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Global stabilization of fixed points using predictive control

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We analyze the global stability properties of some methods of predictive control. We particularly focus on the optimal control function introduced by de Sousa Vieira and Lichtenberg [Phys. Rev. E **54**, 1200 (1996)]. We rigorously prove that it is possible to use this method for the global stabilization of a discrete system $x_{n+1}=f(x_n)$ into a positive equilibrium for a class of maps commonly used in population dynamics. Moreover, the controlled system is globally stable for all values of the control parameter for which it is locally asymptotically stable. Our study highlights the difficulty of obtaining global stability results for other methods of predictive control, where higher iterations of f are used in the control scheme. © *2010 American Institute of Physics*. [doi:10.1063/1.3432558]

In many situations, the mechanisms of control of chaos and targeting seek not only to suppress any possible chaotic behavior, leading the system to a suitable equilibrium, but also to make its basin of attraction as large as possible. While local stability analysis is not difficult to address, in general, global stability results are often based on numerical observance. We tackle this problem for some methods of predictive control and succeed in proving a sharp result of global stabilization valid for a wide family of maps usually employed in the mathematical modeling of discrete systems.

I. INTRODUCTION

We discuss the issue of global stabilization of a chaotic dynamical system into an equilibrium employing prediction-based control (PBC), that is, techniques for control of chaos that use predicted iterations of the system. As far as we know, these methods were introduced for discrete dynamical systems by Ushio and Yamamoto.[1] However, we point out that in the case of stabilization of fixed points, the same method was first suggested by de Sousa Vieira and Lichtenberg;[2] using a combination of a nonlinear delayed feedback control (DFC) method[3] and a technique of Socolar *et al.*,[4] they arrived at an "optimal" control technique that overcomes some admitted drawbacks of the DFC method. Some interesting generalizations of PBC methods were introduced by Polyak and Maslov.[5,6]

One of the advantages of predictive control methods is that conditions for local stabilization of fixed points in terms of the control parameter are easy to find.[1,2] However, the problem of global stabilization is more difficult to address, and, except for very particular maps,[2,7] only conclusions based on simulations can be found in literature so far.[5,6] As explained by Bocaletti *et al.* (Ref. 8, Sec 4.1), when talking about the problem of targeting, in many practical situations, it is important to drive most trajectories of a dynamical system to a periodic orbit that yields superior performance over the others according to some criteria. Thus, the problem of targeting consists not only of choosing an appropriate attractor but also making its basin of attraction as large as possible.

Among the dynamical systems exhibiting chaos, a well-known family is given by one-dimensional maps of the form

$$x_{n+1}=f(x_n), \tag{1}$$

where f is a unimodal function with a unique positive equilibrium K. Examples of such maps employed in the modeling of population dynamics are the quadratic map $f(x)=rx(1-x)$, the exponential (Ricker) map $f(x)=xe^{r(1-x)}$, and the generalized Beverton–Holt map $f(x)=rx/(1+x^{\gamma})$ (see Ref. 9). Here, r and γ are positive parameters intrinsic to the model, such as the natural growth rate. It is worth noticing that, although these three maps can be considered discrete approximations of the continuous logistic equation, the Ricker and the Beverton–Holt models have more biological meaning (see, e.g., Ref. 10).

One important property, shared by these and other models, is that they exhibit chaos for some values of the parameters,[11] but multistability is not possible; this means that even if they have an infinite number of periodic orbits, at most one of them can be attracting. In particular, if the unique positive equilibrium K is asymptotically stable [that is, $|f'(K)|<1$], then it is a global attractor. A very interesting mathematical derivation of this property is due to Singer,[12] who used the Schwarzian derivative as a fundamental tool.

In this setting, an important control problem consists of stabilizing the unique positive equilibrium, keeping the global property, that is, in such a way that all positive trajectories converge to the equilibrium. The aim of most of the proposed mechanisms of control of chaos consists of stabilizing one of the unstable periodic orbits (UPOs) embedded in a chaotic attractor either by adjusting a parameter of the model, or modifying the state variable, by introducing an

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external parameter that one can control to drive the chaotic system to a stable situation. The first strategy has its origin in the seminal work of Ott, Grebogi, and Yorke;[13] although it was successfully applied in physics, it is difficult to implement in ecological systems (for more comments and references, see Ref. 8, Sec. 2). The second group of methods seems to be the most appropriate when we try to manage biological systems since intrinsic parameters such as the growth rate are, in general, difficult to modify, while external parameters such as migration, culling, or enrichment are more easily controllable (for further discussion and references see, e.g., Ref. 14).

One of the methods suggested in this context is the constant feedback (CF) method,[15] which consists of modifying the one-dimensional system (1) to

$$x_{n+1} = f(x_n) - C, \tag{2}$$

where C is a positive constant. Gueron[16] proved analytically that for a general family of unimodal maps, it is always possible to find values of C for which Eq. (2) has a stable positive equilibrium. However, the region of attraction is usually small and many trajectories are driven to zero. Biologically, this means that only intermediate values of the population size survive after control, while the others go to extinction due to the Allee effect.[10,17]

The first author recently proved[18] a result of global stabilization for the proportional feedback (PF) control method, which modifies Eq. (1) to

$$x_{n+1} = f(\gamma x_n), \tag{3}$$

where $\gamma \in (0,1)$. In population dynamics, this means that a percentage of the population is removed by migration or harvesting.

The other interesting strategy of control is based on a threshold mechanism. The idea of incorporating a self-regulatory threshold dynamics on a chaotic system goes back to the work of Sinha and Biswas[19] and Glass and Zheng[20] (see also Refs. 21 and 22). This strategy of control has also a clear interpretation in biological terms, corresponding to control measures such as culling of a stock population, hunting or catching of a managed population stock, or treatment of infectious diseases (for more comments and further references, see Ref. 23).

In this paper, we focus on global stabilization of system (1) using predictive feedback control. Methods of predictive control to stabilize an unstable T-periodic orbit of the discrete-time system (1) have the form[1]

$$x_{n+1} = g(x_n, u_n),$$

where the control input u_n is determined by the difference between the predicted state $f^T(x_n)$ and the current state x_n, that is,

$$u_n = f^T(x_n) - x_n.$$

As usual, f^T is defined as the Tth iteration of f, and it is used as a prediction of x_{n+T}. For $T=1$ (stabilization of fixed points), the simplest scheme of PBC is written as

$$x_{n+1} = f(x_n) - \alpha(f(x_n) - x_n), \tag{4}$$

where α is a real control parameter. Equation (4) is exactly the control proposed by de Sousa Vieira and Lichtenberg (Ref. 2, Sec. III) to avoid some handicaps of the DFC method, such as the increase in dimensionality of the system[3] and the so-called odd limitation number.[24]

Some additional good properties of the control law (4) underlined in Ref. 2 are that the fixed points of the controlled system are the same as those in the uncontrolled system, knowledge of the location of the UPO is not necessary, and it is easy to implement in the sense that the control term contains only the amplified versions of the input and output of the dynamical system.

