Proceedings of the Edinburgh Mathematical Society **Submitted Paper**
Paper 30 March 2004

AN EIGENVALUE PROBLEM FOR NON-BOUNDED QUASI-LINEAR OPERATOR

José Carmona[1] and Antonio Suárez[2]

[1] *Dpto. de Álgebra y Análisis Matemático, Facultad de Ciencias, Cañada de San Urbano, Almería, Spain*
e-mail: jcarmona@ual.es

[2] *Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Sevilla, Spain*
e-mail: suarez@us.es

(Received)

Abstract In this paper we study the eigenvalues associated with a positive eigenfunction of a quasi-linear elliptic problem with a not necessarily bounded operator. For that, we use the bifurcation theory and obtain the existence of positive solution for a range of values of the bifurcation parameter.

AMS 2000 *Mathematics subject classification:* Primary 35J60, 35J25
Secondary 35D05

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and let $A(x,s)$ be a real symmetric matrix which coefficients, $a_{ij} : \overline{\Omega}\times\mathbb{R}_0^+ \to \mathbb{R}$, are Carathéodory functions.

We assume that there exists a positive constant α satisfying for every $(x,s,\xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$,

$$A(x,s)\xi\cdot\xi \geq \alpha|\xi|^2. \tag{A_1}$$

In this paper we analyze the nonlinear eigenvalue problem

$$\left\{\begin{array}{rcll} -\operatorname{div}(A(x,u)\nabla u) & = & \lambda u, & x \in \Omega, \\ u & = & 0, & x \in \partial\Omega, \end{array}\right. \tag{P_λ}$$

where, we say that λ is an eigenvalue for this problem if (P_λ) admits a positive and nontrivial solution, that is, if there exists $u \in H_0^1(\Omega)$, $u \geq 0$, $u \not\equiv 0$, such that $A(x,u)\nabla u \in (L^2(\Omega))^N$ and

$$\int_\Omega A(x,u)\nabla u\cdot\nabla v = \lambda\int_\Omega uv,\ \forall v \in H_0^1(\Omega).$$

In addition to the interest itself in the study of (P_λ), this kind of equation has been used to model a species inhabiting in Ω where its diffusion depends on the density of the species, which arises in more realistic models, see [3] and references therein.

Problem (P_λ) is well known when A does not depend on s, i.e., when $A(x,s) = B(x)$ with $B = (b_{ij})$ and $b_{ij} \in L^\infty(\Omega)$, $b_{ij} \geq b_0 > 0$ in Ω. In this case, there exists the principal eigenvalue, denoted by $\lambda_1(B)$, for the problem:

$$\left\{ \begin{array}{rcll} -\operatorname{div}(B(x)\nabla u) & = & \lambda u, & x \in \Omega, \\ u & = & 0, & x \in \partial\Omega, \end{array} \right. \tag{1.1}$$

being the unique eigenvalue with a positive eigenfunction, see for instance [5].

In [2], assuming that A satisfies (A_1) and

$$|A(x,s)| \leq \beta, \quad \text{for each } (x,s) \in \Omega \times \mathbb{R}, \tag{A_2}$$

the author proved that for each $r > 0$, there exists $\lambda_r > 0$ and a positive solution $u_r \in H_0^1(\Omega)$, of (P_{λ_r}) such that $\|u_r\|_2 = r$. Moreover, denoting by

$$\lambda_0 := \lambda_1(A(x,0)),$$

he showed that if $r \to 0$, then $\lambda_r \to \lambda_0$ and $\dfrac{u_r}{r}$ converges to a positive eigenfunction associated to λ_0 in $H_0^1(\Omega)$. Finally, if A also verifies

$$\lim_{s\to\infty} A(x,s) = A_\infty(x), \;\; \text{uniformly in } x \in \Omega, \tag{A_3}$$

then $\lambda_r \to \lambda_\infty$ and $\dfrac{u_r}{r}$ goes to a positive eigenfunction associated to λ_∞ in $H_0^1(\Omega)$ as $r \to \infty$, where

$$\lambda_\infty := \lambda_1(A_\infty(x)).$$

In [4], a slightly modification of (P_λ) is analyzed. Under conditions (A_{1-3}), $\lambda u + h(x)$ for some $0 \leq h \in L^2(\Omega)$ is considered instead of λu. But the arguments used to prove the existence of solution leads to the trivial one in the case $h \equiv 0$.

