Iterative Methods for Approximating Fixed Points of Bregman Nonexpansive Operators

Victoria Martín-Márquez[*], Simeon Reich[†] and Shoham Sabach[‡]

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Abstract

Diverse notions of nonexpansive type operators have been extended to the more general framework of Bregman distances in reflexive Banach spaces. We study these classes of operators, mainly with respect to the existence and approximation of their (asymptotic) fixed points. In particular, the asymptotic behavior of Picard and Mann type iterations is discussed for quasi-Bregman nonexpansive operators. We also present parallel algorithms for approximating common fixed points of a finite family of Bregman strongly nonexpansive operators by means of a block operator which preserves the Bregman strong nonexpansivity. All the results hold, in particular, for the smaller class of Bregman firmly nonexpansive operators, a class which contains the generalized resolvents of monotone mappings with respect to the Bregman distance.

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[*]Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160, 41080 Sevilla, Spain. Email: victoriam@us.es.

[†]Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel. Email: sreich@tx.technion.ac.il.

[‡]Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel. Email: ssabach@tx.technion.ac.il.

1 Introduction

It is well known that many nonlinear problems can be reduced to the search for fixed points of nonlinear operators. See, for example, [19, 25, 31] and the references therein. Iterative methods are often used for finding and approximating such fixed points (see [5, 17] and their references).

Let K be a nonempty, closed and convex subset of a Banach space X and let $T : K \to K$ be an operator. The most well-known method for solving the fixed point equation $Tx = x$ is perhaps the Picard successive iterations method when T is a strict contraction, that is,

$$\|Tx - Ty\| \leq c \|x - y\|$$

for some $0 \leq c < 1$ and all $x, y \in K$.

Picard's method generates a sequence $\{x_n\}_{n\in\mathbb{N}}$ successively by

$$x_{n+1} = Tx_n \tag{1}$$

for each $n \geq 0$, with x_0 chosen arbitrarily in K. This sequence then converges in norm to the unique fixed point of T. However, if T is not a strict contraction (for instance, even if T is nonexpansive with a unique fixed point), then Picard's successive iterations method fails, in general, to converge. To see this, it suffices, for example, to take for T a rotation of the unit disc in the plane about the origin of coordinates.

Krasnosel'ski [24], however, has shown that in this example, one can obtain a convergent sequence of successive approximations if instead of T one takes the auxiliary nonexpansive operator $(1/2)(I + T)$, where I denotes the identity operator of X, that is, if the sequence of successive approximations is defined, for arbitrary $x_0 \in K$, by

$$x_{n+1} = \frac{1}{2}(I + T)x_n \tag{2}$$

for each $n \geq 0$. It is easy to check that the operators T and $(1/2)(I + T)$ have the same fixed point sets, so that the limit of a convergent sequence defined by (2) is necessarily a fixed point of T.

A more general iterative scheme is the following one:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Tx_n \tag{3}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, 1)$ is a sequence satisfying appropriate conditions. The sequence $\{x_n\}_{n\in\mathbb{N}}$, generated by (3), is referred to as the Mann sequence in light of [26]. In an infinite-dimensional Hilbert space, the Mann iterative scheme only yields weak convergence in general (see [4, 20]).

Our first purpose in this paper is to study (modifications of) these two methods in reflexive Banach spaces. We are also interested in finding variants of these methods for approximating common fixed points of finitely many operators. A commonly used approach to tackling this problem arose in the following way.

The problem of finding a point in the intersection of a finite number of closed and convex subsets $\{K_i\}_{i=1}^N$ of a Banach space is a frequently appearing problem in diverse areas of mathematical and physical sciences. This problem is commonly referred to as the convex feasibility problem (CFP). A classical method for solving this problem is the cyclic projection one. In 1965 Bregman [7, Theorem 1] showed that in a Hilbert space H, for every initial point $x_0 \in H$, the sequence $\{x_n\}_{n\in\mathbb{N}}$, generated by the cyclic projection algorithm

$$x_{n+1} = P_{n(\mathrm{mod}\, N)+1} x_n, \tag{4}$$

where P_i denotes the metric projection onto K_i and the mod N function takes values in $\{0, \ldots, N-1\}$, converges weakly to a point in $K := \bigcap_{i=1}^N K_i \neq \emptyset$.

In order to obtain a similar result in a general Banach space X, Bregman [8] introduced a distance-like function which later was given the name Bregman distance (see (9)) by Censor and Lent [15]. Bregman proved that if in (4) the metric projection is replaced with the Bregman projection (where the norm is replaced with the Bregman distance) then the cluster points of $\{x_n\}_{n\in\mathbb{N}}$ are in K (see [8, Equation (1.2) and Theorem 1]).

When a parallel computer is available, it may be more convenient to use another method, called the parallel scheme, for constructing an approximating sequence $\{x_n\}_{n\in\mathbb{N}}$. To this end, at the n-th iteration a set of N positive real numbers $\{w_n^i\}_{i=1}^N$ (the weights) with $\sum_{i=1}^N w_n^i = 1$ is chosen, and, analogously to the sequential scheme, the new point x_{n+1} is created by computing a convex combination of all the projections, namely,

$$x_{n+1} = \sum_{i=1}^N w_n^i P_i x_n. \tag{5}$$

Over the years, the sequential and parallel algorithmic schemes have been extended to more flexible block-iterative methods in which only a block $\{K_i\}_{i\in J}$ of the sets is activated at the n-th iteration, where J is a subset of $\{1, \ldots, N\}$. The block-iterative methods evolved further to include the so-called relaxation methods for solving the convex feasibility problem, which go back to Kaczmarz [22] and Cimmino [18]. These methods are of special interest because of their relatively easy implementation and computational efficiency in solving extremely large and sparse problems. Contributions to the study of relaxation methods are surveyed in [14]. Aharoni and Censor [1] discuss a block iterative projection method which incorporates as special cases many of the earlier relaxation techniques.

Butnariu and Censor [10] studied the following iterative procedure in a Hilbert space H:

$$x_{n+1} = \sum_{i=1}^{N} w_n^i \left(\alpha_i x_n + (1-\alpha_i) P_i x_n\right), \tag{6}$$

where $x_0 \in H$, $\{w_n^i\}_{i=1}^N \subset [0,1]$ $(n \in \mathbb{N})$ such that $\sum_{i=1}^N w_n^i = 1$ $(n \in \mathbb{N})$ and $\{\alpha_i\}_{i=1}^N \subset (-1,1)$. They proved that $\{x_n\}_{n\in\mathbb{N}}$ converged strongly to an element of K.

Let K be a nonempty, closed and convex subset of a Hilbert space H. An operator $T : K \to K$ is called *nonexpansive* (or 1-Lipschitz) if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$. It turns out that nonexpansive fixed point theory can be applied to the solution of a variety of problems such as finding zeroes of monotone operators and solutions to certain evolution equations, and to solving convex feasibility (CFP), variational inequality (VIP) and equilibrium problems (EP). Kikkawa and Takahashi [23] have recently applied method (6) to the problem of finding common fixed points of a finite family of nonexpansive mappings in Banach spaces. More precisely, they studied the algorithm

$$x_{n+1} = \sum_{i=1}^{N} w_n^i \left(\alpha_n^i x_n + \left(1-\alpha_n^i\right) T_i x_n\right), \tag{7}$$

where $x_0 \in X$, $\{\alpha_n^i\}_{i=1}^N$ and $\{w_n^i\}_{i=1}^N$ are sequences in $[0,1]$ $(n \in \mathbb{N})$ such that $\sum_{i=1}^N w_n^i = 1$, $(n \in \mathbb{N})$ and $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive operators from K into itself. They prove that under certain conditions on the Banach space X and the sequences $\{\alpha_n^i\}_{i=1}^N$ and $\{w_n^i\}_{i=1}^N$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to an element of $F := \bigcap_{i=1}^N \operatorname{Fix}(T_i)$, where $\operatorname{Fix}(T)$ stands for the fixed point set of the operator T.

