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Potential operators, maximal functions, and generalizations of A_∞

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1 Introduction

The purpose of this paper is to prove two kinds of weighted norm inequalities for integral operators of potential type and their associated maximal operators in spaces of homogeneous type. The first kind of result is an extension of a result of Muckenhoupt and Wheeden, which showed that for A_∞ weights in Euclidean space, a weighted L^p norm of the classical Riesz fractional integral operator $I_\alpha f$ of a function f is equivalent to the same norm of the fractional maximal function $M_\alpha f$. Our extension of this result involves fairly general integral operators of potential type and their associated maximal operators in spaces of homogeneous type, and classes of weights which are more general than A_∞. The second kind of inequality that we will prove gives two-weight L^p, L^q norm estimates for such maximal operators, assuming again an appropriate (but substantially weakened) version of A_∞. An important point here is that we completely avoid using the "good-lambda inequality" technique of Burkholder and Gundy. Finally, we can combine these two results to obtain two-weight L^p, L^q norm estimates for potential operators that improve similar ones derived by Sawyer and Wheeden.

In the usual n-dimensional Euclidean space R^n, if $0 < \alpha < n$, let

$$I_\alpha f(x) = \int_{R^n} f(y) \frac{1}{|x-y|^{n-\alpha}} \, dy$$

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AMS Subject Classification 2000: 42B25, 44A15 denote the Riesz fractional integral of f of order α, and let

$$M_\alpha f(x) = \sup_{B: x \in B} r(B)^{\alpha-n} \int_B |f(y)|\, dy$$

be the corresponding fractional maximal function of f, where B denotes a Euclidean ball and $r(B)$ is the radius of B. The pointwise inequality

$$M_\alpha f(x) \le C\, I_\alpha f(x),$$

with C independent of x and f, follows easily from the definitions. On the other hand, the reverse pointwise inequality is false, but it is shown in [MW] that $I_\alpha f$ and $M_\alpha f$ are often comparable in norm. To be more precise, if $w \in A_\infty$ and $0 < p < \infty$, then it is shown in [MW] that

$$\int_{R^n} |I_\alpha f(x)|^p w(x)\, dx \le c \int_{R^n} M_\alpha f(x)^p w(x)\, dx \tag{1}$$

with c independent of f. Here A_∞ denotes the collection of weights w on R^n with the property that there exist constants $C, \delta > 0$ such that if E is a Lebesgue measurable subset of a ball B, then

$$\frac{w(E)}{w(B)} \le C \left(\frac{|E|}{|B|} \right)^\delta,$$

where $w(E) = \int_E w(x)\, dx$ and $|E|$ is the Lebesgue measure of E .

This estimate has had applications in potential theory, such as in the proof of the Hedberg–Wolff theorem [AH] concerning nonlinear potentials. The corresponding theorem for potential operators of convolution type plays a role in the proof of the characterization by Kerman and Sawyer of trace type inequalities, which in turn have applications to eigenvalues estimates for Schrödinger operators (see [KS]). Inequality (1) is also related to the fact that the positive cone for the Triebel-Lizorkin space $F_p^{\alpha,q}$, $\alpha < 0$, is independent of q; see [JPW] and [AH].

An example of the first kind of result that we will prove is an estimate similar to (1) but with R^n replaced by a space $\mathcal{S}$ of homogeneous type, I_α replaced by a potential operator $T = T_K$ of the form

$$Tf(x) = T(fd\mu)(x) = \int_{\mathcal{S}} f(y)K(x,y)d\mu(y) \tag{2}$$

where μ is the underlying doubling measure on $\mathcal{S}$ (see §3 for the exact definitions of a space of homogeneous type and a doubling measure), and with $M_\alpha f$ replaced by

$$M_\varphi f(x) = M_\varphi(fd\mu)(x) = \sup_{B:x\in B} \varphi(B) \int_B |f(y)|d\mu(y) \tag{3}$$

where $\varphi(B) = \varphi_K(B)$ is a functional which acts on balls and is defined by

$$\varphi(B) = \sup_{\substack{x,y\in B \\ d(x,y)\geq cr(B)}} K(x,y) \tag{4}$$

for a sufficiently small positive geometric constant c (see [SW1]). Here $d(x,y)$ denotes the quasimetric associated with $\mathcal{S}$. For example, in the case of the Riesz potential we have $K(x,y) = |x-y|^{\alpha-n}, 0 < \alpha < n$, so that $\varphi(B) \approx r(B)^{\alpha-n}$, and then M_φ reduces to the fractional maximal operator M_α. Other examples of operators of the forms (2) or (3) arise from important differential operators; see the next section for more details.

In our generalization of (1), we will assume that the weight satisfies a condition that is analogous to the A_∞ condition described above but with Lebesgue measure replaced by a notion of *content*, such as Hausdorff content, on the right-hand side.

An example of the second kind of result that we will prove is the two-weight estimate

$$\left(\int_{\mathcal{S}} \{M_\varphi(fd\mu)\, w\}^q d\mu\right)^{1/q} \leq C \left(\int_{\mathcal{S}} (|f|\, v)^p d\mu\right)^{1/p}$$

with $1 < p \leq q < \infty$ and C independent of f. Such estimates have been studied extensively, but the nature of the condition that we shall impose on the weights is different from elsewhere. In particular, in addition to the necessary condition

$$\varphi(B) \left(\int_B w^q d\mu\right)^{1/q} \left(\int_B v^{-p'} d\mu\right)^{1/p'} \leq c,$$

$p' = p/(p-1)$, for all balls B, we shall assume that $v^{-p'}$ satisfies an appropriate A_∞ condition of the content type. For comparison purposes, we note that the two-weight boundedness of M_φ is studied in [PW] under a different kind of strengthening of the necessary condition, such as a Fefferman–Phong condition of the type

$$\varphi(B)\,\mu(B)^{1/p'}\left(\int_B w^q d\mu\right)^{1/q}\left(\frac{1}{\mu(B)}\int_B v^{-rp'}d\mu\right)^{1/rp'} \le c$$

for some $r > 1$. For example, weights for which just *reverse* doubling conditions are valid may satisfy an A_∞ condition of content type but not a Fefferman–Phong condition.

As mentioned earlier, it is possible to combine the two kinds of results that we will prove. In this way, we obtain two-weight norm estimates for potential operators Tf assuming that both weights satisfy content conditions.

2 Statements of the main results

Following [SW1], we consider potential operators of the form

$$T(fd\sigma)(x) = \int_{\mathcal{S}} f(y)K(x,y)d\sigma(y), \tag{5}$$

where $\mathcal{S}$ is a space of homogeneous type with underlying doubling measure μ, and σ is any Borel measure on $\mathcal{S}$. This definition agrees with (2) in case $\sigma = \mu$. The exact definition of a space of homogeneous type is given is §3; by a doubling measure, we mean a Borel measure μ with the property that there is a constant C such that for every "ball" $B \subset \mathcal{S}$,

$$\mu(2B) \le C\mu(B).$$

As usual, $2B$ denotes the ball with the same center as B but twice the radius. If $d(x,y)$ is the corresponding quasimetric in $\mathcal{S}$, we will always assume that the kernel $K(x,y)$ is nonnegative and satisfies the following growth conditions: there exist constants C_1, C_2 strictly larger than 1 such that

$$K(x,y) \le C_1 K(x',y) \quad \text{if} \quad d(x',y) \le C_2 d(x,y), \tag{6}$$

$$K(x,y) \le C_1 K(x,y') \quad \text{if} \quad d(x,y') \le C_2 d(x,y).$$

The main classical examples of such operators are the Riesz integrals $I_\alpha f$ mentioned in the introduction. An important class of examples for metrics other than the usual Euclidean metric consists of potential operators related to the regularity of subelliptic differential equations. In particular, vector fields of Hörmander type ([H]) as well as the classes of nonsmooth vector fields studied in [FL] and [SW2] lead to integral operators of the type we will study. In addition, the differential operators of Grushin type considered in [FGuW] (at least in the simplest case of Lebesgue measure) are related to integrals of type (2). In fact, for all these examples the associated potential operator has the form

$$Tf(x) = \int_{\mathcal{S}} f(y) \, \frac{d(x,y)}{\mu(B(x,d(x,y)))} \, d\mu(y) \tag{7}$$

where $d(x,y)$ is a distance function that is naturally related to the vector fields and $B(x,r)$ denotes the corresponding ball with center x and radius r.

Associated with K is the functional $\varphi = \varphi_K$ defined in (4) by

$$\varphi(B) = \sup_{\substack{x,y \in B \\ d(x,y) \ge cr(B)}} K(x,y)$$

for a sufficiently small positive geometric constant c. As mentioned earlier, $\varphi(B) \approx r(B)^{\alpha - n}$ in the case of the Riesz fractional integral I_α. In the subelliptic case (7), note that $\varphi(B) \approx r(B)/\mu(B)$.

The conditions (6) on K lead to useful growth properties of φ. If B is a ball and $\theta > 0$, let θB denote the ball concentric with B whose radius is $\theta r(B)$. It is shown in [SWZ, (4.2) and

(4.3)] that if $\theta > 1$, then there is a constant C depending only on θ, C_1, C_2, the constant c in (4) and geometric properties of $\mathcal{S}$ so that

$$\varphi(B) \leq C\varphi(\theta B) \text{ for all balls } B \subset \mathcal{S}. \tag{8}$$

Also, for such a constant C (but now one that is independent of θ),

$$\varphi(B) \leq C\varphi(B') \text{ for all pairs of balls } B' \subset B. \tag{9}$$

We shall assume in some of our results that φ satisfies the following additional condition for some $\epsilon > 0$:

$$\varphi(B_1)\mu(B_1) \leq C \left(\frac{r(B_1)}{r(B_2)}\right)^{\epsilon} \varphi(B_2)\mu(B_2) \quad \text{if} \quad B_1 \subset B_2. \tag{10}$$

For example, in the case of the fractional integrals I_α, we can pick $\epsilon = \alpha$ in (10); for the operator in (7), we can choose $\epsilon = 1$.

For any Borel measure σ, we define the maximal operator

$$M_\varphi(fd\sigma)(x) = \sup_{B:x\in B} \varphi(B) \int_B |f|\, d\sigma. \tag{11}$$

Note that if $\sigma = \mu$, this definition agrees with (3).

The pointwise estimate $T(fd\sigma)(x) \geq cM_\varphi(fd\sigma)(x)$ is easy to show by using the assumptions on K. Our first main theorem will show that the opposite inequality often holds in norm. Thus the theorem generalizes (1) for classical fractional integrals. In fact, the result improves (1) not only by extending it to spaces of homogeneous type but also by allowing a larger class of weights even in the usual Euclidean case. In order to state the result, we first define a suitable class of measures in a space of homogeneous type. The definition is motivated by a similar one in [SW1] for the usual Euclidean case, and it is most conveniently phrased in terms of a grid $\mathcal{D}_m$ of dyadic cubes Q. Here m is a large negative integer which indexes the edgelengths $\ell(Q)$ of the smallest cubes $Q \in \mathcal{D}_m$, namely, the smallest edgelengths are ρ^m for an appropriate geometric constant $\rho > 1$, and each cube in the grid has edgelength ρ^k for some $k \geq m$ (see §3 for details about the grid). The cubes in $\mathcal{D}_m$ have the important nestedness property that if $Q_1, Q_2 \in \mathcal{D}_m$ and $Q_1 \cap Q_2 \neq \emptyset$, then either $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. Also, each cube $Q \in \mathcal{D}_m$ is contained in a ball $B(Q)$ whose radius is comparable to the edgelength of Q; $B(Q)$ is called the *containing ball* of Q.