In [1], assuming in addition the existence of an Osgood function $\omega : \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$|A(x,s_1) - A(x,s_2)| \leq \omega(|s_1 - s_2|), \tag{A_4}$$

for every $(x,s_1),(x,s_2) \in \Omega \times \mathbb{R}$, using a bifurcation analysis, the authors study a more general problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(A(x,u)\nabla u) & = & f(\lambda,x,s), & x \in \Omega, \\ u & = & 0, & x \in \partial\Omega, \end{array} \right.$$

for $f : \mathbb{R} \times \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ and A satisfying (A_{1-4}). In the particular case $f(\lambda,x,s) = \lambda s$, from their results it can be deduced the existence of an unbounded continuum (closed and connected subset) of positive solutions bifurcating from the trivial solution at $\lambda = \lambda_0$ and meeting with infinity at the value $\lambda = \lambda_\infty$. Thus, as a consequence, there exists positive solution of (P_λ) for $\lambda \in (\lambda_0, \lambda_\infty)$ or $(\lambda_\infty, \lambda_0)$. In the following section we complete this study for A satisfying (A_{1-4}) by giving sufficient conditions for the uniqueness of positive solution.

The main goal of this work (see Section 3) is to analyze (P_λ) when A is not necessarily bounded and/or does not satisfy (A_3). In this case, we show that there exists an unbounded continuum of positive solutions bifurcating from the trivial one at $\lambda = \lambda_0$. If, in addition there exists a continuous function $g : \mathbb{R}_0^+ \to \mathbb{R}$, with $\lim\limits_{s\to+\infty} g(s) = +\infty$, satisfying for every $(x, s, \xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$,

$$A(x, s)\xi \cdot \xi \geq g(s)|\xi|^2 \geq \alpha|\xi|^2. \tag{A_∞}$$

then, the bifurcation from infinity at $\lambda = \lambda_\infty$ (which exists in the bounded case) “disappears”. Specifically, there exists at least a positive solution u_λ for $\lambda \in (\lambda_0, \infty)$ and $\|u_\lambda\| \to \infty$ as $\lambda \to \infty$. However, if A is bounded in a subset of Ω, then again a bifurcation to infinity exists.

Along the work we will use the following notation:

- $H_0^1(\Omega)$ and $E = C_0(\overline{\Omega})$ are the usual Sobolev space and the space of the continuous functions in $\overline{\Omega}$ vanishing on $\partial\Omega$ endowed with the norms $\|u\| = \|\nabla u\|_2$ and $\|u\|_0 = \sup_\Omega |u|$, respectively.
- $\mathrm{cl}(D)$ denotes the closure of the set D.
- $\mathcal{S}$ denotes the set

  $$\mathcal{S} = \mathrm{cl}\{(\lambda, u) \in \mathbb{R} \times E :\ u \text{ is solution for } (P_\lambda),\ u \geq 0,\ u \not\equiv 0\}.$$

  Any continuum subset of $\mathcal{S}$ will be called a continuum of positive solutions of (P_λ), although it may contain the trivial solution $(\lambda, 0)$ for some value of $\lambda > 0$.
- I will denote both the identity matrix and the identity operator.
- Given square matrices B_1, B_2 we say that $B_1 > 0$ (respect. $B_1 \geq 0$) if the quadratic form induced by B_1 is definite positive (respect. semidefinite positive). We say that $B_1 < B_2$ (respect. $B_1 \leq B_2$) if $B_2 - B_1 > 0$ (respect. $B_2 - B_1 \geq 0$).
- The map $\mathrm{Proj}_\mathbb{R} : \mathbb{R} \times E \mapsto \mathbb{R}$ stands for the projection of the product space $\mathbb{R} \times E$ onto $\mathbb{R}$.

