Existence and regularity of the pressure for the stochastic Navier-Stokes equations

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April 25, 2003

Abstract

We prove, on one hand, that for a convenient body force with values in the distribution space $(H^{-1}(D))^d$, where D is the geometric domain of the fluid, there exist a velocity u and a pressure p solution of the stochastic Navier-Stokes equation in dimension 2, 3 or 4.

On the other hand, we prove that, for a body force with values in the dual space V' of the divergence free subspace V of $(H_0^1(D))^d$, in general it is not possible to solve the stochastic Navier-Stokes equations. More precisely, although such body forces have been considered, there is no topological space in which Navier-Stokes equations could be meaningful for them.

AMS Subject Classification (2000): 60H15, 60H30, 35R15, 35Q30

1 Introduction

Let D be a connected and bounded open subset of $\mathbb{R}^d$, where $d = 2$, 3 or 4, with a regular enough boundary ∂D. Let us fix a final time $T > 0$, and consider the following system of stochastic Navier-Stokes equations with homogeneous Dirichlet boundary condition:

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$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = F(\cdot,u) + G(\cdot,u)\dot{W}_t, \text{ in } D\times(0,T), \\ \nabla\cdot u = 0, \text{ in } D\times(0,T), \\ u = 0, \text{ on } \partial D\times(0,T), \end{cases} \tag{1}$$

where $u = (u_1, \dots, u_d)$ and p are unknown random fields on $D \times [0,T]$, representing respectively the velocity and the pressure of an incompressible fluid filling the domain D, in each point of $D\times[0,T]$ (in fact, p is the sum of the pressure and of some potential q corresponding to the part of forces of the form ∇q). Here, the body force F is a given measurable sublinear mapping from $[0,T]\times(L^2(D))^d$ into $(H^{-1}(D))^d$, W is a cylindrical K-valued Wiener process on a complete probability space $(\Omega, \mathcal{F}, P)$, where K is a fixed separable Hilbert space, G is a given measurable sublinear mapping from $[0,T]\times(L^2(D))^d$ into $\mathbb{L}^2(K;(L^2(D))^d)$ and $\nu > 0$ is the kinematic viscosity of the fluid, which is constant.

Existence result. As we will see in Theorem 2.2, for such data, there exist a solution (u,p) to (1). Similar results were obtained in [1], [3], [4], [5] and [6] among others. Our contribution here is that we obtain the pressure p without any regularity assumption on u, contrarily to [5] or [6] in which its values are assumed to be in $(H^2(D))^d$, and that we get equation (1) in the distribution sense, contrarily to the case where the body force is valued in V' in which case no satisfactory sense can be given, see Theorem 6.1.

As in most quoted papers, we obtain the solution in two steps. First, we consider a velocity u satisfying P-a.s. the following so called "variational N-S equation" (in which the pressure is eliminated): for all $t\in[0,T]$ and for all $v\in(\mathcal{D}(D))^d$ such that $\nabla\cdot v = 0$,

$$\begin{aligned} \int_D u(t)\cdot v\,dx = \int_D u(0)\cdot v\,dx - \nu\int_0^t\int_D \nabla u(s)\cdot\nabla v\,dxds \\ - \int_0^t\int_D (u(s)\cdot\nabla u(s))\cdot v\,dxds + \int_0^t \langle F(s,u(s)),v\rangle_{H^{-1}(D)\times H_0^1(D)}\,ds \\ + \int_D\int_0^t G(s,u(s))\,dW_s\cdot v\,dx. \end{aligned} \tag{2}$$

The existence of such a u is proved in [6]; the first result in this direction was given in [2] in the case of $K = \mathbb{R}$ and G constant; it was extended to a multiplicative noise and to an infinite-dimensional K in [1], [3], [4], [5] and [6] among others.

In the second step, we associate a pressure p to such a u by using a generalization of de Rham theorem to processes, see Theorem 4.1.

Non-existence result. Various authors considered a body force F with values in the dual V' of the space $V = \{v \in (H_0^1(D))^d : \nabla \cdot v = 0\}$ instead of $(H^{-1}(D))^d$ as above. Then they solve, again for all $v \in (\mathcal{D}(D))^d$ such that $\nabla \cdot v = 0$ (or equivalently for all $v \in V$),

$$\begin{aligned}\int_D u(t) \cdot v \, dx = \int_D u(0) \cdot v \, dx - \nu \int_0^t \int_D \nabla u(s) \cdot \nabla v \, dx ds \\ - \int_0^t \int_D (u(s) \cdot \nabla u(s)) \cdot v \, dx ds + \int_0^t \langle F(s, u(s)), v \rangle_{V' \times V} \, ds \\ + \int_D \int_0^t G(s, u(s)) \, dW_s \cdot v \, dx,\end{aligned} \tag{3}$$

that is (2) in which the duality $H^{-1} \times H_0^1$ is replaced by the duality $V' \times V$.

Unfortunately, as we will see in Theorem 6.1, it cannot exist any p corresponding to such u, or more exactly to such F, such that the first equation in (1) be satisfied. Indeed, F being valued in V' while other terms are valued in $(H^{-1}(\Omega))^d$, it would be necessary to imbed these two spaces in a same Hausdorff space, which is impossible.

Similar existence and non-existence results for the deterministic Navier–Stokes equations may be found in [12].

2 Existence of a solution (u, p) of the stochastic Navier–Stokes equations.

In all the sequel, let

$$D \text{ be a bounded, connected and Lipschitz open subset of } \mathbb{R}^d, \tag{4}$$

$$d \in \{2, 3, 4\}, \tag{5}$$

$$K \text{ be a separable Hilbert space.} \tag{6}$$

Let F and G be two mappings such that

$$\begin{cases} F \text{ is measurable from } [0, T] \times (L^2(D))^d \text{ into } (H^{-1}(D))^d, \\ G \text{ is measurable from } [0, T] \times (L^2(D))^d \text{ into } \mathbb{L}^2(K; (L^2(D))^d), \end{cases} \tag{7}$$

and, for all $t \in [0, T]$, $w \in (\mathcal{D}(D))^d$ and $e \in K$,

$$\begin{cases} v \mapsto \langle F(t, v), w \rangle_{(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d} \text{ and } v \mapsto \displaystyle\int_D G(t, v) e \cdot w \, dx \\ \text{are continuous from } (L^2(D))^d \text{ into } \mathbb{R}, \end{cases} \tag{8}$$

and such that there exists a positive number c_1 such that, for all $t \in [0, T]$ and $v \in (L^2(D))^d$,

$$\begin{cases} \|F(t,v)\|_{(H^{-1}(D))^d} \le c_1(1 + \|v\|_{(L^2(D))^d}), \\ \|G(t,v)\|_{\mathbf{L}^2(K;(L^2(D))^d)} \le c_1(1 + \|v\|_{(L^2(D))^d}). \end{cases} \tag{9}$$