Given an integer m and a nonnegative functional $\tau(B)$ of balls B, we define the notion of the τ *content* of a set E, $||E||_{\tau,m}$, as follows. If $E \subset \mathcal{S}$, let

$$\|E\|_{\tau,m} = \inf\{\sum_i \tau(B(Q_i)) : Q_i \in \mathcal{D}_m, E \subset \cup Q_i\}.$$

Thus $\|E\|_{\tau,m}$ is a sort of Hausdorff content of E associated with τ and the grid $\mathcal{D}_m$. Typical choices of τ are $\tau(B) = r(B)^\beta$ for some $\beta > 0$, and also $\tau(B) = \mu(B)$. The first choice corresponds to Hausdorff content of E of dimension β.

By the nestedness property of dyadic cubes, if $E \subset Q \in \mathcal{D}_m$ then in the definition of $\|E\|_{\tau,m}$ we may assume that all $Q_i \subset Q$. We say that a measure $\nu \in A_\infty^{dy}(\tau)$ if there are positive constants C, δ independent of E, Q and m such that

$$\frac{\nu(E)}{\nu(Q)} \leq C \left(\frac{\|E\|_{\tau,m}}{\tau(B(Q))}\right)^\delta \quad \text{if} \quad E \subset Q \in \mathcal{D}_m, \tag{12}$$

where $\nu(E)$ denotes the ν-measure of E.

In case ν is absolutely continuous with respect to μ, say $d\nu = w\,d\mu$, we say that $w \in A_\infty^{dy}(\tau)$ if $w\,d\mu \in A_\infty^{dy}(\tau)$. We also use the notation $w(E)$ for the $w\,d\mu$-measure of E: $w(E) = \int_E w\,d\mu$. If $d\nu = w\,dx$ on R^n, it is not hard to see that when τ is chosen so that $\tau(B) = r(B)^n (\approx |B|)$, then $w \in A_\infty^{dy}(\tau)$ is the same as $w \in A_\infty$ if w is a doubling weight.

If we choose $\tau(B) = r(B)^\beta$, $\beta > 0$, for all balls B, then the class $A_\infty^{dy}(\tau)$ was defined in [SW1]. In this case, we will use the notation A_∞^β instead of $A_\infty^{dy}(\tau)$, i.e.,

$$A_\infty^\beta = A_\infty^{dy}(\tau) \quad \text{when } \tau(B) = r(B)^\beta.$$

Related notions based on balls instead of dyadic cubes can be obtained by defining

$$H_{\tau,B}(E) = \inf\{\sum_i \tau(B_i) : E \subset \cup B_i \subset B\} \quad \text{if} \quad E \subset B$$

and replacing (12) by the assumption

$$\frac{\nu(E)}{\nu(B)} \leq C \left(\frac{H_{\tau,B}(E)}{\tau(B)} \right)^{\delta} \quad \text{if} \quad E \subset B. \tag{13}$$

If ν is a doubling measure, these two notions turn out to be the same for many functionals τ; see §3 for details. Recall that we always assume that the underlying measure μ is a doubling measure.

We say that a Borel measure ν satisfies the *reverse doubling condition* of order β, $\beta > 0$, if there is a positive constant c such that

(14)

We write $\nu \in RD_\beta$ for such a measure ν. Any doubling measure belongs to RD_β for some $\beta > 0$ by [W, p.269]. We will show in §3 that if ν is a doubling measure and $\nu \in RD_\beta$, then $\nu \in A_\infty^\beta$ for the same value of β. In fact, the assumption that ν is a doubling measure is not needed in order to conclude that $\nu \in A_\infty^\beta$ if ν satisfies the following dyadic reverse doubling condition uniformly in m:

$$\nu(Q_2) \geq c \left(\frac{\ell(Q_2)}{\ell(Q_1)} \right)^{\beta} \nu(Q_1) \quad \text{if} \quad Q_1, Q_2 \in \mathcal{D}_m, Q_1 \subset Q_2. \tag{15}$$

We write $\nu \in RD_\beta^{dy}$ for such ν.

We can now state our first main theorem. In it, we assume that $\tau(B)$ is a functional for which there are constants $c, \epsilon > 0$ with

$$\begin{aligned} &a) \qquad \tau(B_1) \leq c\tau(B_2) \quad \text{if} \quad B_1 \subset B_2 \\ &b) \qquad \tau(2B) \leq c\tau(B) \quad \text{for all balls } B \\ &c) \qquad \varphi(B_1)\tau(B_1) \leq c \left(\frac{r(B_1)}{r(B_2)} \right)^{\epsilon} \varphi(B_2)\tau(B_2) \quad \text{if} \quad B_1 \subset B_2. \end{aligned} \tag{16}$$

Theorem 2.1 *Let σ and ω be Borel measures on $\mathcal{S}$, $T(fd\sigma)$ be defined by (5) for a kernel which satisfies (6), and $M_\varphi(fd\sigma)$ be defined by (11) with φ as in (4). Let τ be a nonnegative functional which satisfies (16). If $\omega \in A_\infty^{dy}(\tau)$ and $1 \le p < \infty$, then*

$$\int_{\mathcal{S}} |T(fd\sigma)|^p d\omega \le C \int_{\mathcal{S}} (M_\varphi(fd\sigma))^p d\omega \tag{17}$$

with C independent of f. In particular, if τ satisfies (16) and $w \in A_\infty^{dy}(\tau)$, then

$$\int_{\mathcal{S}} |T(fd\mu)|^p w\, d\mu \le C \int_{\mathcal{S}} (M_\varphi(fd\mu))^p w\, d\mu. \tag{18}$$

In the important special case when $\tau(B) = r(B)^\beta$ for $\beta > 0$, Theorem 2.1 leads to the next corollary assuming that μ satisfies the doubling condition of order D, i.e., that

$$\mu(B) \le C \left(\frac{r(B)}{r(\tilde{B})} \right)^D \mu(\tilde{B}) \text{ for all balls } \tilde{B} \subset B.$$

Any doubling measure satisfies the doubling condition of order D for some $D > 0$.

Corollary 2.2 *Let $1 \le p < \infty$, μ be a doubling measure of order D, and K be a kernel such that φ satisfies (10) for some $\epsilon > 0$. If ω and σ are Borel measures on $\mathcal{S}$ and $\omega \in A_\infty^\beta$ with $\beta + \epsilon > D$, then*

$$\int_{\mathcal{S}} |T(fd\sigma)|^p\, d\omega \le C \int_{\mathcal{S}} |M_\varphi(fd\sigma)|^p\, d\omega. \tag{19}$$

In particular, (19) holds if either $\omega \in RD_\beta^{dy}$ with $\beta + \epsilon > D$, or if ω is a doubling measure and $\omega \in RD_\beta$ with $\beta + \epsilon > D$.

For example, in the usual n-dimensional Euclidean situation with $d\mu = dx$ (so that $D = n$), Corollary 2.2 includes the result (1) from [MW] by choosing $d\omega = w(x)dx$ and $\beta = n$ since $A_\infty \subset A_\infty^n$. In fact, Corollary 2.2 improves the result in (1) by showing that (1) is valid if $w \in A_\infty^\beta$ for any $\beta > n - \alpha$, since the value of ϵ in (10) for the Riesz operator I_α is α. Corollary 2.2 thus extends the result of [R] showing that (1) is valid for any doubling weight w which satisfies the reverse doubling condition of order β for some $\beta > n - \alpha$.

Our proof of Theorem 2.1 is not based on the good-lambda technique of Burkholder and Gundy [BG] used in [MW] but instead follows the line of the proof of Theorem 2.2 in [PW] (see also [P1]) . The method is flexible enough to allow us to also derive the following two-weight inequalities without any restriction on the weight which appears on the left. We use the notation Mf for the Hardy–Littlewood maximal function of f defined by

$$Mf(x) = \sup_{B:x\in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y), \tag{20}$$

and if k is a positive integer, $M^k f$ denotes the k-fold iterate $M(M(\ldots(Mf)\ldots))$. Also, $[p]$ denotes the integral part of p.

Theorem 2.3 *Let*

$$T(f d\mu)(x) = \int_{\mathcal{S}} f(y)\, K(x,y)\, d\mu(y)$$

be an integral operator of type (2) with a kernel K such that (6) holds and φ satisfies (10).

i) If $1 < p < \infty$, there is a constant C such that for any weight w and all f,

$$\int_{\mathcal{S}} |T(f d\mu)|^p\, w\, d\mu \le C \int_{\mathcal{S}} (M_\varphi(f d\mu))^p\, M^{[p]+1} w\, d\mu. \tag{21}$$

ii) If $p \le 1$, there is a constant C such that for any weight w and all f,

$$\int_{\mathcal{S}} |T(f d\mu)|^p\, w\, d\mu \le C \int_{\mathcal{S}} (M_\varphi(f d\mu))^p\, M w\, d\mu. \tag{22}$$

See [P2] for the Euclidean version except in the case $p < 1$.

Our proof works only for potential operators and not for singular integrals. A point of interest is that the result cannot be proved by means of good-lambda inequalities. The proof of the cases $p > 1$ and $p < 1$ is based on "extrapolation" type techniques for non-A_∞ weights very much in the spirit of [CP1].

We will also derive an analogue of Theorem 2.1 for fractional maximal operators. Instead of restricting ourselves just to M_φ, we can consider similar operators formed by using any nonnegative function $\psi(B)$ of balls $B \subset \mathcal{S}$ which satisfies the following conditions:

$$\begin{aligned}
&a) \qquad \psi(B_1) \le c\,\psi(B_2) \quad \text{if} \quad B_1 \subset B_2 \subset c\,B_1 \\
&b) \qquad \psi(B_1)\,\mu(B_1) \le c\,\psi(B_2)\,\mu(B_2) \quad \text{if} \quad B_1 \subset B_2 \\
&c) \qquad \text{if } \mathcal{S} \text{ is unbounded, then} \quad \lim_{r(B)\to\infty} \psi(B) = 0, \quad \text{in the sense that} \\
&\qquad\quad \text{given } \epsilon > 0, \text{ there exists } N > 0 \text{ such that } \psi(B) < \epsilon \text{ if } r(B) > N.
\end{aligned} \tag{23}$$

Note that condition b) corresponds to the case $\epsilon = 0$ in (10), and hence b) is weaker than (10). In some of our results, we assume that ψ also satisfies the doubling condition $\psi(2B) \le C\psi(B)$ for all balls B.

The main example of such a functional is $\psi(B) = r(B)^\alpha/\mu(B)$ with $\alpha > 0$, and in this case condition c) is true if α strictly exceeds the reverse doubling order of μ.

If σ is any Borel measure on $\mathcal{S}$ (σ may not be absolutely continuous with respect to μ), we define the maximal function $M_\psi(f\,d\sigma)$ by (11) with φ replaced by ψ:

$$M_\psi(f\,d\sigma)(x) = \sup_{B:x\in B} \psi(B) \int_B |f(y)|\,d\sigma(y). \tag{24}$$

The next theorem concerns two-weight L^p, L^q estimates for M_ψ.

