots$.

To conclude the proof of the theorem, we are left with showing that for arbitrary $1 < p < \infty$, the inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)dx \le C\int_{\mathbb{R}^n} |f(x)|^p\, M^{[p]}w(x)dx, \tag{29}$$

is false in general. To prove this assertion we consider the Hilbert transform

$$Hf(x) = pv\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy.$$

Then, by duality (29) is equivalent to

$$\int_{\mathbb{R}} |Hf(x)|^{p'}\, M^{[p]}w(x)^{1-p'}dx \le C\int_{\mathbb{R}} |f(x)|^{p'}\, w(x)^{1-p'}dx. \tag{30}$$

Let $f = w = \chi_{(-1,1)}$. A standard computation shows that

$$M^k f(x) \approx \frac{\log^{k-1}(1+|x|)}{|x|}, \qquad |x| \ge e$$

for each $k = 1, 2, 3, \cdots$. Then, we have

$$\int_{\mathbb{R}} |Hf(x)|^{p'}\, M^{[p]}w(x)^{1-p'}dx \ge C\int_{x>e}\left(\frac{1}{x}\right)^{p'}\left(\frac{\log^{[p]-1}(x)}{x}\right)^{1-p'}dx \approx$$

$$\approx \int_{x>e}\log^{([p]-1)(1-p')}(x)\frac{dx}{x} = \infty,$$

since $([p]-1)(1-p')+1 \geq 0$. However, the right hand side of (30) equals $\int_{\mathbb{R}} f(y)dy = 2 < \infty$.

$\square$

Proof of Theorem 1.2:

As above, we shall assume that $M^2 w$ is finite almost everywhere. For $0 < \epsilon < 1$ set as before $A_\epsilon(t) = t\log^\epsilon(1+t)$. Then, the inequality

$$\int_Q w(y)\log^\epsilon(1+w(y))\,dy \leq C\int_Q Mw(y)\,dy,$$

whose proof is analogue to that of (28) using that the derivative of $A_\epsilon(t)$ is less than of $1/t$, implies exactly as in the proof of Theorem 1.1 that

$$M_{A_\epsilon} w(x) \leq C\,M^2 w(x).$$

This concludes the proof of Theorem 1.2.

$\square$

Proof of Theorem 1.4: By an standard argument, it is enough to show that there is a costant C such that

$$\int_{\mathbb{R}^n} |Ta(y)|\,w(y)dy \leq C$$

for each Mw–atom a. To prove this, suppose that $supp(a) \subset Q$ for some cube Q. Then

$$\int_{\mathbb{R}^n} |Ta(y)|\,w(y)dy = \int_{3Q} |Ta(y)|\,w(y)dy + \int_{\mathbb{R}^n\setminus 3Q} |Ta(y)|\,w(y)dy = I + II.$$

Now, II is majorized, as in the proof of Theorem 1.6, by using Lemma 3.3, p. 413 of [5]

$$II \leq C\int_{\mathbb{R}^n} |a(y)|\,Mw(y)dy \leq \frac{C}{Mw(Q)}\int_Q Mw(y)dy = C,$$

where C is independent of a.

Fir I we use the fact that any singular integral operator $T : L^\infty(Q, \frac{dx}{|Q|}) \to L_{L_{\exp}}(Q, \frac{dx}{|Q|})$. Then

$$I = |3Q|\frac{1}{|3Q|}\int_{3Q} |Ta(y)|\,w(y)dy \leq C|Q|\|Ta\|_{_{Lexp,3Q}}\|w\|_{_{LlogL,3Q}} \leq$$

$$\leq C\,|Q| \|a\|_{\infty,3Q} \frac{1}{|3Q|} \int_{3Q} Mw(y)dy \leq C,$$

by (28) and by the definition of Mw–atom. This finishes the proof of Theorem 1.4.

□

We shall end this section by disproving inequality

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p\, M^{[p]}w(y)dy \tag{31}$$

from remark 1.3, whenever p is greater than one but not an integer.

