NON-NEGATIVE SOLUTIONS FOR A HETEROGENEOUS DEGENERATE COMPETITION MODEL

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Abstract

This paper deals with the existence, uniqueness and qualitative properties of nonnegative and nontrivial solutions of a spatially heterogeneous Lotka-Volterra competition model with nonlinear diffusion. We give conditions in terms of the coefficients involved in the setting of the problem which assure the existence of nonnegative solutions as well as uniqueness of positive solution. In order to obtain the results we employ monotonicity methods, singular spectral theory and a fixed point index.

Short title: Degenerate competition problem

1. Introduction

In this work we are mainly concerned with the existence and uniqueness of nonnegative solutions for the problem

$$\begin{cases} L_1(w^m) = w(\lambda - a(x)w - b(x)z) & \text{in } \Omega, \\ L_2(z^n) = z(\mu - d(x)z - c(x)w) & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with regular boundary $\partial\Omega$, L_k, $k = 1, 2$ are two second order uniformly elliptic operators of the form

$$L_k := -\sum_{i,j=1}^{N} a_{ij}^k(x) D_i D_j + \sum_{i=1}^{N} b_i^k(x) D_i \qquad k = 1, 2, \tag{2}$$

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with $a_{ij}^k, b_i^k \in C^1(\overline{\Omega})$; $m, n > 1; \lambda, \mu \in \mathbb{R}$ and $a, b, c, d \in C^1(\overline{\Omega})$ nonnegative and nontrivial.

Problem (1) provides us with the steady-state solutions to a related evolutionary problem, which models the behaviour of two competing species, with populations densities $w(x)$ and $z(x)$, inhabiting Ω. We refer to [14] for the meaning of each coefficient and details about the model.

When $m = n = 1$ (linear diffusion), (1) has been extensively studied in the recent years. In the case that a, b, c and d are strictly positive functions, see for example [6], [7], [8], [10], [12], [13], [19], [20], [21], [25], [28], [32] and the references therein. When b and/or c vanish in a domain of Ω (that means that, for instance, z does not interact with w in the set $B_0 := \{x \in \Omega : b(x) = 0\}$); problem (1) was studied in [22], [26] and [28]. And finally, recently the case a vanishes in a part of Ω but all other coefficients functions are strictly positive over Ω has been analysed in [18] and [27], where essential qualitative changes occur. Observe that in this case positive constants are not supersolutions of (1) and, in fact, it is shown that the a priori bounds are lost for some values of λ and μ appearing a new kind of positive solutions (which are infinite over a region of Ω and finite on the rest of Ω) that govern the behaviour of a related evolutionary problem.

However, model (1) is less known when $m, n > 1$, and it has been only analysed under more restrictive hypotheses, with constant coefficients (homogeneous environmental case) in [14] and when a and d are strictly positive in [9] and [31], all of them with $L_1 = L_2 = -\Delta$. These new parameters (m, n) were introduced in [23] and [29] by describing the dynamics of biological population whose mobility depends upon their density. In this context, it means that the diffusion, the rate of movement of the species from high density regions to low ones, is slower than in the linear case, giving more realistic results. Mathematically, this has mainly three consequences which distinguish this system from the one with $m = n = 1$: the strong maximum principle does not apply (and so, unlike the linear case, there can exist nonnegative and nontrivial solutions which are not positive in all Ω), a-priori bounds for all the solutions of (1) and for all the values of λ and μ, even when a or d vanishes, exist and that the linearized method cannot be applied directly.

In order to study (1) we make the appropriate change of variables $w^m = u$ and $z^n = v$, which transforms (1) into

$$\begin{cases} L_1 u = u^{1/m}(\lambda - a(x)u^{1/m} - b(x)v^{1/n}) & \text{in } \Omega, \\ L_2 v = v^{1/n}(\mu - d(x)v^{1/n} - c(x)u^{1/m}) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Since only nonnegative solutions have physical interest, there are four types of solutions: the trivial one, the semitrivial solutions $(u, 0)$ and $(0, v)$, those with both components strictly positive, the *coexistence states*, and those where at least one component could vanish in a part of Ω, the *semicoexistence states*. Observe that a semicoexistence state could be a coexistence one (see Proposition 3.3). Sometimes, we are able to prove that a semicoexistence state vanishes in a region of Ω (see Theorem 3.4), and so it is not a coexistence state.

