Existence and Approximation of Fixed Points of Right Bregman Nonexpansive Operators

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November 6, 2012

Dedicated to Jonathan Borwein on his 60th birthday

Abstract

We study the existence and approximation of fixed points of right Bregman nonexpansive operators in reflexive Banach space. We present, in particular, necessary and sufficient conditions for the existence of fixed points and an implicit scheme for approximating them.

2010 Mathematics Subject Classification: 26B25, 46T99, 47H04, 47H05, 47H09, 47H10, 47J05, 47J25, 52A41, 54C15.

Key words and phrases. Bregman distance, Bregman firmly nonexpansive operator, monotone mapping, nonexpansive operator, reflexive Banach space, resolvent, retraction, totally convex function.

1 Introduction

The study of nonexpansive operators in Banach spaces has been an important topic in Nonlinear Functional Analysis and Optimization Theory for almost fifty years

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now [?, ?, ?, ?]. There are several significant classes of nonexpansive operators which enjoy remarkable properties not shared by all such operators. We refer, for example, to firmly nonexpansive operators [?, ?]. These operators are of utmost importance in fixed point, monotone mapping, and convex optimization theories in view of Minty's Theorem regarding the correspondence between firmly nonexpansive operators and maximally monotone mappings [?, ?, ?, ?]. The largest class of nonexpansive operators comprises the quasi-nonexpansive operators. These operators still enjoy relevant fixed point properties although nonexpansivity is only required about each fixed point [?].

In this paper we are concerned with certain analogous classes of operators which are, in some sense, nonexpansive not with respect to the norm, but with respect to Bregman distances [?, ?, ?, ?]. Since these distances are not symmetric in general, it seems natural to distinguish between left and right Bregman nonexpansive operators. Some left classes, so to speak, have already been studied and applied quite intensively [?, ?, ?, ?, ?, ?]. We have recently introduced and studied several classes of right Bregman nonexpansive operators in reflexive Banach spaces [?, ?]. In these two papers we focused on the properties of their fixed point sets. Our main aim in the present paper is to study the existence and approximation of fixed points of these operators.

Our paper is organized as follows. In Section 2 we discuss several pertinent facts of Convex Analysis and Bregman operator theory. In the next section we present necessary and sufficient conditions for right quasi-Bregman nonexpansive operators to have (asymptotic) fixed points in general reflexive Banach spaces. The fourth section is devoted to a study of a Browder type implicit algorithm [?] for computing fixed points of right Bregman firmly nonexpansive operators. Finally, in the last section we use the implicit method proposed in Section 4 to approximate zeroes of monotone mappings.

2 Preliminaries

All the results in this paper are set in a real reflexive Banach space X. The norms of X and X^*, its dual space, are denoted by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. The pairing $\langle \xi, x \rangle$ is defined by the action of $\xi \in X^*$ at $x \in X$, that is, $\langle \xi, x \rangle := \xi(x)$. The set of all real numbers is denoted by $\mathbb{R}$ and $\overline{\mathbb{R}} = (-\infty, +\infty]$ is the extended real line, while $\mathbb{N}$ stands for the set of nonnegative integers. The closure of a subset K of X is denoted by $\overline{K}$. The (effective) *domain* of a convex function $f : X \to \overline{\mathbb{R}}$ is defined to be

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

When $\operatorname{dom} f \neq \emptyset$ we say that f is *proper*. The *Fenchel conjugate* function of f is the convex function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(\xi) = \sup \{ \langle \xi, x \rangle - f(x) : x \in X \}.$$

It is not difficult to check that when f is proper and lower semicontinuous, so is f^*. The function f is called *cofinite* if $\operatorname{dom} f^* = X^*$.

In this section we present the basic notions and facts that are needed in the sequel. We divide this section into three parts in the following way. The first one (Subsection **??**) is devoted to admissible functions, while the second (Subsection **??**) concern certain types of Bregman nonexpansive operators.

2.1 Admissible functions

Let $x \in \operatorname{int} \operatorname{dom} f$, that is, let x belong to the interior of the domain of the convex function $f : X \to \overline{\mathbb{R}}$. For any $y \in X$, we define the *right-hand derivative* of f at the point x by

$$f^\circ(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{1}$$

If the limit as $t \to 0$ in (**??**) exists for each y, then the function f is said to be *Gâteaux differentiable at* x. In this case, the *gradient* of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y \rangle := f^\circ(x, y)$ for all $y \in X$ [**?**, Definition 1.3, page 3]. The function f is called *Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in \operatorname{int} \operatorname{dom} f$. When the limit as $t \to 0$ in (**??**) is attained uniformly for any $y \in X$ with $\|y\| = 1$, we say that f is *Fréchet differentiable* at x.

The function f is called *Legendre* if it satisfies the following two conditions.

(L1) $\operatorname{int} \operatorname{dom} f \neq \emptyset$ and the subdifferential ∂f is single-valued on its domain.

(L2) $\operatorname{int} \operatorname{dom} f^* \neq \emptyset$ and ∂f^* is single-valued on its domain.

The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [**?**]. Their definition is equivalent to conditions (L1) and (L2) because the space X is assumed to be reflexive (see [**?**, Theorems 5.4 and 5.6, page 634]). It is well known that in reflexive spaces $\nabla f = (\nabla f^*)^{-1}$ (see [**?**, page 83]). When this fact is combined with conditions (L1) and (L2), we obtain

$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^* \quad \text{and} \quad \operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f.$$

It also follows that f is Legendre if and only if f^* is Legendre (see [**?**, Corollary 5.5, page 634]) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. When the Banach space X is smooth and strictly convex, in particular, a Hilbert space, the function $(1/p)\left\|\cdot\right\|^p$ with $p \in (1, \infty)$ is Legendre (*cf.* [**?**, Lemma 6.2, page 639]). For examples and more information regarding Legendre functions, see, for instance, [**?**, **?**].

Throughout this paper, $f : X \to \overline{\mathbb{R}}$ is always an *admissible* function, that is, a proper, lower semicontinuous, convex and Gâteaux differentiable function. Under these conditions we know that f is continuous in $\operatorname{int}\operatorname{dom} f$ (see [**?**, Fact 2.3, page 619]).

The bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$, which is defined by

$$D_f\left(y, x\right) := f\left(y\right) - f\left(x\right) - \left\langle \nabla f\left(x\right), y - x\right\rangle, \tag{2}$$

is called the *Bregman distance* (*cf.* [**?**, **?**]).

The Bregman distance does not satisfy the well-known properties of a metric, but it does enjoy the following two important properties.