2. The case of bounded matrices A

In order to study problem (P_λ), let us recall that, for matrices A satisfying $(A_{1,2})$, if $u \in H_0^1(\Omega)$ is solution of (P_λ) then using the De Giorgi-Stampacchia Theorem ([8, Théorème 7.3] and [6, Theorem I] or [7, Theorem 8.29]), $u \in C^{0,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$. Moreover, if the coefficients of the matrix A satisfy

$$a_{ij} \in C^{1,\gamma'}(\overline{\Omega} \times \mathbb{R}), \text{ for some } 0 < \gamma' < 1, \tag{2.1}$$

then by Theorem 15.17 in [7] we have that $u \in C_0^{2,\gamma\gamma'}(\overline{\Omega})$.

We also recall that for every $(\lambda, u) \in \mathcal{S}$ with $u \in \mathcal{C}^1(\overline{\Omega})$ and $u \not\equiv 0$, using the Hopf maximum principle, we have that $u > 0$ in Ω and the normal exterior derivative $\frac{\partial u}{\partial n_e}$ is negative in $\partial\Omega$.

The following lemma provides us necessary conditions in $\lambda \in \mathbb{R}$ for which (P_λ) admits solution in some special cases.

Lemma 2.1. *Assume $(A_{1,3})$ and that (P_λ) admits a positive solution. Then*

1. *$\lambda_0 \le \lambda$ (respect. $<, \ge, >$) if for every $s \in \mathbb{R}^+$, $A(x,0) \le A(x,s)$ (respect. $<, \ge, >$).*
2. *$\lambda_\infty \ge \lambda$ (respect. $>, \le, <$) if for every $s \in \mathbb{R}^+$, $A_\infty(x) \ge A(x,s)$ (respect. $>, \le, <$).*

Proof. The result follows from the fact that for given symmetric matrices $B_1(x)$, $B_2(x)$ for which there exist $\lambda_1(B_1)$ and $\lambda_1(B_2)$, with $0 < B_1 \le B_2$ then

$$\lambda_1(B_1) = \inf\left\{\int_\Omega B_1(x)\nabla u \cdot \nabla u,\ u \in H_0^1(\Omega),\ \|u\|_2 = 1\right\} \le \lambda_1(B_2).$$

Thus, if $u \in H_0^1(\Omega)$ is a solution of (P_λ), we conclude by taking into account that $\lambda = \lambda_1(A(x,u))$. □

The main result of this section is the following:

Theorem 2.2. *Assume (A_{1-4}). We have that λ_0 and λ_∞ are the only bifurcation points from the trivial solution and from infinity, respectively, and there exists a continuum $\Sigma \subset \mathcal{S}$ of positive solutions meeting $(\lambda_0, 0)$ and (λ_∞, ∞), in particular, (P_λ) possesses a positive solution for every $\lambda \in (\lambda_0, \lambda_\infty)$ or $\lambda \in (\lambda_\infty, \lambda_0)$. Moreover,*

- *the bifurcation from λ_0 is subcritical (resp. supercritical) if there exists $s_0 > 0$ such that*
$$A(x,s) < A(x,0),\ (\text{respect. } A(x,s) > A(x,0)),\ \forall s \in (0, s_0),$$
- *the bifurcation from λ_∞ is subcritical (resp. supercritical) if*
$$A(x,s) < A_\infty(x),\ (\text{resp. } A(x,s) > A_\infty(x)),\ \forall s \in \mathbb{R}^+.$$

Furthermore,

- *if $A(x,0) < A(x,s) < A_\infty(x)$ for every $s \in \mathbb{R}^+$, then there exists nontrivial solution for (P_λ) if, and only if, $\lambda \in (\lambda_0, \lambda_\infty)$, in particular $Proj_{\mathbb{R}}\Sigma = [\lambda_0, \lambda_\infty)$. If, in addition, $A(x,s)$ is increasing in s and it verifies (2.1), the solution is unique.*
- *If $A(x,0) > A(x,s) > A_\infty(x)$ for every $s \in \mathbb{R}^+$, then there exists nontrivial solution for (P_λ) if, and only if, $\lambda \in (\lambda_\infty, \lambda_0)$, in particular $Proj_{\mathbb{R}}\Sigma = (\lambda_\infty, \lambda_0]$.*

Proof. The existence of the continuum Σ of positive solutions follows by Theorem 5.1 in [1], and so the existence of positive solutions for every λ in $(\lambda_0, \lambda_\infty)$ or in $(\lambda_\infty, \lambda_0)$.