In our main result, we prove that for a family of unimodal maps, including the quadratic map, the Ricker map, and the generalized Beverton–Holt map, the optimal nonlinear control of de Sousa Vieira and Lichtenberg leads to the global stabilization of a chaotic system into its positive equilibrium. Moreover, this is true for all values of the parameters for which local stabilization is possible. We also discuss the case of more general methods of predictive control, showing that this result of global stability is no longer valid.

This paper is organized as follows: in Sec. II, we state our main assumptions and recall some related results. Section III is devoted to establish the main global stability result. In Sec. IV, the issue of local and global stability is discussed for other PBC methods. Finally, in Sec. V, we discuss the main conclusions and directions for future research. The proofs and auxiliary results are placed in Appendixes A and B.

II. STATEMENT OF THE PROBLEM

First, we introduce some general properties and notation that will be used throughout the paper from now on. Denote by $I=[0,b]$ a real interval ($b=+\infty$ is allowed). The map $f: I \to I$ that defines the dynamical system (1) will be a C^3 function satisfying the following properties:

(A1) f has only two fixed points: $x=0$ and $x=K>0$, with $f'(0)>1$.
(A2) f has a unique critical point $c<K$ in such a way that $f'(x)>0$ for all $x\in(0,c)$, $f'(x)<0$ for all $x\in(c,b)$.
(A3) $(Sf)(x)<0$ for all $x\neq c$, where

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

is the Schwarzian derivative of f.
(A4) $f''(x)<0$ for all $x\in(0,c)$.

These assumptions are motivated by the fact that many maps usually employed in discrete models fulfill them. In particular, the quadratic map for all $r>1$, the exponential map for all $r>0$, and the generalized Beverton–Holt map for $r>1$ and $\gamma\geq 2$ (see Ref. 12). We notice that if f satisfies (A3) and $f''(0)<0$, then condition (A4) holds; otherwise, f' would have a positive local minimum, which contradicts the maximum principle (Ref. 12, Proposition 2.4). Thus, the family of maps satisfying conditions (A1)–(A4) is essentially

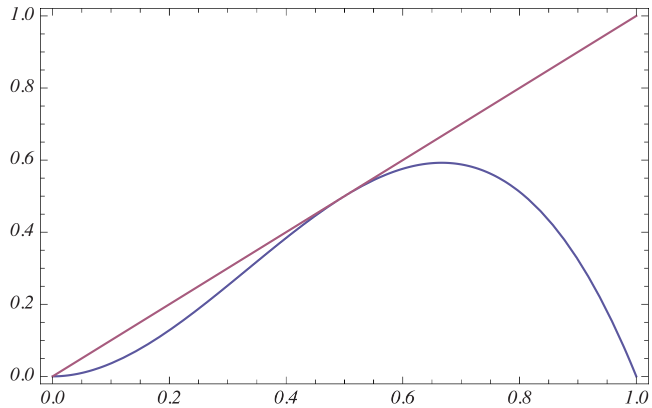

FIG. 1. (Color online) Graph of the map $f(x)=4x^2(1-x)$ and the line $y=x$. The iterates of initial points $x_0<1/2$ converge to zero (Allee effect).

the family of S-unimodal maps considered by Collet and Eckmann (Ref. 25, Sec. II.4) (see also Ref. 26, Sec. 5.3). Similar conditions were assumed for CF control in Refs. 16 and 17 and for PF in Ref. 18. Besides the above-mentioned maps, there are other functions satisfying conditions (A1)–(A4), which do not come from population dynamics; an example is the sine map $f(x)=r\sin(\pi x)$, $0\le x\le 1$, with $r\in(1/\pi,1)$ (Ref. 27, p. 369).

When used in population models, maps satisfying condition (A4) correspond to *compensation* models, where the per capita production is a decreasing function of the population density [see, e.g., Ref. 28 (Sec. 1.2) and Ref. 29 (Sec. 1.4)]. This condition is not satisfied by *depensation* models, for which the per capita production is smaller than one for low values of the population density. The latter models exhibit the so-called *Allee effect*, which means that if the initial population size is below a critical parameter, the population will die out. An example is the generalization of the quadratic map $f(x)=rx^2(1-x)$ (see Ref. 28). In Fig. 1, we plot the graph of this map with $r=4$; although it has a unique positive fixed point and a unique critical point (maximum), it does not satisfy (A4) and exhibits the Allee effect.

Condition (A3) is a more technical assumption. The Schwarzian derivative was first introduced into the study of one-dimensional dynamical systems by Singer,[12] and it became a valuable tool (see, e.g., Refs. 25 and 26). Although it is difficult to give a biological interpretation of it, it is a remarkable fact (already observed by Singer) that the most usual maps employed in the modeling of population dynamics satisfy condition (A3). We recall[12] that if f satisfies conditions (A1) and (A3), and $f'(K)\ge -1$, then K is a global attractor of system (1).

We notice that, for $\alpha\in[0,1]$, the method (4) is well-defined in the sense that, since $f:I\to I$, the modified map

$$F_\alpha(x) := f(x) - \alpha(f(x)-x) = \alpha x + (1-\alpha)f(x)$$

also maps I into I because $F_\alpha(x)$ is a convex combination of x and $f(x)$. This property does not hold for general DFC and PBC methods. If we think of population models, this is important because negative values of the population do not make sense. We limit our discussion to this range of values of the control parameter α.

The method (4) was applied in Ref. 2 to the quadratic map, with $r>3$ [to ensure that K is unstable for $f(x)=rx(1-x)$]. The authors observed that the positive fixed point $K=1-1/r$ is locally stable for the controlled system if $\alpha>(r-3)/(r-1):=\alpha_0$. Moreover, the equilibrium K is a global attractor for system (4) for all $\alpha\in(\alpha_0,1)$. However, as pointed out later by McGuire *et al.*,[7] the quadratic map is a very particular case since the modified map F_α is still quadratic. Hence, it trivially shares the global stability properties of f.

For general unimodal maps, function F_α does not inherit properties (A2) and (A3); usually, the shape of F_α is no longer one-humped for $\alpha>0$. For example, the exponential map $f(x)=xe^{r(1-x)}$ is bimodal for $0<\alpha<e^{r-2}/(1+e^{r-2})$ and increasing for $e^{r-2}/(1+e^{r-2})<\alpha<1$.[30]

Thus, although numerical experiments suggest that K is still globally attracting for F_α when $|F'_\alpha(K)|<1$, an analytic proof of this result is not available. Our main result in this paper fills this gap for the family of maps satisfying conditions (A1)–(A4).

III. GLOBAL STABILIZATION

In this section, we state our main result. As before, we denote $F_\alpha(x)=\alpha x+(1-\alpha)f(x)$.

We begin with a preliminary result, which is of independent interest.

Proposition 1: Assume that $f'(x)<0$ and $F'_\alpha(x)<0$ for all x in an interval $J\subset I$. If $(Sf)(x)<0$ for all $x\in J$, then $(SF_\alpha)(x)<0$ for all $x\in J$ and all $\alpha\in(0,1)$.

As an immediate consequence of Proposition 1 and Theorem 5 in Ref. 31, we get the following corollary.

Corollary 1: If f satisfies conditions (A1)–(A3), and $f'(K)<-1$, then the family of maps $\{F_\alpha:\alpha\in(0,1)\}$ undergoes a period-doubling bifurcation at $\alpha_0=(f'(K)+1)/(f'(K)-1)$, where $F'_{\alpha_0}(K)=-1$.