- The *three point identity*: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$, we have

$$D_f\left(x, y\right) + D_f\left(y, z\right) - D_f\left(x, z\right) = \left\langle \nabla f\left(z\right) - \nabla f\left(y\right), x - y\right\rangle. \tag{3}$$

- The *four point identity*: for any $y, w \in \operatorname{dom} f$ and $x, z \in \operatorname{int}\operatorname{dom} f$, we have

$$D_f\left(y, x\right) - D_f\left(y, z\right) - D_f\left(w, x\right) + D_f\left(w, z\right) = \left\langle \nabla f\left(z\right) - \nabla f\left(x\right), y - w\right\rangle. \tag{4}$$

According to [**?**, Section 1.2, page 17] (see also [**?**]), the *modulus of total convexity of* f is the bifunction $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$, which is defined by

$$\upsilon_f\left(x, t\right) := \inf\left\{D_f\left(y, x\right) : y \in \operatorname{dom} f,\ \left\|y - x\right\| = t\right\}.$$

The function f is called *totally convex at a point* $x \in \operatorname{int}\operatorname{dom} f$ if $\upsilon_f\left(x, t\right) > 0$ whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point $x \in \operatorname{int}\operatorname{dom} f$. This property is less stringent than uniform convexity (see [**?**, Section 2.3, page 92]).

Examples of totally convex functions can be found, for instance, in [**?**, **?**, **?**]. We remark in passing that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [**?**, Theorem 2.10, page 9]).

2.2 Right Bregman operators

Let $f : X \to \overline{\mathbb{R}}$ be admissible and let K be a nonempty subset of X. The *fixed point set* of an operator $T : K \to X$ is the set $\{x \in K : Tx = x\}$. It is denoted by $\operatorname{Fix}(T)$. Recall that a point $u \in K$ is said to be an *asymptotic fixed point* [**?**] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in K such that $x_n \rightharpoonup u$ (that is, $\{x_n\}_{n\in\mathbb{N}}$ is weakly convergent to u) and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. We denote the asymptotic fixed point set of T by $\widehat{\operatorname{Fix}}(T)$.

We first list significant types of nonexpansivity with respect to the Bregman distance.

Definition 2.1 (Right Bregman nonexpansivity)**.** Let K and S be nonempty subsets of $\operatorname{dom} f$ and $\operatorname{int}\operatorname{dom} f$, respectively. An operator $T : K \to \operatorname{int}\operatorname{dom} f$ is said to be:

(i*) *right Bregman firmly nonexpansive* (R-BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(Tx) - \nabla f(Ty), x - y\rangle \tag{5}$$

for all $x, y \in K$, or equivalently,

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(x, Tx) + D_f(y, Ty) \\ \leq D_f(x, Ty) + D_f(y, Tx). \end{aligned} \tag{6}$$

(ii*) *Right quasi-Bregman firmly nonexpansive* (R-QBFNE) with respect to S if

$$0 \leq \langle \nabla f(p) - \nabla f(Tx), Tx - x\rangle \tag{7}$$

for all $x \in K$ and $p \in S$, or equivalently,

$$D_f(Tx, p) + D_f(x, Tx) \leq D_f(x, p). \tag{8}$$

(iii*) *Right quasi-Bregman nonexpansive* (R-QBNE) with respect to S if

$$D_f(Tx, p) \leq D_f(x, p), \ \forall\, x \in K,\ p \in S. \tag{9}$$

(iv*) *Right Bregman strongly nonexpansive* (R-BSNE) with respect to S if it is R-QBNE with respect to S and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded, $p \in S$, and

$$\lim_{n\to\infty} \left(D_f(x_n, p) - D_f(Tx_n, p)\right) = 0, \tag{10}$$

it follows that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0. \tag{11}$$

For the sake of completeness we give here also the definitions of left Bregman nonexpansivity.

Definition 2.2 (Left Bregman nonexpansivity)**.** Let K and S be nonempty subsets of $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively. An operator $T : K \to \operatorname{int}\operatorname{dom} f$ is said to be:

(i) *left Bregman firmly nonexpansive* (L-BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \tag{12}$$

for any $x, y \in K$, or equivalently,

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ \le D_f(Tx, y) + D_f(Ty, x). \end{aligned} \tag{13}$$

(ii) *Left quasi-Bregman firmly nonexpansive* (L-QBFNE) with respect to S if

$$0 \le \langle \nabla f(x) - \nabla f(Tx), Tx - p\rangle \tag{14}$$

for any $x \in K$ and $p \in S$, or equivalently,

$$D_f(p, Tx) + D_f(Tx, x) \le D_f(p, x). \tag{15}$$

(iii) *Left quasi-Bregman nonexpansive* (L-QBNE) with respect to S if

$$D_f(p, Tx) \le D_f(p, x) \ \ \forall\, x \in K,\, p \in S. \tag{16}$$

(iv) *Left Bregman strongly nonexpansive* (L-BSNE) with respect to S if it is L-QBNE with respect to S and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded, $p \in S$, and

$$\lim_{n\to\infty} \left(D_f(p, x_n) - D_f(p, Tx_n)\right) = 0, \tag{17}$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0. \tag{18}$$

Remark 2.3 (Types of Bregman nonexpansivity with respect to S)**.** As in [?], we distinguish between two types of Bregman nonexpansivity, depending on the set S, in such a way that if $S = \operatorname{Fix}(T)$ we say that T is properly Bregman nonexpansive, whereas if $S = \widehat{\operatorname{Fix}}(T)$ we say that T is strictly Bregman nonexpansive, according to the different notions of Bregman nonexpansivity. The connections among all these classes of right Bregman nonexpansive operators are presented in Table **??**. $\Diamond$

		strictly R-QBFNE	$\Rightarrow$	strictly R-BSNE	$\Rightarrow$	strictly R-QBNE
		$\Downarrow$		$\Downarrow$		$\Downarrow$
R-BFNE	$\Rightarrow$	properly R-QBFNE	$\Rightarrow$	properly R-BSNE	$\Rightarrow$	properly R-QBNE

Table 1: Connections among types of right Bregman nonexpansivity

The following result [**?**] is essential for the proof of our approximation result in Section **??**. It shows that the operator $I - T$ has a certain demiclosedness property. Before formulating this result, we recall that a mapping $B : X \to X^*$ is said to be *weakly sequentially continuous* if the weak convergence of $\{x_n\}_{n\in\mathbb{N}} \subset X$ to x implies the weak* convergence of $\{Bx_n\}_{n\in\mathbb{N}}$ to Bx.

Proposition 2.4 (Asymptotic fixed point set of R-BFNE operators)**.** *Let $f : X \to \mathbb{R}$ be Legendre and uniformly continuous on bounded subsets of X, and let ∇f be weakly sequentially continuous. Let K be a nonempty subset of $\operatorname{dom} f$ and let $T : K \to \operatorname{int}\operatorname{dom} f$ be an R-BFNE operator. Then $\operatorname{Fix}(T) = \widehat{\operatorname{Fix}}(T)$.*

In [**?**] we studied properties of several classes of right Bregman nonexpansive operators from the point of view of their fixed point sets. A useful tool for such a study is the following operator.