The description $Proj_{\mathbb{R}}\Sigma$, in the cases $A(x,0) < A(x,s) < A_\infty(x)$ or $A(x,0) < A(x,s) < A_\infty(x)$ for every $s \in \mathbb{R}^+$, follows directly from Lemma 2.1. Moreover, arguing as in that lemma we get the laterality of the bifurcations.

Now, assume that $A(x,s)$ is increasing in s and (2.1) is satisfied. In order to prove the uniqueness of solution for (P_λ), let us suppose that there exist $\lambda \in (\lambda_0, \lambda_\infty)$ and $u_1, u_2 \in E$, solutions of (P_λ) with $u_1 \not\equiv u_2$. We claim that u_1, u_2 can be chosen such that $u_1 \le u_2$. Indeed, this is a consequence of the existence of a sequence (λ_n, u_n) with $\lambda_n \to \lambda_0$ and $u_n \to 0$ in E. In fact, by regularity results, $u_n \to 0$ in $\mathcal{C}^1(\overline{\Omega})$. Thus, for $\lambda_n < \lambda$, u_n is a subsolution for (P_λ) and for large n, $u_n \le \min\{u_1, u_2\}$. Then, by the sub and supersolution method, there exits $w \in E$ solution of (P_λ) with

$$u_n \le w \le u_1, \quad u_n \le w \le u_2.$$

This implies that $w \not\equiv u_1$ or $w \not\equiv u_2$, and the claim is proved by taking $u_1 = w$ and $u_2 = u_i$ for some $i = 1, 2$.

Now we take $v = \frac{u_2^2}{u_1}$ as test function in the equation satisfied by u_1 and $v = u_2$ in that satisfied by u_2. Thus, subtracting both equalities we have that:

$$\begin{aligned} 0 &= \int_\Omega A(x,u_1)\nabla u_1 \cdot \nabla\left(\frac{u_2^2}{u_1}\right) - \int_\Omega A(x,u_2)\nabla u_2 \cdot \nabla u_2 \\ &= -\int_\Omega A(x,u_1)\left(\frac{u_2}{u_1}\nabla u_1 - \nabla u_2\right)\cdot\left(\frac{u_2}{u_1}\nabla u_1 - \nabla u_2\right) \\ &\quad -\int_\Omega \left(A(x,u_2) - A(x,u_1)\right)\nabla u_2 \cdot \nabla u_2 < 0. \end{aligned}$$

This contradiction gives the uniqueness. □

3. The case of unbounded matrices A

In this section, we study (P_λ) when A is not necessarily bounded and does not satisfy (A_3). We prove firstly that every solution of (P_λ) is bounded. More precisely we have

Lemma 3.1. *Let $A(x,s)$ satisfy (A_1) and $u \in H_0^1(\Omega)$ be a solution of (P_λ), then $u \in E$. Moreover, there exist positive constants $c_1, c_2, \gamma_1, \gamma_2$ such that*

$$\|u\|_0^{\gamma_1} \le c_1 + c_2\|u\|^{\gamma_2}. \tag{3.1}$$

Proof. Once we know that $u \in L^\infty(\Omega)$, and $\|u\|_\infty^{\gamma_1} \le c_1 + c_2\|u\|^{\gamma_2}$ for some positive constants $c_1, c_2, \gamma_1, \gamma_2$, then the result follows directly from the De Giorgi-Stampacchia Theorem. Let us prove the $L^\infty(\Omega)$-estimate. We consider for every $k \in \mathbb{R}^+$ the function $G_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by

$$G_k(s) = \begin{cases} 0 & 0 \le s \le k, \\ s - k & s > k. \end{cases}$$

Thus, we can take $v = G_k(u)$ as test function in the weak equation satisfied by u and using (A_1) we have

$$\alpha\|\nabla G_k(u)\|_2^2 \leq \int_\Omega A(x,u)\nabla u \nabla G_k(u) \leq \lambda \int_{\Omega_k} u G_k(u), \tag{3.2}$$

where $\Omega_k \equiv \{x \in \Omega : u(x) > k\}$.