In the present paper we study a generalization of Algorithm (7) in the more general framework of Bregman distances. In this connection we introduce the block operator corresponding to a finite family of Bregman operators of nonexpansive type and prove several results concerning the relations between the common fixed points of the family and the block operator. In order to approximate fixed points of such operators, we use two well-known iterative methods, namely, the Picard and Mann iterations. For both methods we prove diverse results in several scenarios.

Our paper is organized in the following way. We start (see Section 2) with basic definitions, results and remarks concerning our main objects of studying, for example, Bregman distances, totally convex functions and Bregman nonexpansive type operators. In the following part (see Section 3) we prove several auxiliary results which are essential in our later analysis. We show, in particular, that any Bregman strongly nonexpansive operator is asymptotically regular (see Proposition 11). The third and the fourth sections are devoted to the analysis of Picard and Mann iterations, respectively. In these two sections we prove convergence results for Bregman nonexpansive operators. In the last, but not least, section (Section 6) we introduce the block operator and prove several results concerning approximating fixed points of such operators.

2 Preliminaries

Let X denote a real reflexive Banach space with norm $\|\cdot\|$ and let X^* stand for the (topological) dual of X equipped with the induced norm $\|\cdot\|_*$. We denote the value of the functional $\xi \in X^*$ at $x \in X$ by $\langle \xi, x \rangle$. Given $\{x_n\}_{n\in\mathbb{N}}$ and $x \in X$, the strong convergence (weak convergence) of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to x is denoted by $x_n \to x$ ($x_n \rightharpoonup x$).

Let $f : X \to (-\infty, +\infty]$ be a function. The *domain* of f is defined to be

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

When $\operatorname{dom} f \neq \emptyset$ we say that f is *proper*. We denote by $\operatorname{int}\operatorname{dom} f$ the *interior* of the domain of f.

The *Fenchel conjugate* of f is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) = \sup\{\langle \xi, x\rangle - f(x) : x \in X\}.$$

Let $x \in \operatorname{int}\operatorname{dom} f$. For any $y \in X$, we define the *right-hand derivative* of f at x by

$$f^\circ(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{8}$$

If the limit as $t \to 0$ in (8) exists for any y, then the function f is said to be *Gâteaux differentiable at x*. In this case, the *gradient* of f at x is the function $\nabla f(x) : X \to (-\infty, +\infty]$ defined by $\langle \nabla f(x), y\rangle = f^\circ(x, y)$ for any $y \in X$. The function f

is called *Gâteaux differentiable* if it is Gâteaux differentiable at any $x \in \operatorname{int} \operatorname{dom} f$. Throughout this paper, the function $f : X \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function which is also Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f$.

Definition 1 (Legendre function)**.** The function f is called *Legendre* if it satisfies the following two conditions:

(L1) the interior of the domain of f, $\operatorname{int} \operatorname{dom} f$, is nonempty, f is Gâteaux differentiable and

$$\operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f;$$

(L2) the interior of the domain of f^*, $\operatorname{int} \operatorname{dom} f^*$, is nonempty, f^* is Gâteaux differentiable and

$$\operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*.$$

Since X is reflexive, we always have $\nabla f = (\nabla f^*)^{-1}$ (see [6, p. 83]). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*$$

and

$$\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f.$$

Conditions (L1) and (L2), in conjunction with [3, Theorem 5.4, p. 634], imply that the functions f and f^* are strictly convex on the interior of their respective domains.

Definition 2 (Bregman distance)**.** The bifunction $D_f : \operatorname{dom} f \times \operatorname{int} \operatorname{dom} f \to [0, +\infty)$ given by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \tag{9}$$

is called the *Bregman distance with respect to* f (*cf.* [15]).

With the function f we associate the bifunction $W^f : \operatorname{dom} f^* \times \operatorname{dom} f \to [0, +\infty)$ defined by

$$W^f(\xi, x) := f(x) - \langle \xi, x \rangle + f^*(\xi). \tag{10}$$

Proposition 1 (Properties of W^f)**.** *Let f be a Legendre function. Then the following statements hold:*

(i) The function $W^f(\cdot, x)$ is convex for any $x \in \operatorname{dom} f$.

(ii) $W^f(\nabla f(x), y) = D_f(y, x)$ for any $x \in \operatorname{int} \operatorname{dom} f$ and $y \in \operatorname{dom} f$.

(iii) For any $\xi, \eta \in \operatorname{dom} f^*$ *and* $x \in \operatorname{dom} f$*, we have*

$$W^f(\xi, x) + \langle \eta, (\nabla f^*)(\xi) - x \rangle \leq W^f(\xi + \eta, x).$$

Proof. (i) This is clear since f^* is convex.

(ii) Let $x \in \operatorname{int}\operatorname{dom} f$ and let $y \in \operatorname{dom} f$. It is known that

$$f(x) + f^*(\nabla f(x)) = \langle \nabla f(x), x \rangle.$$

Therefore

$$\begin{aligned} W^f(\nabla f(x), y) &= f(y) - \langle \nabla f(x), y \rangle + f^*(\nabla f(x)) \\ &= f(y) - \langle \nabla f(x), y \rangle + [\langle \nabla f(x), x \rangle - f(x)] \\ &= f(y) - f(x) - \langle \nabla f(x), y - x \rangle \\ &= D_f(y, x). \end{aligned}$$

(iii) Let $x \in \operatorname{dom} f$ be given. Define the function $g : X^* \to (-\infty, +\infty]$ by $g(\xi) = W^f(\xi, x)$. Then

$$\nabla g(\xi) = \nabla (f^* - \langle \cdot, x \rangle)(\xi) = (\nabla f^*)(\xi) - x.$$

Hence

$$g(\xi + \eta) - g(\xi) \geq \langle \eta, (\nabla f^*)(\xi) - x \rangle,$$

that is,

$$W^f(\xi, x) + \langle \eta, (\nabla f^*)(\xi) - x \rangle \leq W^f(\xi + \eta, x)$$

for all $\xi, \eta \in \operatorname{dom} f^*$.

□

We now recall the definition of a totally convex function which was introduced in [11, 12].

Definition 3 (Total convexity)**.** The function f is called *totally convex at a point* $x \in \operatorname{int}\operatorname{dom} f$ if its *modulus of total convexity at* x, $\upsilon_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$, defined by

$$\upsilon_f(x, t) := \inf \{ D_f(y, x) : y \in \operatorname{dom} f, \ \|y - x\| = t \},$$

is positive whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point of $\operatorname{int}\operatorname{dom} f$.

Definition 4 (Total convexity on bounded subsets)**.** The function f is called *totally convex on bounded sets* if, for any nonempty bounded set $E \subset X$, the *modulus of total convexity of* f on E, $\upsilon_f(E,t)$, is positive for any $t > 0$, where $\upsilon_f(E,\cdot) : [0,+\infty) \to [0,+\infty]$ is defined by

$$\upsilon_f(E,t) := \inf \{\upsilon_f(x,t) : x \in E \cap \operatorname{int}\operatorname{dom} f\}.$$

Recall that, according to Censor and Lent [15], the *Bregman projection with respect to* f of a point $x \in X$ onto the closed convex set K is the (necessarily unique) minimizer over K of the functional $D_f(\cdot,x) : X \to [0,+\infty]$; *i.e.*,

$$\operatorname{proj}_K^f(x) := \arg\min \{D_f(y,x) : y \in K\}.$$

The following characterization was proved in [13, Corollary 4.4, p. 23].