Definition 2.5 (Conjugate operator)**.** Let $f : X \to \overline{\mathbb{R}}$ be Legendre and let $T : K \subset \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ be an operator. We define the *conjugate operator* associated with T by

$$T_f^* := \nabla f \circ T \circ \nabla f^* : \nabla f(K) \to \operatorname{int}\operatorname{dom} f^*.$$

When there is no danger of confusion we use the notation T^* for T_f^*. We also denote $\left(T_f^*\right)_{f^*}^*$ by T^{**}. It is very natural to ask what the connections between left and right classes of Bregman nonexpansivity are. This question can be answered by using the following proposition [**?**, Proposition 2.7].

Proposition 2.6 (Properties of the conjugate operator)**.** *Let $f : X \to \overline{\mathbb{R}}$ be Legendre and let $T : K \subset \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ be an operator. Then the following properties hold.*

(i) $\operatorname{dom} T^* = \nabla f\left(\operatorname{dom} T\right)$ *and* $\operatorname{ran} T^* = \nabla f\left(\operatorname{ran} T\right)$.

(ii) *T is R-BFNE if and only if T^* is L-BFNE.*

(iii) $\operatorname{Fix}(T) = \nabla f^* (\operatorname{Fix}(T^*))$.

(iv) *T is R-QBFNE (R-QBNE or R-BSNE) if and only if T^* is L-QBFNE (L-QBNE or L-BSNE).*

(v) $T^{**} = T$.

(vi) *If, in addition, ∇f and ∇f^* are uniformly continuous on bounded subsets of* $\operatorname{int}\operatorname{dom} f$ *and* $\operatorname{int}\operatorname{dom} f^*$*, respectively, then*

$$\widehat{\operatorname{Fix}}(T^*) = \nabla f\left(\widehat{\operatorname{Fix}}(T)\right).$$

This connection between left and right Bregman nonexpansive operators allows us to get properties of right Bregman nonexpansive operators from their left counterparts (*cf.* [**?**]). The following result is an example of this.

Proposition 2.7 ($\nabla f(\operatorname{Fix}(T))$ of an R-QBNE operator is closed and convex)**.** *Let $f : X \to \overline{\mathbb{R}}$ be a Legendre function and let K be a nonempty subset of* $\operatorname{int}\operatorname{dom} f$ *such that $\nabla f(K)$ is closed and convex. If $T : K \to \operatorname{int}\operatorname{dom} f$ is an R-QBNE operator, then $\nabla f(\operatorname{Fix}(T))$ is closed and convex.*

Proof. Since T is R-QBNE, the conjugate operator T^* is L-QBNE with respect to f^* (see Proposition **??**(iv)). Moreover, f^* is Legendre, and the domain of T^* is $\nabla f(K)$, which is closed and convex by assumption. Applying [**?**, Lemma 15.5, page 307] and Proposition **??**(iii), we get that $\operatorname{Fix}(T^*) = \nabla f(\operatorname{Fix}(T))$ is closed and convex, as asserted. □

The *right Bregman projection* (*cf.* [**?**, **?**]) with respect to f of $x \in \operatorname{int}\operatorname{dom} f$ onto a nonempty, closed and convex set $K \subset \operatorname{int}\operatorname{dom} f$ is defined by

$$\overrightarrow{\operatorname{proj}}_K^f(x) := \operatorname*{argmin}_{y \in K} \{D_f(x,y)\} = \{z \in K : D_f(x,z) \leq D_f(x,y)\ \forall y \in K\}. \tag{19}$$

It is not clear *a priori* that the right Bregman projection is well defined because D_f is not convex in its second variable. However, Bauschke *et al.* (*cf.* [**?**, Proposition 7.1, page 9]) proved that

$$\overrightarrow{\operatorname{proj}}_K^f = \nabla f^* \circ \overleftarrow{\operatorname{proj}}_{\nabla f(K)}^{f^*} \circ \nabla f, \tag{20}$$

where $\overleftarrow{\operatorname{proj}}_K^f$ stands for the left Bregman projection onto K with respect to f (see [**?**, **?**] for more information). As a consequence, one is able to prove that the right Bregman projection with respect to functions with admissible and totally convex conjugates has a variational characterization (*cf.* [**?**, Proposition 4.10]) as long as $\nabla f(K)$ is closed and convex.

Proposition 2.8 (Characterization of the right Bregman projection)**.** *Let $f : X \to \mathbb{R}$ be a function such that f^* is admissible and totally convex. Let $x \in X$ and let K be a subset in* $\operatorname{int}\operatorname{dom} f$ *such that $\nabla f(K)$ is closed and convex. If $\hat{x} \in K$, then the following conditions are equivalent.*

(i) *The vector $\hat{x}$ is the right Bregman projection of x onto K with respect to f.*

(ii) *The vector $\hat{x}$ is the unique solution of the variational inequality*

$$\langle \nabla f(z) - \nabla f(y), z - x \rangle \geq 0 \qquad \forall y \in K.$$

(iii) *The vector $\hat{x}$ is the unique solution of the inequality*

$$D_f(z, y) + D_f(x, z) \leq D_f(x, y) \qquad \forall y \in K.$$

Given two subsets $K \subset C \subset X$, an operator $R : C \to K$ is said to be a *retraction of C onto K* if $Rx = x$ for each $x \in K$. A retraction $R : C \to K$ is said to be *sunny* (see [**?**, **?**]) if

$$R(Rx + t(x - Rx)) = Rx$$

for each $x \in C$ and any $t \geq 0$, whenever $Rx + t(x - Rx) \in C$.

Under certain conditions on f, it turns out that the right Bregman projection is the unique sunny R-QBNE retraction of X onto its range (*cf.* [**?**, Corollary 4.6]).

Proposition 2.9 (Properties of the right Bregman projection)**.** *Let $f : X \to \mathbb{R}$ be a Legendre, cofinite and totally convex function, and assume that f^* is totally convex. Let K be a nonempty subset of X.*

(i) *If $\nabla f(K)$ is closed and convex, then the right Bregman projection,*

$$\overrightarrow{\operatorname{proj}}_K^f = \nabla f^* \circ \overleftarrow{\operatorname{proj}}_{\nabla f(K)}^{f^*} \circ \nabla f,$$

is the unique sunny R-QBNE retraction of X onto K.

(ii) *If K is a sunny R-QBNE retract of X, then $\nabla f(K)$ is closed and convex, and $\overrightarrow{\operatorname{proj}}_K^f$ is the unique sunny R-QBNE retraction of X onto K.*

The previous result yields the fact that the fixed point set of any R-QBNE operator is a sunny R-QBNE retract of X and the corresponding retraction is uniquely defined by the right Bregman projection onto the fixed point set (*cf.* [**?**, Corollary 4.7]).

Proposition 2.10 ($\operatorname{Fix}(T)$ is a sunny R-QBNE retract)**.** *Let $f : X \to \mathbb{R}$ be Legendre, cofinite and totally convex, with a totally convex conjugate f^*. If $T : X \to X$ is an R-QBNE operator, then there exists a unique sunny R-QBNE retraction of X onto* $\operatorname{Fix}(T)$*, and this is the right Bregman projection onto* $\operatorname{Fix}(T)$*.*

3 Existence of fixed points

In this section we obtain necessary and sufficient conditions for R-QBNE operators to have (asymptotic) fixed points in general reflexive Banach spaces. We begin with a necessary condition for a strictly R-QBNE operator to have an asymptotic fixed point.