Corollary 1 solves an open problem posed in Ref. 7. There, the authors proposed to investigate the conditions under which period doubling occurs for general unimodal maps, which are being stabilized at a fixed point using control (4). They showed that if f has a negative Schwarzian derivative, then the same property is inherited by F_α when the positive feedback parameter α is sufficiently small. Corollary 1 proves that period doubling occurs regardless of the value of $\alpha\in(0,1)$ necessary for stabilization.

Next we formulate our main result.

Theorem 1: *Assume that f satisfies conditions (A1)–A4), and define $F_\alpha(x)=f(x)-\alpha(f(x)-x)$. If $F'_\alpha(K)\ge -1$, then the positive equilibrium K is a global attractor of Eq. (4) on $(0,b)$.*

The proofs of Proposition 1 and Theorem 1 can be found in Appendix A.

As an example, consider the Ricker map $f(x)=xe^{r(1-x)}$, which satisfies conditions (A1)–(A4) for all $r>0$. The positive equilibrium $K=1$ is globally stable for $r\le 2$, then it undergoes a typical sequence of period-doubling bifurcations, and it becomes chaotic for $r>2.6924$.[11]

According to Corollary 1 and Theorem 1, for all $r>2$, the family of maps $\{F_\alpha(x)=\alpha x+(1-\alpha)xe^{r(1-x)}:\alpha\in[0,1)\}$

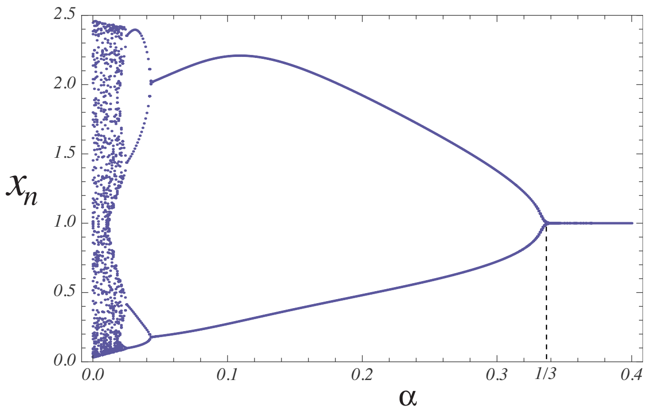

FIG. 2. (Color online) Bifurcation diagram for the controlled system (5) with $r=3$ and α as the bifurcation parameter. The value $\alpha=0$ corresponds to the uncontrolled (chaotic) system; the equilibrium becomes globally asymptotically stable after the bifurcation point $\alpha=1/3$ (vertical dashed line).

experiences a period-halving bifurcation at $\alpha=(r-2)/r$ in such a way that the positive equilibrium is globally attracting for the controlled system

$$x_{n+1}=x_n e^{r(1-x_n)}-\alpha(x_n e^{r(1-x_n)}-x_n) \tag{5}$$

for all $\alpha\in[(r-2)/r,1)$. We show the bifurcation diagram for $r=3$ in Fig. 2. The equilibrium is stabilized in a period-halving bifurcation at $\alpha=1/3$ (see the vertical dashed line). Theorem 1 ensures that, for $\alpha\geq 1/3$, all orbits starting at $x_0>0$ converge to the positive equilibrium $K=1$.

Next we give a numerical validation of Theorem 1 by means of a statistical approach applied to Eq. (5) with $r=3$; this method not only illustrates our main result but also provides significant rates of convergence, not dependent on the initial condition.

First we choose randomly 2000 initial conditions in the interval [0,2.5], which is invariant and attracting for the map f. See Fig. 3, where we show the distribution of this sample and its evolution after 15 iterations of f. Since the uncontrolled system is chaotic, no periodic pattern is appreciated. We notice that, in Figs. 3–8, n denotes the number of iterations.

Figure 4 shows the distribution of the random sample after 15 iterations of the controlled system for $\alpha=0.35$, which is close to the bifurcation point. It can be observed that the iterates of all initial conditions approach the positive equilibrium. The convergence is faster for larger values of α, as it is shown for $\alpha=0.4$ and the same number of iterations.

Next we estimate the probability density function of the random variable provided by the sample we chose using a kernel density estimation. We use the kernel density approximation

$$\phi(x)=\frac{1}{hm}\sum_{i=1}^{m} k\left(\frac{x-x_i}{h}\right),$$

where $(x_i)_{i=1}^{m}$ is the sample, k is the standard Gaussian function

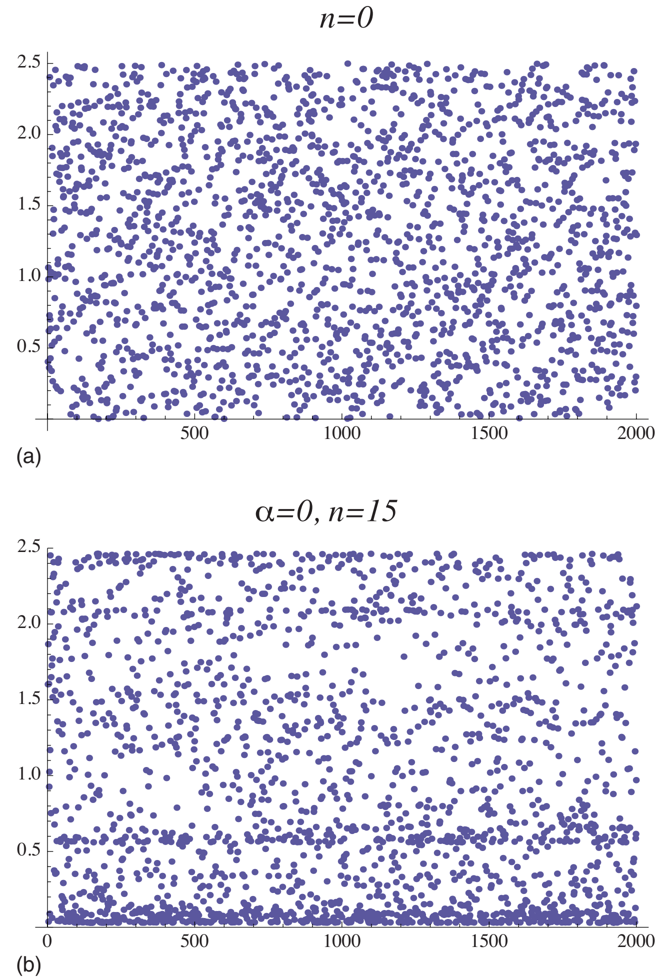

FIG. 3. (Color online) Distribution of a sample of 2000 random initial conditions in the interval [0,2.5], and the evolution of the uncontrolled system ($\alpha=0$) after 15 iterations.

$$k\left(\frac{x-x_i}{h}\right)=\frac{1}{\sqrt{2\pi}}e^{-(x-x_i)^2/2h^2},$$

and h is a smoothing bandwidth. We use

$$h=\left(\max_{i=1}^{m}\{x_i\}-\min_{i=1}^{m}\{x_i\}\right)\Big/ 4.$$

Our simulations with different samples of random initial conditions show that, when evolved under the equations of the controlled system (5), the approximation of the probability density function converges to a delta function located at the positive fixed point.