Theorem 2.4 *Let ψ satisfy (23) and the doubling condition. Suppose that $1 < p \le q < \infty$, and let ω and σ be Borel measures such that $\sigma \in A_\infty^{dy}(\psi^{-1})$ (i.e., $\sigma \in A_\infty^{dy}(\tau)$ with $\tau(B) = 1/\psi(B)$ for all B) and*

$$\psi(B)\,\omega(B)^{1/q}\sigma(B)^{1/p'} \le C \tag{25}$$

for all balls B. If $M_\psi(f\,d\sigma)$ is the maximal function defined by (24), then

$$\left(\int_{\mathcal{S}} M_\psi(f\,d\sigma)^q\,d\omega\right)^{1/q} \le C\left(\int_{\mathcal{S}} |f|^p\,d\sigma\right)^{1/p}. \tag{26}$$

In particular, if w and v are weight functions which satisfy $v^{-p'} \in A_\infty^{dy}(\psi^{-1})$ and

$$\psi(B)\left(\int_B w^q\, d\mu\right)^{1/q}\left(\int_B v^{-p'}\, d\mu\right)^{1/p'} \le C, \tag{27}$$

then

$$\left(\int_{\mathcal{S}} \{M_\psi(f d\mu)\, w\}^q\, d\mu\right)^{1/q} \le C\left(\int_{\mathcal{S}} (|f|\, v)^p\, d\mu\right)^{1/p}. \tag{28}$$

It is easy to see (by choosing f to be the characteristic function of a ball) that (25) is a necessary condition for (26). Similarly, (27) is a necessary condition for (28). Note that the second statement of Theorem 2.4 follows from the first by choosing $d\omega = w^q\, d\mu$ and $d\sigma = v^{-p'}\, d\mu$, and by replacing f by $fv^{p'}$ in the conclusion of the first part.

In the special case when τ is given by $\tau(B) = r(B)^\beta$, $\beta > 0$, Theorem 2.4 leads to the following corollary.

Corollary 2.5 *Let ψ be a doubling functional that satisfies (23)(a),(c) and the condition*

$$\psi(B_1)\mu(B_1) \le c\left(\frac{r(B_1)}{r(B_2)}\right)^\epsilon \psi(B_2)\mu(B_2), \quad B_1 \subset B_2,$$

for some $\epsilon > 0$. Suppose that $1 < p \le q < \infty$, and let ω and σ be Borel measures such that

$$\psi(B)\, \omega(B)^{1/q}\, \sigma(B)^{1/p'} \le C$$

for all balls B. If $\sigma \in A_\infty^\beta$ with $\beta + \epsilon \ge D$, where D is the doubling order of μ, then

$$\left(\int_{\mathcal{S}} M_\psi(f d\sigma)^q\, d\omega\right)^{1/q} \le c\left(\int_{\mathcal{S}} |f|^p\, d\sigma\right)^{1/p}$$

with c independent of f. In particular, if w and v are weight functions that satisfy $v^{-p'} \in A_\infty^\beta$ with $\beta + \epsilon \ge D$ and

$$\psi(B)\left(\int_B w^q\, d\mu\right)^{1/q}\left(\int_B v^{-p'}\, d\mu\right)^{1/p'} \le C$$

for all balls B, then

$$\left(\int_{\mathcal{S}} \{M_\psi(f d\mu)\, w\}^q\, d\mu\right)^{1/q} \le c\left(\int_{\mathcal{S}} (|f|\, v)^p\, d\mu\right)^{1/p}$$

with c independent of f.

For example, if ψ is the functional which corresponds to the kernel of the operator (7), namely $\psi(B) = r(B)/\mu(B)$, then we have $\epsilon = 1$ in Corollary 2.5, and consequently the restriction on β there becomes $\beta \geq D - 1$.

Note that in Corollary 2.5, the value of $\beta + \epsilon$ is allowed to equal D while strict inequality is assumed in Corollary 2.2. Corollary 2.5 is proved in §5.

For the purpose of comparison, we point out that estimates like (28) are proved in [PW] without assuming that $v^{-p'} \in A_\infty^{dy}(\psi^{-1})$. However, the results there assume a strenghtening of condition (27) in the Fefferman–Phong sense, such as the existence of an index $r > 1$ for which

$$\psi(B) \left(\int_B w^q \, d\mu \right)^{\frac{1}{q}} \mu(B)^{\frac{1}{p'}} \left(\frac{1}{\mu(B)} \int_B v^{-rp'} \, d\mu \right)^{\frac{1}{rp'}} \leq C$$

for all balls B. Strengthenings of (27) which involve other Orlicz norms of v^{-1} are also considered in [PW].

Finally, we can combine Theorems 2.1 and 2.4 (or Corollaries 2.2 and 2.5) to immediately obtain two-weight norm estimates between $T(f d\sigma)$ and f as follows.

Theorem 2.6 *Let ω and σ be Borel measures on $\mathcal{S}$, $T(f d\sigma)$ be defined by (5) for a kernel which satisfies (6), and let $M_\varphi(f d\sigma)$ be defined by (11) with φ as in (4) and satisfying (23)(c). For $1 < p \leq q < \infty$, suppose that*

$$\phi(B)\, \omega(B)^{1/q} \sigma(B)^{1/p'} \leq C$$

for all balls B. Then

$$\left(\int_{\mathcal{S}} |T(f d\sigma)|^q \, d\omega \right)^{1/q} \leq C \left(\int_{\mathcal{S}} |f|^p \, d\sigma \right)^{1/p}$$

with C independent of f provided that either

i) φ satisfies (23)(b), $\sigma \in A_\infty^{dy}(1/\varphi)$ and $\omega \in A_\infty^{dy}(\tau)$ for any τ which satisfies (16), or *ii) φ satisfies (10), $\sigma \in A_\infty^{\beta_1}$ and $\omega \in A_\infty^{\beta_2}$ with $\beta_1 + \epsilon \geq D$ and $\beta_2 + \epsilon > D$ where D is the doubling order of μ and ϵ is as in (10).*

Part ii) of this theorem extends a result proved in [SW1] in the usual Euclidean context. In fact, it also improves the corresponding result in [SW1] by allowing equality in the condition involving β_1.

In the next two sections, we give some background facts about spaces of homogeneous type and Orlicz classes. We will use Orlicz spaces only in the proof of Theorem 2.3. Our main theorems are proved after these sections beginning with the results about maximal functions and then those for integral operators, including relations between integral operators and maximal functions.

3 Spaces of homogeneous type

In this section, we briefly recall some basic definitions and facts about spaces of homogeneous type.

A quasimetric d on a set $\mathcal{S}$ is a function $d : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ which satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all x, y;

(iii) there exists a finite constant $\kappa \geq 1$ such that

$$d(x, y) \leq \kappa(d(x, z) + d(z, y))$$

for all $x, y, z \in \mathcal{S}$.

Given $x \in \mathcal{S}$ and $r > 0$, let $B(x, r) = \{y \in \mathcal{S} : d(x, y) < r\}$ be the ball with center x and radius r. If $B = B(x, r)$ is a ball, we denote its radius r by $r(B)$ and its center x by x_B. If ν is a measure and E is a measurable set, $\nu(E)$ denotes the ν-measure of E.

Definition 3.1 *A space of homogeneous type $(\mathcal{S}, d, \mu)$ is a set $\mathcal{S}$ together with a quasimetric d and a nonnegative Borel measure μ on $\mathcal{S}$ such that the doubling condition*

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)) \tag{29}$$

holds for all $x \in \mathcal{S}$ and $r > 0$.

The balls $B(x, r)$ are not necessarily open, but by a theorem of Macias and Segovia [MS], there is a continuous quasimetric d' which is equivalent to d (i.e., there are positive constants c_1 and c_2 such that $c_1 d'(x, y) \leq d(x, y) \leq c_2 d'(x, y)$ for all $x, y \in \mathcal{S}$) for which every ball is open. We always assume that the quasimetric d is continuous and that balls are open.

If C is the smallest constant for which (29) holds, then the number $D = \log C$ is called the doubling order of μ. By iterating (29), we have

$$\frac{\mu(B)}{\mu(\tilde{B})} \leq C_\mu \left(\frac{r(B)}{r(\tilde{B})} \right)^D \text{ for all balls } \tilde{B} \subset B. \tag{30}$$

We also assume that annuli in $\mathcal{S}$ are not empty, i.e., that $B(x, R) \setminus B(x, r)$ is not empty for all $x \in \mathcal{S}$ and $0 < r < R < \infty$. As mentioned before, any doubling measure μ then satisfies the reverse doubling property: there exist $\delta > 0$ and $c_\mu > 0$ such that

$$\frac{\mu(B)}{\mu(\tilde{B})} \geq c_\mu \left(\frac{r(B)}{r(\tilde{B})} \right)^\delta \text{ for all balls } \tilde{B} \subset B. \tag{31}$$

We shall often use the following observation: if P and B are balls with $P \cap B \neq \emptyset$ and $r(P) \leq \gamma r(B)$ for some $\gamma > 0$, then

$$P \subset c_\gamma B \tag{32}$$

with $c_\gamma = \kappa\gamma + \kappa^2\gamma + \kappa^2$. To verify (32), note that if $z \in B \cap P$ and $y \in P$, then

$$d(y, x_B) \leq \kappa[d(y, x_P) + d(x_P, x_B)] \leq \kappa[r(P) + \kappa(d(x_P, z) + d(z, x_B))]$$

$$\leq \kappa[r(P) + \kappa(r(P) + r(B))] \leq \kappa[\gamma r(B) + \kappa(\gamma r(B) + r(B))] = c_\gamma r(B),$$

which implies (32).

We will use a grid of dyadic sets in $\mathcal{S}$ which are "almost balls", as constructed in [SW1]. In fact, the following has been proved there:

If $\rho = 8\kappa^5$, then for any (large negative) integer m, there are points $\{x_j^k\}$ and a family $\mathcal{D}_m = \{\mathcal{E}_j^k\}$ of sets for $k = m, m+1, \cdots$ and $j = 1, 2, \cdots$ such that

- $B(x_j^k, \rho^k) \subset \mathcal{E}_j^k \subset B(x_j^k, \rho^{k+1})$
- For each $k = m, m+1, \cdots$, the family $\{\mathcal{E}_j^k\}$ is pairwise disjoint in j, and $\mathcal{S} = \cup_j \mathcal{E}_j^k$.
- If $m \le k < l$, then either $\mathcal{E}_j^k \cap \mathcal{E}_i^l = \emptyset$ or $\mathcal{E}_j^k \subset \mathcal{E}_i^l$.

We call the family $\mathcal{D} = \cup_{m \in \mathbf{Z}} \mathcal{D}_m$ a dyadic cube decomposition of $\mathcal{S}$ and refer to the sets in $\mathcal{D}$ as dyadic cubes. A dyadic cube will usually be denoted by Q, and Q^* will denote the containing ball described above with $\frac{1}{\rho} Q^* \subset Q \subset Q^*$; thus, if $Q = \mathcal{E}_j^k$ then $Q^* = B(x_j^k, \rho^{k+1})$. We set $\ell(Q) = r(Q^*)/\rho$ and call $\ell(Q)$ the "sidelength" of Q. We note that while the cubes in each $\mathcal{D}_m$ have the dyadic properties listed above, no nestedness properties of the cubes in $\mathcal{D}_{m_1}$ relative to the cubes in $\mathcal{D}_{m_2}$ are assumed if m_1 and m_2 are different.