Proposition 3.1 (Necessary condition for $\widehat{\mathrm{Fix}}\,(T)$ to be nonempty)**.** *Let $f : X \to \overline{\mathbb{R}}$ be an admissible and totally convex function. Let $T : K \subset \operatorname{int}\operatorname{dom} f \to K$ be an operator. The following assertions hold.*

(i) *If T is strongly R-QBNE and $\widehat{\mathrm{Fix}}\,(T)$ is nonempty; or*

(ii) *if T is weakly R-QBNE and $\mathrm{Fix}\,(T)$ is nonempty,*

then $\{T^n x\}_{n\in\mathbb{N}}$ is bounded for each $x \in K$.

Proof. (i) We know from (**??**) that

$$D_f\,(Tx, p) \leq D_f\,(x, p)$$

for any $p \in \widehat{\mathrm{Fix}}\,(T)$ and $x \in K$. Therefore

$$D_f\,(T^n x, p) \leq D_f\left(T^{n-1}x, p\right) \leq \cdots \leq D_f\,(x, p)$$

for any $p \in \widehat{\mathrm{Fix}}\,(T)$ and $x \in K$. This inequality shows that the nonnegative sequence $\{D_f\,(T^n x, p)\}_{n\in\mathbb{N}}$ is bounded. Now the boundedness of the sequence $\{T^n x\}_{n\in\mathbb{N}}$ follows from [**?**, Lemma 3.1, page 31].

(ii) This result is a consequence of the arguments in assertion (i) when $p \in \widehat{\mathrm{Fix}}\,(T)$ is replaced with $p \in \mathrm{Fix}\,(T)$.

□

A left variant of Proposition **??**(ii) has already been proved in [**?**, Theorem 15.7, page 307]. Note that this left variant result can be rewritten as follows, where the conditions on f, T and K are somewhat different.

Proposition 3.2 (Necessary condition for $\mathrm{Fix}\,(T)$ to be nonempty (left variant))**.** *Let $f : X \to \overline{\mathbb{R}}$ be an admissible function and assume that ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$. Let $T : K \subset \operatorname{int}\operatorname{dom} f \to K$ be a properly L-QBNE operator. If $\mathrm{Fix}\,(T)$ is nonempty, then $\{T^n x\}_{n\in\mathbb{N}}$ is bounded for each $x \in K$.*

Using this result and the properties of the conjugate operator, we can now obtain a variant of Proposition **??**(ii) under different assumptions on f.

Proposition 3.3 (Necessary condition for $\mathrm{Fix}\,(T)$ to be nonempty (second version))**.** *Let $f : X \to \overline{\mathbb{R}}$ be a function such that f^* is admissible, and assume that ∇f and ∇f^* are bounded on bounded subsets of* $\mathrm{int}\,\mathrm{dom}\, f$ *and* $\mathrm{int}\,\mathrm{dom}\, f^*$*, respectively. Let $T : K \subset \mathrm{int}\,\mathrm{dom}\, f \to K$ be a properly R-QBNE operator. If* $\mathrm{Fix}\,(T)$ *is nonempty, then $\{T^n x\}_{n\in\mathbb{N}}$ is bounded for each $x \in K$.*

Proof. Since T is a properly R-QBNE operator with $\mathrm{Fix}\,(T) \neq \emptyset$, it follows from Proposition **??**(iii) and (iv) that

$$T^* := \nabla f \circ T \circ \nabla f^* : \nabla f\,(K) \to \nabla f\,(K) \tag{21}$$

is a properly L-QBNE operator with respect to f^* with $\mathrm{Fix}\,(T^*) = \nabla f\,(\mathrm{Fix}\,(T)) \neq \emptyset$. Since the assumptions of Proposition **??**(ii) hold, the sequence $\{(T^*)^n \xi\}_{n\in\mathbb{N}}$ is bounded for each $\xi \in \nabla f\,(K)$.

Next we note that

$$(T^*)^n = T^* \circ \cdots \circ T^* = \nabla f \circ T^n \circ \nabla f^* = (T^n)^*\,. \tag{22}$$

Therefore $\{(T^n)^* \xi\}_{n\in\mathbb{N}}$ is bounded for each $\xi \in \nabla f\,(K)$, which means that the sequence $\{\nabla f\,(T^n x)\}_{n\in\mathbb{N}}$ is bounded for each $x \in K$. Now the desired result follows because ∇f^* is bounded on bounded subsets of $\mathrm{int}\,\mathrm{dom}\, f^*$. $\square$

Given an operator $T : K \subset \mathrm{int}\,\mathrm{dom}\, f \to K$, we let

$$S_n^f\,(z) := (1/n) \sum_{k=1}^{n} \nabla f\left(T^k z\right), \quad z \in K. \tag{23}$$

Using these f-averages, we now present a sufficient condition for R-BFNE operators to have a fixed point. We start by proving this result directly.

Proposition 3.4 (Sufficient condition for $\mathrm{Fix}\,(T)$ to be nonempty)**.** *Let $f : X \to \overline{\mathbb{R}}$ be an admissible function. Let K be a nonempty subset of* $\mathrm{int}\,\mathrm{dom}\, f$ *such that $\nabla f\,(K)$ is closed and convex, and let $T : K \to K$ be an R-BFNE operator. If there exists $x \in K$ such that $\left\|S_n^f\,(x)\right\| \nrightarrow \infty$ as $n \to \infty$, then* $\mathrm{Fix}\,(T)$ *is nonempty.*

Proof. Assume there exists $x \in K$ such that $\left\|S_n^f\,(x)\right\| \nrightarrow \infty$ as $n \to \infty$. Let $y \in K$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$ be given. Since T is R-BFNE, we have (see (**??**))

$$D_f\left(T^{k+1}x, Ty\right) + D_f\left(Ty, T^{k+1}x\right) \leq D_f\left(y, T^{k+1}x\right) + D_f\left(T^k x, Ty\right), \tag{24}$$

where $T^0 = I$, the identity operator. From the three point identity (see (**??**)) and (**??**) we get

$$
\begin{aligned}
D_f\left(T^{k+1}x, Ty\right) + D_f\left(Ty, T^{k+1}x\right) \le\ & D_f\left(T^k x, Ty\right) + D_f\left(Ty, T^{k+1}x\right) \\
& + D_f\left(y, Ty\right) \\
& + \left\langle \nabla f\left(T^{k+1}x\right) - \nabla f\left(Ty\right), Ty - y\right\rangle .
\end{aligned}
$$

This implies that

$$
\begin{aligned}
0 \le\ & D_f\left(y, Ty\right) + D_f\left(T^k x, Ty\right) - D_f\left(T^{k+1}x, Ty\right) \\
& + \left\langle \nabla f\left(T^{k+1}x\right) - \nabla f\left(Ty\right), Ty - y\right\rangle .
\end{aligned}
$$