In Fig. 5, we represent the initial density function and its evolution after 50 iterations of the uncontrolled system [Eq. (5) with $\alpha=0$]. As observed before, there is no periodic pattern. The evolution of the density function for $\alpha=0.35$ (close to the bifurcation point $\alpha=1/3$) after 15 and 50 iterations is shown in Fig. 6. The convergence to the delta function is faster for larger values of α, as shown in Fig. 7 for $\alpha=0.4$: after only 15 iterations, the density function is highly concentrated at the positive equilibrium point $K=1$.

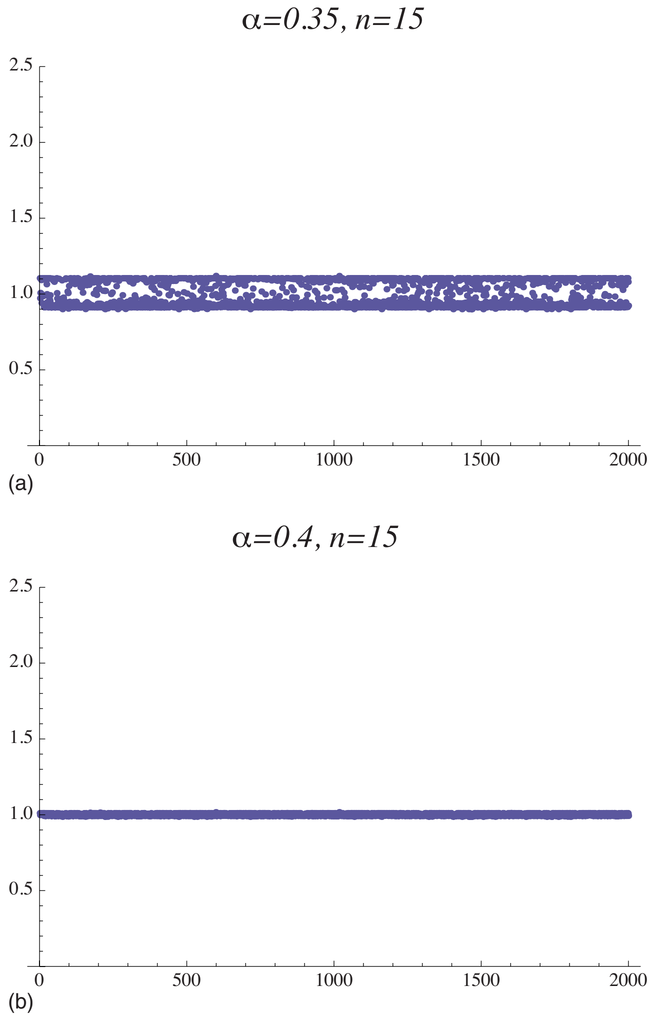

FIG. 4. (Color online) Evolution of the random sample of 2000 initial conditions in the interval [0,2.5] after 15 iterations of the controlled system for $\alpha=0.35$ and $\alpha=0.4$.

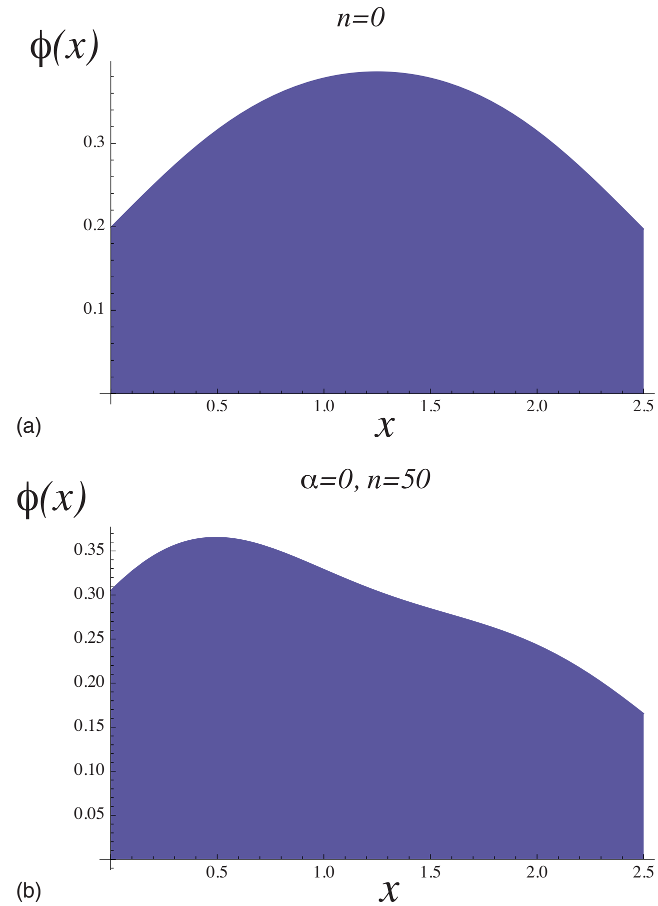

FIG. 5. (Color online) Approximation of the probability density function for the random variable corresponding to a sample of 2000 random initial conditions in the interval [0,2.5], and its evolution after 50 iterations under the uncontrolled system (5) with $r=3$ and $\alpha=0$.

To further illustrate this numerical validation, we show in Fig. 8 the time series of the solution of Eq. (5), with $r=3$ and initial point $x_0=0.5$, for $\alpha=0.35$ and $\alpha=0.4$.

IV. OTHER METHODS OF PREDICTIVE CONTROL

The global control attained with the method of de Sousa Vieira and Lichtenberg has the restriction that, in general, the perturbation required to stabilize all trajectories around the positive equilibrium is relatively large. In this regard, we emphasize that, while targeting problems usually require small perturbations, there are situations where large perturbations are more appropriate. For example, this is the case of interventions in population dynamics, where the control problem can be seen as a strategy for sustainable development (e.g., by culling or enrichment).[14,23,32,33]

In order to reduce the control perturbation, an interesting generalization of PBC methods was introduced by Polyak and Maslov.[5,6] Their main idea consists of using higher iterations of the map f as predicted states. To stabilize a fixed point, this method has the form

$$x_{n+1}=f(x_n)+(-1)^{m+1}\varepsilon(f^{m+1}(x_n)-f^m(x_n)), \tag{6}$$

where m is a positive integer and $\varepsilon>0$. Notice that the case $m=0$ is equivalent to Eq. (4), with the usual notation $f^0(x):=x$.

Now, assume that f is a map satisfying conditions (A1)–(A4) and K is an unstable equilibrium of Eq. (1). Then, $f'(K)<-1$, and it is easy to check that K is asymptotically stable for Eq. (6) if

$$\varepsilon>\left(\frac{-1}{f'(K)}\right)^m\frac{f'(K)+1}{f'(K)-1}:=\varepsilon_m.$$

Then, the bifurcation point α_0 defined in the statement of Corollary 1 for Eq. (4) is replaced by $\varepsilon_m=(-1/f'(K))^m\alpha_0$ when control (6) is applied. For example, for the chaotic exponential map $f(x)=xe^{3(1-x)}$, we have $f'(K)=2$ and $\alpha_0=1/3$. Thus, $\varepsilon_m=\alpha_0/2^m$, which means that the perturbation of f in Eq. (6) can be made as small as desired, choosing a sufficiently big value of m.