Using the Sobolev and Hölder inequalities, in the case $N > 2$, by (3.2) we yield, for $u \in L^r(\Omega)$ with $r > \frac{2^*}{2^*-1}$, and some positive constant c,

$$\|G_k(u)\|_{2^*}^2 \leq c\|u\|_r \|G_k(u)\|_{2^*} (\text{meas } \Omega_k)^{(1-1/r-1/2^*)}. \tag{3.3}$$

Taking into account that, for every $h > k$, $G_k(u) \geq h - k$ in Ω_h, (3.3) implies that

$$(h-k)(\text{meas } \Omega_h)^{1/2^*} \leq c\|u\|_r (\text{meas } \Omega_k)^{(1-1/r-1/2^*)},$$

or equivalently

$$\text{meas } \Omega_h \leq \frac{c\|u\|_r^{2^*} (\text{meas } \Omega_k)^{2^*-1-2^*/r}}{(h-k)^{2^*}}. \tag{3.4}$$

We can now apply the Stampacchia Lemma ([8, Lemma 4.1]) to deduce that:

i) if $u \in L^r(\Omega)$ with $r > \frac{N}{2}$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq c\|u\|_r$,

ii) if $u \in L^r(\Omega)$ with $r = \frac{N}{2}$, then $u \in L^t(\Omega)$ for $t \in [1, \infty)$ and $\|u\|_t^t \leq c + c'\|u\|_r^t$,

iii) if $u \in L^r(\Omega)$ with $r < \frac{N}{2}$, then $u \in L^t(\Omega)$ for $t = \frac{2^* r}{(2-2^*)r+2^*} - \delta$ and $\delta > 0$ arbitrarily small. Moreover, $\|u\|_t^t \leq c + c'\|u\|_r^{t+\delta}$.

Since $u \in L^{2^*}(\Omega)$ and $2^* > \frac{2^*}{2^*-1}$, we can argue as before for $r_0 = 2^*$. Thus, if $2^* > \frac{N}{2}$ we conclude by item i). In the case $2^* = \frac{N}{2}$ we use item ii) in order to take $r_1 > \frac{N}{2}$ and conclude again by item i). Finally, in the case $2^* < \frac{N}{2}$ we can take

$$r_1 = \frac{2^* r_0}{(2-2^*)r_0 + 2^*} - \delta_1 > r_0.$$

As before, if $r_1 \geq \frac{N}{2}$ we easily conclude. In other case we take

$$r_2 = \frac{2^* r_1}{(2-2^*)r_1 + 2^*} - \delta_2.$$

By an iterative argument we conclude after a finite number of steps. Indeed, in other case, we have that r_n is bounded, where r_n is defined recurrently by

$$\begin{cases} r_0 = 2^* \\ r_{n+1} = \dfrac{2^* r_n}{(2-2^*)r_n + 2^*} - \delta_{n+1}. \end{cases}$$

where $\lim_{n\to\infty}\delta_n = 0$. Moreover, r_n is non decreasing and so it converges to $r \in (2^*, \frac{N}{2}]$ that satisfies

$$r = \frac{2^* r}{(2-2^*)r + 2^*},$$

that is, $2^* = (2-2^*)r + 2^*$, which implies that $r = 0$ and this is a contradiction.

Observe that the estimate (3.1) follows, after this finite number of steps, from estimates in items i)-iii), and the Sobolev embedding.

Finally, in the case $N = 2$ we can choose $r > \frac{q}{q-2}$ for any $q > 2$ and argue as before with 2^* replaced by q. In this case we finish by item i). □

Along this section, we assume, instead of (A_2), that for each $s_0 \in \mathbb{R}^+$ there exists $\beta(s_0)$ such that

$$|A(x,s)| \le \beta(s_0), \tag{$\tilde{A}_2$}$$

for $(x,s) \in \overline{\Omega} \times [0, s_0]$.