Consider $T = H$ the Hilbert tranform as above. For $\lambda > 0$, we let $f = \chi_{(1,e^\lambda)}$, and $w = \chi_{(0,1)}$. Then for $y \neq 1, e^\lambda$

$$Hf(y) = \log |\tfrac{y-1}{y-e^\lambda}|.$$

When $y \in (0,1)$ we have

$$|Hf(y)| = |\log |\tfrac{y-1}{y-e^\lambda}|| = \log \frac{e^\lambda - y}{1-y} > \log e^\lambda = \lambda.$$

Then, assuming that (31) holds for all λ we had

$$1 = \int_0^1 w(y)dy \leq w(\{y \in (0,1) : |Hf(y)| > \lambda\}) \leq$$

$$\leq \frac{C}{\lambda^p} \int_{\mathbb{R}} |f(y)|^p\, M^{[p]}w(y)dy = \frac{C}{\lambda^p} \int_1^{e^\lambda} M^{[p]}w(y)dy \approx$$

$$\approx \frac{1}{\lambda^p} \int_1^{e^\lambda} \log^{[p]-1} w(y)dy \approx \lambda^{[p]-p}.$$

By letting $\lambda \to \infty$ we see that this a contradiction when p is not an integer.

There is another argument due to S. Hofmann, and is as follows. Since p is not an integer we can find an small $\epsilon > 0$ such that $[p] < p - \epsilon < p < p + \epsilon < [p] + 1$. Then, (31) implies that M is at once of weak type $(p-\epsilon, p-\epsilon)$ and $(p+\epsilon, p+\epsilon)$ with respect to the weights $(w, M^{[p]}w)$. Then, by the Marcinkiewicz interpolation theorem M is of strong type (p,p) with respect to the weights $(w, M^{[p]}w)$. But this is a contradiction as shown in Theorem 1.1.

3 Calderón–Zygmund operators

In this section we shall state our main results for the more general Calderón–Zygmund operators.

We recall the definition of a Calderón–Zygmund operator in $\mathbb{R}^n$.

A kernel on $\mathbb{R}^n \times \mathbb{R}^n$ will be a locally integrable complex–valued fuction K, defined on $\Omega = \mathbb{R}^n \times \mathbb{R}^n \setminus$ diagonal. A kernel K on $\mathbb{R}^n$ satisfies the standard estimates, if there exist $\delta > 0$ and $C < \infty$ such that for all distinct $x, y \in \mathbb{R}^n$ and all z such that $|x-z| < |x-y|/2$:

(i) $|K(x,y)| \leq C\,|x-y|^{-n}$;

(ii) $|K(x,y) - K(z,y)| \leq C \left(\dfrac{|x-z|}{|x-y|}\right)^{\delta} |x-y|^{-n}$;

(iii) $|K(y,x) - K(y,z)| \leq C \left(\dfrac{|x-z|}{|x-y|}\right)^{\delta} |x-y|^{-n}$.

We say that a linear and continuous operator $T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is associated with a kernel K, if

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)g(x)f(y)\,dxdy,$$

whenever $f, g \in C_0^\infty(\mathbb{R}^n)$ with $supp(f) \cap supp(g) = \emptyset$.

We say that T is a Calderón–Zygmund operator if the associated kernel K satisfies the standard estimates, and if it extends to a bounded linear operator in $L^2(\mathbb{R}^n)$.

Theorem 3.1: Let $1 < p < \infty$, and let T be a Calderón–Zygmund operator. Then, there exists a constant C such that for each weight w

$$\int_{\mathbb{R}^n} |Tf(y)|^p\, w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p\, M^{[p]+1}w(y)dy, \tag{32}$$

and there exists another constant C such that for all $\lambda > 0$

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(y)|\, M^2 w(y)dy. \tag{33}$$

The proof of Theorem 3.1 is essentially the same as Theorems 1.1 and 1.2, after observing that the adjoint T^* of any Calderón–Zygmund operator T is also a Calderón–Zygmund operator with kernel $K^*(x,y) = K(y,x)$.

There are corresponding results to Theorems 1.2, 1.4, 1.5, and for 1.6 for any Calderón– Zygmund operator. We shall omit the obvious statements.

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