Proposition 2 (Characterization of the Bregman projection)**.** *Let* $f : X \to (-\infty,+\infty]$ *be a Gâteaux differentiable and totally convex function. Let* $x \in \operatorname{int}\operatorname{dom} f$ *and let* $K \subset \operatorname{int}\operatorname{dom} f$ *be a nonempty, closed and convex set. If* $\hat{x} \in K$*, then the following statements are equivalent:*

(i) the vector $\hat{x}$ *is the Bregman projection of* x *onto* K, $\operatorname{proj}_K^f(x)$*;*

(ii) the vector $\hat{x}$ *is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0, \ \forall y \in K; \tag{11}$$

(iii) the vector $\hat{x}$ *is the unique solution of the inequality*

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \qquad \forall y \in K. \tag{12}$$

Let $K \subset \operatorname{int}\operatorname{dom} f$ be a nonempty set. The *fixed point set* of an operator $T : K \to \operatorname{int}\operatorname{dom} f$ is the set $\{x \in K : Tx = x\}$ and is denoted by $\operatorname{Fix}(T)$. A point p in the closure of K is said to be an *asymptotic fixed point* of T (*cf.* [16, 28]) if K contains a sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{\operatorname{Fix}}(T)$.

We next list significant types of nonexpansivity with respect to the Bregman distance.

Definition 5 (Bregman nonexpansivity)**.** Let $S \subset \operatorname{dom} f$ be a nonempty subset. The operator $T : K \to \operatorname{int}\operatorname{dom} f$ is said to be:

(i) *Bregman nonexpansive* (BNE) if

$$D_f(Tx, Ty) \leq D_f(x, y), \ \forall\, x, y \in K; \tag{13}$$

(ii) *quasi-Bregman nonexpansive* (QBNE) with respect to S if

$$D_f(p, Tx) \leq D_f(p, x), \ \forall\, x \in K,\, p \in S; \tag{14}$$

(iii) *Bregman firmly nonexpansive* (BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \tag{15}$$

for any $x, y \in K$, or equivalently,

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ \leq D_f(Tx, y) + D_f(Ty, x); \end{aligned} \tag{16}$$

(iv) *quasi-Bregman firmly nonexpansive* (QBFNE) with respect to S if

$$0 \leq \langle \nabla f(x) - \nabla f(Tx), Tx - p \rangle \ \forall\, x \in K,\, p \in S, \tag{17}$$

or equivalently,

$$D_f(p, Tx) + D_f(Tx, x) \leq D_f(p, x); \tag{18}$$

(v) *Bregman strongly nonexpansive* (BSNE) with respect to S if

$$D_f(p, Tx) \leq D_f(p, x) \ \forall\, x \in K,\, p \in S, \tag{19}$$

and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded, $p \in S$, and

$$\lim_{n\to+\infty} \left(D_f(p, x_n) - D_f(p, Tx_n) \right) = 0, \tag{20}$$

it follows that

$$\lim_{n\to+\infty} D_f(Tx_n, x_n) = 0. \tag{21}$$

Remark 1 (Types of quasi-Bregman nonexpansivity)**.**

(i) An operator which satisfies (14) (or (18)) with respect to $S := \operatorname{Fix}(T)$ is called *weakly QBNE* (or *weakly QBFNE*).

(ii) An operator which satisfies (14) (or (18)) with respect to $S := \widehat{\text{Fix}}\,(T)$ is called *strongly QBNE* (or *strongly QBFNE*).

(iii) An operator which satisfies (14) (or (18)) with respect to $S := \text{Fix}\,(T) = \widehat{\text{Fix}}\,(T)$ is called *QBNE* (or *QBFNE*).

$\lozenge$

Remark 2 (Types of Bregman strong nonexpansivity)**.**

(i) An operator which satisfies (19)-(21) with respect to $S := \text{Fix}\,(T)$ is called *weakly BSNE*.

(ii) An operator which satisfies (19)-(21) with respect to $S := \widehat{\text{Fix}}\,(T)$ is called *strongly BSNE* (this class of operators was first defined in [28]).

(iii) An operator which satisfies (19)-(21) with respect to $S := \text{Fix}\,(T) = \widehat{\text{Fix}}\,(T)$ is called *BSNE*.

$\lozenge$

The relations among all these classes of Bregman nonexpansive operators are summarized in the following scheme (see Table 1).

$$
\begin{array}{ccccccc}
 & & \text{strongly QBFNE} & \Rightarrow & \text{strongly BSNE} & \Rightarrow & \text{strongly QBNE} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{BFNE} & \Rightarrow & \text{weakly QBFNE} & \Rightarrow & \text{weakly BSNE} & \Rightarrow & \text{weakly QBNE}
\end{array}
$$

Table 1: Implications between the Bregman nonexpansive types

An interesting particular case of Bregman nonexpansive operators is the following one: Assume now that $f = (1/2)\left\|\cdot\right\|^2$ and the space X is a Hilbert space H, so that $\nabla f = I$ (the identity operator) and $D_f\,(y, x) = (1/2)\left\|x - y\right\|^2$. Thence, Definition 5(i)-(iv) takes the form presented in Definition 6(i')-(iv'). The analog of Definition 5(v) is presented in Definition 6(v'). This latter class of operators was first studied in [9]. Since the norm variant does not follow from the Bregman case as do the other classes we emphasize the connection between these two classes in Remark 3.

Definition 6. In this case we assume that $S := \text{Fix}\,(T)$. We say that $T : K \to H$ is:

(i') *nonexpansive* (NE) if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall \ x, y \in K; \tag{22}$$

(ii') *quasi-nonexpansive* (QNE) if

$$\|Tx - p\| \leq \|x - p\|, \ \forall \ x \in K, \ p \in \operatorname{Fix}(T); \tag{23}$$

(iii') *firmly nonexpansive* (FNE) if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \ \forall \ x, y \in K; \tag{24}$$

(iv') *quasi-firmly nonexpansive* (QFNE) if

$$\|Tx - p\|^2 + \|Tx - x\|^2 \leq \|x - p\|^2, \ \forall \ x \in K, \ p \in \operatorname{Fix}(T), \tag{25}$$

or equivalently, $0 \leq \langle x - Tx, Tx - p\rangle$.

(v') *strongly nonexpansive* (SNE) if T is nonexpansive and for any bounded sequence $\{x_n - y_n\}_{n\in\mathbb{N}}$ satisfying

$$\lim_{n\to+\infty} (\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0, \tag{26}$$

it follows that

$$\lim_{n\to+\infty} ((x_n - y_n) - (Tx_n - Ty_n)) = 0. \tag{27}$$

Remark 3 (Connection between BSNE and SNE operators)**.** When $f = (1/2)\|\cdot\|^2$ and $S = \operatorname{Fix}(T)$, definition 5(v) means that $T : K \to H$ is BSNE with respect to $\operatorname{Fix}(T)$ if T is QNE (definition 6(ii')) and if for any bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying

$$\lim_{n\to\infty} \left(\|x_n - p\|^2 - \|Tx_n - p\|^2\right) = 0 \tag{28}$$

for all $p \in \operatorname{Fix}(T)$, it follows that

$$\lim_{n\to\infty} (x_n - Tx_n) = 0. \tag{29}$$

One is able to show that, in this case, strong nonexpansivity implies Bregman strong nonexpansivity. Indeed, if T is SNE, the quasi-nonexpansivity is guaranteed by definition. Now, given a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying (28) for some $p \in \operatorname{Fix}(T)$, we have

$$\lim_{n\to\infty} (\|x_n - p\| - \|Tx_n - p\|) = 0. \tag{30}$$

By taking in definition 6(v') the sequence $\{y_n\}_{n\in\mathbb{N}}$ to be the constant sequence defined by $y_n = p$ for all $n \in \mathbb{N}$, we see that (29) follows from (27), so T is weakly BSNE, as claimed. The converse does not hold in general, mainly because nonexpansivity is required.

Note that if $S = \widehat{\mathrm{Fix}}\,(T)$, the previous implication is no longer true. However, in the finite dimensional case, $H = \mathbb{R}^n$, if T is continuous, then $\mathrm{Fix}\,(T) = \widehat{\mathrm{Fix}}\,(T)$. This happens, in particular, when T is SNE. Therefore, in finite dimension, any SNE mapping (called *paracontraction* in [16]) is also strongly BSNE.

To sum up, we can say that Bregman strong nonexpansivity turns out to be a generalization of strong nonexpansivity. $\diamondsuit$

The following two results emphasize the advantage of strongly BSNE operators over other types of Bregman nonexpansive operators (*cf.* [28, Lemma 1, p. 314] and [28, Lemma 2, p. 314], respectively).