We consider the truncated problems

$$\begin{cases} -\operatorname{div}(A(x, T_n(u))\nabla u) &= \lambda u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{cases} \tag{$P_{\lambda,n}$}$$

being $T_n(s)$ the map defined, for each $n \in \mathbb{N}$, by

$$T_n(s) = \begin{cases} s & 0 \le s \le n, \\ n & s > n. \end{cases}$$

By Theorem 2.2, there exist Σ_n unbounded maximal continua of positive solutions such that $(\lambda_0, 0) \in \Sigma_n$ for each $n \in \mathbb{N}$. Now, we can prove

Theorem 3.2. *Suppose that A satisfies $(A_{1,4})$ and $(\tilde{A}_2)$. Then, there exists an unbounded continuum $\Sigma \subset \mathcal{S}$ such that $(\lambda_0, 0) \in \Sigma$.*

Proof. Firstly, we denote by Σ_k^n the connected component of $\Sigma_k \cap (\mathbb{R} \times \overline{B}_n(0))$ containing $(\lambda_0, 0)$. We claim that

$$\Sigma_k^n = \Sigma_n^n \quad \text{for } k \ge n. \tag{3.5}$$

Indeed, if $k \ge n$ and $(\lambda, u) \in \Sigma_k^n$ then u is solution of $(P_{\lambda,n})$. Thus, Σ_k^n is a closed and connected subset of

$$\operatorname{cl}\{(\lambda, u) \in \mathbb{R} \times E : \ u \text{ is solution non-trivial of } (P_{\lambda,n})\}$$

containing $(\lambda_0, 0)$. So, $\Sigma_k^n \subset \Sigma_n$, whence we deduce that $\Sigma_k^n \subset \Sigma_n^n$. We can reason similarly and obtain that $\Sigma_n^n \subset \Sigma_k \cap (\mathbb{R} \times \overline{B}_n(0))$, and so it follows (3.5). So, we get

$$\Sigma_n^n = \lim_k \Sigma_k^n.$$

Therefore, for each $n \in \mathbb{N}$ we have a continuum

$$\Sigma_n^n \subset \operatorname{cl}\{(\lambda, u) \in \mathbb{R} \times E : \ u \text{ is a non-trivial solution of } (P_\lambda)\}$$

containing $(\lambda_0, 0)$ and if $(\lambda, u) \in \Sigma_n^n$ then $\|u\|_0 \leq n$.

Now, we are going to prove that

$$\Sigma_n^n \subset \Sigma_{n+1}^{n+1} \quad \text{for each } n \in \mathbb{N}. \tag{3.6}$$

Indeed, observe that

$$\Sigma_n^n = \Sigma_{n+1}^n \subset \Sigma_{n+1} \cap (\mathbb{R} \times \overline{B}_n(0)) \subset \Sigma_{n+1} \cap (\mathbb{R} \times \overline{B}_{n+1}(0)),$$

so, since Σ_{n+1}^{n+1} is the connected component of $\Sigma_{n+1} \cap (\mathbb{R} \times \overline{B}_{n+1}(0))$ containing $(\lambda_0, 0)$ and Σ_n^n is a connected of such subset containing it, (3.6) follows.

Finally, we show that the set

$$\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n^n$$

satisfies the theorem. Firstly, observe that since Σ_n is unbounded, Σ is also unbounded. Indeed, since $\mathrm{Proj}_{\mathbb{R}} \Sigma_n$ is bounded, so there exists a connected subset of $\Sigma_n \cap (\mathbb{R} \times \overline{B}_n(0))$ containing $(\lambda_0, 0)$ and intersecting with $\mathbb{R} \times \partial \overline{B}_n(0)$ for each $n \in \mathbb{N}$; i.e., for each $n \in \mathbb{N}$ there exists $(\lambda_n, u_n) \in \Sigma_n^n$, with $\|u_n\|_0 = n$.

On the other hand, since Σ_n^n is connected and $(\lambda_0, 0) \in \Sigma_n^n$ for each $n \in \mathbb{N}$, it follows that Σ is connected.