Some disadvantages of this control method are that (a) it is computationally more expensive than Eq. (4), especially for large m, and (b) it is not ensured (see the example below) that the modified map

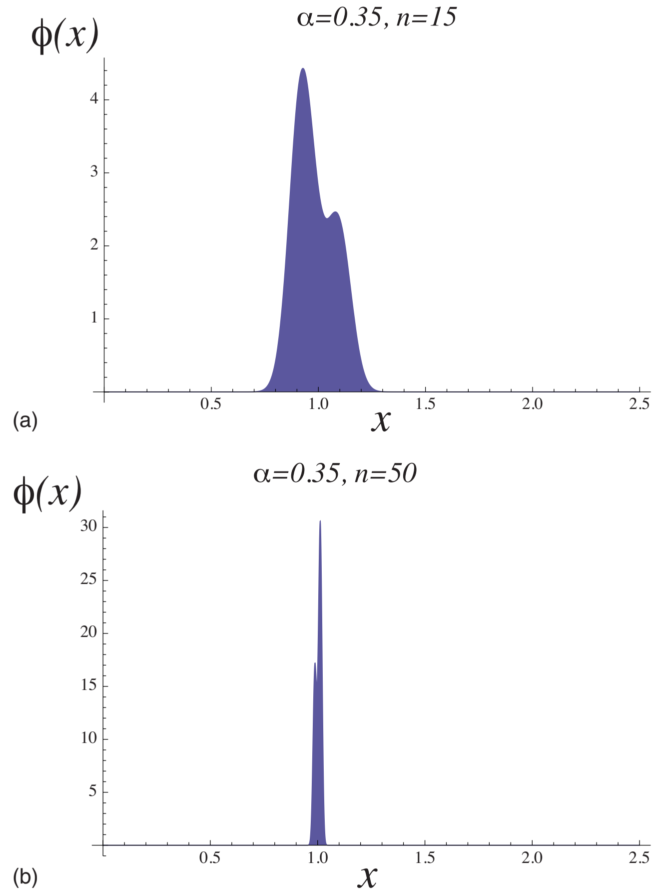

FIG. 6. (Color online) Evolution of the approximation of the probability density function for the random variable corresponding to a sample of 2000 random initial conditions in the interval [0,2.5] after 15 and 50 iterations under the controlled system (5) with $r=3$ and $\alpha=0.35$.

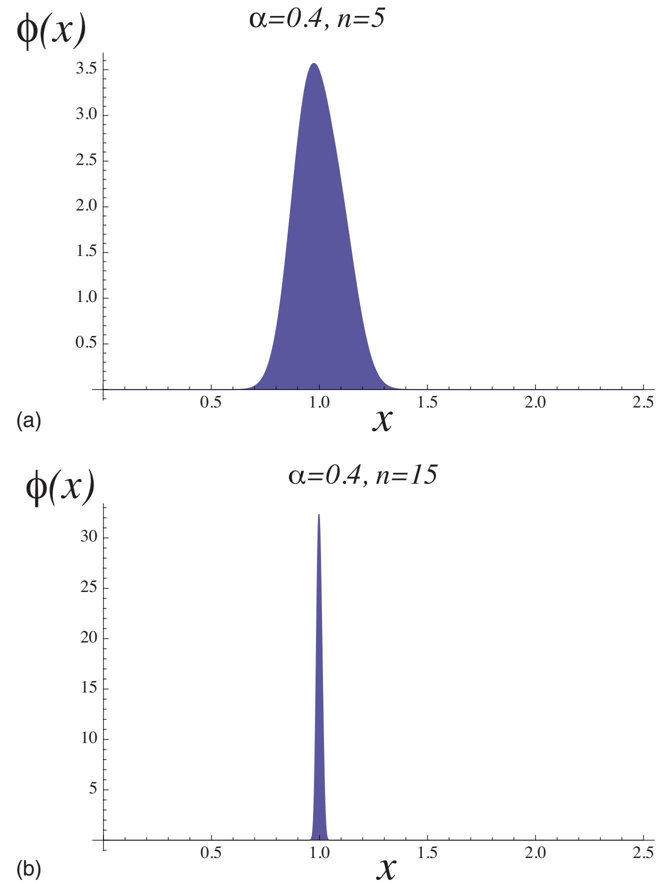

FIG. 7. (Color online) Evolution of the approximation of the probability density function for the random variable corresponding to a sample of 2000 random initial conditions in the interval [0,2.5] after 5 and 15 iterations under the controlled system (5) with $r=3$ and $\alpha=0.4$.

$$F_{\varepsilon,m}(x) = f(x) + (-1)^{m+1}\varepsilon(f^{m+1}(x) - f^{m}(x))$$

is well-defined for $m \geq 2$. We notice that the scheme for $m=1$ is still well-defined for $\varepsilon \in [0,1]$ because

$$F_{\varepsilon,1}(x) = f(x) + \varepsilon(f^2(x) - f(x)) = (1-\varepsilon)f(x) + \varepsilon f^2(x)$$

is a convex combination of $f(x)$ and $f^2(x)$.

A natural question is whether or not the global stability result valid for $m=0$ still holds for Eq. (6) when $m>0$. In Ref. 6, it is claimed that this is an expected property, confirmed by simulation results using the quadratic family. However, we show that, in general, the asymptotic stability of the equilibrium is only a local property. Indeed, even when the equilibrium is locally stable, system (6) can have other stable periodic orbits, including new fixed points.

For example, for $m=2$ and the chaotic quadratic map $f(x)=3.9x(1-x)$, the fixed point $K=0.7435$ is locally stable for Eq. (6) when $\varepsilon > \varepsilon_2 = 0.086$. However, for any ε arbitrarily close to ε_2, there is a new fixed point K_1 in such a way that all orbits starting at a point $x_0 < K_1$ converge to zero. This means that the control method induces an Allee effect. In Fig. 9(a), we plot the function

$$F_{\varepsilon,2}(x) = f(x) - \varepsilon(f^3(x) - f^2(x)),$$

with $\varepsilon=0.1$ (solid line); for the sake of comparison, we also include the graph of function $f(x)=3.9x(1-x)$ corresponding to the uncontrolled system (dashed line) and the line $y=x$. A magnification of the graph of $F_{\varepsilon,2}$ between 0 and 0.05 is included in Fig. 9(b) to emphasize the new positive fixed point $K_1 \approx 0.016\,06$ and the Allee effect.

In Appendix B, we prove that a global stability result holds for $m=1$ in Eq. (6) when f is the quadratic map, but this is an exception. In general, even for $m=1$, the conclusions of Corollary 1 and Theorem 1 are no longer valid.