Let us denote by H the closure of the set

$$\mathcal{V} = \{v \in (\mathcal{D}(D))^d : \nabla \cdot v = 0 \text{ in } D\}$$

in $(L^2(D))^d$, and by V the closure of $\mathcal{V}$ in $(H^1(D))^d$. Then, H is a Hilbert space equipped with the inner product of $(L^2(D))^d$, and V is a Hilbert space equipped with the inner product of $(H^1(D))^d$. Finally, let

$$\begin{cases} \mu \text{ be a probability measure on } H \text{ such that, for all } r \in [1, \infty), \\ \displaystyle\int_H \|v\|_H^r \, d\mu(v) < \infty. \end{cases} \tag{10}$$

Definition 2.1 *A* **martingale solution** *to* (1) *starting from* μ *is a set* $\{(\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}_{t\in[0,T]}, W, u, p\}$, *such that:*

$$(\Omega, \mathcal{F}, P) \text{ is a probability space,}$$
$$\{\mathcal{F}_t\}_{t\in[0,T]} \text{ is a normal filtration on } (\Omega, \mathcal{F}, P),$$
$$W \text{ is a cylindrical } K\text{-valued } \mathcal{F}_t\text{-Wiener process,}$$
$$u \in M^2_{\mathcal{F}_t}(0,T;V) \cap L^r(\Omega, \mathcal{F}, P; L^\infty(0,T;H)), \quad \forall r \in [1,\infty), \tag{11}$$
$$p \in L^1(\Omega, \mathcal{F}_t, P; W^{-1,\infty}(0,t;L^2(D))), \quad \forall t \in (0,T], \tag{12}$$
$$P\{u(0) \in B\} = \mu(B), \quad \forall B \in \mathcal{B}(H), \tag{13}$$

and such that, P-a.s.:

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = F(\cdot,u) + G(\cdot,u)\dot{W}_t, \quad \text{in } (\mathcal{D}'(D_T))^d, \tag{14}$$
$$\nabla \cdot u = 0, \quad \text{in } \mathcal{D}'(D_T), \tag{15}$$
$$\int_D p\,dx = 0, \quad \text{in } \mathcal{D}'(0,T), \tag{16}$$
$$u \in \mathcal{C}([0,T]; H\text{-weak} \cap (H^{-1}(D))^d), \tag{17}$$

where $D_T = (0,T) \times D$.

We are now in position to state the existence result of a solution (u, p) to the stochastic Navier–Stokes equations (1):

Theorem 2.2 *Assuming* (4) *to* (10)*, there exists a martingale solution to* (1) *starting from* μ*.*

Remark 2.3 *In view of Theorem 4.1, to any such velocity* u*, it corresponds a unique pressure* p*.*

Many papers consider martingale solutions of the variational equation (2) *in which* p *is eliminated, see for example [6]. Again in view of Theorem 4.1, assuming* (4) *to* (10)*, it is equivalent to the existence of a martingale solution to* (1)*.*

3 Some definitions.

In this section, we recall the definitions of some above used properties.

3.1 Lipschitz domain, used in (4)

A non empty open subset D of $\mathbb{R}^d$ is said Lipschitz if it is locally the epigraph of a Lipschitz function. Or, more precisely, if there exist two positive numbers a and κ such that, for each point $\xi \in \partial D$, there exists a system of cartesian coordinates $(x_1, \ldots, x_d)$ with origin at ξ and a real function ψ defined on $O' = \{x' \in \mathbb{R}^{d-1} : |x'| < a\}$, where $x' = (x_1, \ldots, x_{d-1})$, such that, for all x' and y' in O',

$$|\psi(x') - \psi(y')| \leq \kappa |x' - y'|,$$

and, for all $x \in O = \{x = (x', x_d) \in \mathbb{R}^d : |x'| < a,\ |x_d| < a\}$,

$$x \in D \iff x_d < \psi(x'),$$
$$x \in \partial D \iff x_d = \psi(x').$$

3.2 Normal filtration and Wiener process, used in Def. (2.1)

A normal filtration on a propability space $(\Omega, \mathcal{F}, P)$ is an increasing and right continuous family $\{\mathcal{F}_t\}_{t\in[0,T]}$ of sub σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the P null sets of $\mathcal{F}$.

Given a separable Hilbert space K, a cylindrical K-valued $\mathcal{F}_t$-Wiener process is any "process" W formally defined as

$$W_t = \sum_{i=1}^{\infty} \beta_t^i e_i,$$

where $(\beta_t^i : t \geq 0, \; i = 1, 2, ...)$ are mutually independent standard real $\mathcal{F}_t$-Wiener processes defined on $(\Omega, \mathcal{F}, P)$, and $\{e_i : i = 1, 2, ...\}$ is an orthonormal basis of K. It is well known that the series defining W does not converge in K, but rather in any Hilbert space $\tilde{K}$ such that $K \subset \tilde{K}$ and the injection of K in $\tilde{K}$ is Hilbert-Schmidt (see e.g. [7]).

3.3 Mesurability and $\mathbb{L}^2$ space, used in (7)

Let X and Y be Banach spaces and let $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ be the σ-algebras of Borel subsets respectively of X and Y. A map from $h : X \to Y$ is said measurable if $h^{-1}(B) \in \mathcal{B}(X)$ for all $B \in \mathcal{B}(Y)$.

Given now another (than K) separable Hilbert space U, with inner product $(\cdot, \cdot)_U$, we denote by $\mathbb{L}^2(K; U)$ the space of Hilbert-Schmidt operators from K into U provided with the Hilbert norm associated to the scalar product defined, for all A and B in $\mathbb{L}^2(K; U)$ by

$$(A, B)_{\mathbb{L}^2(K;U)} = \sum_{k=1}^{\infty} (Ae_k, Be_k)_U$$

where $\{e_k\}_{k=1}^{\infty}$ is a Hilbert basis of K.