Summing up these inequalities with respect to $k = 0, 1, \ldots, n-1$, we now obtain

$$
\begin{aligned}
0 \le\ & nD_f\left(y, Ty\right) + D_f\left(x, Ty\right) - D_f\left(T^n x, Ty\right) \\
& + \left\langle \sum_{k=0}^{n-1} \nabla f\left(T^{k+1}x\right) - n\nabla f\left(Ty\right), Ty - y\right\rangle .
\end{aligned}
$$

Dividing this inequality by n, we get

$$
\begin{aligned}
0 \le\ & D_f\left(y, Ty\right) + \frac{1}{n}\left[D_f\left(x, Ty\right) - D_f\left(T^n x, Ty\right)\right] \\
& + \left\langle \frac{1}{n}\sum_{k=0}^{n-1} \nabla f\left(T^{k+1}x\right) - \nabla f\left(Ty\right), Ty - y\right\rangle
\end{aligned}
$$

and hence

$$
0 \le D_f\left(y, Ty\right) + \frac{1}{n} D_f\left(x, Ty\right) + \left\langle S_n^f\left(x\right) - \nabla f\left(Ty\right), Ty - y\right\rangle . \tag{25}
$$

Since $\left\|S_n^f\left(x\right)\right\| \nrightarrow \infty$ as $n \to \infty$ by assumption, we know that there exists a subsequence $\left\{S_{n_k}^f\left(x\right)\right\}_{k\in\mathbb{N}}$ of $\left\{S_n^f\left(x\right)\right\}_{n\in\mathbb{N}}$ such that $S_{n_k}^f\left(x\right) \rightharpoonup \xi \in X^*$ as $k \to \infty$. Substituting n_k for n in (**??**) and letting $k \to \infty$, we obtain

$$
0 \le D_f\left(y, Ty\right) + \left\langle \xi - \nabla f\left(Ty\right), Ty - y\right\rangle . \tag{26}
$$

Since $\nabla f\left(K\right)$ is closed and convex, we know that $\xi \in \nabla f\left(K\right)$. Therefore there exists $p \in K$ such that $\nabla f\left(p\right) = \xi$ and from (**??**) we obtain

$$
0 \le D_f\left(y, Ty\right) + \left\langle \nabla f\left(p\right) - \nabla f\left(Ty\right), Ty - y\right\rangle . \tag{27}
$$

Setting $y = p$ in (**??**), we get from the four point identity (see (**??**)) that

$$\begin{aligned} 0 &\le D_f(p, Tp) + \langle \nabla f(p) - \nabla f(Tp), Tp - p \rangle \\ &= D_f(p, Tp) + D_f(p, p) - D_f(p, Tp) - D_f(Tp, p) + D_f(Tp, Tp) \\ &= -D_f(Tp, p). \end{aligned}$$

Hence $D_f(Tp, p) \le 0$ and so $D_f(Tp, p) = 0$. It now follows from [**?**, Lemma 7.3(vi), p. 642] that $Tp = p$. That is, $p \in \operatorname{Fix}(T)$. □

At this point we recall the left variant of this result [**?**, Theorem 15.8, page 310], where

$$S_n(z) := (1/n) \sum_{k=1}^{n} T^k z, \quad z \in K. \tag{28}$$

Proposition 3.5 (Sufficient condition for $\operatorname{Fix}(T)$ to be nonempty (left variant))**.** *Let $f : X \to \overline{\mathbb{R}}$ be an admissible function. Let K be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$*, and let $T : K \to K$ be an L-BFNE operator. If there exists $x \in K$ such that $\|S_n(x)\| \nrightarrow \infty$ as $n \to \infty$, then* $\operatorname{Fix}(T)$ *is nonempty.*

Using this result, we obtain a second version of Proposition **??** under different assumptions on the function f.

Proposition 3.6 (Sufficient condition for $\operatorname{Fix}(T)$ to be nonempty (second version))**.** *Let $f : X \to \overline{\mathbb{R}}$ be a function such that f^* is admissible. Let K be a nonempty subset of* $\operatorname{int}\operatorname{dom} f$ *such that $\nabla f(K)$ is closed and convex, and let $T : K \to K$ be an R-BFNE operator. If there exists $x \in K$ such that $\left\|S_n^f(x)\right\| \nrightarrow \infty$ as $n \to \infty$, then* $\operatorname{Fix}(T)$ *is nonempty.*

Proof. Since T is an R-BFNE operator, we obtain from Proposition **??**(ii) that T^* is an L-BFNE operator. In addition, from (**??**) we get the following connection between the f-average operator S_n^f (see (**??**)) and the operator S_n (defined by (**??**) for the operator T) with respect to the conjugate operator T^*, which here we denote by $S_n^{T^*}$. Given $x \in K$ and $\xi := \nabla f(x) \in \nabla f(K)$,

$$\begin{aligned} S_n^f(x) &= \frac{1}{n} \sum_{k=1}^{n} \nabla f\left(T^k x\right) = \frac{1}{n} \sum_{k=1}^{n} \nabla f\left(T^k\left(\nabla f^*(\xi)\right)\right) \\ &= \frac{1}{n} \sum_{k=1}^{n} \left(\nabla f \circ T \circ \nabla f^*(\xi)\right)^k = \frac{1}{n} \sum_{k=1}^{n} \left(T^*(\xi)\right)^k := S_n^{T^*}(\xi). \end{aligned}$$

Hence the assumption that there exists $x \in K$ such that $\left\|S_n^f(x)\right\| \nrightarrow \infty$ as $n \to \infty$ is equivalent to the assumption that there exists $\xi \in \nabla f(K)$ such that $\left\|S_n^{T^*}(\xi)\right\| \nrightarrow \infty$ as $n \to \infty$. Now we apply Proposition **??** to f^* and T^* on $\nabla f(K)$, which is assumed to be closed and convex, and get that $\operatorname{Fix}(T^*)$ is nonempty. From Proposition **??**(iii) we obtain that $\operatorname{Fix}(T)$ is nonempty too. $\square$

From Propositions **??** and **??** we deduce the following result which says that every nonempty set K such that $\nabla f(K)$ is bounded, closed and convex has the fixed point property for R-BFNE self-operators

Corollary 3.7. *Let $f : X \to \overline{\mathbb{R}}$ be either an admissible function or a function such that f^* is admissible. Let K be a nonempty subset of* $\operatorname{int}\operatorname{dom} f$ *such that $\nabla f(K)$ is bounded, closed and convex, and let $T : K \to K$ be an R-BFNE operator. Then* $\operatorname{Fix}(T)$ *is nonempty.*

4 Approximation of fixed points

In this section we study the convergence of a Browder type implicit algorithm [**?**] for computing fixed points of R-BFNE operators with respect to a Legendre function f.