For any Borel measure ω, define

$$M_{\omega,m} g(x) = \sup_{\substack{B: x \in B \\ r(B) \ge \rho^m}} \frac{1}{\omega(B)} \int_B |g| \, d\omega,$$

and also the dyadic version

$$M_{\omega,m}^{dy} g(x) = \sup_{\substack{Q: x \in Q \\ Q \in \mathcal{D}_m}} \frac{1}{\omega(Q)} \int_Q |g| \, d\omega.$$

If ω is absolutely continuous with respect to μ, i.e., $d\omega = w \, d\mu$, then we write $M_{w,m} g$ and $M_{w,m}^{dy} g$ instead of $M_{w d\mu, m} g$ and $M_{w d\mu, m}^{dy} g$. As usual, we say that w is a weight if $w(x)$ is a nonnegative locally integrable function with respect to μ, and for a measurable set E, we write $w(E) = \int_E w(x)\, d\mu(x)$.

Let us show, as mentioned in §2, that (12) and (13) are identical notions for many functionals τ if ν is a doubling measure. In fact, we will show this is the case if τ just satisfies (16)(a),(b). Let ν be a doubling measure. First, suppose that (13) holds for ν, and let $E \subset Q \in \mathcal{D}_m$. Given $\eta > 0$, select cubes $\{Q_i\}$ in $\mathcal{D}_m$ with $E \subset \cup Q_i \subset Q$ and $\sum \tau(B(Q_i)) < \|E\|_{\tau,m} + \eta$, where $B(Q_i)$ denotes the containing ball of Q_i. Note that $E \subset \cup B(Q_i) \subset cB(Q)$ for some geometric constant c which is independent of m, E, Q, and $\{Q_i\}$. By (13),

$$\frac{\nu(E)}{\nu(cB(Q))} \leq C \left(\frac{H_{\tau,cB(Q)}(E)}{\tau(cB(Q))} \right)^{\delta}$$
$$\leq C \left(\frac{\sum \tau(B(Q_i))}{\tau(cB(Q))} \right)^{\delta}.$$

By the properties of containing balls, and since ν is doubling and (16)(a) holds, we obtain

$$\frac{\nu(E)}{\nu(Q)} \leq C \left(\frac{\sum \tau(B(Q_i))}{\tau(B(Q))} \right)^{\delta} \leq C \left(\frac{\|E\|_{\tau,m} + \eta}{\tau(B(Q))} \right)^{\delta},$$

and (12) follows by letting $\eta \to 0$.

Conversely, suppose that (12) holds for a doubling measure ν, and let $E \subset \cup B_i \subset B$. We want to show that

$$\frac{\nu(E)}{\nu(B)} \leq C \left(\frac{\sum \tau(B_i)}{\tau(B)} \right)^{\delta}$$

for some $\delta > 0$, i.e., that (13) holds. Let m be so large that $\rho^m << r(B)$. Cover B by a finite number N_1 of disjoint dyadic cubes $Q \in \mathcal{D}_m$ with $\ell(Q) \approx r(B)$, where N_1 and the constants of equivalence are independent of B and the grid $\mathcal{D}_m$. By the doubling of ν, $\nu(B) \approx \nu(Q)$ for each such Q, with similar constants of equivalence, assuming as we may that each Q touches B. By considering the sets $E \cap Q$ individually and adding, we may assume that $E \subset Q$ for one such Q. Also, by considering those B_i with $r(B_i) \geq \rho^m$, and eventually letting $m \to -\infty$, we may assume that all $r(B_i) \geq \rho^m$. Cover each B_i by a finite number N_2 (independent of i and m) of cubes $\{Q_{ij}\}_{j=1}^{N_2}$ in $\mathcal{D}_m$ with $r(B_i) \approx \ell(Q_{ij})$ uniformly in i, j, m. It follows easily from the properties of the quasimetric that $Q_{ij} \subset cB_i$ for a uniform positive constant c. Since $E \subset \cup_{i,j} Q_{ij}$ and we may discard any Q_{ij} not contained in Q, it follows from (12) that

$$\frac{\nu(E)}{\nu(Q)} \leq C \left(\frac{\sum_{i,j} \tau(B(Q_{ij}))}{\tau(B(Q))} \right)^{\delta}$$

$$= C \left(\frac{\sum_i \sum_j \tau(B(Q_{ij}))}{\tau(B(Q))} \right)^{\delta} \leq C \left(\frac{N_2 \sum_i \tau(B_i)}{\tau(B)} \right)^{\delta}$$

because the number of j's is at most N_2 and (16)(a),(b) hold. Since $\nu(B) \approx \nu(Q)$, a similar estimate holds with $\nu(Q)$ replaced by $\nu(B)$ in the denominator on the left, which gives the desired inequality.

As usual, we say that a measure ν belongs to $A_\infty(\mu)$ if there are positive constants C and η such that

$$\nu(E) \leq C \left(\frac{\mu(E)}{\mu(B)} \right)^{\eta} \nu(B)$$

for every ball B and every measurable set $E \subset B$. It is not difficult to show that if $\nu \in A_\infty(\mu)$ then $\nu \in A_\infty^{dy}(\mu)$ (in the sense of definition (12)). In fact, let E be a set with $E \subset \cup Q_i \subset Q$ for $Q_i, Q \in \mathcal{D}_m$. If $\nu \in A_\infty(\mu)$, it follows easily that ν is doubling, and then

$$\frac{\nu(E)}{\nu(Q)} \leq c \left(\frac{\mu(E)}{\mu(Q)} \right)^{\eta}$$

for some $\eta > 0$ since Q can be included in a ball of comparable measure. Therefore, since $E \subset \cup B(Q_i)$,

$$\frac{\nu(E)}{\nu(Q)} \leq c \left(\sum \frac{\mu(B(Q_i))}{\mu(Q)} \right)^{\eta} \leq c \left(\sum \frac{\mu(B(Q_i))}{\mu(B(Q))} \right)^{\eta}$$

since μ is doubling. Consequently, $\nu \in A_\infty^{dy}(\mu)$.

Let us now show that any measure ν which satisfies the dyadic reverse doubling condition RD_β^{dy} (see (15)) belongs to A_∞^β for the same value of β. A similar fact was shown in [SW1] in the Euclidean case. If $E \subset \cup Q_i \subset Q$ with $Q_i, Q \in \mathcal{D}_m$, then

$$\nu(E) \le \sum_i \nu(Q_i) \le c \sum_i \left(\frac{\ell(Q_i)}{\ell(Q)} \right)^\beta \nu(Q)$$

since $\nu \in RD_\beta^{dy}$. Defining τ by $\tau(B) = r(B)^\beta$ for all B, it follows that

$$\nu(E) \le c \sum_i \frac{\tau(B(Q_i))}{\tau(B(Q))} \nu(Q),$$

where $B(Q_i)$ and $B(Q)$ are the containing balls of Q_i and Q. Taking the infimum over all such coverings $\{Q_i\}$ of E, we obtain that

$$\nu(E) \le c \frac{\|E\|_{\tau,m}}{\tau(B(Q))} \nu(Q),$$

so that (12) holds with $\delta = 1$ and thus $\nu \in A_\infty^\beta$. The conclusion that $\nu \in A_\infty^\beta$ also holds if the hypothesis that $\nu \in RD_\beta^{dy}$ is replaced by the assumptions that $\nu \in RD_\beta$ and ν is a doubling measure; in fact, these assumptions are easily seen to imply that $\nu \in RD_\beta^{dy}$.

In passing, we note that if the underlying measure $\mu \in A_\infty^{dy}(\tau)$ and if $\nu \in A_\infty(\mu)$, namely, for some $\eta > 0$,

$$\frac{\nu(E)}{\nu(B)} \le C \left(\frac{\mu(E)}{\mu(B)} \right)^\eta \quad \text{if} \quad E \subset B,$$

then also $\nu \in A_\infty^{dy}(\tau)$ for the same τ (although the value of δ for ν equals η times the value of δ for μ). This follows easily from the definitions since μ and ν are doubling measures.

4 Orlicz spaces and Orlicz maximal functions

To prove Theorem 2.3, we will use some facts about Orlicz spaces which we recall here, referring to [RR] and [BS] for a complete account.

A function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. It follows that $\Phi(t)/t$ is increasing, and in particular that

$$\Phi(\gamma t) \geq \gamma \Phi(t) \quad \text{if } \gamma \geq 1 \text{ and } t \geq 0.$$

Sometimes we will also assume that Φ satisfies the doubling condition $\Phi(2t) \leq C\Phi(t)$.

For Orlicz norms we are usually only concerned about the behavior of Young functions for t large. By definition, the Orlicz space L_Φ consists of all measurable functions f such that

$$\int_{\mathcal{S}} \Phi\left(\frac{|f|}{\lambda}\right) d\mu < \infty$$

for some positive λ. Note that if $0 < \lambda_1 < \lambda_2$, then

$$\Phi\left(\frac{|f|}{\lambda_2}\right) \leq \frac{\lambda_1}{\lambda_2}\Phi\left(\frac{|f|}{\lambda_1}\right),$$

so that

$$\lim_{\lambda\to\infty} \int_{\mathcal{S}} \Phi\left(\frac{|f|}{\lambda}\right) d\mu = 0 \quad \text{if } f \in L_\Phi.$$

The space L_Φ is a Banach function space with the Luxemburg norm

$$\|f\|_\Phi = \|f\|_{\Phi,\mu} = \inf\{\lambda > 0 : \int_{\mathcal{S}} \Phi(\frac{|f|}{\lambda})\, d\mu \leq 1\}.$$

Each Young function Φ has an associated complementary Young function $\bar{\Phi}$ satisfying

$$t \leq \Phi^{-1}(t)\bar{\Phi}^{-1}(t) \leq 2\,t \tag{33}$$

for all $t > 0$, where Φ^{-1} stands for the inverse function of Φ. The function $\bar{\Phi}$ is called the conjugate of Φ, and the space $L_{\bar{\Phi}}$ is called the conjugate space of L_Φ. For example, if $\Phi(t) = t^p$ for $1 < p < \infty$ then $\bar{\Phi}(t) = t^{p'}, p' = p/(p-1)$, and the conjugate space of $L^p(\mu)$ is $L^{p'}(\mu)$. An example that we will need is $\Phi(t) \approx t^p(\log t)^{-1-\epsilon}$ for large t, $1 < p < \infty$, $\epsilon > 0$, with complementary function $\bar{\Phi}(t) \approx t^{p'}(\log t)^{(p'-1)(1+\epsilon)}$ for large t (cf. [O], p.275).

A very important property of Orlicz spaces is the generalized Hölder inequality

$$\int_{\mathcal{S}} |fg|\, d\mu \leq \|f\|_\Phi \|g\|_{\bar{\Phi}}. \tag{34}$$

In order to define another maximal function which will play a role in the proof of Theorem 2.3, we need local versions of Orlicz norms. If Φ is a Young function and B is a ball, let

$$\|f\|_{\Phi,B} = \|f\|_{\Phi,B,\mu} = \inf\{\lambda > 0 : \frac{1}{\mu(B)} \int_B \Phi(\frac{|f|}{\lambda})\, d\mu \leq 1\}.$$

For this norm, the local version of the generalized Hölder inequality (34) is

$$\frac{1}{\mu(B)} \int_B fg\, d\mu \leq \|f\|_{\Phi,B} \|g\|_{\bar{\Phi},B}. \tag{35}$$

Define a maximal function corresponding to Φ by

$$\mathcal{M}_\Phi f(x) = \sup_{B:x\in B} \|f\|_{\Phi,B}. \tag{36}$$

This maximal function has been used in the usual Euclidean context in [P1] and also in the case $\Phi(t) \approx t \log t$ in the work of T. Iwaniec and Greco [GI] and in [WW]. The norm behavior of $\mathcal{M}_\Phi f$ is closely related to the next definition.