3.4 L^r and $M^2_{\mathcal{F}_t}$ spaces, used in (11)

Let again X and Y be Banach spaces. Given a σ-algebra $\mathcal{G} \subset \mathcal{F}$, we denote by $\mathcal{L}^0(\Omega, \mathcal{G}, P; Y)$ the vector space of all the mappings $h : \Omega \to Y$ that are $\mathcal{G}$-measurables, i.e., such that $h^{-1}(B) \in \mathcal{G}$ for all $B \in \mathcal{B}(Y)$. We denote by $L^0(\Omega, \mathcal{G}, P; Y)$ the vector space of equivalence classes of mappings in $\mathcal{L}^0(\Omega, \mathcal{G}, P; Y)$, differing only on a P-null set. For a given $r \in [1, \infty)$, we denote

$$L^r(\Omega, \mathcal{G}, P; Y) = \{h \in L^0(\Omega, \mathcal{G}, P; Y) : E(\|h\|_Y^r) < \infty\} \tag{18}$$

where E stands for the expectation. Analogously, we denote

$$L^\infty(\Omega, \mathcal{G}, P; Y) = \{h \in L^0(\Omega, \mathcal{G}, P; Y) : \|h\|_Y \in L^\infty(\Omega, \mathcal{G}, P)\}.$$

In particular, as usual, given an open subset $\mathcal{O}$ of $\mathbb{R}^n$, we denote $L^r(\mathcal{O}; Y) = L^r(\mathcal{O}, \mathcal{B}(\mathcal{O}), dx; Y)$, where dx is the Lebesgue measure, $L^r(\mathcal{O}) = L^r(\mathcal{O}; \mathbb{R})$ and $L^r(0, T; Y) = L^r((0, T); Y)$.

Now, let us recall that an $\mathcal{F}_t$-progressively measurable stochastic process with values in X is any stochastic process $z : \Omega \times [0, T] \to X$ such that, for all $t \in [0,T]$, the restriction of z to $\Omega \times [0,t]$ is $\mathcal{F}_t \times \mathcal{B}([0,t])$-measurable. More generally, given $z \in L^0(\Omega, \mathcal{F}, P; L^1(0,T;X))$, we say that z is $\mathcal{F}_t$-progressively measurable if there exists an $\mathcal{F}_t$-progressively measurable stochastic process $\hat{z}$ with values in X such that $\hat{z} = z$, $dP \times dt$-a.e. We denote

$$M^2_{\mathcal{F}_t}(0,T;X) = \{z \in L^2(\Omega \times (0,T), dP \times dt; X) : z \text{ is } \mathcal{F}_t\text{-progr. meas.}\}.$$

If X is a Hilbert space, then the space $M^2_{\mathcal{F}_t}(0,T;X)$ is a Hilbert subspace of $L^2(\Omega \times (0,T), dP \times dt; X)$.

3.5 Vector-valued distributions, used in (12)

We now define distribution spaces because $W^{-1,\infty}(0,T;L^2(D))$, used in (12), will next be defined as a subspace of such a space.

Let again $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and let $\mathcal{Y}$ be a complete lcstvs, that is a locally convex separated topological vector space (the case where it is not a Banach space is used in (23)). The space of $\mathcal{Y}$-valued distributions on $\mathcal{O}$ is defined by

$$\mathcal{D}'(\mathcal{O};\mathcal{Y}) = \mathcal{L}_c(\mathcal{D}(\mathcal{O});\mathcal{Y})$$

where $\mathcal{L}_c$ stands for linear continuous (here, it is equivalent to sequentially continuous) and $\mathcal{D}(\mathcal{O})$ is the space of indefinitely differentiable functions with a compact support included in $\mathcal{O}$. As usually, we denote $\mathcal{D}'(\mathcal{O}) = \mathcal{D}'(\mathcal{O};\mathbb{R})$, $\mathcal{D}'(0,T;\mathcal{Y}) = \mathcal{D}'((0,T);\mathcal{Y})$ and $\mathcal{D}(0,T) = \mathcal{D}((0,T))$. Given $f \in \mathcal{D}'(\mathcal{O};\mathcal{Y})$ and $\varphi \in \mathcal{D}(\mathcal{O})$, we frequently denote $\langle f, \varphi \rangle_{\mathcal{D}'(\mathcal{O};\mathcal{Y}) \times \mathcal{D}(\mathcal{O})} = f(\varphi)$.

Given $f \in \mathcal{C}(\mathcal{O};\mathcal{Y})$ we identify it to the distribution $\dot{f}$ defined by

$$\langle \dot{f}, \varphi \rangle_{\mathcal{D}'(\mathcal{O};\mathcal{Y}) \times \mathcal{D}(\mathcal{O})} = \int_{\mathcal{O}} \varphi f \, dx, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}). \tag{19}$$

This provides a topological imbedding $\mathcal{C}(\mathcal{O};Y) \subset \mathcal{D}'(\mathcal{O};Y)$. The completeness (it could be relaxed in sequential completeness) of $\mathcal{Y}$ is assumed in order to get this imbedding which is essential (else, the space $\mathcal{L}_c(\mathcal{D}(\mathcal{O});\mathcal{Y})$ is still defined but it no longer "contains“ continuous functions, and therefore it must not be denoted $\mathcal{D}'$ and its elements must not be named “distributions”).

3.6 Sobolev spaces, used in (7) and (12)

Let again $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, Y be a Banach space, and $r \in [1,\infty]$. Given $f \in L^r(\mathcal{O};Y)$ we identify it to the distribution $\dot{f}$ again defined by (19).

This provides a topological imbedding $L^r(\mathcal{O};Y) \subset \mathcal{D}'(\mathcal{O};Y)$ and allows to define the derivatives of f to be $\partial_i \dot{f}$, where ∂_i stands for $\partial/\partial x_i$.

Now, we can define

$$W^{1,r}(\mathcal{O};Y) = \{f \in L^r(\mathcal{O};Y) : \partial_i f \in L^r(\mathcal{O};Y),\ 1 \le i \le n\},$$

$$W^{-1,r}(\mathcal{O};Y) = \Big\{f \in \mathcal{D}'(\mathcal{O};Y) : f = f_0 + \sum_{i=1}^{n} \partial_i f_i,\ f_i \in L^r(\mathcal{O};Y),\ 0 \le i \le n\Big\}.$$

These spaces are respectively endowed with the norms

$$\|f\|_{W^{1,r}(\mathcal{O};Y)} = \Big((\|f\|_{L^r(\mathcal{O};Y)})^r + \sum_{i=1}^{n} (\|\partial_i f\|_{L^r(\mathcal{O};Y)})^r \Big)^{1/r},$$

$$\|f\|_{W^{-1,r}(\mathcal{O};Y)} = \inf \Big(\sum_{i=0}^{n} (\|f_i\|_{L^r(\mathcal{O};Y)})^r \Big)^{1/r},$$

where the infimum is taken for all the decompositions of f. As usual, H^1 stands for $W^{1,2}$, H^{-1} for $W^{-1,2}$, and $H^s(\mathcal{O})$ for $H^s(\mathcal{O};\mathbb{R})$.