Now we describe the parts of this work stating their main results. Observe that the semitrivial solutions satisfy the following equation, the reason for our study in Section 2,

$$\begin{cases} Lw = f(x)w^{1/r} - g(x)w^{2/r} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where L is an operator of the form (2), $f, g \in C^1(\overline{\Omega})$ with $g \geq 0$, $g \not\equiv 0$, f can change sign and $r = m$ or n. Although the semitrivial solutions give $f \equiv \lambda$ (or μ) and so constant, it will be very useful to study (4) when f changes sign. This equation has been previously studied in [3], [14], [15], [24] and [30] assuming more restrictions in the data of (4). We collect the main results of these works, and as a consequence we obtain that the semitrivial solution $(u, 0) =$ (resp. $(0, v)$) exists and it is unique if, and only if, $\lambda > 0$ (resp. $\mu > 0$).
Then, we study the existence of *dead cores* (see [17]) of the solutions of (4). Given a solution w of (4); we call the set $\Omega_0 := \{x \in \Omega : w(x) = 0\}$, if this is nonempty, a dead core of w. We demonstrate a result which assures the existence of a dead core for any nonnegative solution of (4) under suitable hypotheses (see Theorem 2.4). A direct consequence of our result is that any nonnegative solution of (4) has dead core if the maximum of f is small. To our knowledge, the above results concerning to the existence of dead core have been obtained when $L = -\Delta$, see [3], [14], [17] and [30], with their proofs being based on the radial properties of the Laplacian. In this way our result generalises previous ones.

In Section 3 we carry out an analysis of the existence of semicoexistence, coexistence states and dead cores of the system (3). Using the results of Section 2 and monotonicity methods we obtain results which can be summarized as follows: take $\lambda \in \mathbb{R}$,

- Assume $\lambda \leq 0$: if $\mu \in (-\infty, 0]$ only the trivial solution exists, if $\mu \in (0, \infty)$ only the trivial and the semitrivial solutions $(0, v)$ exist;
- Assume $\lambda > 0$: there exist positive values $\mu_*(\lambda)$, $\mu^*(\lambda)$, $\mu_1(\lambda)$, $\mu_2(\lambda)$ with

  $$\mu_1(\lambda) < \min\{\mu_*(\lambda), \mu^*(\lambda)\} \quad \text{and} \quad \mu_2(\lambda) > \max\{\mu_*(\lambda), \mu^*(\lambda)\}$$

  such that

  - If $\mu \in (-\infty, 0]$ only the trivial and semitrivial solution $(u, 0)$ exist;

- If $\mu \in (0, \mu_1(\lambda))$ there exists at least a semicoexistence state (u, v) and the component v has dead core;
- If $\mu \in (\mu_1(\lambda), \mu_2(\lambda))$ there exists at least a semicoexistence state;
- If $\mu \in (\mu_2(\lambda), \infty)$ there exists at least a semicoexistence state (u, v) and the component u has dead core;
- If, moreover $\mu_*(\lambda) < \mu^*(\lambda)$, then if $\mu \in (\mu_*(\lambda), \mu^*(\lambda))$ there exists at least a coexistence state.

Analogous results can be obtained when we fix the parameter μ. It's worth mentioning that the existence of $\mu_1(\lambda) > 0$ was shown in [14] when all the coefficients were positive constants. To our knowledge, the existence of $\mu_2(\lambda) > 0$ is new. In Remark 3.1 we give a biological interpretation of this result.