Proposition 3 (Common asymptotic fixed points of a composition)**.** *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of X. Let K be a nonempty, closed and convex subset of X. If each T_i, $1 \leq i \leq N$, is a strongly BSNE self-mapping of K, and the set*

$$\widehat{F} := \bigcap \left\{ \widehat{\mathrm{Fix}}\,(T_i) : 1 \leq i \leq N \right\}$$

is not empty, then $\widehat{\mathrm{Fix}}\,(T_N T_{N-1} \cdots T_1) \subset \widehat{F}$.

Proposition 4 (Composition of strongly BSNE operators)**.** *Let $f : X \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of X. Let K be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$. Let each T_i, $1 \leq i \leq N$, be a strongly BSNE self-mapping of K, and let $T = T_N T_{N-1} \cdots T_1$. If the sets $\widehat{F}$ and $\widehat{\mathrm{Fix}}\,(T)$ are not empty, then T is also strongly BSNE.*

Remark 4. For each $1 \leq i \leq N$, let T_i be a strongly BSNE operator with respect to $\widehat{\mathrm{Fix}}\,(T_i) = \mathrm{Fix}\,(T_i)$, and let $T = T_N T_{N-1} \cdots T_1$. If $F = \bigcap \{\mathrm{Fix}\,(T_i) : 1 \leq i \leq N\}$ is nonempty, then T is also strongly BSNE with respect to $\mathrm{Fix}\,(T) = \widehat{\mathrm{Fix}}\,(T)$.

Indeed, from Proposition 3 we get that

$$\mathrm{Fix}\,(T) \subset \widehat{\mathrm{Fix}}\,(T) \subset \widehat{F} = F \subset \mathrm{Fix}\,(T)\,,$$

which implies that $\mathrm{Fix}\,(T) = \widehat{\mathrm{Fix}}\,(T)$, as claimed.

In addition, in this case, it follows from Proposition 3 that

$$\operatorname{Fix}(T) = \bigcap \{\operatorname{Fix}(T_i) : 1 \leq i \leq N\}. \tag{31}$$

$\Diamond$

Proposition 5 (*cf.* [30, Lemma 15.6, p. 306])**.** *Let $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of X. Let K be a nonempty, closed and convex subset of X, and let $T : K \to K$ be a BFNE operator. Then $\operatorname{Fix}(T) = \widehat{\operatorname{Fix}}(T)$.*

Proposition 6 (*cf.* [30, Theorem 15.7, p. 307])**.** *Let $f : X \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$. Let K be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$ and let $T : K \to K$ be a weakly QBNE operator. If $\operatorname{Fix}(T)$ is nonempty, then $\{T^n y\}_{n\in\mathbb{N}}$ is bounded for each $y \in K$.*

Remark 5. Based on the implications described in Table 1, we see that Proposition 6 holds for all Bregman nonexpansive type operators appearing in the table. $\Diamond$

Definition 7 (Sequentially consistent)**.** The function f is called *sequentially consistent* (see [13]) if for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded,

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \;\Rightarrow\; \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Proposition 7 (*cf.* [12, Lemma 2.1.2, p. 67])**.** *The function $f : X \to (-\infty, +\infty]$ is totally convex on bounded subsets of X if and only if it is sequentially consistent.*

Proposition 8 (*cf.* [29, Proposition 2.1, p. 474])**.** *If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

3 Auxiliary Results

Definition 8 (Weakly sequentially continuous mapping)**.** A mapping $A : X \to X^*$ is called *weakly sequentially continuous* if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, $x_n \rightharpoonup x$ implies that $Ax_n \overset{*}{\rightharpoonup} Ax$.

Proposition 9 (Weak convergence)**.** *Let $f : X \to (-\infty, +\infty]$ be a Legendre function such that ∇f is weakly sequentially continuous. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded and that*

$$\lim_{n\to\infty} D_f(u, x_n) \tag{32}$$

exists for any weak subsequential limit u of $\{x_n\}_{n\in\mathbb{N}}$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to u.

Proof. It suffices to prove the uniqueness of weak subsequential limits of $\{x_n\}_{n\in\mathbb{N}}$ because, since $\{x_n\}_{n\in\mathbb{N}}$ is bounded and X is reflexive, we know that there is at least one. Assume that u and v are any two weak subsequential limits of $\{x_n\}_{n\in\mathbb{N}}$. From (32) we know that

$$\lim_{n\to\infty} \left(D_f(u, x_n) - D_f(v, x_n)\right)$$

exists. From the definition of the Bregman distance (see (9)) we get

$$\begin{aligned} D_f(u, x_n) - D_f(v, x_n) &= [f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n\rangle] \\ &- [f(v) - f(x_n) - \langle \nabla f(x_n), v - x_n\rangle] \\ &= f(u) - f(v) + \langle \nabla f(x_n), v - u\rangle \end{aligned}$$

and therefore

$$\lim_{n\to+\infty} \langle \nabla f(x_n), v - u\rangle$$

exists. Since u and v are weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$, there are subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_{n_m}\}_{m\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup u$ and $x_{n_m} \rightharpoonup v$. Since ∇f is weakly sequentially continuous, we know that $\nabla f(x_{n_k}) \overset{*}{\rightharpoonup} \nabla f(u)$ and $\nabla f(x_{n_m}) \overset{*}{\rightharpoonup} \nabla f(v)$. Thus

$$\begin{aligned} \langle \nabla f(u), v - u\rangle &= \lim_{k\to+\infty} \langle \nabla f(x_{n_k}), v - u\rangle = \lim_{n\to+\infty} \langle \nabla f(x_n), v - u\rangle \\ &= \lim_{m\to+\infty} \langle \nabla f(x_{n_m}), v - u\rangle = \langle \nabla f(v), v - u\rangle . \end{aligned}$$

Hence $\langle \nabla f(v) - \nabla f(u), v - u\rangle = 0$, which implies that $u = v$ because f is strictly convex. □

Proposition 10 (Boundedness Property)**.** *Let $f : X \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$. Let $x \in \operatorname{int}\operatorname{dom} f$. If $\{D_f(x, x_n)\}_{n\in\mathbb{N}}$ is bounded, so is the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. Let β be an upper bound of the sequence $\{D_f(x, x_n)\}_{n\in\mathbb{N}}$. Then from the definition of W^f (see (10)) we obtain that

$$f(x) - \langle \nabla f(x_n), x\rangle + f^*(\nabla f(x_n)) = W^f(\nabla f(x_n), x) = D_f(x, x_n) \le \beta.$$

This implies that the sequence $\{\nabla f(x_n)\}_{n\in\mathbb{N}}$ is contained in the sublevel set, $\operatorname{lev}_{\le}^{\psi}(\beta - f(x))$, of the function $\psi = f^* - \langle \cdot, x\rangle$. Since the function f^* is proper and lower semicontinuous, an application of the Moreau-Rockafellar Theorem (see [2, Fact 3.1, p. 623]) shows that ψ is coercive. Consequently, all sublevel sets of ψ are bounded. Hence the sequence $\{\nabla f(x_n)\}_{n\in\mathbb{N}}$ is bounded. By hypothesis, ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$. Therefore the sequence $x_n = \nabla f^*(\nabla f(x_n))$, $n \in \mathbb{N}$, is bounded too, as claimed. □

Definition 9 (Asymptotic regularity)**.** An operator $T : K \to K$ is called *asymptotically regular* if, for any $x \in K$, we have

$$\lim_{n\to\infty} \left\| T^{n+1}x - T^n x \right\| = 0. \tag{33}$$

In the following result we prove that any BSNE operator is asymptotically regular.