Theorem 4.1 (Implicit method for approximating fixed points)**.** *Let $f : X \to \mathbb{R}$ be a Legendre and positively homogeneous function of degree $\alpha > 1$, which is uniformly continuous on bounded subsets of X. Assume that ∇f is weakly sequentially continuous and f^* is totally convex. Let K be a nonempty and bounded subset of X such that $\nabla f(K)$ is bounded, closed and convex with $0^* \in \nabla f(K)$, and let $T : K \to K$ be an R-BFNE operator. Then the following two assertions hold.*

(i) *For each $t \in (0,1)$, there exists a unique $u_t \in K$ satisfying $u_t = tTu_t$.*

(ii) *The net $\{u_t\}_{t \in (0,1)}$ converges strongly to $\overrightarrow{\operatorname{proj}}^f_{\operatorname{Fix}(T)}(0)$ as $t \to 1^-$.*

Proof. (i) Fix $t \in (0,1)$ and let S_t be the operator defined by $S_t = tT$. Note that, since ∇f is positively homogeneous of degree $\alpha - 1 > 0$, we have $\nabla f(0) = 0^* \in \nabla f(K)$. This implies that S_t is an operator from K into K. Indeed, it is easy to see that for any $x \in K$, since $t^{\alpha-1} \in (0,1)$ and $\nabla f(K)$ is convex, we have

$$\nabla f^*\left(t^{\alpha-1}\nabla f(Tx) + \left(1 - t^{\alpha-1}\right)\nabla f(0)\right) \in K.$$

On the other hand,

$$\begin{aligned}\nabla f^*\left(t^{\alpha-1}\nabla f\left(Tx\right)+\left(1-t^{\alpha-1}\right)\nabla f\left(0\right)\right)&=\nabla f^*\left(t^{\alpha-1}\nabla f\left(Tx\right)\right)\\&=\nabla f^*\left(\nabla f\left(tTx\right)\right)\\&=tTx.\end{aligned}$$

Hence $S_tx\in K$ for any $x\in K$. Next we show that S_t is an R-BFNE operator. Given $x,y\in K$, since T is R-BFNE, we have

$$\begin{aligned}\left\langle\nabla f\left(S_tx\right)-\nabla f\left(S_ty\right),S_tx-S_ty\right\rangle&=t^{\alpha}\left\langle\nabla f\left(Tx\right)-\nabla f\left(Ty\right),Tx-Ty\right\rangle\\&\le t^{\alpha}\left\langle\nabla f\left(Tx\right)-\nabla f\left(Ty\right),x-y\right\rangle\\&=t\left\langle\nabla f\left(S_tx\right)-\nabla f\left(S_ty\right),x-y\right\rangle\\&\le\left\langle\nabla f\left(S_tx\right)-\nabla f\left(S_ty\right),x-y\right\rangle.\end{aligned}$$

Thus S_t is indeed R-BFNE. Since $\nabla f\left(K\right)$ is bounded, closed and convex, it follows from Corollary **??** that S_t has a fixed point. Furthermore, $\operatorname{Fix}\left(S_t\right)$ consists of exactly one point. Indeed, if $u,u'\in\operatorname{Fix}\left(S_t\right)$, then it follows from the right Bregman firm nonexpansivity of S_t that

$$\begin{aligned}\left\langle\nabla f\left(u\right)-\nabla f\left(u'\right),u-u'\right\rangle&=\left\langle\nabla f\left(S_tu\right)-\nabla f\left(S_tu'\right),S_tu-S_tu'\right\rangle\\&\le\left\langle\nabla f\left(S_tu\right)-\nabla f\left(S_tu'\right),u-u'\right\rangle\\&=t^{\alpha-1}\left\langle\nabla f\left(u\right)-\nabla f\left(u'\right),u-u'\right\rangle,\end{aligned}$$

which means that

$$\left\langle\nabla f\left(u\right)-\nabla f\left(u'\right),u-u'\right\rangle\le 0.$$

Since f is Legendre, we know that f is strictly convex and therefore ∇f is strictly monotone. Hence $u=u'$. Thus there exists a unique point $u_t\in K$ such that $u_t=S_tu_t$.

(ii) Note that, since T is R-BFNE, it follows from Corollary **??** that $\operatorname{Fix}\left(T\right)$ is nonempty. Furthermore, since T is R-QBNE (see Table **??**), from Proposition **??** we know that $\nabla f\left(\operatorname{Fix}\left(T\right)\right)$ is closed and convex. Therefore Proposition **??** shows that $\overrightarrow{\operatorname{proj}}^f_{\operatorname{Fix}(T)}$ is well defined and has a variational characterization. Let $\left\{t_n\right\}_{n\in\mathbb{N}}$ be an arbitrary sequence in the real interval $(0,1)$ such that $t_n\to 1^-$ as $n\to\infty$. Denote $x_n=u_{t_n}$ for all $n\in\mathbb{N}$. It suffices to show that $x_n\to\overrightarrow{\operatorname{proj}}^f_{\operatorname{Fix}(T)}\left(0\right)$ as $n\to\infty$. Since K is bounded, there is a subsequence $\left\{x_{n_k}\right\}_{k\in\mathbb{N}}$ of $\left\{x_n\right\}_{n\in\mathbb{N}}$ such that $x_{n_k}\rightharpoonup v$ as $k\to\infty$. From the definition of x_n, we see that $\left\|x_n-Tx_n\right\|=\left(1-t_n\right)\left\|Tx_n\right\|$ for all $n\in\mathbb{N}$. So, we have $\left\|x_n-Tx_n\right\|\to 0$ as

$n \to \infty$ and hence $v \in \widehat{\mathrm{Fix}}\,(T)$. Proposition **??** now implies that $v \in \mathrm{Fix}\,(T)$. We next show that $x_{n_k} \to v$ as $k \to \infty$. Fix $n \in \mathbb{N}$. Since T is properly R-QBFNE (see Table **??**), we have

$$0 \leq \langle \nabla f\,(Tx_n) - \nabla f\,(v)\,, x_n - Tx_n \rangle\,.$$

Since $x_n - Tx_n = (t_n - 1)\,Tx_n$, we also have

$$0 \leq \langle \nabla f\,(Tx_n) - \nabla f\,(v)\,, (t_n - 1)\,Tx_n \rangle\,.$$

This yields

$$0 \leq \langle \nabla f\,(Tx_n) - \nabla f\,(v)\,, -Tx_n \rangle \tag{29}$$

and

$$\langle \nabla f\,(Tx_n) - \nabla f\,(v)\,, Tx_n - v \rangle \leq \langle \nabla f\,(Tx_n) - \nabla f\,(v)\,, -v \rangle\,. \tag{30}$$

Since $x_{n_k} \rightharpoonup v$ and $\|x_{n_k} - Tx_{n_k}\| \to 0$ as $k \to \infty$, it follows that $Tx_{n_k} \rightharpoonup v$. From the weak sequential continuity of ∇f we obtain that $\nabla f\,(Tx_{n_k}) \overset{*}{\rightharpoonup} \nabla f\,(v)$ as $k \to \infty$. Hence it follows from the monotonicity of ∇f and from (**??**) that

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty} \langle \nabla f\,(Tx_{n_k}) - \nabla f\,(v)\,, Tx_{n_k} - v \rangle \\ &\leq \limsup_{k\to\infty} \langle \nabla f\,(Tx_{n_k}) - \nabla f\,(v)\,, -v \rangle \\ &= 0. \end{aligned} \tag{31}$$

Thus

$$\lim_{k\to\infty} \langle \nabla f\,(Tx_{n_k}) - \nabla f\,(v)\,, Tx_{n_k} - v \rangle = 0.$$

Since

$$D_f\,(v, Tx_{n_k}) + D_f\,(Tx_{n_k}, v) = \langle \nabla f\,(Tx_{n_k}) - \nabla f\,(v)\,, Tx_{n_k} - v \rangle\,,$$

it follows that

$$\lim_{k\to\infty} D_f\,(v, Tx_{n_k}) = \lim_{k\to\infty} D_f\,(Tx_{n_k}, v) = 0.$$

From [**?**, Proposition 2.2, page 3] we get that $\|Tx_{n_k} - v\| \to 0$ as $k \to \infty$.