Finally, we will prove that Σ is closed. Let $(\lambda, u) \in \overline{\Sigma}$. Since $\overline{\Sigma}$ is connected, there exists a connected and bounded set $\Sigma' \subset \overline{\Sigma}$ containing $(\lambda_0, 0)$ and (λ, u). Thus, there exists $n \in \mathbb{N}$ such that

$$\Sigma' \subset \mathrm{cl}\{(\lambda, u) \in \mathbb{R} \times E : \ \|u\|_0 \leq n, \ u \text{ is non-trivial solution of } (P_{\lambda,n})\}.$$

In particular, $\Sigma' \subset \Sigma_n \cap (\mathbb{R} \times \overline{B}_n(0))$ whence $\Sigma' \subset \Sigma_n^n$ and so, $(\lambda, u) \in \Sigma_n^n \subset \Sigma$. □

Remark 3.3. 1. We would like to point out that the above result is true even in the case that the limit of $A(x, s)$ does not exist as $s \to \infty$.

2. In the case A bounded in some subset of Ω, then we can conclude that $\mathrm{Proj}_{\mathbb{R}} \Sigma$ is bounded. Indeed, assume that $|A(x, s)| \leq \gamma$ if $x \in B$, where B is a ball such that $B \subset \Omega$, then using the monotony of the principal eigenvalue with respect to the domain, we obtain

$$\lambda = \lambda_1(A(x, u)) \leq \lambda_1^B(A(x, u)) \leq \lambda_1^B(\gamma I) = \gamma \lambda_1^B(I).$$

3. In this case we can obtain a similar result to the main one in [2]. Indeed, for each $r > 0$ there exists $\lambda_r > 0$ and $u_r \in H_0^1(\Omega)$ solution of (P_λ) with $\|u\|_0 = r$.

In the next result we show that when $A(x, s)$ tends to infinity as $s \to \infty$ in the sense of (A_∞), then the bifurcation at infinity disappears, in some sense $\lambda_\infty \to +\infty$ when $A(x, s)$ tends to infinity.

Theorem 3.4. *Assume that A satisfies (A_4), $(\tilde{A}_2)$ and (A_∞). Then, there exists a continuum $\Sigma \subset \mathcal{S}$ such that $(\lambda_0, 0) \in \Sigma$. Moreover, the interval $(\lambda_0, +\infty) \subset \mathrm{Proj}_{\mathbb{R}}\Sigma$ and*

$$\lim_{\substack{\lambda \to +\infty \\ (\lambda, u_\lambda) \in \Sigma}} \|u_\lambda\|_0 = +\infty.$$

Proof. The existence of the continuum unbounded Σ bifurcating from $(\lambda_0, 0)$ follows by Theorem 3.2. Since $\lambda = \lambda_1(A(x,u)) \geq \lambda_1(\alpha I) = \alpha\lambda_1(I)$, there do not exist positive solutions for λ small. So, it suffices to prove that it is not possible bifurcation from infinity. In order to do that we observe that problem (P_λ) can be written as

$$\begin{cases} -\operatorname{div}(B(x,u)g(u)\nabla u) = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where g is given by hypothesis (A_∞) and

$$B(x,u) := \frac{A(x,u)}{g(u)}.$$

Moreover, if we perform the change of variable

$$w = \tilde{g}(u) = \int_0^u g(t)dt,$$

problem (P_λ) is equivalent to

$$\begin{cases} -\operatorname{div}(C(x,w)\nabla w) = \lambda f(w), & x \in \Omega, \\ w = 0, & x \in \partial\Omega, \end{cases} \tag{Q_λ}$$

where

$$C(x,w) := B(x, \tilde{g}^{-1}(w)) \qquad \text{and} \qquad f(w) := \tilde{g}^{-1}(w).$$