The spaces $W^{1,r}(\mathcal{O};Y)$ and $W^{-1,r}(\mathcal{O};Y)$ are Banach spaces; they are separables if Y is separable and $r < \infty$. The spaces $H^1(\mathcal{O};Y)$ and $H^{-1}(\mathcal{O};Y)$ are Hilbert spaces if Y is a Hilbert space.

Let $H_0^1(\mathcal{O})$ be the closure of $\mathcal{D}(\mathcal{O})$ in $H^1(\mathcal{O})$. Then, their dual spaces satisfy $(H_0^1(\mathcal{O}))' \subset (\mathcal{D}(\mathcal{O}))'$ with continuous imbedding. In fact, see for example [8] Theorem 5.1 p. 19,

$$(H_0^1(\mathcal{O}))' = H^{-1}(\mathcal{O}). \tag{20}$$

Moreover, the duality pairing satisfies, for all $f \in H^{-1}(D)$ and $\varphi \in H_0^1(\mathcal{O})$,

$$\langle f, \varphi \rangle_{H^{-1}(D)\times H_0^1(D)} = \langle f, \varphi \rangle_{\mathcal{D}'(D)\times \mathcal{D}(D)}. \tag{21}$$

This duality property is frequently used to define H^{-1}. More generally, if $1 < r < \infty$ and Y is reflexive, then $W^{-1,r}(\mathcal{O};Y) = (W_0^{1,r'}(\mathcal{O};Y'))$, where Y' stands the dual space of Y and $1/r' + 1/r = 1$; it is not convenient to use this equality as a general definition for $W^{-1,r}$ because it does not provide the right space if $r = 1$ or Y is not reflexive.

3.7 Linear image of a distribution, used in (16)

Let $\mathcal{Y}$ and $\mathcal{X}$ be two complete lcstvs and let $A \in \mathcal{L}_c(\mathcal{Y};\mathcal{X})$. Given $f \in \mathcal{D}'(\mathcal{O};\mathcal{Y})$, its image $Af \in \mathcal{D}'(\mathcal{O};\mathcal{X})$ is defined by

$$(Af)(\varphi) = A(f(\varphi)), \quad \forall \varphi \in \mathcal{D}(\mathcal{O}). \tag{22}$$

In the case of Banach spaces, A maps continuously $L^r(\mathcal{O};\mathcal{Y})$, $W^{1,r}(\mathcal{O};\mathcal{Y})$ and $W^{-1,r}(\mathcal{O};\mathcal{Y})$ respectively into $L^r(\mathcal{O};\mathcal{X})$, $W^{1,r}(\mathcal{O};\mathcal{X})$ and $W^{-1,r}(\mathcal{O};\mathcal{X})$.

Now let us examine (16). It reads $\int_D p(\omega)\,dx = 0$ for P-almost all $\omega \in \Omega$. It is meaningful since, $p(\omega)$ lying in $W^{-1,\infty}(0,T;L^2(D))$ by (12), its image by the map $\int_D \in \mathcal{L}_c(L^2(D);\mathbb{R})$ is defined by (22) and satisfies

$$\int_D p(\omega)\,dx \in W^{-1,\infty}(0,T;\mathbb{R}).$$

3.8 Separation of variables, used in (11) and (12)

The separation of variable for functions, which maps $\mathcal{C}((0,T)\times D;\mathcal{Y})$ onto $\mathcal{C}(0,T;\mathcal{C}(D;\mathcal{Y}))$, extends by continuity in a one-to-one bicontinuous map from $\mathcal{D}'((0,T)\times D;\mathcal{Y})$ onto $\mathcal{D}'(0,T;\mathcal{D}'(D;\mathcal{Y}))$ (the surjectivity, which is the hard point, is related to Schwartz's kernel theorem, see [9] for real values). Using this map to identify the spaces, we get the topological equality

$$\mathcal{D}'((0,T)\times D;\mathcal{Y}) = \mathcal{D}'(0,T;\mathcal{D}'(D;\mathcal{Y})) \tag{23}$$

This identity allows us to consider $u(\omega)$ and $p(\omega)$ either as distributions on $(0,T)\times D$, as in (14) and (15), or as distributions on $(0,T)$ with values in a space of distributions on D, as in (11) and (12).

3.9 Nonlinear term $(u\cdot\nabla)u$, used in (14)

We denote $\nabla = (\partial_1,\dots,\partial_d)$ the spatial gradient. Then, given $u = (u_1,\dots,u_d)$, its divergence, used in (15), is $\nabla\cdot u = \partial_1 u_1 + \cdots + \partial_d u_d$.

Similarly, $u\cdot\nabla = u_1\partial_1 + \cdots + u_d\partial_d$. Then, given u and v in $(H^1(D))^d$, we define $(u\cdot\nabla)v$ to be the vector function which components are the $\sum_{j=1}^d u_j\partial_j v_i$, for $i = 1, \dots, d$. Since $d \le 4$ and D is Lipschitz, by Sobolev theorem, $H^1(D) \subset L^4(D)$ with continuous injection. Then, by Hölder inequality, the map $(u,v)\mapsto(u\cdot\nabla)v$ is bilinear continuous from $((H^1(D))^d)^2$ into $(L^{4/3}(D))^d$, and therefore from $(L^2(0,T;V))^2$ into $L^1(0,T;(L^{4/3}(D))^d)$.

3.10 Time-derivative $G\dot{W}_t$, used in (14)

Let U be a separable Hilbert space. Given $G \in M^2_{\mathcal{F}_t}(0,T;\mathbb{L}^2(K;U))$, its Itô's stochastic integral with respect to the cylindrical Wiener process W, denoted $\{\int_0^t G_s\,dW_s : 0 \le t \le T\}$ is defined to be the unique continuous U-valued $\mathcal{F}_t$-martingale such that, for all $g \in U$ and $t \in [0,T]$,

$$\Big(\int_0^t G_s\,dW_s, g\Big)_U = \sum_{i=1}^{\infty}\int_0^t (G_s e_k, g)_U\,d\beta^i_s, \tag{24}$$

where each stochastic integral in the series is understood as an Itô's stochastic integral with respect to the corresponding real valued Wiener process. The series in (24) converges in $L^2(\Omega, \mathcal{F}_t, P; \mathcal{C}([0,t]))$, for each $t \in (0,T]$, see [7] for details. Since $\int_0^\cdot G_s\, dW_s \in L^2(\Omega, \mathcal{F}_t, P; \mathcal{C}([0,t]; U))$, its time-derivative, that formally we will denote $G\dot{W}_t$, satisfies

$$G\dot{W}_t \in L^2(\Omega, \mathcal{F}_t, P; W^{-1,\infty}(0,t;U)), \quad \forall t \in (0,T],$$

because ∂_t is linear continuous from $\mathcal{C}$, and therefore from L^∞, into $W^{-1,\infty}$.