Definition 4.1 *Let $1 < p < \infty$. A nonnegative function $\Phi(t), t > 0$, satisfies the B_p condition if there is a constant $c > 0$ such that*

$$\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty. \tag{37}$$

Simple examples of functions which satisfy B_p are $t^{p-\delta}$ and $t^p(log(1+t))^{-1-\delta}$, both when $\delta > 0$.

The relevance of condition B_p stems from its relationship to the boundedness of $\mathcal{M}_\Phi$ as stated in the next theorem from [PW].

Theorem 4.2 *Let $1 < p < \infty$ and Φ be a doubling Young function. Then the following statements are equivalent.*

i) $\Phi \in B_p$, i.e., there is a constant $c > 0$ such that

$$\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty. \tag{38}$$

ii) There is a constant $C > 0$ such that

$$\int_{\mathcal{S}} \mathcal{M}_\Phi f(x)^p \, d\mu(x) \leq C \int_{\mathcal{S}} f(x)^p \, d\mu(x) \tag{39}$$

for all nonnegative f.

iii) There is a constant $C > 0$ such that

$$\int_{\mathcal{S}} \mathcal{M}_\Phi f(x)^p \, w(x) \, d\mu(x) \leq C \int_{\mathcal{S}} f(x)^p \, Mw(x) \, d\mu(x) \tag{40}$$

for all nonnegative f and w, where Mw is the Hardy–Littlewood maximal function defined in (20).

For example, in the standard case when $\Phi(t) = t^r$ with $r \geq 1$, so that $\|f\|_{\Phi,B} = \left(\mu(B)^{-1} \int_B |f|^r \, d\mu\right)^{1/r}$, the equivalence of (38) and (39) reduces to the well-known fact that the mapping

$$f \to \sup_{B: x \in B} \left(\frac{1}{\mu(B)} \int_B |f|^r \, d\mu \right)^{1/r}$$

is bounded on $L^p(\mathcal{S}, \mu)$ if and only if $p > r$. The characterization of B_p given above was proved in the Euclidean context in [P3] and used to derive sharp two weight estimates for the classical Hardy–Littlewood maximal function. In the general case, the characterization of B_p plays a main role in some of the results in [PW]. For other applications to different operators from harmonic analysis, see [P1], [P4] [P5], [P6], [CP1] and [CP2].

5 Proofs of Theorem 2.4 and Corollary 2.5

Recall that $M_\psi(f d\sigma)$ is defined by

$$M_\psi(f d\sigma)(x) = \sup_{B: x \in B} \psi(B) \int_B |f(y)| \, d\sigma(y)$$

for any measure σ and any measurable f, where $\psi(B)$ is assumed to be nonnegative and to satisfy (23) and also the doubling condition. To prove Theorem 2.4, we may assume that f is nonnegative, bounded and has bounded support. We may also assume by a limiting argument that $M_\psi(f\,d\sigma)(x)$ is formed by taking the supremum only over balls containing x of radius at least ρ^m for fixed m, where ρ is the constant used in §3 to construct the dyadic grid $\mathcal{D}_m$. For each integer k we let

$$\Omega_k = \{x \in S : M_\psi(f\,d\sigma)(x) > \gamma^k\}$$

where $\gamma > 1$ is a constant to be chosen. For each $x \in \Omega_k$, there is a ball B_x containing x with

$$\psi(B_x)\int_{B_x} f\,d\sigma > \gamma^k.$$

Now we claim the following:

> To each B_x, there corresponds a dyadic cube $Q_x \in \mathcal{D}_m$ (Q_x may not contain x, although B_x does) of size comparable to B_x with $Q_x \cap B_x \neq \emptyset$ and
>
> $$\psi(B(Q_x))\int_{Q_x} f\,d\sigma > \frac{\gamma^k}{c},$$
>
> where c is a geometric constant.

Recall here that if Q is any dyadic cube, then $B(Q)$ denotes the containing ball of Q; the radius of $B(Q)$ is comparable to the edgelength of Q.

To prove the claim, first note that we can cover B_x by a fixed number N (independent of x, γ, k) of disjoint dyadic cubes Q of size comparable to B_x. Indeed, let k_0 be the integer such that $\rho^{k_0} \le r(B_x) < \rho^{k_0+1}$, and consider any m with $m < k_0$ (here ρ is the constant used in §3) to construct the dyadic grid $\mathcal{D}_m$. We make the following subclaim:

> In $\mathcal{D}_m$, there are at most N cubes $\{\mathcal{E}_j^{k_0}\}_j$ meeting B_x (i.e., there are at most N cubes of sidelength ρ^{k_0} meeting B_x), where N is a structural constant which is independent of B_x and m, provided that $m < k_0$.

To prove the subclaim, fix $m < k_0$ and denote those cubes $\{\mathcal{E}_j^{k_0}\}_j$ in $\mathcal{D}_m$ which have nonempty intersection with B_x by Q_j, $j = 1, \cdots, N$. The $\{Q_j\}$ are disjoint, and they are contained in some fixed enlargement of B_x since they touch B_x and their radii are comparable to $r(B_x)$. Thus, the sum of the $\mu(Q_j)$ is at most $c\mu(B_x)$. Since the size of each Q_j is comparable to the size of B_x, each $\mu(Q_j)$ exceeds a fixed multiple of $\mu(B_x)$ by doubling (in fact, the measures are comparable). Therefore, the number N of Q_j's must be at most a fixed geometric constant, which proves the subclaim.

Consequently, for any fixed $m < k_0$, if $Q_j, j = 1, \ldots, N$, are the cubes in $\mathcal{D}_m$ mentioned above, then $\chi_{B_x} \leq \sum_{j=1}^N \chi_{Q_j}$, and so

$$\gamma^k < \psi(B_x) \int_{B_x} f \, d\sigma \leq \psi(B_x) \sum_{j=1}^N \int_{Q_j} f \, d\sigma.$$

Thus for some j_0,

$$\gamma^k < N \, \psi(B_x) \int_{Q_{j_0}} f \, d\sigma.$$

Pick Q_x to be Q_{j_0}. Since the sizes of Q_x, B_x and $B(Q_x)$ are comparable, the claim follows by using $Q_x \cap B_x \neq \emptyset$ and the properties of ψ.

Choosing $\gamma > c$ we have

$$\psi(B(Q_x)) \int_{Q_x} f \, d\sigma > \gamma^{k-1}. \tag{41}$$

Define

$$\widetilde{\Omega}_k = \left\{ x : \sup_{Q \in \mathcal{D}_m : x \in Q} \psi(B(Q)) \int_Q f \, d\sigma > \gamma^{k-1} \right\}$$

and let $\{Q_j^k\}_j$ be the maximal cubes in $\mathcal{D}_m$ with

$$\gamma^{k-1} < \psi(B(Q_j^k)) \int_{Q_j^k} f \, d\sigma.$$

Then $\widetilde{\Omega}_k = \cup_j Q_j^k$. Observe that if Q_x is the dyadic cube from (41) then $Q_x \subset Q_j^k$ for some j, and hence since there exists $c_0 > 1$ such that $x \in c_0 B(Q_j^k)$, we obtain that $\Omega_k \subset \cup_j c_0 B(Q_j^k)$.

Now if $\tilde{Q}_j^k$ is the next largest dyadic cube containing Q_j^k, then

$$\psi(B(\tilde{Q}_j^k)) \int_{\tilde{Q}_j^k} f\,d\sigma \le \gamma^{k-1}$$

and therefore, by using the doubling property of ψ,

$$\gamma^{k-1} < \psi(B(Q_j^k)) \int_{Q_j^k} f\,d\sigma \le c_\psi \gamma^{k-1}. \tag{42}$$

Let $M_{\sigma,m}^{dy}$ be defined as in Section 3. To prove the theorem it will enough to show that

$$\left(\int_{\mathcal{S}} M_\psi(f d\sigma)^q\,d\omega\right)^{1/q} \le c\left(\int_{\mathcal{S}} (M_{\sigma,m}^{dy} f)^p\,d\sigma\right)^{1/p} \tag{43}$$

with c independent of m and f, since $M_{\sigma,m}^{dy}$ is bounded on $L^p(d\sigma)$ uniformly in m for $1 < p < \infty$ (see for example [W], Lemma 3.8).

Now, we start with

$$\int_{\mathcal{S}} M_\psi(f d\sigma)^q\,d\omega = \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} M_\psi(f d\sigma)^q d\omega$$

$$\le \sum_k \gamma^{(k+1)q}\omega(\Omega_k) \le \gamma^{2q} \sum_k \gamma^{(k-1)q} \sum_j \omega(c_0 B(Q_j^k))$$

$$\le \gamma^{2q} \sum_{k,j} \left(\psi(B(Q_j^k)) \int_{Q_j^k} f\,d\sigma\right)^q \omega(c_0 B(Q_j^k)).$$

Since $\psi(B(Q_j^k)) \le \psi(c_0 B(Q_j^k))$ by (23)(a), if we use (25) for $c_0 B(Q_j^k)$, we can continue with

$$\le C \sum_{k,j} \sigma(B(Q_j^k))^{-q/p'} \left(\int_{Q_j^k} f\,d\sigma\right)^q \le C \sum_{k,j} \sigma(Q_j^k)^{-q/p'} \left(\int_{Q_j^k} f\,d\sigma\right)^q,$$

$$\le C\left(\sum_{k,j} \sigma(Q_j^k)^{1-p} \left(\int_{Q_j^k} f\,d\sigma\right)^p\right)^{q/p},$$

where we have used the facts that $Q_j^k \subset B(Q_j^k)$ and that $p \le q$.

Recall that $\widetilde{\Omega}_k = \cup_j Q_j^k$. Now let $E_j^k = Q_j^k \setminus \widetilde{\Omega}_{k+1}$. Then $E_j^k \subset \widetilde{\Omega}_k \setminus \widetilde{\Omega}_{k+1}$ and the sets E_j^k are disjoint in both j, k. We wish to show that $\sigma(Q_j^k) \le c\,\sigma(E_j^k)$ if γ is sufficiently large. It is enough to show that $\sigma(Q_j^k \cap \widetilde{\Omega}_{k+1}) < \frac{1}{2}\,\sigma(Q_j^k)$. If $Q_j^k \cap Q_i^{k+1} \neq \emptyset$, we claim that $Q_i^{k+1} \subset Q_j^k$. By the dyadic structure, the only other possibility is that Q_j^k is a proper subset of Q_i^{k+1}. But then by definition of Q_i^{k+1},

$$\begin{aligned} \gamma^k &< \psi(B(Q_i^{k+1})) \int_{Q_i^{k+1}} f\, d\sigma \\ &\leq \gamma^{k-1} \end{aligned}$$

by the maximality of Q_j^k. Since $\gamma > 1$, this is impossible and the claim follows. Thus

$$\sigma(Q_j^k \cap \widetilde{\Omega}_{k+1}) = \sigma\left(\cup_{i:Q_j^k \cap Q_i^{k+1} \neq \emptyset} Q_j^k \cap Q_i^{k+1}\right) = \sigma\left(\cup_{i:Q_i^{k+1} \subset Q_j^k} Q_i^{k+1}\right).$$

Now, since $\sigma \in A_\infty^{dy}(\psi^{-1})$ by hypothesis, if we apply (12) with Q and E there taken to be Q_j^k and $\cup_{i:Q_i^{k+1} \subset Q_j^k} Q_i^{k+1}$, then it follows that there exists $\delta > 0$ such that

$$\frac{\sigma\left(\cup_{i:Q_i^{k+1} \subset Q_j^k} Q_i^{k+1}\right)}{\sigma(Q_j^k)} \leq c \left(\frac{\sum_{i:Q_i^{k+1} \subset Q_j^k} \psi(B(Q_i^{k+1}))^{-1}}{\psi(B(Q_j^k))^{-1}}\right)^\delta.$$

By (42), the last expression is at most

$$c\left(\psi(B(Q_j^k)) \sum_{i:Q_i^{k+1} \subset Q_j^k} \frac{1}{\gamma^k} \int_{Q_i^{k+1}} f\, d\sigma\right)^\delta$$

$$\leq c\left(\frac{\psi(B(Q_j^k))}{\gamma^k} \int_{Q_j^k} f\, d\sigma\right)^\delta \leq c\,(\frac{c_\psi}{\gamma})^\delta < \frac{1}{2}$$

if we choose γ large enough. We can now conclude the proof of (43) and hence the proof of Theorem 2.4. In fact, since

$$\sigma(Q_j^k)^{1-p} = \sigma(Q_j^k)\frac{1}{\sigma(Q_j^k)^p} \leq c\sigma(E_j^k)\frac{1}{\sigma(Q_j^k)^p}$$

by what we showed above, then

$$\left(\sum_{k,j} \sigma(Q_j^k)^{1-p} \left(\int_{Q_j^k} f\, d\sigma\right)^p\right)^{q/p} \leq C\left(\sum_{k,j} \sigma(E_j^k) \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} f\, d\sigma\right)^p\right)^{q/p}$$

$$\leq C \left(\int_{\mathcal{S}} (M^{dy}_{\sigma,m} f)^p \, d\sigma \right)^{q/p}$$

since the E_j^k are pairwise disjoint in j and k.