Proposition 11 (BSNE operators are asymptotically regular)**.** *Assume that $f : X \to (-\infty, +\infty]$ is a Legendre function which is totally convex on bounded subsets of* $\operatorname{int}\operatorname{dom} f$ *and assume that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$*. Let K be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$*. Let T be a strongly (weakly) BSNE operator from K into itself such that* $\widehat{\operatorname{Fix}}(T) \neq \emptyset$ $(\operatorname{Fix}(T) \neq \emptyset)$*. Then T is asymptotically regular.*

Proof. Assume that T is strongly BSNE. Let $u \in \widehat{\operatorname{Fix}}(T)$ and let $x \in K$. From (19) we get that

$$D_f\left(u, T^{n+1}x\right) \le D_f(u, T^n x) \le \ldots \le D_f(u, Tx).$$

Thus $\lim_{n\to\infty} D_f(u, T^n x)$ exists and the sequence $\{D_f(u, T^n x)\}_{n\in\mathbb{N}}$ is bounded. Now Proposition 10 implies that $\{T^n x\}_{n\in\mathbb{N}}$ is also bounded for any $x \in K$. Since the limit $\lim_{n\to\infty} D_f(u, T^n x)$ exists, we have

$$\lim_{n\to\infty} \left(D_f(u, T^n x) - D_f\left(u, T^{n+1}x\right)\right) = 0.$$

From (20) and (21) we get

$$\lim_{n\to\infty} D_f\left(T^{n+1}x, T^n x\right) = 0.$$

Since $\{T^n x\}_{n\in\mathbb{N}}$ is bounded, we now obtain from Proposition 7 that

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| = 0.$$

In other words, T is asymptotically regular. The proof when T is weakly BSNE is identical when we take $u \in \operatorname{Fix}(T)$. □

Remark 6. From the proof of Proposition 11 we see that Proposition 6 can be improved for strongly BSNE operators since the result remains true if the assumption $\operatorname{Fix}(T) \neq \emptyset$ is replaced by the assumption $\widehat{\operatorname{Fix}}(T) \neq \emptyset$. ◊

4 Picard Iterations for Bregman Nonexpansive Operators

The main result in this section is the following one.

Theorem 1 (Picard Iteration)**.** *Let $f : X \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$. Let K be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$ and let $T : K \to K$ be a strongly QBNE operator. Then the following assertions hold:*

(i) if $\widehat{\operatorname{Fix}}(T)$ is nonempty, then $\{T^n x\}_{n\in\mathbb{N}}$ is bounded for each $x \in K$;

(ii) if, furthermore, T is asymptotically regular, then, for each $x \in K$, any weak subsequential limit of $\{T^n x\}_{n\in\mathbb{N}}$ belongs to $\widehat{\operatorname{Fix}}(T)$;

(iii) if, furthermore, ∇f is weakly sequentially continuous, then $\{T^n x\}_{n\in\mathbb{N}}$ converges weakly to an element in $\widehat{\operatorname{Fix}}(T)$ for each $x \in K$.

Proof. (i) See Proposition 6 and Remark 6.

(ii) Since $\{T^n x\}_{n\in\mathbb{N}}$ is bounded (by point (i)), there is a subsequence $\{T^{n_k} x\}_{k\in\mathbb{N}}$ which converges weakly to some u. Define $x_n = T^n x$ for any $n \in \mathbb{N}$. Since T is asymptotically regular, it follows from (33) that $x_n - Tx_n \to 0$. Therefore we have $x_{n_k} \rightharpoonup u$ and $x_{n_k} - Tx_{n_k} \to 0$, which means that $u \in \widehat{\operatorname{Fix}}(T)$.

(iii) From point (ii) and since T is strongly QBNE, we already know that the limit $\lim_{n\to\infty} D_f(u, T^n x)$ exists for any weak limit u of the sequence $\{T^n x\}_{n\in\mathbb{N}}$. The result now follows immediately from Proposition 9.

□

Corollary 1. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function which is totally convex on bounded subsets of X. Suppose that ∇f is weakly sequentially continuous and ∇f^* is bounded on bounded subsets of* $\operatorname{int} \operatorname{dom} f^*$. *Let K be a nonempty, closed and convex subset of* $\operatorname{int} \operatorname{dom} f$. *Let $T : K \to K$ be a BSNE operator with respect to* $\operatorname{Fix}(T) = \widehat{\operatorname{Fix}}(T) \neq \emptyset$. *Then $\{T^n x\}_{n \in \mathbb{N}}$ converges weakly to an element in* $\operatorname{Fix}(T)$ *for each $x \in K$.*

Proof. The result follows immediately from Theorem 1 and Proposition 11. □

Remark 7. If $\operatorname{Fix}(T) \neq \widehat{\operatorname{Fix}}(T)$, but $\widehat{\operatorname{Fix}}(T) \neq \emptyset$, then we only know that, for a strongly BSNE operator T, $\{T^n x\}_{n \in \mathbb{N}}$ converges weakly to an element in $\widehat{\operatorname{Fix}}(T)$ for each $x \in K$. This result was previously proved in [28, Lemma 4, p. 315] under somewhat different assumptions. ♢

Remark 8. Let $f : X \to \mathbb{R}$ be a uniformly Fréchet differentiable function which is bounded on bounded subsets of X. From Proposition 5 and Corollary 1 we get that Theorem 1 holds for BFNE operators. It is well known that in Hilbert spaces, Picard iterations of firmly nonexpansive operators converge weakly to a fixed point of the operator (see, for instance, [21]). ♢

Remark 9 (Common fixed point - Composition case)**.** Let $f : X \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of X. Suppose that ∇f is weakly sequentially continuous and ∇f^* is bounded on bounded subsets of $\operatorname{int} \operatorname{dom} f^*$. Let K be a nonempty, closed and convex subset of $\operatorname{int} \operatorname{dom} f$.

Let $\{T_i : 1 \leq i \leq N\}$ be N BSNE operators with respect to $\widehat{\operatorname{Fix}}(T_i) = \operatorname{Fix}(T_i) \neq \emptyset$ for each $1 \leq i \leq N$ and let $T = T_N T_{N-1} \cdots T_1$. From Proposition 4 and Remark 4 we obtain that if $\bigcap \{\operatorname{Fix}(T_i) : 1 \leq i \leq N\} \neq \emptyset$, then T is also strongly BSNE with respect to $\widehat{\operatorname{Fix}}(T) = \operatorname{Fix}(T) = \bigcap \{\operatorname{Fix}(T_i) : 1 \leq i \leq N\}$.

From Theorem 1 we now get that $\{T^n x\}_{n \in \mathbb{N}}$ converges weakly to a common fixed point of the family of strongly BSNE operators. Similarly, if we just assume that each T_i is strongly BSNE, $1 \leq i \leq N$, with $\widehat{\operatorname{Fix}}(T_i) \neq \emptyset$, then we get weak convergence of the sequence $\{T^n x\}_{n \in \mathbb{N}}$ to a common asymptotic fixed point. ♢

5 Mann Iterations for Bregman Nonexpansive Operators

In the following theorem we study a different iterative method, which is defined by using convex combinations with respect to the convex function f, a concept which was first introduced in [16].

Theorem 2 (Mann Iteration)**.** *Let $T : X \to X$ be a strongly BSNE operator with $\widehat{\mathrm{Fix}}\,(T) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a Legendre function which is totally convex on bounded subsets of X. Suppose that ∇f is weakly sequentially continuous and ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$*. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence generated by the iterative scheme*

$$x_{n+1} = \nabla f^* \left(\alpha_n \nabla f\left(x_n\right) + (1-\alpha_n)\nabla f\left(Tx_n\right)\right), \tag{34}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ satisfies $\limsup_{n\to\infty}\alpha_n < 1$. Then, for each $x_0 \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a point in $\widehat{\mathrm{Fix}}\,(T)$.

Proof. We divide the proof into 3 steps.