This applies here with $U = (L^2(D))^d$, because, thanks to (7), (8), (9) and (11), $G(\cdot,u) \in M^2_{\mathcal{F}_t}(0,T; \mathbb{L}^2(K; (L^2(D))^d))$. Then,

$$G(\cdot,u)\dot{W}_t \in L^2(\Omega, \mathcal{F}_t, P; W^{-1,\infty}(0,t; (L^2(D))^d)), \quad \forall t \in (0,T]. \tag{25}$$

4 A generalization of de Rham theorem to processes.

We will associate a pressure p to a solution u of the variational Navier–Stokes equation by the following result.

Theorem 4.1 *Let D satisfy (4), $(\Omega, \mathcal{G}, P)$ be a complete probability space and, given $r_0 \in [1,\infty]$, $r_1 \in [1,\infty]$ and $s_1 \in \mathbb{Z}$, let*

$$h \in L^{r_0}(\Omega, \mathcal{G}, P; W^{s_1,r_1}(0,T; (H^{-1}(D))^d)) \tag{26}$$

be such that, for all $v \in (\mathcal{D}(D))^d$ such that $\nabla \cdot v = 0$, P-a.s.,

$$\langle h, v\rangle_{(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d} = 0, \quad \text{in } \mathcal{D}'(0,T). \tag{27}$$

Then, there exists a unique

$$p \in L^{r_0}(\Omega, \mathcal{G}, P; W^{s_1,r_1}(0,T; L^2(D))) \tag{28}$$

such that, P-a.s.,

$$\nabla p = h, \quad \text{in } (\mathcal{D}'((0,T)\times D))^d, \tag{29}$$

$$\int_D p\, dx = 0, \quad \text{in } \mathcal{D}'(0,T). \tag{30}$$

Moreover, there exists a positive number $c_2(D)$, independent of h, such that, P-a.s.,

$$\|p\|_{W^{s_1,r_1}(0,T;L^2(D))} \le c_2(D)\, \|h\|_{W^{s_1,r_1}(0,T;(H^{-1}(D))^d)}. \tag{31}$$

Proof. Let

$$E = \{w \in (H^{-1}(D))^d : \langle w, v\rangle_{(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d} = 0,\ \forall v \in \mathcal{V}\}$$

be equipped with the norm of $(H^{-1}(D))^d$. Given $w \in E$, there exists a unique $q \in L^2(D)$ such that $\nabla q = w$ and $\int_D q\,dx = 0$ and there exists a positive number $c_2(D)$, independent of w, such that

$$\|q\|_{L^2(D)} \leq c_2(D)\,\|w\|_{(H^{-1}(D))^d}. \tag{32}$$

Indeed, by de Rham theorem, see for example [10], there exists $q_1 \in L^2(D)$ such that $\nabla q_1 = w$. Moreover, see for example Theorem 14 in [11], thanks to hypothesis (4) on D, $\nabla q_1 \in (H^{-1}(D))^d$ implies that $q_1 \in L^2(D)$ and $\|q_1 - \frac{1}{|D|}\int_D q_1\|_{L^2(D)} \leq c_2(D)\,\|w\|_{(H^{-1}(D))^d}$. Then, $q = q_1 - \frac{1}{|D|}\int_D q_1$ satisfies (32). Its uniqueness is obvious since D is connected.

Then, we define a continuous linear map A from E into $L^2(D)$ by $Aw = q$. It satisfies, for all $w \in E$,

$$\nabla Aw = w, \quad \int_D Aw\,dx = 0. \tag{33}$$

Now, let us give two properties that hold for P-almost all $\omega \in \Omega$. First, (27) gives, by definition (22), for all $\varphi \in \mathcal{D}(0,T)$,

$$\langle (h(\omega))(\varphi), v\rangle_{(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d} = 0,$$

that is, $(h(\omega))(\varphi) \in E$. Second, by (26), $h(\omega) \in W^{s_1,r_1}(0,T;(H^{-1}(D))^d)$. Since E is closed in $(H^{-1}(D))^d$, these two properties give, by the first property of the following Lemma 4.2, for P-almost all $\omega \in \Omega$,

$$h(\omega) \in W^{s_1,r_1}(0,T;E). \tag{34}$$

Since $W^{s_1,r_1}(0,T;E)$ is closed in $W^{s_1,r_1}(0,T;(H^{-1}(D))^d)$, (26) and (34) give, now by the second property of Lemma 4.2,

$$h \in L^{r_0}(\Omega, \mathcal{G}, P; W^{s_1,r_1}(0,T;E)).$$

Then, thanks to (32) and (33), its image $p = Ah$ satisfies (28) to (31), see Section 3.7. $\square$

To complete the proof, it remains to check the following properties.

Lemma 4.2 *Given a closed subspace F of a Banach space Y, $r \in [1, \infty]$ and $s \in \mathbb{Z}$,*

$$L^r(\Omega, \mathcal{G}, P; F) = \{h \in L^r(\Omega, \mathcal{G}, P; Y) : h(\omega) \in F, \quad P\text{-a.s. } \omega \in \Omega\}, \qquad (35)$$

$$W^{s,r}(0,T;F) = \{h \in W^{s,r}(0,T;Y) : h(\varphi) \in F, \quad \text{for all } \ \varphi \in \mathcal{D}(0,T)\}. \quad (36)$$

Proof. In view of definition (18), (35) follows from the fact that, if $h(\omega) \in F$, then it belongs to Y and $\|h(\omega)\|_Y = \|h(\omega)\|_F$ and from the identity

$$\mathcal{L}^0(\Omega, \mathcal{G}, P; F) = \{h : \Omega \to Y, \ h \in \mathcal{L}^0(\Omega, \mathcal{G}, P; Y)\}.$$

This identity is a consequence of the two following properties:
— First, if h is measurable into Y, it is measurable into F since $\mathcal{B}(F) \subset \mathcal{B}(Y)$ (this imbedding holds because the σ-algebra $\mathcal{B}$ is generated by closed sets, and every closed set of F is closed in Y).
— Second, if h is measurable into F, it is measurable into Y since, given $B \in \mathcal{B}(Y)$, then $h^{-1}(B) = h^{-1}(B \cap F)$ and $B \cap F \in \mathcal{B}(F)$ (this holds because, if B is closed in Y, then $B \cap F$ is closed in F).