Now we argue by contradiction, and assume that there exists a sequence of solutions (λ_n, u_n) of (P_{λ_n}) such that $\lambda_n \to \overline{\lambda} > 0$ and $\|u_n\|_0 \to \infty$. Then, by (3.1) we have that $\|u_n\| \to \infty$ and taking $w_n = \tilde{g}(u_n)$, it is clear that $\|w_n\|_0 \to \infty$. In addition, since (A_∞) implies that $\alpha^2\|u_n\|^2 \leq \|w_n\|^2$, we also have that $\|w_n\| \to \infty$. For the normalized sequence $z_n := \frac{w_n}{\|w_n\|}$ we know the existence of $z \in H_0^1(\Omega)$, such that

$$z_n \to z \quad \text{strongly in } L^2(\Omega), \text{ and a.e. in } \Omega.$$

and so, taking $w_n/\|w_n\|^2$ as a test function in (Q_{λ_n}), we obtain that

$$\alpha \leq \int_\Omega C(x,w_n)\nabla z_n \cdot \nabla z_n = \lambda_n \int_\Omega \frac{f(w_n)}{\|w_n\|} z_n. \tag{3.7}$$

Now, taking into account that

$$\frac{f(s)}{s} \to 0 \quad \text{as } s \to \infty,$$

and that $f(s) \leq \frac{1}{\alpha}s$ for each $s \in \mathbb{R}^+$, we can argue as Theorem 5.5 in [1] and conclude that
$$\int_\Omega \frac{f(w_n)}{\|w_n\|} z_n \to 0, \quad \text{as } n \to \infty.$$
Indeed, we can write for every $n \in \mathbb{N}$
$$\begin{aligned}\int_\Omega \frac{f(w_n)}{\|w_n\|} z_n &= \int_\Omega \frac{f(w_n)}{\|w_n\|}(z_n - z) + \int_\Omega \frac{f(w_n)}{\|w_n\|} z \\ &\leq \frac{1}{\alpha}\|z_n\|_2 \|z_n - z\|_2 + \int_{\Omega_0} \frac{f(w_n)}{\|w_n\|} z,\end{aligned}$$
where $\Omega_0 = \{x \in \Omega : \ z(x) \neq 0\}$. Thus, we only have to prove that
$$\lim_{n\to\infty} \int_{\Omega_0} \frac{f(w_n)}{\|w_n\|} z = 0,$$
which is a direct consequence of the Lebesgue Theorem, since for a.e. $x \in \Omega_0$, $w_n(x) = z_n(x)\|w_n\| \to +\infty$ and then
$$\frac{f(w_n(x))}{\|w_n\|} z(x) \to 0, \ \text{a.e. } x \in \Omega_0.$$
Thus, taking limits in (3.7), we have that $\alpha \leq 0$, which is a contradiction. □

This work was developed while the first author was visiting the Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla. He gratefully acknowledges the whole department for the warm hospitality and the friendly atmosphere. First author partially supported by D.G.E.S. Ministerio de Educación y Ciencia (Spain) n. PB98-1283. The second supported by the Spanish Ministry of Science and Technology under Grant BFM2000-0797.

References

1. Arcoya, D., Carmona, J., Pellacci, B.: *Bifurcation for Quasilinear Operator*, *Proc. Roy. Soc. Edinb.* **131 A**, (2001), 733-765.
2. Boccardo, L.: *Positive eigenfunctions for a class of quasili-linear operator*, *Bollettino B. M. I.* **18-B** (1981), 951-959.
3. Cantrell R.S., Cosner C.: *Diffusive logistic equations with indefinite weights: population models in disrupted environments, Part I and II*, *Proc. R. Soc. Edinb.*, **112A**, 293-318 (1989) and *SIAM J. Math. Anal.*, **22**, 1043-1064 (1991).
4. Chabrowski, J.: *Some Existence Theorems for the Dirichlet Problem for Quasilinear Elliptic Equations*, *Annali di Matematica pura ed applicata.*, **158**, 391-398 (1991).
5. De Figueiredo D.G.: *Positive solutions of semilinear elliptic problems*, Lectures Notes in *Math.*, **957**, (1982).
6. De Giorgi, E. : *Sulla Differenziabilità e l'Analiticità degli Estremali degli Integrali Multipli Regolari. Mem. Accad. Sci. Torino* (1957), 25-43.
7. Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations of Second Order. Springer*, 1983.
8. Stampacchia, G.: *Le Problème de Dirichlet pour les Équations Elliptiques du Second Ordre À Coefficients Discontinus. Ann. Inst. Fourier Grenoble.* **117** (1965), 138-152.