Step 1. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded.*

Let $p \in \widehat{\mathrm{Fix}}\,(T)$. From Proposition 1(i), (ii) and (19) we have for all $n \in \mathbb{N}$,

$$\begin{aligned} D_f\left(p, x_{n+1}\right) &= D_f\left(p, \nabla f^*\left(\alpha_n \nabla f\left(x_n\right) + (1-\alpha_n)\nabla f\left(Tx_n\right)\right)\right) \\ &= W^f\left(\alpha_n \nabla f\left(x_n\right) + (1-\alpha_n)\nabla f\left(Tx_n\right), p\right) \\ &\leq \alpha_n W^f\left(\nabla f\left(x_n\right), p\right) + \left(1-\alpha_n\right) W^f\left(\nabla f\left(Tx_n\right), p\right) \\ &= \alpha_n D_f\left(p, x_n\right) + \left(1-\alpha_n\right) D_f\left(p, Tx_n\right) \\ &\leq \alpha_n D_f\left(p, x_n\right) + \left(1-\alpha_n\right) D_f\left(p, x_n\right) \\ &= D_f\left(p, x_n\right). \end{aligned} \tag{35}$$

This shows that the nonnegative sequence $\{D_f\left(p, x_n\right)\}_{n\in\mathbb{N}}$ is decreasing, thus bounded, and $\lim_{n\to\infty} D_f\left(p, x_n\right)$ exists. From Proposition 10 we obtain that $\{x_n\}_{n\in\mathbb{N}}$ is bounded, as claimed.

Step 2. *Every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to $\widehat{\mathrm{Fix}}\,(T)$.*

For any $p \in \widehat{\mathrm{Fix}}\,(T)$ we have by the first inequality of (35),

$$D_f\left(p, x_{n+1}\right) \leq D_f\left(p, x_n\right) + \left(1-\alpha_n\right)\left(D_f\left(p, Tx_n\right) - D_f\left(p, x_n\right)\right).$$

Hence

$$\left(1-\alpha_n\right)\left(D_f\left(p, x_n\right) - D_f\left(p, Tx_n\right)\right) \leq D_f\left(p, x_n\right) - D_f\left(p, x_{n+1}\right) \tag{36}$$

for all $n \in \mathbb{N}$. We already know that $\lim_{n\to\infty} D_f\left(p, x_n\right)$ exists. Since $\limsup_{n\to\infty}\alpha_n < 1$, it follows that

$$\lim_{n\to\infty}\left(D_f\left(p, x_n\right) - D_f\left(p, Tx_n\right)\right) = 0.$$

Now, since T is strongly BSNE and $p \in \widehat{\operatorname{Fix}}(T)$, we obtain

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, Proposition 7 implies that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Therefore, if there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to some $v \in X$ as $k \to +\infty$, then $v \in \widehat{\operatorname{Fix}}(T)$.

Step 3. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a point in $\widehat{\operatorname{Fix}}(T)$.*

Since ∇f is weakly sequentially continuous, the result follows immediately from Proposition 9 since $\lim_{n\to\infty} D_f(u, x_n)$ exists for any weak subsequential limit u of the sequence $\{x_n\}_{n\in\mathbb{N}}$ by Step 2. □

Corollary 2. *Let $T : X \to X$ be a BSNE operator with respect to $\operatorname{Fix}(T) = \widehat{\operatorname{Fix}}(T) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a Legendre function which is totally convex on bounded subsets of X. Suppose that ∇f is weakly sequentially continuous and ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence generated by (34), where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$. Then, for each $x_0 \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a point in $\operatorname{Fix}(T)$.*

Remark 10. When $f = (1/2)\|\cdot\|^2$ and X is a Hilbert space, since both ∇f and ∇f^* are the identity operator, the iterative scheme (34) coincides with the Mann iteration the weak convergence of which for nonexpansive mappings is well known, even in more general Banach spaces, under the assumption that $\sum_{n\in\mathbb{N}} \alpha_n (1-\alpha_n) = \infty$ (see [27]).

Remark 11 (Common fixed point - Composition case)**.** Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of X. Suppose that ∇f is weakly sequentially continuous and ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$.

Let $\{T_i : 1 \leq i \leq N\}$ be N BSNE operators with respect to $\widehat{\operatorname{Fix}}(T_i) = \operatorname{Fix}(T_i) \neq \emptyset$ for each $1 \leq i \leq N$ and let $T = T_N T_{N-1} \cdots T_1$. Then from Proposition 4 and Remark 4 we obtain that, if $\bigcap \{\operatorname{Fix}(T_i) : 1 \leq i \leq N\} \neq \emptyset$, T is also BSNE with respect to $\widehat{\operatorname{Fix}}(T) = \operatorname{Fix}(T) = \bigcap \{\operatorname{Fix}(T_i) : 1 \leq i \leq N\}$.

Now combining Theorem 2, Proposition 3 and (31), we get that the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (34) converges weakly to an element in $\bigcap \{\operatorname{Fix}(T_i) : 1 \leq i \leq N\}$ for each $x \in X$.

In the case where each T_i is strongly BSNE with $\widehat{\mathrm{Fix}}\,(T_i) \neq \emptyset$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ weakly converges to a common asymptotic fixed point of the family $\{T_i : 1 \leq i \leq N\}$ whenever such a point exists. $\Diamond$

6 Block Iterative Algorithms

We begin this section with three simple observations which are essential for our later study of the block operator.

Lemma 1. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function and let $\{t_i\}_{i=1}^N \subset (0,1)$ satisfy $\sum_{i=1}^N t_i = 1$. Let $\{x_i\}_{i=1}^N$ be a subset of X and assume that*

$$f\left(\sum_{i=1}^N t_i x_i\right) = \sum_{i=1}^N t_i f\left(x_i\right). \tag{37}$$

Then $x_1 = x_2 = \ldots = x_N$.

Proof. If $x_k \neq x_l$ for some $k, l \in \{1, 2, \ldots, N\}$, then from the strict convexity of f we get

$$f\left(\frac{t_k}{t_k + t_l} x_k + \frac{t_l}{t_k + t_l} x_l\right) < \frac{t_k}{t_k + t_l} f\left(x_k\right) + \frac{t_l}{t_k + t_l} f\left(x_l\right).$$

Using this inequality, we obtain

$$\begin{aligned}
f\left(\sum_{i=1}^N t_i x_i\right) &= f\left(\left(t_k + t_l\right)\left(\frac{t_k}{t_k + t_l} x_k + \frac{t_l}{t_k + t_l} x_l\right) + \sum_{i \neq k,l} t_i x_i\right) \\
&\leq \left(t_k + t_l\right) f\left(\frac{t_k}{t_k + t_l} x_k + \frac{t_l}{t_k + t_l} x_l\right) + \sum_{i \neq k,l} t_i f\left(x_i\right) \\
&< \left(t_k + t_l\right)\left(\frac{t_k}{t_k + t_l} f\left(x_k\right) + \frac{t_l}{t_k + t_l} f\left(x_l\right)\right) + \sum_{i \neq k,l} t_i f\left(x_i\right) \\
&= \sum_{i=1}^N t_i f\left(x_i\right).
\end{aligned}$$

This contradicts assumption (37). $\square$

As a direct consequence of Lemma 1 we have the following result.

Corollary 3. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function and let $\{t_i\}_{i=1}^N \subset (0,1)$ satisfy $\sum_{i=1}^N t_i = 1$. Let $\{\xi_i\}_{i=1}^N$ be a finite subset of X^* and assume that*

$$f^*\left(\sum_{i=1}^N t_i \xi_i\right) = \sum_{i=1}^N t_i f^*(\xi_i).$$

Then $\xi_1 = \xi_2 = \ldots = \xi_N$.

Proof. Since the function f is Legendre, its conjugate f^* is also Legendre. Thus the result follows immediately from Lemma 1. □

The following lemma concerns the Bregman distance.

Lemma 2. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function and let $\{t_i\}_{i=1}^N \subset (0,1)$ satisfy $\sum_{i=1}^N t_i = 1$. Let $z \in X$. Let $\{x_i\}_{i=1}^N$ be a finite subset of X and assume that*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) = \sum_{i=1}^N t_i D_f(z, x_i). \tag{38}$$

Then $x_1 = x_2 = \ldots = x_N$.