Let us now prove (36) in three steps.
— First,

$$\mathcal{D}'(0,T;F) = \{h \in \mathcal{D}'(0,T;Y) : h(\varphi) \in F, \ \forall \varphi \in \mathcal{D}(0,T)\}. \qquad (37)$$

This is obvious since $\mathcal{D}'(0,T;Y) = \mathcal{L}_c(\mathcal{D}(0,T);Y)$.
— Second,

$$L^r(0,T;F) = L^r(0,T;Y) \cap \mathcal{D}'(0,T;F). \qquad (38)$$

Indeed, given mollifiers $(\rho_n)_{n \in \mathbb{N}}$ and a localizing sequence $(\alpha_n)_{n \in \mathbb{N}}$ (that is $\alpha_n \in \mathcal{C}^\infty$, $\alpha_n = 0$ outside $(1/n, T - 1/n)$, $\alpha_n = 1$ in $[2/n, T - 2/n]$), let $h_n = (\alpha_n h) \star \rho_n$. Then, $h_n \in \mathcal{D}(0,T;F)$, and therefore $\|h_n - h_m\|_{L^r(0,T;F)} = \|h_n - h_m\|_{L^r(0,T;Y)}$, and $h_n \to h$ in $L^r(0,T;Y)$. Therefore $(h_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^r(0,T;F)$ which is complete, and then $h \in L^r(0,T;F)$. The converse being obvious, (38) holds.

In fact, this proof does not hold if $r = \infty$; in this case, it suffices to remark that $L^\infty(0,T;Y) = \{f \in L^1(0,T;Y) : \|f\|_Y \in L^\infty(0,T)\}$.
— Third,

$$W^{s,r}(0,T;F) = W^{s,r}(0,T;Y) \cap \mathcal{D}'(0,T;F). \qquad (39)$$

For $s = 0$, it is (38).

For $s > 0$, it follows from (38) since $W^{s,r} = \{h : \partial_t^n h \in L^r, \ n \le s\}$.

For $s = -1$, let $h \in W^{-1,r}(0,T;Y) \cap \mathcal{D}'(0,T;F)$. By definition, $h = h_0 + \partial_t h_1$ where $h_i \in L^r(0,T;Y)$, and then $h = \partial_t g$ where $g = h_1 + \int_0^\cdot h_0 \in L^r(0,T;Y)$. Let $\xi \in \mathcal{D}(0,T)$ be such that $\int_0^T \xi = 1$, and let $k = g - \int_0^T g\xi$. Obviously, $k \in L^r(0,T;Y)$. Assume for a moment that

$$k \in \mathcal{D}'(0,T;F). \tag{40}$$

Then, $k \in L^r(0,T;F)$ by (38) and, since $\partial_t k = \partial_t g = h$, it follows that $k \in W^{-1,r}(0,T;Y)$. The converse being obvious, (39) holds for $s = -1$.

Now, let us check (40). Given $\varphi \in \mathcal{D}(0,T)$,

$$\begin{aligned} \langle k, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} &= \int_0^T \Big(g(t) - \int_0^T g(s)\xi(s)\,ds \Big) \varphi(t)\,dt \\ &= \int_0^T g(t) \Big(\varphi(t) - \xi(t) \int_0^T \varphi(s)\,ds \Big)\,dt \\ &= \int_0^T g(t) \partial_t \psi(t)\,dt \end{aligned}$$

where $\psi = \int_0^\cdot (\varphi - \xi \int_0^T \varphi)$. But ψ lies in $\mathcal{D}(0,T)$ since it is differentiable, it cancels at 0 and T, and it is constant on a neighbourhood of these two points. Then,

$$\langle k, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} = \langle g, \partial_t \psi \rangle_{\mathcal{D}' \times \mathcal{D}} = -\langle \partial_t g, \psi \rangle_{\mathcal{D}' \times \mathcal{D}} = -\langle h, \psi \rangle_{\mathcal{D}' \times \mathcal{D}}$$

which lies in F since $h \in \mathcal{D}'(0,T;F)$. With (37), this proves (40).

Finally, a similar proof may be given for $s \leq -2$. It is left to the reader since this case is not used in the present work. □

Remark 4.3 *In view of the proof of Theorem 4.1, $(H^{-1}(D))^d$ may be replaced by any Sobolev space $(W^{s_2,r_2}(D))^d$ or by $(\mathcal{D}'(D))^d$ in (26), provided that $L^2(D)$ be replaced by $W^{s_2+1,r_2}(D)$ or by $\mathcal{D}'(D)$ in (28) and that $\int_D$ be replaced by any linear form on this space in (30).*

Moreover, Theorem 4.1 extends to any open subset D of $\mathbb{R}^d$, instead of assumptions (4), provided that $L^2(D)$ be replaced by $L^2_{\text{loc}}(D)$ in (28) and that (30) be replaced, for any connected component D_i of D, by $\int_{d_i} p\,dx = 0$, where d_i is a non-empty open bounded set in $\mathbb{R}^d$ such that $\overline{d_i} \subset D_i$.

Finally, Theorem 4.1 extends to all $s_1 \in \mathbb{R}$ by interpolation.

5 Proof of Theorem 2.2.

Let us remind, see for example [6], that there exists a process u satisfying (11) and (17) — and therefore

$$u \in L^2(\Omega, \mathcal{F}_t, P; L^2(0,t;V) \cap L^\infty(0,t;H)), \quad \forall t \in (0,T], \tag{41}$$

— (13) and, P-a.s., the variational equation (2) for all $t \in [0, T]$ and $v \in \mathcal{V}$. Differentiating (2) with respect to t ($\omega \in \Omega$ being fixed), we get, in $\mathcal{D}'(0, T)$,

$$-\int_D \partial_t u \cdot v\, dx - \nu \int_D \nabla u \cdot \nabla v\, dx - \int_D ((u \cdot \nabla) u) \cdot v\, dx$$
$$+ \langle F(\cdot, u), v\rangle_{(H^{-1}(D))^d \times (H_0^1(D))^d} + \int_D G(\cdot, u) \dot{W}_t \cdot v\, dx = 0.$$

Since $v \in (\mathcal{D}(D))^d$, thanks to (19), (21) and $\langle \nabla u, \nabla v\rangle = -\langle \Delta u, v\rangle$, this reads

$$\langle -\partial_t u + \nu \Delta u - (u \cdot \nabla) u + F(\cdot, u) + G(\cdot, u)\dot{W}_t, v\rangle_{(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d} = 0. \quad (42)$$

Let us denote $h = -\partial_t u + \nu \Delta u - (u \cdot \nabla) u + F(\cdot, u) + G(\cdot, u)\dot{W}_t$. As we will check next, (41) implies

$$h \in L^1(\Omega, \mathcal{F}_t, P; W^{-1,\infty}(0, t; (H^{-1}(D))^d)), \quad \forall t \in (0, T], \quad (43)$$

therefore Theorem 4.1 provides p satisfying (12), (16) and $\nabla p = h$, that is the Navier–Stokes equation (14).