In Sections 4 and 5 we study the uniqueness of coexistence states of (3). For that we use the fixed point index. Observe that because $m, n > 1$ the linearization of (3) around the trivial or semitrivial solutions do not exist, so we cannot apply the results in [11] (see also [25] and [28]) to compute their indices. So, we will build appropriate homotopies for that. To compute the index of a coexistence state we can use a linearization. In this case the linearization of (3) around a coexistence state leads us to a eigenvalue problem of the form

$$\begin{cases} \mathcal{L}U + MU = \sigma U & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\mathcal{L} = \mathrm{diag}(L_1, L_2)$ and $M = (m_{ij})$, $1 \leq i, j \leq 2$ with $m_{ij} \geq 0$ for $i \neq j$ and m_{ij} blowing up near $\partial\Omega$ in a controlled way. Following [16] and [28] we define a specific order and establish the existence of the principal eigenvalue of (5) as well as a characterization of its positivity by means the existence of a supersolution. Now, we prove that, again with fixed $\lambda > 0$, there exists a unique coexistence state when μ belongs to a subset of $(\mu_*(\lambda), \mu^*(\lambda))$. Furthermore, if $m = n$ and a, d are strictly positive functions we have uniqueness of coexistence state if b_M or c_M is small. The results about uniqueness of coexistence state of (3) are also, we believe, new.

2. Preliminaries. The degenerate logistic equation

We consider the Banach space $X := C_0^1(\overline{\Omega})$ ordered by its cone of non-negative functions P, whose interior is

$$\text{int } (P) := \{u \in X : u(x) > 0 \text{ for all } x \in \Omega \text{ and } \partial u/\partial n < 0 \text{ on } \partial\Omega\},$$

where n denotes the outward unit normal on $\partial\Omega$. We say that $u \in X$ is nonnegative, $u \geq 0$, if $u \in P$, and u is positive, $u > 0$, if $u \in \text{int } (P)$.

Given $q \in L^\infty(\Omega)$ and L an operator of the form (2), we denote by $\sigma_1(L+q)$ the principal eigenvalue of $L+q$ subject to homogeneous Dirichlet boundary conditions. Moreover, if we denote by $\varphi \in \text{int } P$ the unique positive eigenfunction associated with $\sigma_1(L+q)$ normalized such that $\|\varphi\|_\infty = 1$, then it is well known that

$$\frac{\partial \varphi}{\partial \nu} < 0 \qquad \text{on } \partial\Omega, \tag{6}$$

for ν any direction out of Ω. Recall that as positive constants are supersolutions of L, then

$$\sigma_1(L) > 0. \tag{7}$$

Finally, for $f \in Y := C^0(\overline{\Omega})$ we write

$$f_M := \max_{x\in\overline{\Omega}} f(x), \qquad f_L := \min_{x\in\overline{\Omega}} f(x).$$

2.1. Existence of solutions In this section we study the semitrivial solutions of (3). Observe that if the solutions of (3) are of the form $(u, 0)$ and $(0, v)$, then satisfy equations of the following type

$$\begin{cases} Lw = f(x)w^q - g(x)w^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where L is an operator of the form (2), $f, g \in C^1(\overline{\Omega})$ with $g \geq 0$, $g \not\equiv 0$, f can change sign and q and p satisfy

(H) $\qquad 0 < q < 1, \qquad p > q.$

Our first result gives us the existence of nonnegative solution of (8) and lists some useful properties. For a proof of this result see [15] for instance.

Theorem 2.1. *Assume (H). The following assertions are true:*

1. *There exists a maximal nonnegative and nontrivial solution of* (8) *if, and only if, $f_M > 0$. We denote it by $\theta_{[L,q,p,f,g]}$.*
2. *The following estimates hold:*

$$\begin{gathered} \theta_{[L,q,p,f,g]}(x) \leq f_M^{1/(1-q)} e_M^{q/(1-q)} e(x) \qquad x \in \Omega, \\ (\theta_{[L,q,p,f,g]})_M \leq (f_M e_M)^{1/(1-q)}, \end{gathered} \tag{9}$$

   where $e \in C^2(\overline{\Omega})$ is the unique solution of

$$\begin{cases} Le = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

3. *If $\underline{w} \in C^1(\overline{\Omega})$ is a nonnegative subsolution of* (8)*, then $\underline{w} \leq \theta_{[L,q,p,f,g]}$.*

4. *Let* $f_i \in C^1(\overline{\Omega})$, $i = 1, 2$ *be such that* $f_1 \le f_2$, *then* $\theta_{[L,q,p,f_1,g]} \le \theta_{[L,q,p,f_2,g]}$.