□

Next, we prove Corollary 2.5. In fact, it is enough to show that the hypotheses on σ and ψ in Corollary 2.5 imply that $\sigma \in A^{dy}_\infty(1/\psi)$ since the corollary then follows from Theorem 2.4. The hypotheses are that $\sigma \in A^\beta_\infty$ with $\beta + \epsilon \geq D$ where D is the doubling order of μ and ϵ is given by

$$\psi(B_1)\mu(B_1) \leq c \left(\frac{r(B_1)}{r(B_2)} \right)^\epsilon \psi(B_2)\mu(B_2) \quad \text{if} \quad B_1 \subset B_2.$$

To show that $\sigma \in A^{dy}_\infty(1/\psi)$, let E be a set and suppose that $E \subset \cup Q_i \subset Q$ for $Q_i, Q \in \mathcal{D}_m$. Write B_i and B for the corrsponding containing balls: $B_i = B(Q_i)$ and $B = B(Q)$. By hypothesis, there exists $\delta > 0$ such that

$$\left(\frac{\sigma(E)}{\sigma(Q)} \right)^{1/\delta} \leq c \sum \left(\frac{r(B_i)}{r(B)} \right)^\beta$$

$$\leq c \sum \frac{\mu(B_i)}{\mu(B)} \left(\frac{r(B_i)}{r(B)} \right)^{\beta - D} \qquad \text{by the doubling of } \mu$$

$$= c \sum \frac{\mu(B_i)\psi(B_i)}{\mu(B)\psi(B)} \frac{\psi(B)}{\psi(B_i)} \left(\frac{r(B_i)}{r(B)} \right)^{\beta - D}$$

$$\leq c \sum \left(\frac{r(B_i)}{r(B)} \right)^{\epsilon + \beta - D} \frac{\psi(B)}{\psi(B_i)} \qquad \text{by the condition on } \psi$$

$$\leq c \sum \frac{\psi(B)}{\psi(B_i)} \qquad \text{since } \epsilon + \beta - D \geq 0$$

$$= c \sum \frac{1/\psi(B_i)}{1/\psi(B)}.$$

Taking the infimum over all Q and $\{Q_i\}$, and raising both sides to the power δ, we obtain

$$\frac{\sigma(E)}{\sigma(Q)} \leq c \left(\frac{||E||_{1/\psi,m}}{1/\psi(B)} \right)^\delta.$$

This shows that $\sigma \in A^{dy}_\infty(1/\psi)$, and the proof is complete.

□

6 Proofs of Theorem 2.1 and Corollary 2.2

The proof of Theorem 2.1 is a variant of that of Theorem 2.2 from [PW]. For $f, g \geq 0$, we start by showing that

$$\int_{\mathcal{S}} T_m(f\,d\sigma)\,g\,d\omega \leq c \sum_{Q \in D_m} \varphi(B(Q)) \int_{2\kappa B(Q)} f\,d\sigma \int_Q g\,d\omega$$

where T_m is the truncation of T defined by

$$T_m(f\,d\sigma)(x) = \int_{d(x,y)>\rho^m} f(y)\,K(x,y)\,d\sigma(y),$$

ρ being the constant that is used in the definition of the dyadic grid $\mathcal{D}_m$. To prove this, momentarily fix x, y with $d(x,y) > \rho^m$ and pick the integer $\ell \geq m$ for which $\rho^\ell < d(x,y) \leq \rho^{\ell+1}$. Select $Q \in \mathcal{D}_m$ with $l(Q) = \rho^\ell$ and $x \in Q$. Let $B(Q)$ denote the containing ball of Q, and let x_Q denote its center $x_{B(Q)}$. Thus, $\frac{1}{\rho}B(Q) \subset Q \subset B(Q)$ and $r(B(Q)) = \rho^{\ell+1}$. We then have

$$d(y, x_Q) \leq \kappa(d(y,x) + d(x, x_Q)) \leq \kappa(\rho^{\ell+1} + \rho^{\ell+1}) = 2\kappa r(B(Q)),$$

so that $y \in 2\kappa B(Q)$. Since $d(x,y) > \rho^\ell = r(2\kappa B(Q))/2\kappa\rho$, then by definition and property (9) of φ,

$$K(x,y) \leq \varphi(2\kappa B(Q)) \leq C\varphi(B(Q)).$$

Hence,

$$K(x,y) \leq c\varphi(B(Q))\chi_Q(x)\chi_{2\kappa B(Q)}(y) \leq c \sum_{Q \in D_m} \varphi(B(Q))\chi_Q(x)\chi_{2\kappa B(Q)}(y),$$

where the last estimate holds for all x, y with $d(x,y) > \rho^m$. Therefore,

$$T_m(f\,d\sigma)(x) \leq c \sum_{Q \in \mathcal{D}_m} \varphi(B(Q))\chi_Q(x) \int_{2\kappa B(Q)} f(y)d\sigma(y),$$

and then if $g \geq 0$, we obtain the desired estimate

$$\int_S T_m(f\,d\sigma)\,g\,d\omega \leq c \sum_{Q\in\mathcal{D}_m} \varphi(B(Q)) \int_{2\kappa B(Q)} f\,d\sigma \int_Q g\,d\omega.$$

By hypothesis, the measure $\omega \in A_\infty^{dy}(\tau)$ where τ satisfies (16), i.e., where

$$\varphi(B_1)\tau(B_1) \leq c \left(\frac{r(B_1)}{r(B_2)}\right)^\epsilon \varphi(B_2)\tau(B_2) \quad \text{if } B_1 \subset B_2$$

for some $\epsilon > 0$. Also, by hypothesis, τ is doubling and satisfies the monotonicity condition $\tau(B_1) \leq c\tau(B_2)$ if $B_1 \subset B_2$.

For $k \in \mathbf{Z}$ and $\gamma > 1$ to be chosen, let

$$\mathcal{C}^k = \{Q \in \mathcal{D}_m : \gamma^k < \frac{1}{\tau(B(Q))}\int_Q g\,d\omega \leq \gamma^{k+1}\},$$

and let $\{Q_j^k\}_j$ be the maximal cubes in $\mathcal{D}_m$ with

$$\gamma^k < \frac{1}{\tau(B(Q_j^k))}\int_{Q_j^k} g\,d\omega.$$

If $\tilde{Q}_j^k$ is the next largest dyadic cube containing Q_j^k, then

$$\frac{1}{\tau(B(\tilde{Q}_j^k))}\int_{\tilde{Q}_j^k} g\,d\omega \leq \gamma^k,$$

and therefore, by the doubling property of τ,

$$\gamma^k < \frac{1}{\tau(B(Q_j^k))}\int_{Q_j^k} g\,d\omega \leq c_{\tau,\rho}\gamma^k.$$

Hence, if we select γ with $\gamma \geq c_{\tau,\rho}$ then

$$\gamma^k < \frac{1}{\tau(B(Q_j^k))}\int_{Q_j^k} g\,d\omega \leq \gamma^{k+1}. \tag{44}$$

Thus $Q_j^k \in \mathcal{C}^k$. Since every cube $Q \in \mathcal{D}_m$ lies in some $\mathcal{C}^k$, every Q must be contained in some Q_j^k. Of course, the $\{Q_j^k\}_j$ are pairwise disjoint for fixed k. Then

$$\int_S T_m(f\,d\sigma)\,g\,d\omega \leq c\sum_k \sum_{Q\in\mathcal{C}^k} \varphi(B(Q))\,\tau(B(Q)) \int_{2\kappa B(Q)} f\,d\sigma\,\frac{1}{\tau(B(Q))}\int_Q g\,d\omega$$

$$\leq c\sum_k \gamma^{k+1}\sum_j \sum_{Q\in\Delta_m(Q_j^k)} \varphi(B(Q))\,\tau(B(Q)) \int_{2\kappa B(Q)} f\,d\sigma, \tag{45}$$

where for any $Q_0 \in \mathcal{D}_m$ we denote

$$\Delta_m(Q_0) = \{Q \in \mathcal{D}_m : Q \subset Q_0\}.$$

We need the following variant of Lemma 7.1 of [PW].

Lemma 6.1 *If (16) holds, there is a constant C such that for any $f \geq 0$ and any dyadic cube $Q_0 \in D_m$,*

$$\sum_{Q\in\Delta_m(Q_0)} \varphi(B(Q))\,\tau(B(Q)) \int_{2\kappa B(Q)} f\,d\sigma \leq C\varphi(B(Q_0))\,\tau(B(Q_0)) \int_{\kappa(2\kappa+1)B(Q_0)} f\,d\sigma. \tag{46}$$

Proof: By (16) and the doubling of φ and τ, the left side of (46) is bounded by

$$c\sum_{Q\in\Delta_m(Q_0)} \left(\frac{\ell(Q)}{\ell(Q_0)}\right)^{\epsilon} \varphi(B(Q_0))\,\tau(B(Q_0)) \int_{2\kappa B(Q)} f\,d\sigma$$

$$= c\,\varphi(B(Q_0))\,\tau(B(Q_0)) \sum_{Q\in\Delta_m(Q_0)} \left(\frac{\ell(Q)}{\ell(Q_0)}\right)^{\epsilon} \int_{2\kappa B(Q)} f\,d\sigma$$

$$= c\,\varphi(B(Q_0))\,\tau(B(Q_0)) \sum_{\ell=0}^{\infty} \sum_{\substack{Q\in\Delta_m(Q_0)\\ \ell(Q)=\rho^{-\ell}\ell(Q_0)}} \rho^{-\epsilon\ell} \int_{2\kappa B(Q)} f\,d\sigma$$

$$= c\,\varphi(B(Q_0))\,\tau(B(Q_0)) \sum_{\ell=0}^{\infty} \rho^{-\epsilon\ell} \sum_{\substack{Q\in\Delta_m(Q_0)\\ \ell(Q)=\rho^{-\ell}\ell(Q_0)}} \int_{2\kappa B(Q)} f\,d\sigma. \tag{47}$$