Consider $m=1$ in Eq. (6), that is,

$$x_{n+1} = f(x_n) + \varepsilon(f^2(x_n) - f(x_n)), \quad (7)$$

with $f(x)=xe^{3(1-x)}$. The family of maps $F_{\varepsilon,1}(x)=f(x) + \varepsilon(f^2(x)-f(x))$ undergoes a period-doubling bifurcation at $\varepsilon = 1/6 \approx 0.166$, where $F'_{\varepsilon,1}(1)=-1$. The fixed point $K=1$ is locally stable for $\varepsilon > 1/6$, but this equilibrium coexists with an attracting 2-cycle until this cycle disappears in a tangent bifurcation at $\varepsilon \approx 0.1958$. After this value, the positive fixed point seems to be globally attracting, but only until a new fixed point appears at $\varepsilon \approx 0.4605$. In Fig. 10, we show the bifurcation diagram for $\varepsilon \in [0,0.3]$. The unstable fixed point is represented by a dashed line between $\varepsilon=0$ and $\varepsilon=1/6$. The other dashed lines represent an unstable 2-cycle that disappears (with the attracting 2-cycle) after a saddle-node bifurcation at $\varepsilon \approx 0.1958$.

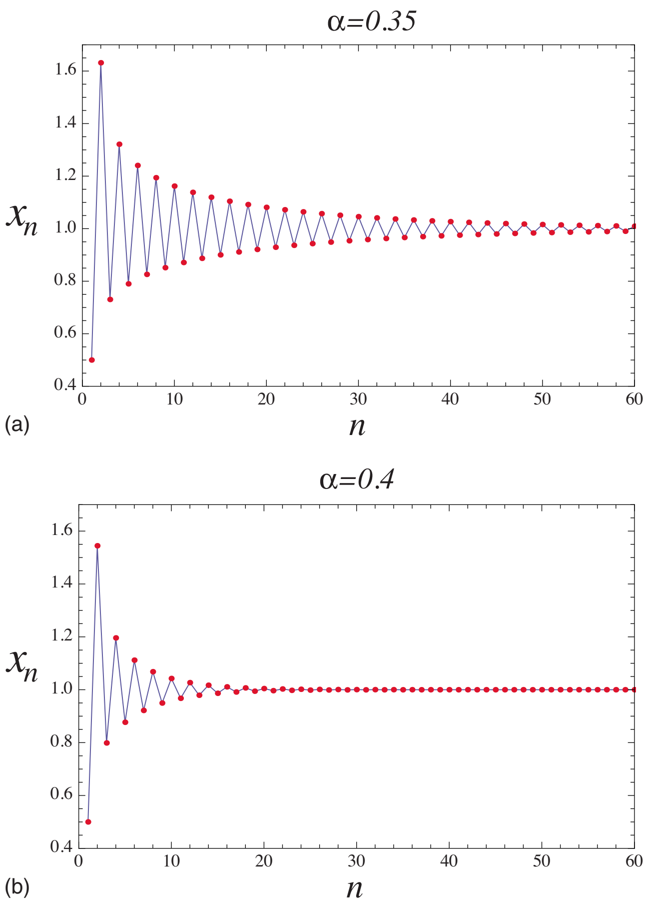

FIG. 8. (Color online) Time series for the controlled system (5) with $r=3$; the values of the control parameter are $\alpha=0.35$ and $\alpha=0.4$. In both cases, the initial point is $x_0=0.5$, and 60 iterations were made.

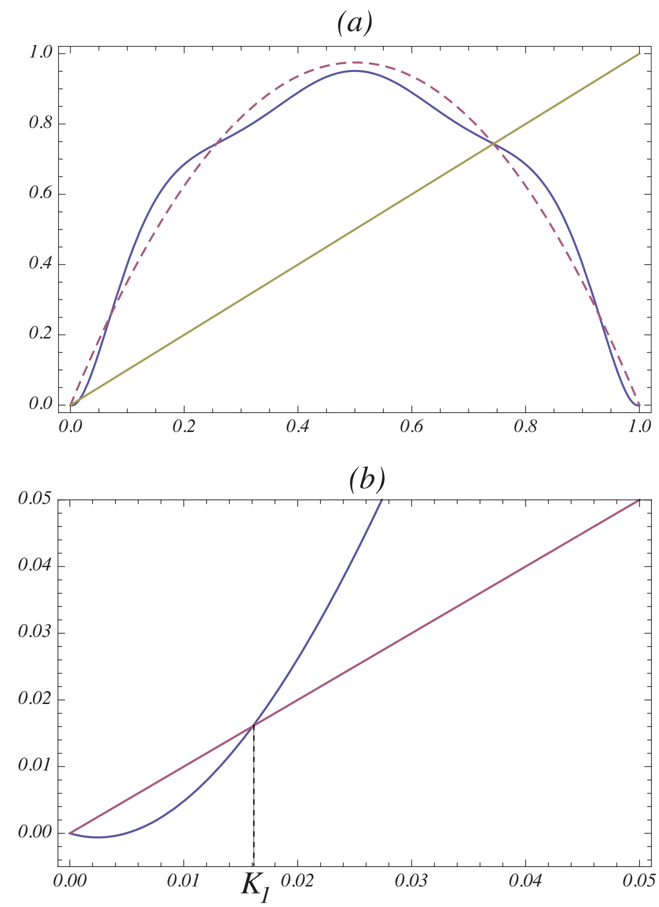

FIG. 9. (Color online) (a) Graphs of functions $f(x)=3.9x(1-x)$ (dashed line) and $F_{\varepsilon,2}(x)=f(x)-\varepsilon(f^3(x)-f^2(x))$, with $\varepsilon=0.1$ (solid line). The fixed points are obtained as the intersections with the line $y=x$. (b) Magnification of the graph of $F_{\varepsilon,2}$ between 0 and 0.05; the fixed point $K_1\approx 0.016\,06$ induces an Allee effect: all initial conditions below K_1 are driven to zero under successive iterations of $F_{\varepsilon,2}$.

V. CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

We have analyzed the global stability properties of some methods of predictive control. We particularly focused on the optimal control function introduced by de Sousa Vieira and Lichtenberg.[2] Based on their analysis of the logistic map, they affirmed that this method has good global stabilizing properties, thus decreasing the sensitivity to noise. Our main result (Theorem 1) comes up with a rigorous analytic proof of this observation for a family of maps commonly used in modeling of discrete-time systems. Moreover, we have given a positive answer to some questions related to the preservation of the period-doubling route to chaos in the controlled system posed in Ref. 7.