Finally, we claim that $v = \overrightarrow{\operatorname{proj}}^f_{\operatorname{Fix}(T)}(0)$. Indeed, note that inequality (**??**) holds when we replace v with any $p \in \operatorname{Fix}(T)$. Then, since $\nabla f(Tx_{n_k}) \overset{*}{\rightharpoonup} \nabla f(v)$ and $Tx_{n_k} \to v$ as $k \to \infty$, letting $k \to \infty$ in this inequality, we get

$$0 \leq \langle \nabla f(v) - \nabla f(p), -v \rangle$$

for any $p \in \operatorname{Fix}(T)$. In other words,

$$0 \leq \langle \nabla f(v) - \nabla f(p), 0 - v \rangle$$

for any $p \in \operatorname{Fix}(T)$. Now we obtain from Proposition **??** that $v = \overrightarrow{\operatorname{proj}}^f_{\operatorname{Fix}(T)}(0)$, as asserted.

□

Here is the left variant of this result [**?**].

Proposition 4.2 (Implicit method for approximating fixed point (left variant))**.** *Let $f : X \to \mathbb{R}$ be a Legendre and totally convex function, which is positively homogeneous of degree $\alpha > 1$, uniformly Fréchet differentiable and bounded on bounded subsets of X. Let K be a nonempty, bounded, closed and convex subset of X with $0 \in K$, and let $T : K \to K$ be an L-BFNE operator. Then the following two assertions hold.*

(i) *For each $t \in (0, 1)$, there exists a unique $u_t \in K$ satisfying $u_t = tTu_t$;*

(ii) *The net $\{u_t\}_{t \in (0,1)}$ converges strongly to $\overleftarrow{\operatorname{proj}}^f_{\operatorname{Fix}(T)}(\nabla f^*(0^*))$ as $t \to 1^-$.*

Again using the left variant and the conjugation properties, we can obtain a right variant under somewhat different conditions.

Theorem 4.3 (Implicit method for approximating fixed points (second version))**.** *Let $f : X \to \overline{\mathbb{R}}$ be a Legendre and cofinite function. Assume that f^* is totally convex, positively homogeneous of degree $\alpha > 1$, and uniformly Fréchet differentiable and bounded on bounded subsets of X^*. Let K be a nonempty subset of $\operatorname{int} \operatorname{dom} f$ such that $\nabla f(K)$ is bounded, closed and convex with $0^* \in \nabla f(K)$. Let $T : K \to K$ be an R-BFNE operator. Then the following two assertions hold.*

(i) *For each $t \in (0, 1)$, there exists a unique $u_t \in K$ satisfying $u_t = tTu_t$.*

(ii) *The net $\{u_t\}_{t \in (0,1)}$ converges strongly to $\overrightarrow{\operatorname{proj}}^f_{\operatorname{Fix}(T)}(0)$ as $t \to 1^-$.*

Proof. (i) Since T is an R-BFNE operator, we obtain from Proposition **??**(ii) that the conjugate operator $T^* : \nabla f(K) \to \nabla f(K)$ is an L-BFNE operator with respect to f^*. Now we apply Proposition **??**(i) to T^* and get that for each $t \in (0,1)$, there exists a unique $\xi_t \in \nabla f(K)$ satisfying $\xi_t = tT^*\xi_t$. Denote $u_t = \nabla f^*(\xi_t) \in K$. Then from the definition of conjugate operators we get

$$\begin{aligned}\xi_t = tT^*\xi_t \;&\Leftrightarrow\; \nabla f(u_t) = tT^*\nabla f(u_t)\\ &\Leftrightarrow\; \nabla f(u_t) = t\left(\nabla f \circ T \circ \nabla f^*\right)(\nabla f(u_t))\\ &\Leftrightarrow\; \nabla f(u_t) = t\nabla f(Tu_t).\end{aligned}$$

Note that, since ∇f^* is positively homogeneous of degree $\alpha-1>0$, the gradient ∇f is positively homogeneous of degree $1/(\alpha-1)>0$. Hence

$$\nabla f(u_t) = \nabla f\left(t^{\alpha-1}Tu_t\right).$$

So, for each $t \in (0,1)$, there exists a unique $u_t \in K$ satisfying $u_t = t^{\alpha-1}Tu_t$, which yields assertion (i) because $\alpha - 1 > 0$ and $0 < t < 1$.

(ii) From the positive homogeneity, we deduce that $\nabla f^*(0^*) = 0$. Therefore, applying Proposition **??**(ii) to f^* and the conjugate operator T^* on $\nabla f(K)$, we get that the net $\{\xi_t\}_{t\in(0,1)}$ converges strongly to

$$\overleftarrow{\mathrm{proj}}^{f^*}_{\mathrm{Fix}(T^*)}(\nabla f(0)) = \overleftarrow{\mathrm{proj}}^{f^*}_{\mathrm{Fix}(T^*)}(0^*)$$

as $t \to 1^-$. Now, since $u_t = \nabla f^*(\xi_t) \in K$ for all $t \in (0,1)$, it follows from (**??**) that

$$\begin{aligned}\lim_{t\to 1^-} \nabla f(u_t) &= \overleftarrow{\mathrm{proj}}^{f^*}_{\mathrm{Fix}(T^*)}(0^*)\\ &= \nabla f\left(\overrightarrow{\mathrm{proj}}^{f}_{\mathrm{Fix}(T)}(\nabla f^*(0^*))\right)\\ &= \nabla f\left(\overrightarrow{\mathrm{proj}}^{f}_{\mathrm{Fix}(T)}(0)\right). \qquad (32)\end{aligned}$$

Since f^* is uniformly Fréchet differentiable and bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$, we know that ∇f^* is uniformly continuous on bounded subsets of X^* [**?**, Proposition 2.1]. Since $\{\xi_t = \nabla f(u_t)\}_{t\in(0,1)}$ is bounded as a convergent sequence, it now follows from (**??**) that $\{u_t\}_{t\in(0,1)}$ converges strongly to $\overrightarrow{\mathrm{proj}}^{f}_{\mathrm{Fix}(T)}(0)$ as $t \to 1^-$.