To estimate the last expression, first observe that if $Q \subset Q_0$ and $\ell(Q) \leq \ell(Q_0)$, then $2\kappa B(Q) \subset \kappa(2\kappa+1)B(Q_0)$, since if $y \in 2\kappa B(Q)$ then

$$d(y, x_{Q_0}) \leq \kappa[d(y, x_Q) + d(x_Q, x_{Q_0})] \leq \kappa[2\kappa r(B(Q)) + r(B(Q_0))]$$

$$= \kappa[2\kappa\rho\ell(Q) + r(B(Q_0))] \leq \kappa[2\kappa\rho\ell(Q_0) + r(B(Q_0))] = \kappa(2\kappa+1)\,r(B(Q_0)).$$

Thus (47) is at most

$$c\,\varphi(B(Q_0))\,\tau(B(Q_0))\sum_{\ell=0}^{\infty}\rho^{-\epsilon\ell}\int_{\kappa(2\kappa+1)B(Q_0)}\sum_{\substack{Q\in D_m\\ \ell(Q)=\rho^{-\ell}\ell(Q_0)}}\chi_{2\kappa B(Q)}(x)\,f(x)\,d\sigma(x),$$

and therefore (46) will follow if we show that

$$\sum_{\substack{Q\in D_m\\ \ell(Q)=\rho^{-\ell}\ell(Q_0)}}\chi_{2\kappa B(Q)}(x)\leq C \tag{48}$$

uniformly in x,j,k,l,m. To prove this, fix x,j,k,l,m and write $r=\rho^{-\ell}\ell(Q_0)$. If $Q\in\mathcal{D}_m$, $\ell(Q)=r$ and $x\in 2\kappa B(Q)$, then for any $y\in Q$ we have

$$\begin{aligned}d(x,y)&\leq\kappa[d(x,x_Q)+d(x_Q,y)]\leq\kappa[2\kappa r(B(Q))+r(B(Q))]\\&\leq\kappa(2\kappa+1))\rho\ell(Q)=c_1r,\end{aligned}$$

so that $Q\subset B(x,c_1r)$. But those $Q\in D_m$ with $\ell(Q)=r$ are disjoint, and consequently by doubling, since each Q has sidelength comparable to the radius of $B(x,c_1r)$, the number of such $Q\subset B(x,c_1r)$ is bounded uniformly in x and r. This proves (48) and so also the lemma.

To complete the proof of Theorem 2.1, note that by (45) and (46),

$$\begin{aligned}\int_{\mathcal{S}}T_m(f\,d\sigma)g\,d\omega&\leq c\sum_{j,k}\gamma^{k+1}\varphi(B(Q_j^k))\,\tau(B(Q_j^k))\int_{\kappa(2\kappa+1)B(Q_j^k)}f\,d\sigma\\&\leq c\gamma\sum_{j,k}\varphi(B(Q_j^k))\int_{\kappa(2\kappa+1)B(Q_j^k)}f\,d\sigma\int_{Q_j^k}g\,d\omega\end{aligned}\tag{49}$$

by (44). We also have

$$\begin{aligned}\sum_{i:Q_i^l\subset Q_j^k}\tau(B(Q_i^l))&\leq\sum_{i:Q_i^l\subset Q_j^k}\gamma^{-\ell}\int_{Q_i^l}g\,d\omega\leq\gamma^{-\ell}\int_{Q_j^k}g\,d\omega\\&\leq c\gamma^{k-l}\tau(B(Q_j^k)),\end{aligned}$$

and if Q_i^l is a proper subset of Q_j^k, then $\ell>k$ since

$$\gamma^k<\frac{1}{\tau(B(Q_j^k))}\int_{Q_j^k}g\,d\omega\leq\gamma^\ell$$

by the maximality of Q_i^l. Then, since the Q_i^l are disjoint in i for fixed l, and since $\omega \in A_\infty^{dy}(\tau)$,

$$\sum_{i:Q_i^l \subset Q_j^k} \omega(Q_i^l) = \omega(\cup_{i:Q_i^l \subset Q_j^k} Q_i^l) \leq c \left(\frac{\left\| \cup_{i:Q_i^l \subset Q_j^k} Q_i^l \right\|_{\tau,m}}{\tau(B(Q_j^k))} \right)^\delta \omega(Q_j^k)$$

$$\leq c \left(\frac{\sum_{i:Q_i^l \subset Q_j^k} \tau(B(Q_i^l))}{\tau(B(Q_j^k))} \right)^\delta \omega(Q_j^k) \leq c\gamma^{(k-l)\delta} \omega(Q_j^k). \tag{50}$$

Now let

$$\Omega_k = \{x : \sup_{Q \in \mathcal{D}_m : x \in Q} \frac{1}{\tau(B(Q))} \int_Q g \, d\omega > \gamma^k\},$$

and set $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Then $E_j^k \subset \Omega_k \setminus \Omega_{k+1}$ and the sets E_j^k are disjoint in both j,k. Note that $\Omega_k = \cup_j Q_j^k$. We wish to show that $\omega(Q_j^k) \leq c\omega(E_j^k)$ if γ is sufficiently large. It is enough to show that $\omega(Q_j^k \cap \Omega_{k+1}) < \frac{1}{2}\omega(Q_j^k)$. But by the dyadic structure and our earlier observations,

$$\omega(Q_j^k \cap \Omega_{k+1}) = \sum_{i:Q_j^k \cap Q_i^{k+1} \neq \emptyset} \omega(Q_j^k \cap Q_i^{k+1}) = \sum_{i:Q_i^{k+1} \subset Q_j^k} \omega(Q_i^{k+1})$$

$$\leq c\gamma^{-\delta} \omega(Q_j^k) \quad \text{by (50) with} \quad l = k+1$$

$$< \frac{1}{2} \omega(Q_j^k) \quad \text{if } \gamma \text{ is large.}$$

Next, we rewrite the sum on the right side of (49) as

$$\sum_{j,k} \varphi(B(Q_j^k)) \int_{\kappa(2\kappa+1)B(Q_j^k)} f \, d\sigma \, \omega(Q_j^k) \, \frac{1}{\omega(Q_j^k)} \int_{Q_j^k} g \, d\omega$$

and apply Hölder's inequality to obtain

$$\int_{\mathcal{S}} T_m(f d\sigma) \, g \, d\omega \leq$$

$$c \left(\sum_{j,k} \left[\varphi(B(Q_j^k)) \int_{\kappa(2\kappa+1)B(Q_j^k)} f \, d\sigma \right]^p \omega(Q_j^k) \right)^{1/p} \left(\sum_{j,k} \left[\frac{1}{\omega(Q_j^k)} \int_{Q_j^k} g \, d\omega \right]^{p'} \omega(Q_j^k) \right)^{1/p'}.$$

We may replace $\omega(Q_j^k)$ by $\omega(E_j^k)$ in the numerators of both of these sums and then use the disjointness of the E_j^k to majorize the last expression by

$$c\left(\int_{\mathcal{S}} M_\varphi(f\,d\sigma)^p\,d\omega\right)^{1/p}\left(\int_{\mathcal{S}}(M^{dy}_{\omega,m}g)^{p'}\,d\omega\right)^{1/p'},$$

where $M^{dy}_{\omega,m}g$ is the dyadic maximal function defined in §3. As mentioned in §5, $M^{dy}_{\omega,m}$ is bounded on $L^p(d\omega)$ uniformly in m for $1<p<\infty$. Thus, we obtain that

$$\int_{\mathcal{S}} T_m(f\,d\sigma)\,g\,d\omega \le c\,\|M_\varphi(f\,d\sigma)\|_{L^p(d\omega)}\|g\|_{L^{p'}(d\omega)},$$

with c independent of m, f and g, and consequently Theorem 2.1 follows from duality by letting $m \to -\infty$.

□

Let us now prove Corollary 2.2. Let τ be defined by $\tau(B) = r(B)^\beta$ for all B, with $\beta > 0$ to be chosen. By hypothesis, there exists $\epsilon > 0$ such that

$$\varphi(B_1)\mu(B_1) \le c\left(\frac{r(B_1)}{r(B_2)}\right)^{\epsilon}\varphi(B_2)\mu(B_2) \quad \text{if} \quad B_1 \subset B_2.$$

Therefore, since μ is assumed to satisfy the doubling condition of order D,

$$\begin{aligned}\varphi(B_1) \le c\left(\frac{r(B_1)}{r(B_2)}\right)^{\epsilon-D}&\varphi(B_2) \quad \text{if} \quad B_1 \subset B_2\\ = c\left(\frac{r(B_1)}{r(B_2)}\right)^{(\beta+\epsilon-D)-\beta}&\varphi(B_2)\\ = c\left(\frac{r(B_1)}{r(B_2)}\right)^{\beta+\epsilon-D}&\varphi(B_2)\,\frac{\tau(B_2)}{\tau(B_1)}.\end{aligned}$$

It follows that (16)(c) holds with ϵ there taken to be $\beta+\epsilon-D$. Clearly, (16)(a), (b) also hold for any $\beta > 0$. Thus, if $\beta+\epsilon-D > 0$, by applying Theorem 2.1 (with $\tau(B) = r(B)^\beta$ as above), the conclusion of Corollary 2.2 follows immediately.

□

7 Proof of Theorem 2.3

We start with the case $p = 1$, where the proof will be a variant of that of Theorem 2.1. The cases $p > 1$ and $p < 1$ will follow from the case $p = 1$ using extrapolation ideas and duality between L^p spaces.

7.1 The case p=1

By a limiting argument we may assume that w is bounded with compact support. For $f \geq 0$, we start with inequality (49) with $d\omega$ replaced by $w\, d\mu$ and $d\sigma$ replaced by $d\mu$:

$$\int_S T_m(f d\mu) w\, d\mu \leq c\gamma \sum_{k,j} \varphi(B(Q_j^k)) \int_{\kappa(2\kappa+1)B(Q_j^k)} f\, d\mu \int_{Q_j^k} w\, d\mu,$$

where the dyadic cubes Q_j^k are now the maximal cubes in $\mathcal{D}_m$ with

$$\gamma^k < \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} w\, d\mu$$

and satisfy

$$\gamma^k < \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} w\, d\mu \leq c_\mu\, \gamma^k.$$

Define

$$\Omega_k = \left\{ x : \sup_{Q \in \mathcal{D}_m : x \in Q} \frac{1}{\mu(Q)} \int_Q w\, d\mu > \gamma^k \right\},$$

and let $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Note that $\Omega_k = \cup_j Q_j^k$. As usual, if $Q_j^k \cap Q_i^{k+1} \neq \emptyset$, then $Q_i^{k+1} \subset Q_j^k$. If γ is large enough, then $\mu(Q_j^k) \leq 2\mu(E_j^k)$ since

$$\mu(Q_j^k \cap \Omega_{k+1}) = \sum_{i : Q_j^k \cap Q_i^{k+1} \neq \emptyset} \mu(Q_j^k \cap Q_i^{k+1})$$

$$= \sum_{i : Q_i^{k+1} \subset Q_j^k} \mu(Q_i^{k+1}) \leq \gamma^{-k-1} \sum_{i : Q_i^{k+1} \subset Q_j^k} \int_{Q_i^{k+1}} w\, d\mu$$

$$\leq \gamma^{-k-1} \int_{Q_j^k} w\, d\mu \leq \gamma^{-k-1} c_\mu \gamma^k \mu(Q_j^k)$$

$$= \frac{c_\mu}{\gamma}\,\mu(Q_j^k) < \frac{1}{2}\mu(Q_j^k)$$

if γ is large.