Proof. Equality (38) can be reformulated as follows:

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) = W^f\left(\sum_{i=1}^N t_i \nabla f(x_i), z\right) = \sum_{i=1}^N t_i D_f(z, x_i).$$

Now from the definition of W^f (see (10)) and the definition of the Bregman distance (see (9)) we get

$$\begin{aligned} \sum_{i=1}^N t_i \left(f(z) - f(x_i) - \langle \nabla f(x_i), z - x_i \rangle\right) &= f(z) + f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right) \\ &- \left\langle \sum_{i=1}^N t_i \nabla f(x_i), z \right\rangle. \end{aligned}$$

Thus

$$f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right) = \sum_{i=1}^N t_i \left(\langle \nabla f(x_i), x_i \rangle - f(x_i)\right).$$

Since $f(x_i) + f^*(\nabla f(x_i)) = \langle \nabla f(x_i), x_i \rangle$ for any $1 \leq i \leq N$, we obtain

$$f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right) = \sum_{i=1}^{N} t_i f^*(\nabla f(x_i)).$$

Corollary 3 now implies that $x_1 = x_2 = \ldots = x_N$, as claimed. □

Definition 10 (Block operator)**.** Let $\{T_i : 1 \leq i \leq N\}$ be N operators from X to X and let $\{w_i\}_{i=1}^N \subset (0,1)$ satisfy $\sum_{i=1}^N w_i = 1$. Then the *block operator* corresponding to $\{T_i : 1 \leq i \leq N\}$ and $\{w_i : 1 \leq i \leq N\}$ is defined by

$$T_B := \nabla f^*\left(\sum_{i=1}^{N} w_i \nabla f(T_i)\right). \tag{39}$$

Remark 12. The following inequality will be essential in our next result:

$$\begin{aligned} D_f(p, T_B x) &= D_f\left(p, \nabla f^*\left(\sum_{i=1}^{N} w_i \nabla f(T_i x)\right)\right) = W^f\left(\sum_{i=1}^{N} w_i \nabla f(T_i x), p\right) \\ &\leq \sum_{i=1}^{N} w_i W^f(\nabla f(T_i x), p) = \sum_{i=1}^{N} w_i D_f(p, T_i x). \end{aligned} \tag{40}$$

It follows from Proposition 1(i),(ii). ♢

In our next result we prove that the block operator defined by (39) is weakly QBNE when each T_i, $1 \leq i \leq N$, is weakly QBNE.

Proposition 12 (Block operator of weakly QBNE operators)**.** *Assume that $f : X \to (-\infty, +\infty]$ is a Legendre function and let $\{T_i : 1 \leq i \leq N\}$ be N weakly QBNE operators from X into X such that $F = \bigcap \{\mathrm{Fix}(T_i) : 1 \leq i \leq N\} \neq \emptyset$. Let $\{w_i\}_{i=1}^N \subset (0,1)$ satisfy $\sum_{i=1}^N w_i = 1$. Then T_B is weakly QBNE with respect to $F = \mathrm{Fix}(T_B)$.*

Proof. Let $p \in F$. Since each T_i, $i = 1, \ldots, N$, is QBNE, we obtain from (40) that

$$D_f(p, T_B x) \leq \sum_{i=1}^{N} w_i D_f(p, T_i x) \leq \sum_{i=1}^{N} w_i D_f(p, x) = D_f(p, x) \tag{41}$$

for all $x \in X$. Thus T_B is a QBNE operator with respect to F. Next we will show that $\mathrm{Fix}(T_B) = F$.

The inclusion $F \subset \operatorname{Fix}(T_B)$ is obvious, so it is enough to show that $\operatorname{Fix}(T_B) \subset F$. To this end, let $u \in \operatorname{Fix}(T_B)$ and $k \in \{1, 2, \ldots, N\}$. For $p \in F$, such that $p \neq u$, we obtain from (40) that

$$D_f(p,u) = D_f(p, T_B u) \leq \sum_{i=1}^{N} w_i D_f(p, T_i u) \leq \sum_{i \neq k} w_i D_f(p,u) + w_k D_f(p, T_k u).$$

Therefore

$$w_k D_f(p,u) = \left(1 - \sum_{i \neq k} w_i\right) D_f(p,u) \leq w_k D_f(p, T_k u),$$

that is,

$$w_k D_f(p,u) \leq w_k D_f(p, T_k u).$$

Since $w_k > 0$, it follows that $D_f(p,u) \leq D_f(p, T_k u)$. On the other hand, since T_k is weakly QBNE and $p \in F \subset \operatorname{Fix}(T_k)$, we have $D_f(p, T_k u) \leq D_f(p,u)$. Thus $D_f(p,u) = D_f(p, T_k u)$ for all $k \in \{1, 2, \ldots, N\}$. Hence

$$D_f\left(p, \nabla f^*\left(\sum_{i=1}^{N} w_i \nabla f(T_i u)\right)\right) = D_f(p, T_B u) = D_f(p,u) = \sum_{i=1}^{N} w_i D_f(p, T_i u). \tag{42}$$

Now Lemma 2 implies that $T_1 u = T_2 u = \ldots = T_n u$. Therefore $u \in F$. □

Proposition 13 (Asymptotic fixed points of block operators)**.** *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. If each T_i, $i = 1, \ldots, N$, is a strongly BSNE operator from X into itself, and the set*

$$\widehat{F} := \bigcap \left\{ \widehat{\operatorname{Fix}}(T_i) : 1 \leq i \leq N \right\}$$

is not empty, then $\widehat{\operatorname{Fix}}(T_B) \subset \widehat{F}$.

Proof. Let $u \in \widehat{F}$ and let $x \in \widehat{\operatorname{Fix}}(T_B)$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ which converges weakly to x such that $\lim_{n \to \infty}(x_n - T_B x_n) = 0$. Since the function f is bounded on bounded subsets of X, ∇f is also bounded on bounded subsets of X (see [12, Proposition 1.1.11, p. 17]). So the sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{\nabla f(T_B x_n)\}_{n \in \mathbb{N}}$ are bounded. Since f is also uniformly Fréchet differentiable, it is uniformly continuous on bounded subsets of X and therefore

$$\lim_{n \to \infty} \left(f(T_B x_n) - f(x_n)\right) = 0. \tag{43}$$

In addition, by Proposition 8 we obtain that ∇f is also uniformly continuous on bounded subsets of X and thus

$$\lim_{n\to\infty}\left(\nabla f\left(T_B x_n\right)-\nabla f\left(x_n\right)\right)=0. \tag{44}$$

From the definition of the Bregman distance (see (9)) we obtain that

$$\begin{aligned}
D_f\left(u,x_n\right)-D_f\left(u,T_B x_n\right) &= \left[f\left(u\right)-f\left(x_n\right)-\left\langle \nabla f\left(x_n\right),u-x_n\right\rangle\right] \\
&- \left[f\left(u\right)-f\left(T_B x_n\right)-\left\langle \nabla f\left(T_B x_n\right),u-T_B x_n\right\rangle\right] \\
&= f\left(T_B x_n\right)-f\left(x_n\right)-\left\langle \nabla f\left(x_n\right),u-x_n\right\rangle \\
&+ \left\langle \nabla f\left(T_B x_n\right),u-T_B x_n\right\rangle \\
&= f\left(T_B x_n\right)-f\left(x_n\right)-\left\langle \nabla f\left(x_n\right)-\nabla f\left(T_B x_n\right),u-x_n\right\rangle \\
&+ \left\langle \nabla f\left(T_B x_n\right),x_n-T_B x_n\right\rangle.
\end{aligned}$$

Combining the facts that $\left\{x_n\right\}_{n\in\mathbb{N}}$ and $\left\{\nabla f\left(T_B(x_n)\right)\right\}_{n\in\mathbb{N}}$ are bounded, (43) and (44), we obtain that

$$\lim_{n\to\infty}\left(D_f\left(u,x_n\right)-D_f\left(u,T_B x_n\right)\right)=0. \tag{45}$$