□

Remark 4.4. Under the hypotheses of Theorem **??**, since $\nabla f(K)$ is closed and convex, if we assume, in addition, that f is totally convex, then Proposition **??** implies that the right Bregman projection onto $\operatorname{Fix}(T)$ is the unique sunny R-QBNE retraction of X onto $\operatorname{Fix}(T)$. In other words, the sequence $\{u_t\}_{t\in(0,1)}$ converges strongly to the value of the unique sunny R-QBNE retraction of X onto $\operatorname{Fix}(T)$ at the origin. In the setting of a Hilbert space, when $f = (1/2)\|\cdot\|^2$, this fact recovers the result of Browder [**?**], which shows that, for a nonexpansive mapping T, the approximating curve $x_t = (1-t)u + tTx_t$ generates the unique sunny nonexpansive retraction onto $\operatorname{Fix}(T)$ when $t \to 1^-$, in the particular case where $u = 0$. $\diamondsuit$

5 Zeroes of monotone mappings

Let $A : X \to 2^{X^*}$ be a set-valued mapping. Recall that the (effective) *domain* of the mapping A is the set $\operatorname{dom} A = \{x \in X : Ax \neq \emptyset\}$. We say that A is *monotone* if for any $x, y \in \operatorname{dom} A$, we have

$$\xi \in Ax \text{ and } \eta \in Ay \quad \Longrightarrow \quad 0 \leq \langle \xi - \eta, x - y\rangle . \tag{33}$$

A monotone mapping A is said to be *maximal* if the graph of A is not a proper subset of the graph of any other monotone mapping.

A problem of great interest in Optimization Theory is that of finding zeroes of set-valued mappings $A : X \to 2^{X^*}$. Formally, the problem can be written as follows:

$$\text{Find } x \in X \text{ such that } 0^* \in Ax. \tag{34}$$

This problem occurs in practice in various forms. For instance, minimizing a lower semicontinuous and convex function $f : X \to \overline{\mathbb{R}}$, a basic problem of optimization, amounts to finding a zero of the mapping $A = \partial f$, where $\partial f(x)$ stands for the subdifferential of f at the point $x \in X$. Finding solutions of some classes of differential equations can also be reduced to finding zeroes of certain set-valued mappings $A : X \to 2^{X^*}$.

In the case of a Hilbert space $\mathcal{H}$, one of the most important methods for solving (**??**) consists of replacing it with the equivalent fixed point problem for the classical resolvent $R_A : \mathcal{H} \to 2^{\mathcal{H}}$ of A, defined by

$$R_A := (I + A)^{-1} .$$

In this case, provided that A satisfies some monotonicity conditions, the resolvent R_A is single-valued, nonexpansive and even firmly nonexpansive. When X is not a

Hilbert space, the classical resolvent R_A is of limited interest and other operators should be employed. For example, in several papers (see, for instance, [**?**, **?**]) the f-resolvent Res_A^f was used for finding zeroes of monotone mappings A in general reflexive Banach spaces. More precisely, given a set-valued mapping $A : X \to 2^{X^*}$, the *f-resolvent* of A is the operator $\mathrm{Res}_A^f : X \to 2^X$ which is defined by

$$\mathrm{Res}_A^f := (\nabla f + A)^{-1} \circ \nabla f. \tag{35}$$

In this paper we consider another variant of the classical resolvent for general reflexive Banach spaces, namely, the conjugate resolvent of a mapping A [**?**].

Definition 5.1 (Conjugate f-resolvent)**.** Let $A : X \to 2^{X^*}$ be a set-valued mapping. The *conjugate resolvent* of A with respect to f, or the conjugate f-resolvent, is the operator $\mathrm{CRes}_A^f : X^* \to 2^{X^*}$ defined by

$$\mathrm{CRes}_A^f := (I + A \circ \nabla f^*)^{-1}. \tag{36}$$

The conjugate resolvent satisfies the following properties [**?**].

Proposition 5.2 (Properties of conjugate f-resolvents)**.** *Let $f : X \to \overline{\mathbb{R}}$ be an admissible function and let $A : X \to 2^{X^*}$ be a mapping such that* $\operatorname{int}\operatorname{dom} f \cap \operatorname{dom} A \neq \emptyset$*. The following statements hold.*

(i) $\operatorname{dom}\mathrm{CRes}_A^f \subset \operatorname{int}\operatorname{dom} f^*$.

(ii) $\operatorname{ran}\mathrm{CRes}_A^f \subset \operatorname{int}\operatorname{dom} f^*$.

(iii) $\nabla f^*\left(\operatorname{Fix}\left(\mathrm{CRes}_A^f\right)\right) = \operatorname{int}\operatorname{dom} f \cap A^{-1}(0^*)$.

(iv) *Suppose, in addition, that A is a monotone mapping. Then the following assertions also hold.*

  (a) *If $f|_{\operatorname{int}\operatorname{dom} f}$ is strictly convex, then the operator CRes_A^f is single-valued on its domain and R-BFNE.*

  (b) *If $f : X \to \mathbb{R}$ is such that* $\operatorname{ran}\nabla f \subset \operatorname{ran}(\nabla f + A)$, *then* $\operatorname{dom}\mathrm{CRes}_A^f = \operatorname{int}\operatorname{dom} f^*$.

According to Proposition **??**(iii) and (iv)(a), we can apply Theorem **??** in the dual space X^* to the conjugate resolvent CRes_A^f and obtain an implicit method for approximating zeroes of monotone mappings.

Theorem 5.3 (Implicit method for approximating zeroes). *Let $f : X \to \mathbb{R}$ be a Legendre and totally convex function such that f^* is positively homogeneous of degree $\alpha > 1$ and uniformly continuous on bounded subsets of X^*. Assume that ∇f^* is weakly sequentially continuous. Let K^* be a nonempty and bounded subset of X^* such that $\nabla f^*(K^*)$ is bounded, closed and convex with $0 \in \nabla f^*(K^*)$. Let λ be any positive real number and let $A : X \to 2^{X^*}$ be a monotone mapping such that $\nabla f(\operatorname{dom} A) \subset K^* \subset \operatorname{ran}(I + \lambda A \circ \nabla f^*)$. Then the following two assertions hold.*

(i) *For each $t \in (0,1)$, there exists a unique $\xi_t \in K^*$ satisfying $\xi_t = t\mathrm{CRes}^f_{\lambda A}\xi_t$.*

(ii) *The net $\{\xi_t\}_{t\in(0,1)}$ converges strongly to $\overrightarrow{\mathrm{proj}}^f_{\nabla f(A^{-1}(0^*))}(0^*)$ as $t \to 1^-$.*

Acknowledgements. The first author was partially supported by DGES (Grant BFM2009-1096-C02-01) and by the Junta de Andalucía (Grant FQM-127); the second and third authors were partially supported by the Israel Science Foundation (Grant 647/07), the Graduate School of the Technion, the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund. Part of the research pertaining to this paper was carried out when the authors were visiting each other's institutions. They thank their colleagues for their kind hospitality.

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