By (41), P-a.s., $u \in L^2(0, T; V)$. Since $\nabla \cdot v = 0$ for all $v \in \mathcal{V}$ and therefore, by continuity, for all $v \in V$, this implies $\nabla \cdot u = 0$, that is (15).

It only remains to check (43). Denoting $\mathcal{E}$ the space in its right-hand side, let us check that all the terms of h belong to it.

— First, ∂_t is linear continuous from $L^\infty(0, T; H)$ into $W^{-1,\infty}(0, T; H)$ and then into $W^{-1,\infty}(0, T; (H^{-1}(D))^d)$, and therefore (41) implies $\partial_t u \in \mathcal{E}$.

— Next, Δ being linear continuous from $(H^1(D))^d$, and then from V, into $(H^{-1}(D))^d$, (41) implies $\nu \Delta u \in L^2(\Omega, \mathcal{F}_T, P; L^2(0, T; (H^{-1}(D))^d))$ which is included in $\mathcal{E}$ due to the topological imbedding

$$L^1(0, T; Y) \subset W^{-1,\infty}(0, T; Y). \quad (44)$$

To get this imbedding it suffices to notice that every $f \in L^1(0, T; Y)$ satisfies $f = \partial_t \int_0^\cdot f$ and $\int_0^\cdot f \in L^\infty(0, T; Y)$, and therefore $f \in W^{-1,\infty}(0, T; Y)$ thanks to its definition in Section 3.6.

— As seen in Section 3.9, the map $v \mapsto (v \cdot \nabla) v$ is bilinear continuous from $V \times V$ into $(L^{4/3}(D))^d$, and then into $(H^{-1}(D))^d$ by Sobolev theorem since $d \le 4$. Therefore, (41) implies $(u \cdot \nabla) u \in L^1(\Omega, \mathcal{F}_T, P; L^1(0, T; (H^{-1}(D))^d))$ which is included in $\mathcal{E}$ by (44).

— Next, $F(\cdot, u) \in \mathcal{E}$ thanks to (7), (8), (9) and (11).

— Finally, $G(\cdot, u)\dot{W}_t \in \mathcal{E}$ by (25). This ends the proof of (43).

Remark 5.1 *This proof shows that there exists a positive number* $c_3(D,T)$ *such that any pair* (u,p) *satisfying* (11) *to* (14) *satisfies, in addition,* P*-a.s.,*

$$\begin{aligned}\|p\|_{W^{-1,\infty}(0,T;L^2(D))} &\le c_3(D,T)\Big(\|u\|_{L^\infty(0,T;(L^2(D))^d)} \\ &\qquad + (\nu + c_1)\|u\|_{L^2(0,T;(H^1(D))^d)} + (\|u\|_{L^2(0,T;(H^1(D))^d)})^2\Big) \\ &\le c_3(D,T)\Big(\|u\|_{L^\infty(0,T;(L^2(D))^d)} + (1 + (\nu + c_1)\|u\|_{L^2(0,T;(H^1(D))^d)})^2\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\|p\|_{L^1(\Omega,\mathcal{F},P;W^{-1,\infty}(0,T;L^2(D)))} &\le c_3(D,T)\Big(\|u\|_{L^1(\Omega,\mathcal{F},P;L^\infty(0,T;(L^2(D))^d))} \\ &\qquad + (1 + (\nu + c_1)\|u\|_{L^2(\Omega,\mathcal{F},P;L^2(0,T;(H^1(D))^d))})^2\Big).\end{aligned}$$

6 Nonexistence result for a body force valued in V'.

From now, the assumptions on the body force F in (7), (8) and (9) are replaced by

$$\begin{cases} F \text{ is measurable from } [0,T] \times (L^2(D))^d \text{ into } V', \\ v \mapsto \langle F(t,v), \phi\rangle_{V'\times V} \text{ is continuous from } (L^2(D))^d \text{ into } \mathbb{R}, \\ \|F(t,v)\|_{V'} \le c_1(1 + \|v\|_{(L^2(D))^d}). \end{cases} \tag{45}$$

Then, with all others assumptions (4) to (10), the proof of the existence of a variational solution given in [6] provides a solution u of (3) instead of (2). It again satisfies (41) and then $f = F(\cdot,u)$ satisfies

$$f \in L^2(\Omega,\mathcal{F},P;L^2(0,T;V')). \tag{46}$$

Let $g = -\partial_t u + \nu\Delta u - (u\cdot\nabla)u + G(\cdot,u)\dot{W}_t$. Proceeding as in the proof of Theorem 2.2, we get

$$g \in L^1(\Omega,\mathcal{F},P;W^{-1,\infty}(0,T;(H^{-1}(D))^d)) \tag{47}$$

and, instead of (42),

$$\langle g, v\rangle_{(\mathcal{D}'(D))^d\times(\mathcal{D}(D))^d} + \langle f, v\rangle_{V'\times V} = 0, \quad \forall v \in V. \tag{48}$$

The existence of a corresponding pressure is ruled out by the following result.

Theorem 6.1 *There is no Hausdorff locally convex topological vector space in which, for arbitrary f and g satisfying* (46), (47) *and* (48)*, the equation*

$$g + f = \nabla p \tag{49}$$

might hold.

Remark 6.2 *With the above definitions of f and g,* (49) *reads*

$$-\partial_t u + \nu \Delta u - (u \cdot \nabla) u + G(\cdot, u)\dot{W}_t + F(\cdot, u) = \nabla p,$$

that is Navier–Stokes equation (1)*. Since it cannot hold,* (3) *should not be named "variational Navier–Stokes equation"; this name should be reserved to* (2)*, the equivalence with* (1) *being lost as $(H^{-1}(D))^d$ is replaced by V'.*

Remark 6.3 *Equation* (48) *equivalently reads, thanks to* (21),

$$\langle g, v \rangle_{(H^{-1}(D))^d \times (H_0^1(D))^d} + \langle f, v \rangle_{V' \times V} = 0.$$

Remark 6.4 *The reader may be surprised since, in contradiction with Theorem 6.1, various papers contain proofs of the existence of pressure for V'-valued body forces. These proofs are wrong, the mistake generally lying in the use of a de Rham type theorem to equation* (48)*, as if it was*

$$\langle g + f, v \rangle_{(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d} = 0$$

or $\langle g + f, v \rangle_{(H^{-1}(D))^d \times (H_0^1(D))^d} = 0$.