Since each operator T_i, $i=1,\ldots,N$, is strongly BSNE, we deduce from (19) and (40) that for any $k=1,\ldots,N$,

$$\begin{aligned}
D_f\left(u,T_B x_n\right) &\leq \sum_{i=1}^{N} w_i D_f\left(u,T_i x_n\right)=w_k D_f\left(u,T_k x_n\right)+\sum_{i\neq k} w_i D_f\left(u,T_i x_n\right) \\
&\leq w_k D_f\left(u,T_k x_n\right)+\sum_{i\neq k} w_i D_f\left(u,x_n\right) \\
&= w_k D_f\left(u,T_k x_n\right)+(1-w_k)D_f\left(u,x_n\right) \\
&= w_k\left(D_f\left(u,T_k x_n\right)-D_f\left(u,x_n\right)\right)+D_f\left(u,x_n\right).
\end{aligned} \tag{46}$$

Hence for any $k\in\{1,\ldots,N\}$, we have

$$\lim_{n\to\infty} w_k\left(D_f\left(u,x_n\right)-D_f\left(u,T_k x_n\right)\right)\leq \lim_{n\to\infty}\left(D_f\left(u,x_n\right)-D_f\left(u,T_B x_n\right)\right)=0.$$

Thence

$$\lim_{n\to\infty}\left(D_f\left(u,x_n\right)-D_f\left(u,T_k x_n\right)\right)=0$$

for any $k\in\{1,\ldots,N\}$. Since each operator T_i, $i=1,\ldots,N$, is strongly BSNE, we get from (20) and (21) that

$$\lim_{n\to\infty} D_f\left(T_i x_n,x_n\right)=0.$$

Since f is totally convex and $\{x_n\}_{n\in\mathbb{N}}$ is bounded, it follows from Proposition 7 that

$$\lim_{n\to\infty}(T_i x_n - x_n) = 0.$$

This means that x belongs to $\widehat{\mathrm{Fix}}\,(T_i)$ because we also know that $x_n \rightharpoonup x$. Therefore $x \in \widehat{F}$, which proves that $\widehat{\mathrm{Fix}}\,(T_B) \subset \widehat{F}$, as claimed. □

Proposition 14 (Block operator of strongly BSNE operators)**.** *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$. *If each T_i, $i = 1, \ldots, N$, is a strongly BSNE operator from X into itself, and the sets*

$$\widehat{F} := \bigcap \left\{\widehat{\mathrm{Fix}}\,(T_i) : 1 \le i \le N\right\}$$

and $\widehat{\mathrm{Fix}}\,(T_B)$ *are not empty, then T_B is also strongly BSNE.*

Proof. If $u \in \widehat{\mathrm{Fix}}\,(T_B)$, then $u \in \widehat{F}$ by Proposition 13. Therefore the fact that each T_i, $i = 1, \ldots, N$, is strongly BSNE, with respect to $\widehat{\mathrm{Fix}}\,(T_i)$, implies that (19) holds for T_B and any $x \in X$.

Now we assume that there exists a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty}(D_f(u, x_n) - D_f(u, T_B x_n)) = 0$$

and therefore, as we proved in Proposition 13, we get

$$\lim_{n\to\infty}(D_f(u, x_n) - D_f(u, T_i x_n)) = 0$$

for any $i \in \{1, \ldots, N\}$. Since each T_i, $i = 1, \ldots, N$, is strongly BSNE and $u \in \widehat{\mathrm{Fix}}\,(T_B) \subset \widehat{\mathrm{Fix}}\,(T_i)$, it follows from (20) and (21) that

$$\lim_{n\to\infty} D_f(T_i x_n, x_n) = 0.$$

Since f is totally convex and $\{x_n\}_{n\in\mathbb{N}}$ is bounded, it follows from Proposition 7 that

$$\lim_{n\to\infty}(T_i x_n - x_n) = 0.$$

Since f is uniformly Fréchet differentiable, it follows from Proposition 8 that ∇f is uniformly continuous on bounded subsets of X and thus

$$\lim_{n\to\infty}(\nabla f(T_i x_n) - \nabla f(x_n)) = 0.$$

By the definition of the block operator (see (39)), we have

$$\nabla f(T_B x_n) - \nabla f(x_n) = \sum_{i=1}^{N} w_i (\nabla f(T_i x_n) - \nabla f(x_n))$$

and therefore

$$\lim_{n\to\infty} (\nabla f(T_B x_n) - \nabla f(x_n)) = 0. \tag{47}$$

On the other hand, from the definition of the Bregman distance (see (9)) we obtain that

$$D_f(T_B x_n, x_n) + D_f(x_n, T_B x_n) = \langle \nabla f(T_B x_n) - \nabla f(x_n), T_B x_n - x_n \rangle. \tag{48}$$

Note that each sequence $\{T_i x_n\}_{n\in\mathbb{N}}$, $i = 1, \ldots, N$, is bounded because so is the sequence $\{x_n\}_{n\in\mathbb{N}}$ and $\lim_{n\to\infty}(T_i x_n - x_n) = 0$. Since ∇f and ∇f^* are bounded on bounded subsets of X and $\operatorname{int}\operatorname{dom} f^*$, respectively, it follows that $\{T_B x_n\}_{n\in\mathbb{N}}$ is bounded too. Thence, combining (47) and (48), we deduce that

$$\lim_{n\to\infty} (D_f(T_B x_n, x_n) + D_f(x_n, T_B x_n)) = 0.$$

Therefore

$$\lim_{n\to\infty} D_f(T_B x_n, x_n) = 0.$$

This means that (20) implies (21) for T_B and this proves that T_B is BSNE, as required. □

Proposition 15 (Block operator of weakly BSNE operators). *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$. *If each T_i, $i = 1, \ldots, N$, is a weakly BSNE operators from X into itself, and the set*

$$F := \bigcap \{\operatorname{Fix}(T_i) : 1 \le i \le N\}$$

is not empty, then T_B is also weakly BSNE and $F = \operatorname{Fix}(T_B)$.

Proof. On the one hand, since each T_i is weakly BSNE, it is weakly QBNE. Then the fact that $F \neq \emptyset$ makes it possible to apply Proposition 12 so that $F = \operatorname{Fix}(T_B)$ and T_B is weakly QBNE, that is, it satisfies inequality (19) for any $p \in \operatorname{Fix}(T_B)$.

On the other hand, given a bounded sequence such that, for any $u \in \operatorname{Fix}(T_B)$,

$$\lim_{n\to\infty} (D_f(u, x_n) - D_f(u, T_B x_n)) = 0,$$

analogously to the argument used in Proposition 14, one is able to deduce that

$$\lim_{n\to\infty} D_f\left(T_B x_n, x_n\right) = 0.$$

Thus T_B is indeed weakly BSNE, as asserted. □

As a consequence of the previous results, we now see that Picard and Mann iterations provide convergent iterative methods for approximating common fixed points of a finite family of BSNE operators.

Remark 13 (Common fixed point - Block iterative algorithm)**.** Let $f : X \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f is weakly sequentially continuous and ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$.

Let $\{T_i : 1 \leq i \leq N\}$ be N BSNE operators with respect to $\widehat{\operatorname{Fix}}\left(T_i\right) = \operatorname{Fix}\left(T_i\right) \neq \emptyset$ and let T_B be the block operator defined by (39). If $F := \bigcap \left\{\operatorname{Fix}\left(T_i\right) : 1 \leq i \leq N\right\}$ and $\operatorname{Fix}\left(T_B\right)$ are nonempty, then from Propositions 14 we know that T_B is strongly BSNE. Furthermore, from Proposition 13 we get that

$$\operatorname{Fix}\left(T_B\right) \subset \widehat{\operatorname{Fix}}\left(T_B\right) \subset F \subset \operatorname{Fix}\left(T_B\right),$$

which implies that $\widehat{\operatorname{Fix}}\left(T_B\right) = \operatorname{Fix}\left(T_B\right) \neq \emptyset$.

Therefore, both Theorem 1 and Theorem 2 apply to guarantee the weak convergence of the sequences $\{T_B^n x\}$ and $\{x_n\}_{n\in\mathbb{N}}$, generated by Algorithm (34), under the appropriate conditions. ♢

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