Thus there are sets E_j^k that are pairwise disjoint in both j and k with $E_j^k \subset Q_j^k$ and $\mu(Q_j^k) \le c\mu(E_j^k)$ for a universal constant c. If we denote $\tilde{Q}_j^k = \kappa(2\kappa+1)B(Q_j^k)$, then

$$\begin{aligned}
&\int_S T_m(f\,d\mu)\,w\,d\mu \\
&\le c\sum_{k,j}\varphi(B(Q_j^k))\int_{\tilde{Q}_j^k} f\,d\mu\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k} w\,d\mu\right)\mu(Q_j^k) \\
&\le c\sum_{k,j}\varphi(\tilde{Q}_j^k)\int_{\tilde{Q}_j^k} f\,d\mu\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k} w\,d\mu\right)\mu(E_j^k) \\
&\le c\sum_{k,j}\int_{E_j^k} M_\varphi(f\,d\mu)\,Mw\,d\mu \le c\int_S M_\varphi(f\,d\mu)\,Mw\,d\mu.
\end{aligned}$$

This concludes the proof of (22) when $p = 1$.

7.2 The case $p > 1$

Now let $p > 1$. As we mentioned above, our argument will be based on duality and the case $p = 1$. In fact we will prove something sharper than (21): if $\delta > 0$, there is a constant C such that for any weight w and all f,

$$\int_S |T(f\,d\mu)|^p\,w\,d\mu \le C\int_S (M_\varphi(f\,d\mu))^p\,\mathcal{M}_{L(\log L)^{p-1+\delta}}(w)\,d\mu, \tag{51}$$

where $\mathcal{M}_{L(\log L)^{p-1+\delta}}$ denotes the maximal function $\mathcal{M}_\Phi$ with

$$\Phi(t) = t(1+\log^+ t)^{p-1+\delta}, \quad t > 0.$$

The fact that this estimate is sharper than (21) will be shown later. By the case $p = 1$, there is a constant c so that for all $f, g \ge 0$ and all m,

$$\int_S T_m(f\,d\mu)\,g\,w^{1/p}\,d\mu \le c\int_S M_\varphi(f\,d\mu)\,M(g\,w^{1/p})\,d\mu.$$

Now by the generalized Hölder inequality (35) for an appropriate Young function Ψ that will be chosen soon, we can continue with

$$\le c \int_S M_\varphi(f d\mu)\, \mathcal{M}_\Psi g\, \mathcal{M}_{\bar\Psi}(w^{1/p})\, d\mu$$

$$\le c \left[\int_S M_\varphi(f d\mu)^p\, (\mathcal{M}_{\bar\Psi}(w^{1/p}))^p\, d\mu\right]^{1/p} \left[\int_S (\mathcal{M}_\Psi g)^{p'}\, d\mu\right]^{1/p'}.$$

To conclude the proof of (51), we use Theorem 4.2 for an appropriate $\Psi \in B_{p'}$. Indeed, as mentioned in §4, we can choose $\Psi(t) \approx t^{p'}(\log t)^{-1-\epsilon}$, $\epsilon > 0$ and t large, with complementary function $\bar\Psi(t) \approx t^p(\log t)^{(p-1)(1+\epsilon)}$ for large t. Then by combining estimates, we obtain

$$\int_S T_m(f d\mu)\, g\, w^{1/p}\, d\mu \le c \left[\int_S M_\varphi(f d\mu)^p\, \mathcal{M}_{L(\log L)^{(p-1)(1+\epsilon)}}(w)\, d\mu\right]^{1/p} \left[\int_S g^{p'}\, d\mu\right]^{1/p'}.$$

Since the constant c is independent of m, (51) follows by duality and letting $m \to -\infty$.

To show that (51) implies (21), we recall the following lemma from [PW] (see also [P1], [GI] and [WW] in the usual Euclidean case).

Lemma 7.1 *Let $k = 1, 2, \cdots$. Then there is a positive constant c such that for any measurable function w,*

$$\|w\|_{L(logL)^k,B} \le \frac{c}{\mu(B)} \int_B M^k w\, d\mu, \tag{52}$$

where $M^k w$ denotes the k-fold iterate of the Hardy–Littlewood maximal function defined in (20) and $\|\cdot\|_{L(logL)^k,B}$ denotes $\|\cdot\|_{\Phi,B}$ with $\Phi(t) = t(1 + (log^+ t)^k)$.

For $k = 1, 2, \cdots$, (52) implies that

$$\|w\|_{L(logL)^k,B} \le \frac{c}{\mu(B)} \int_B M^k w\, d\mu.$$

Thus, if we choose $\epsilon = \frac{[p]}{p-1} - 1 > 0$ in the proof of (51), we obtain

$$\mathcal{M}_{\bar\Psi}(w^{1/p})(x)^p \approx \mathcal{M}_{L(\log L)^{[p]}}(w)(x) \le c\, M^{[p]+1}(w)(x),$$

and part i) of Theorem 2.3 follows.

7.3 The case $p < 1$

Assume now that $0 < p < 1$. We will prove the inequality

$$\int_{\mathcal{S}} |T(f\,d\mu)|^p \, w \, d\mu \leq C \int_{\mathcal{S}} (M_\varphi(f\,d\mu))^p \, Mw \, d\mu \tag{53}$$

by an extrapolation argument, after first proving a strengthened version of the case $p = 1$. We begin with a definition.

Definition 7.2 *A weight v satisfies the $RH_\infty(\mu)$ condition (i.e., the reverse Hölder condition of infinite order) if there is a constant $c > 0$ such that for each ball B,*

$$ess\;sup_B \, v \leq \frac{c}{\mu(B)} \int_B v \, d\mu$$

It is easy to check that $RH_\infty(\mu) \subset A_\infty(\mu)$; in fact we can take $\eta = 1$ in the definition of $A_\infty(\mu)$.

We will prove the following version of the case $p = 1$:

Lemma 7.3 *Let v a weight satisfying the $RH_\infty(\mu)$ condition. Then there is a constant C such that for any weight w and all f,*

$$\int_{\mathcal{S}} |T(f\,d\mu)| \, v \, w \, d\mu \leq C \int_{\mathcal{S}} M_\varphi(f\,d\mu) \, v \, Mw \, d\mu. \tag{54}$$

For the proof of this lemma we proceed as in the case $v = 1$:

$$\int_S T_m(f\,d\mu) \, v \, w \, d\mu \leq c \sum_{k,j} \varphi(\tilde{Q}_j^k) \int_{\tilde{Q}_j^k} f \, d\mu \int_{Q_j^k} v \, w \, d\mu$$

$$\leq c \sum_{k,j} \varphi(\tilde{Q}_j^k) \int_{\tilde{Q}_j^k} f \, d\mu \int_{Q_j^k} w \, d\mu \, (ess\;sup_{Q_j^k} \, v)$$

$$\leq c \sum_{k,j} \varphi(\tilde{Q}_j^k) \int_{\tilde{Q}_j^k} f \, d\mu \left(\frac{1}{\mu(Q_j^k)} \int_{Q_j^k} w \, d\mu \right) \int_{Q_j^k} v \, d\mu \text{ since } v \in RH_\infty(\mu)$$

$$\le c\sum_{k,j}\varphi(\tilde{Q}_j^k)\int_{\tilde{Q}_j^k} f\,d\mu\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k} w\,d\mu\right)\int_{E_j^k} v\,d\mu \text{ since } v\in A_\infty(\mu)$$
$$\le c\sum_{k,j}\int_{E_j^k} M_\varphi(f d\mu)\,Mw\,v\,d\mu \le c\int_{\mathcal{S}} M_\varphi(f d\mu)\,v\,Mw\,d\mu.$$

□

Lemma 7.4 *Let $\alpha > 0$ and g be any function such that Mg is finite a.e. Then $(Mg)^{-\alpha}\in RH_\infty(\mu)$.*

This observation is due to C. Neugebauer [N] where he showed something better: $w\in RH_\infty(\mu)\Leftrightarrow w\approx(Mg)^{-\alpha}$ for some g and some $\alpha>0$.

As usual we will denote $w_B=\frac{1}{\mu(B)}\int_B w\,d\mu$. To prove (53), we first observe that $w^{-1}\in RH_\infty(\mu)$ if $w\in A_1(\mu)$, i.e., if $w_B\le C\,ess\,inf_B\,w$ for all B, since then

$$ess\,sup_B\,w^{-1}=(ess\,inf_B\,w)^{-1}\le c\,(w_B)^{-1}\le c\,(w^{-1})_B,$$

where in the last inequality we have used Hölder's inequality. The constant c is in fact the inverse of the constant C in the definition of $A_1(\mu)$.

The second observation we need is that if $w\in RH_\infty(\mu)$ then $w^\lambda\in RH_\infty(\mu)$ when $\lambda>1$:

$$ess\,sup_B\,w^\lambda=(ess\,sup_B\,w)^\lambda\le c^\lambda(w_B)^\lambda\le c^\lambda(w^\lambda)_B.$$

Actually this is also true when $0<\lambda<1$ but is a bit harder and we don't need it; see [N].

Let us now prove (53) for $0<p<1$. We will use an appropriate duality for the spaces $L^p(d\nu)$ when $p<1$ and ν is a measure: if $f\ge 0$ then

$$\|f\|_{L^p(d\nu)}=\inf\{\int f\,u^{-1}\,d\nu : \|u^{-1}\|_{L^{p'}(d\nu)}=1\}=\int f\,u^{-1}\,d\nu$$

for some $u\ge 0$ such that $\|u^{-1}\|_{L^{p'}(d\nu)}=1$, where $p'=\frac{p}{p-1}<0$. This follows from the "reverse" Hölder inequality

$$\int f\,g\,d\nu\ge\|f\|_{L^p(d\nu)}\|g\|_{L^{p'}(d\nu)},$$

which is a consequence of the usual Hölder inequality. Combining this with the Lebesgue differentiation theorem and both Lemmas 7.3 and 7.4 for the weight $(M(g^\delta))^{-1/\delta} \in RH_\infty(\mu)$, $\delta > 0$ and $g \geq 0$, we have

$$\int_S M_\varphi(f d\mu) \, \frac{Mw}{g} \, d\mu \geq \int_S M_\varphi(f d\mu) \, \frac{Mw}{(M(g^\delta))^{1/\delta}} \, d\mu$$

$$\geq c \int_S T(f d\mu) \, \frac{w}{(M(g^\delta))^{1/\delta}} \, d\mu \geq c \|T(f d\mu)\|_{L^p(w d\mu)} \big\|(M(g^\delta))^{-1/\delta}\big\|_{L^{p'}(w d\mu)}.$$

However, since the integral on the right side of (53) when raised to the power $1/p$ equals

$$\int_S M_\varphi(f d\mu) \, \frac{Mw}{g} \, d\mu$$

for some g, everything is reduced to proving that

$$\big\|(M(g^\delta))^{-1/\delta}\big\|_{L^{p'}(w d\mu)} \geq c \|g^{-1}\|_{L^{p'}(M(w) d\mu)}.$$

Since $p' < 0$, this is equivalent to saying that

$$\int_S (M(g^\delta))^{-p'/\delta} \, w \, d\mu \leq c \int_S g^{-p'} \, Mw \, d\mu.$$

But if we choose δ so that $0 < \delta < -p'$, then $-p'/\delta > 1$ and the last estimate follows from the known (see [FS]) weighted norm inequality

$$\int_S (Mf)^q \, w \, d\mu \leq c \int_S |f|^q \, Mw \, d\mu, \quad q > 1.$$

□

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