We have reviewed as well the local and global properties of stabilization of fixed points of more general PBC methods. An important property of these methods is that they do not increase the dimension of the system; thus, the study of local stability is easier than in other control techniques. However, our study highlights the difficulty of obtaining global stability results when using higher iterations of the map f in the control scheme.

de Sousa Vieira and Lichtenberg proposed a generalization of their optimal method to stabilize UPOs with prime period $T>1$ by using the map $F_{\alpha,T}(x)=f^T(x)-\alpha(f^T(x)-x)$. Assume that we aim to stabilize an UPO of period $T=2^m$ of a chaotic map f using this method. An interesting problem to study is whether or not it is possible to get a globally stable T-periodic orbit in the sense that it attracts all initial conditions, which are not an element of the other UPOs. Numerical simulations suggest that this is true for $T=2$ and f satis-

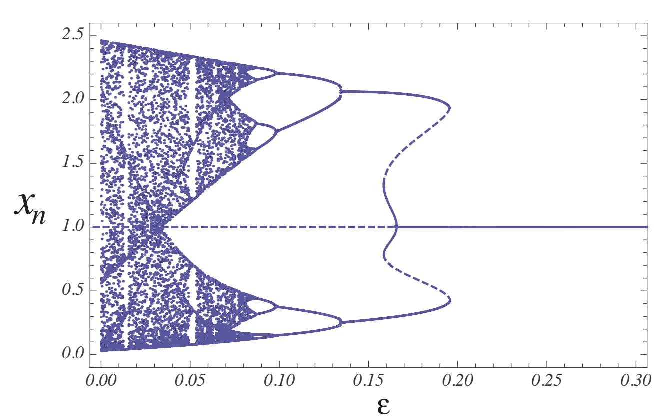

FIG. 10. (Color online) Bifurcation diagram for the controlled system (7) with $f(x)=xe^{3(1-x)}$ and ε as the bifurcation parameter. The fixed point $K=1$ is locally asymptotically stable but not globally stable between 0.166 and 0.1958. The dashed lines indicate instability.

fying conditions (A1)–(A4); to find a proof of this result would be a nice improvement of Theorem 1. However, the answer to this problem is, in general, negative for $T>2$. For example, the quadratic map $f(x)=4x(1-x)$ has two distinct four-periodic orbits with the same multiplier $\mu=-16$; thus, global stabilization of one of them is not possible with this control scheme.

An additional application of the new results proved in this paper is that the controlled map $F_\alpha(x)=\alpha x+(1-\alpha)f(x)$ is used as a model in population dynamics when a probability of surviving the reproductive season (iteroparous populations) is assumed. If f means the nonlinear density growth of the population, the model

$$x_{n+1} = F_\alpha(x_n) = \alpha x_n + (1-\alpha)f(x_n) \tag{8}$$

assumes that a fraction α of energy is invested into adult survivorship rather than reproduction (for more details, see Refs. 10 and 30). Thus, Theorem 1 shows that even if f is chaotic and the survivorship parameter α is large enough, the population is stabilized into the positive equilibrium regardless the initial size of the population. We notice that organisms might evolve into such strategy (survival rather than reproduction) under bad environmental conditions; this means that a population whose fecundity encodes for chaos can stabilize itself at its equilibrium density. For a nice discussion on the evolutionary advantage of self-control, see Ref. 34. As stated there, this makes a difference between ecological systems and physical or chemical systems.

Regarding the application of PBC methods to population problems, an important drawback is that their implementation requires full knowledge of the model equation. Although the exact form of the original map is difficult to obtain, in some cases good census data are available (see, e.g., Ref. 35, where census data of a freshwater fish population were collected every fourth month over 20 years); this allows one to derive a stock-recruitment relationship that fits well into some known function, such as the Ricker map. Then, the model can be validated, and the control method could be implemented to stabilize the population. On the other hand, while another admitted drawback in the application of predictive control in the process industries is that a quick computation upon observation must be performed (see, e.g., Ref. 36), in the setting of population dynamics, the time scales are different; for instance, in a fish farm, the control (4) can provide an assessment on whether next season harvesting should be incremented or, conversely, some enrichment is needed to stabilize the stock size.

We mention that the control method (4) can help not only to stabilize the population but also to avoid extinction (see Ref. 30). For further discussion and references on the application of control methods to population problems, especially to prevent extinction, we refer to Ref. 33. Related results on global stability in Eq. (8) for particular choices of f were proved in Refs. 37 and 38.

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APPENDIX A: PROOF OF THE MAIN RESULT

Proof of Proposition 1: Since $f'(x)<0$ and $(Sf)(x)<0$ for all $x\in J$, we have

$$2\frac{f'''(x)}{f'(x)} < 3\left(\frac{f''(x)}{f'(x)}\right)^2 \Rightarrow 2f'''(x) > 3\frac{(f''(x))^2}{f'(x)}$$

for all $x\in J$. Using that $F'_\alpha(x)<0$ on J, the previous inequality, and the obvious relations

$$F'_\alpha(x) = \alpha + (1-\alpha)f'(x), \quad F''_\alpha(x) = (1-\alpha)f''(x),$$

$$F'''_\alpha(x) = (1-\alpha)f'''(x),$$

we get, for all $x\in J$,

$$2F'''_\alpha(x) = 2(1-\alpha)f'''(x) > 3(1-\alpha)\frac{(f''(x))^2}{f'(x)}$$

$$= 3\frac{(F''_\alpha(x))^2}{F'_\alpha(x)-\alpha} > 3\frac{(F''_\alpha(x))^2}{F'_\alpha(x)}.$$

This inequality implies that $(SF_\alpha)(x)<0$.

To use Proposition 1 to prove our main result, we need the following auxiliary result from Ref. 39.

Lemma 1: (Reference 39, Corollary 2.9) Let $g:(0,b)\to[0,b]$ *be a continuous map with a unique fixed point* K *such that* $(g(x)-x)(x-K)<0$ *for all* $x\neq K$. *Assume that there are points* $0\le a<K<d\le b$ *such that the restriction of* g *to* (a,d) *has at most one turning point and (whenever it makes sense)* $g(x)\le g(a)$ *for every* $x\le a$ *and* $g(x)\ge g(d)$ *for every* $x\ge d$. *If* g *is decreasing at* K, *assume additionally that* $(Sg)\times(x)<0$ *for all* $x\in(a,d)$ *except at most one critical point of* g, *and* $-1\le g'(K)<0$. *Then* K *is a global attractor of* g.

Proof of Theorem 1: Notice that $f'(0)>1$ implies that $F'_\alpha(0)>1$ for all $\alpha\in[0,1)$. On the other hand, $F'_\alpha(x)=0\Leftrightarrow f'(x)=-\alpha/(1-\alpha)$.

If F_α has no critical points, then it is strictly increasing, and therefore K is a global attractor because $F_\alpha(x)>x$ for $x\in(0,K)$ and $F_\alpha(x)<x$ for $x\in(K,b)$. Notice that this happens when $f'(x)>-\alpha/(1-\alpha)$ for all $x\in(0,b)$.

Next, since $(Sf)(x)<0$ for all $x\neq c$, f can have at most one inflexion point in (c,b). First we consider the case when f has no inflexion points in (c,b). Then f' is strictly decreasing in (c,b), and there is at most one point $c_1>c$ such that $F'_\alpha(c_1)=0$. Clearly, c_1 is a local maximum of F_α since $F'_\alpha(x)>0$ on $(0,c_1)$ and $F'_\alpha(x)<0$ on (c_1,b). If $c_1>K$, then $F'_\alpha(K)>0$, and it follows that K is globally attracting. Thus, we can assume that $c<c_1<K$. Since $F_\alpha(x)\le F_\alpha(c_1)$ for all $x\in(0,c_1]$ and, by Proposition 1, $(SF_\alpha)(x)<0$ on (c_1,b), an application of Lemma 1 with $a=c_1$, $d=b$ proves that K is a global attractor of F_α.