The confusion follows from the use, in these proofs, of the same notation $\langle \cdot, \cdot \rangle$ for the duality products in $(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d$, in $(H^{-1}(D))^d \times (H_0^1(D))^d$ and in $V' \times V$.

Proof of Theorem 6.1. Would such a space, say X, exist, it should contain $L^2(\Omega, \mathcal{F}, P; L^2(0, T; V'))$ and $L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; (H^{-1}(D))^d))$, at least in the sense that there should be two linear injective maps i and j from these spaces into X. Then a linear injective map $\widehat{i}$ from V' into X would be defined by $\widehat{i}\varphi = i\widehat{\varphi}$ where $(\widehat{\varphi}(\omega))(t) = \varphi$ for all ω and t. Similarly, we would define a linear injective map $\widehat{j}$ from $(H^{-1}(D))^d$ into X.

Since each $\psi \in (H^{-1}(D))^d$ defines a unique $Q\psi \in V'$ by

$$\langle Q\psi, v \rangle_{V' \times V} = \langle \psi, v \rangle_{(H^{-1}(D))^d \times (H_0^1(D))^d}, \quad \forall v \in V, \tag{50}$$

then ψ and $Q\psi$ should correspond to the same element of X, that is

$$\widehat{j}\psi = \widehat{i} Q\psi.$$

Consider now $\phi = \nabla q$, for a given non constant $q \in L^2(D)$. Then, $\phi \in (H^{-1}(D))^d$, $\phi \neq 0$, and $Q\phi = 0$ since $\langle \phi, v\rangle_{(H^{-1}(D))^d \times (H_0^1(D))^d} = -\langle q, \nabla \cdot v\rangle_{(\mathcal{D}'(D))^d \times (\mathcal{D}(D))^d} = 0$ for all $v \in \mathcal{V}$ and thus, by continuity, for all $v \in V$. Then, $\widehat{i}Q\phi = 0$ and therefore $\widehat{j}\phi = 0$ would hold, in contradiction with the injectivity of $\widehat{j}$. □

The map Q defined by (50) is "canonical" since $Q\psi$ is the restriction to V of the map $\psi \in \mathcal{L}_c((H_0^1(D))^d; \mathbb{R})$. Let us summarize its properties.

Lemma 6.5 *The map Q is linear continuous from $(H^{-1}(D))^d$ onto V' and is not one to one.*

Proof. Continuity holds since, by (50) and (20), $\|Q\psi\|_{V'} \leq \|\psi\|_{((H_0^1(D))^d)'} \leq \|\psi\|_{(H^{-1}(D))^d}$ (thanks to the definition of H^{-1}, this holds for $c = 1$).

The range of Q is V' since by Hann–Banach theorem, any $\varphi \in V'$ possesses an extension $\psi \in ((H_0^1(D))^d)'$, and then $Q\psi = \varphi$.

It is not one to one since, in the proof of Theorem 6.1, we built $\phi \neq 0$ such that $Q\phi = 0$. □

Let us now give strong equations that, instead of Navier–Stokes one, are satisfied by solutions of (3). First, in V', we have, P-a.s.,

$$Q(\partial_t u - \nu\Delta u + (u \cdot \nabla)u) = F(\cdot, u) + Q(G(\cdot, u)\dot{W}_t) \tag{51}$$

in $\mathcal{D}'(0, T; V')$. This is not totally satisfactory since pressure disappeared.

The other possibility is to give an equation in $(H^{-1}(D))^d$. There exist infinitely many $\Phi \in L^1(\Omega, \mathcal{F}, P; L^1(0, T; (H^{-1}(D))^d))$ such that $Q\Phi = F(\cdot, u)$ (for example, a solution is $\Phi = \Delta S(F(\cdot, u))$ where Sf is, for a given $f \in V'$, the solution of $\int_D \nabla Sf \cdot \nabla v = \langle f, v\rangle_{V' \times V}$ for all $v \in V$). For such a Φ, there exists $p \in L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; (H^{-1}(D))^d))$ such that, P-a.s.,

$$\partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \Phi + Q(G(\cdot, u)\dot{W}_t) \tag{52}$$

in $\mathcal{D}'(0, T; (H^{-1}(D))^d)$. This again is not satisfactory, now since Φ and p are not unique (even if p is normalized by $\int_D p = 0$); worst, all p in the above space can be associated to a given u. Indeed, given a pair (Φ_0, p_0) satisfying (52), the pair $(\Phi_0 + \nabla(p - p_0), p)$ is another solution since $Q(\Phi_0 + \nabla(p - p_0)) = Q\Phi_0 = F(\cdot, u)$ because, as seen in the proof of Theorem 6.1, $Q\nabla q = 0$ for all q.

Remark 6.6 *The set spanned by u is the same, wether forces span V' or $(H^{-1}(D))^d$, since Q defined by* (50) *maps $(H^{-1}(D))^d$ onto V'. (Surjectivity follows from $V \subset (H_0^1(D))^d$ since their topologies coincide on V; a strict imbedding $V' \subset W'$ holds when W is a dense subset of V equipped with a strictly finer topology, that is not the case here).*

The only effect of choosing forces in V' instead of $(H^{-1}(D))^d$ is to suppress information on pression, since Q^{-1} may be viewed as a one to one map from V' onto $(H^{-1}(D))^d/\nabla L^2(D)$. Indeed, given $f \in V'$ and $g_0 \in Q^{-1}f$,

$$\begin{aligned} Q^{-1}f &= \{g \in (H^{-1}(D))^d : \langle g - g_0, v\rangle_{(H^{-1}(D))^d \times (H_0^1(D))^d} \text{ for all } v \in V\} \\ &= \{g_0 + \nabla q : q \in L^2(D)\}. \end{aligned}$$

This explains why Navier–Stokes equation, which contains information on pressure, cannot contain terms lying in V'.

Acknowledgements. The authors acknowledge E. Fernández Cara for pointing out the necessity of clarifying the existence of the pressure in the solution of the stochastic Navier-Stokes equations.

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