It remains to consider the case when f has an inflexion point γ in (c,b). It is clear that f' attains a global minimum at $f'(\gamma)$. If $f'(\gamma)\geq-\alpha/(1-\alpha)$, then F_α is strictly increasing, and therefore, K is a global attractor. Thus, we assume that $f'(\gamma)<-\alpha/(1-\alpha)$. In this case, there are at least one, and at most two critical points $c_1<\gamma<c_2$ of F_α. If there is only one, it is a local maximum, and this case is solved as the previous one. If there are two critical points, then F_α is increasing on $(0,c_1)\cup(c_2,b)$ and decreasing on (c_1,c_2). Thus, we can apply again Lemma 1, with $a=c_1$, $d=c_2$. The proof is complete.

APPENDIX B: GLOBAL STABILIZATION FOR THE QUADRATIC MAP

Consider the quadratic map $f(x)=rx(1-x)$; f maps the interval $[0,1]$ into itself if $0\leq r\leq 4$. It is well known that the positive fixed point $K=1-1/r$ is globally stable for $1<r\leq 3$ and unstable for $r>3$. Here, we only consider the unstable case.

Proposition 2: The PBC scheme (7) stabilizes locally the positive equilibrium of $f(x)=rx(1-x)$ *for all* $\varepsilon>\varepsilon_1(r)=(r-3)/(r^2-3r+2)$. *Moreover,* K *is a global attractor on* $[0,1]$ *for all* $\varepsilon\in(\varepsilon_1(r),\varepsilon_2(r))$, *where* $\varepsilon_2(r)=4/(r-1)^2$.

Proof: As mentioned in Sec. IV, the modified map $F_\varepsilon(x)=f(x)+\varepsilon(f^2(x)-f(x))$ maps $[0,1]$ into $[0,1]$, and K is asymptotically stable for the controlled system if $\varepsilon>\varepsilon_1(r)$. Moreover, it is easy to check that $K=1-1/r$ is the unique positive fixed point of F_ε if and only if $\varepsilon<\varepsilon_2(r)=4/(r-1)^2$.

In order to apply Lemma 1, we first prove that $(SF_\varepsilon)(x)<0$ for all noncritical points of F_ε. Indeed, notice that $F_\varepsilon(x)=(H\circ f)(x)$, where $H(x)=x+\varepsilon(f(x)-x)$. Since f and H are quadratic maps, they have a negative Schwarzian derivative. Thus, the result follows from the composition rule $(S(H\circ f))(x)=(SH)(f(x))(f'(x))^2+(Sf)(x)$ (see, e.g., Ref. 12, Theorem 2.1).

Next, observe that F_ε has either only the critical point $c=1/2$ (local maximum) or three critical points $c_1<c<c_2$; in the latest case, $F_\varepsilon(c)$ is a local minimum and $F_\varepsilon(c_1)=F_\varepsilon(c_2)$ are local maxima.

If there is only one critical point, the result follows from Singer's theorem since F_ε is unimodal. If there are three critical points, we complete the proof, applying Lemma 1 to the interval $(a,d)=(c_2,1)$ if $K>c_2$ or to the interval $(a,d)=(c_1,c_2)$ if $K<c_2$.

[1]T. Ushio and S. Yamamoto, Phys. Lett. A **264**, 30 (1999).
[2]M. de Sousa Vieira and A. J. Lichtenberg, Phys. Rev. E **54**, 1200 (1996).
[3]K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[4]J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier, Phys. Rev. E **50**, 3245 (1994).
[5]B. T. Polyak, Autom. Remote Control (Engl. Transl.) **66**, 1791 (2005).
[6]B. T. Polyak and V. P. Maslov, "Controlling chaos by predictive control," in Proceedings of the 16th World Congress of IFAC, Praha, 2005.
[7]J. McGuire, M. T. Batchelor, and B. Davies, Phys. Lett. A **233**, 361 (1997).
[8]S. Boccaletti, C. Grebogi, Y.-C. Lai, H. Mancini, and D. Maza, Phys. Rep. **329**, 103 (2000).
[9]T. S. Bellows, J. Anim. Ecol. **50**, 139 (1981).
[10]H. R. Thieme, *Mathematics in Population Biology*, Princeton Series in Theoretical and Computational Biology (Princeton University Press, Princeton, 2003).
[11]R. M. May, Nature (London) **261**, 459 (1976).
[12]D. Singer, SIAM J. Appl. Math. **35**, 260 (1978).
[13]E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[14]R. V. Solé, J. G. P. Gamarra, M. Ginovart, and D. López, Bull. Math. Biol. **61**, 1187 (1999).
[15]S. Parthasarathy and S. Sinha, Phys. Rev. E **51**, 6239 (1995).
[16]S. Gueron, Phys. Rev. E **57**, 3645 (1998).
[17]S. J. Schreiber, J. Math. Biol. **42**, 239 (2001).
[18]E. Liz, Phys. Lett. A **374**, 725 (2010).
[19]S. Sinha and D. Biswas, Phys. Rev. Lett. **71**, 2010 (1993).
[20]L. Glass and W. Zeng, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 1061 (1994).
[21]S. Sinha, Phys. Rev. E **49**, 4832 (1994).
[22]S. Sinha, Phys. Rev. E **63**, 036212 (2001).
[23]F. M. Hilker and F. H. Westerhoff, Phys. Rev. E **73**, 052901 (2006).
[24]T. Ushio, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **43**, 815 (1996).
[25]P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Progress in Physics (Birkhäuser, Boston, 1980).
[26]A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak, and V. V. Fedorenko, *Dynamics of One-Dimensional Maps*, Mathematics and Its Applications (Kluwer Academic, Dordrecht, 1997).
[27]S. H. Strogatz, *Nonlinear Dynamics and Chaos*, Studies in Nonlinearity (Westview, Cambridge, 1994).
[28]C. W. Clark, *Mathematical Bioeconomics*, Pure and Applied Mathematics, 2nd ed. (Wiley, New York, 1990).
[29]F. Brauer and C. Castillo-Chávez, *Mathematical Models in Population Biology and Epidemiology*, Texts in Applied Mathematics (Springer-Verlag, New York, 2001).
[30]E. Liz, "Complex dynamics of survival and extinction in simple population models with harvesting," Theor. Ecol. (2009).
[31]H. A. El-Morshedy, V. Jiménez López, and E. Liz, Nonlinear Anal.: Real World Appl. **9**, 776 (2008).
[32]F. M. Hilker and F. H. Westerhoff, Phys. Lett. A **362**, 407 (2007).
[33]F. M. Hilker and F. H. Westerhoff, Am. Nat. **170**, 232 (2007).
[34]M. Doebeli, Philos. Trans. R. Soc. London, Ser. B **254**, 281 (1993).
[35]J. Lobón-Cerviá, Can. J. Fish. Aquat. Sci. **64**, 1429 (2007).
[36]D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, Automatica **36**, 789 (2000).
[37]E. Liz, Discrete Contin. Dyn. Syst., Ser. B **7**, 191 (2007).
[38]E. Liz, J. Math. Anal. Appl. **330**, 740 (2007).
[39]H. A. El-Morshedy and V. Jiménez López, J. Difference Equ, Appl. **14**, 391 (2008).