5. *If* $f_L > 0$, *then any nonnegative solution of* (8) *is positive. Moreover, in this case there exists a unique positive solution and it satisfies*

$$\varepsilon\varphi(x) \le \theta_{[L,q,p,f,g]}(x) \qquad x \in \Omega, \tag{11}$$

where ε *is the unique positive root of*

$$\sigma_1(L)\varepsilon^{1-q} + g_M \varepsilon^{p-q} = f_L. \tag{12}$$

Remark. If we consider f_L as a real parameter, then it is easy to prove that as $f_L \to \infty$, $\varepsilon(f_L) = O(f_L^{1/(1-q)})$ when $p \le 1$ and $\varepsilon(f_L) = O(f_L^{1/(p-q)})$ when $p > 1$.

2.2. Existence of dead cores In order to state and prove the main result, we need some preliminary ones.

Lemma 2.2. *Let* $R > 0$ *and* $\gamma > 0$. *Consider the problem*

$$\begin{cases} Lw = -Rw^q - g(x)w^p & \text{in } \Omega, \\ w = \gamma & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Then, there exists a unique nonnegative solution of (13).

Proof. For the existence we use the sub-supersolution method. Indeed, it is easy to prove that $(\underline{w}, \overline{w}) = (0, \gamma)$ is a sub-supersolution of (13). For the uniqueness we can apply Theorem 2 in [1].

The following technical result is fundamental in our study. Moreover, it generalizes Lemma 7 in [30] and Lemma 2.5 in [3], where a similar result was proved when $L = -\Delta$ and $g(x) \equiv 0$.

Lemma 2.3. *We fix* $\gamma > 0$ *and* $\beta > 2/(1-q)$. *Let* δ_0 *be such that for all* $x, x_0 \in \mathbb{R}^N$ *such that* $0 \le |x - x_0| \le \delta_0$

$$\begin{aligned} &|x - x_0|^{\beta q} + \beta|x - x_0|^{\beta-1} L(|x - x_0|) + \\ &+\beta(1-\beta)|x - x_0|^{\beta-2} \sum_{i,j=1}^{N} a_{ij}(x) D_i(|x - x_0|) D_j(|x - x_0|) \ge 0. \end{aligned} \tag{14}$$

Then, for all $0 < \delta < dist(x_0, \partial\Omega)$, *the unique nonnegative solution,* w, *of* (13) *in* $B(x_0, \delta)$ *is such that* $w(x_0) = 0$ *provided that*

$$R \ge \left(\frac{\gamma}{\min\{\delta, \delta_0\}^\beta}\right)^{1-q}. \tag{15}$$

Remark. Observe that since $\beta > 2/(1-q)$, then $\beta q < \beta - 2 < \beta - 1$, and so the existence of δ_0 satisfying (14) is guaranteed. Moreover, since $\beta > 2$ (14) can be considered in a classical sense.

Proof. Consider the function

$$\Phi(x) := \begin{cases} \Phi_1(x) := R^{1/(1-q)}|x-x_0|^\beta & \text{if } x \in \overline{B}(x_0, \delta_0), \\ \Phi_2(x) := R^{1/(1-q)}\delta_0^\beta & \text{if } x \in B(x_0, \delta) \setminus B(x_0, \delta_0), \end{cases}$$

with $\Phi \equiv \Phi_1$ if $\delta \leq \delta_0$. By the choice of β, we have that $\Phi_1 \in H^2(B(x_0, \delta_0))$. Moreover,

$$\frac{\partial \Phi_1}{\partial n_L} \geq 0, \quad \text{on } \partial B(x_0, \delta_0),$$

where n_L stands for the conormal associated with L, i.e., $(n_L)_i := \sum_{j=1}^N a_{ij} n_j$. Indeed, for $x \in \partial B(x_0, \delta_0)$ we have

$$\frac{\partial \Phi_1}{\partial n_L}(x) = R^{1/(1-q)}\beta |x-x_0|^{\beta-3}\Big(\sum_{i,j=1}^N a_{ij}(x)(x^i - x_0^i)(x^j - x_0^j)\Big) \geq 0.$$

Moreover,

$$\begin{aligned}
&L(\Phi_1) + R\Phi_1^q + g(x)\Phi_1^p = R^{1/(1-q)}(\beta|x-x_0|^{\beta-1}L(|x-x_0|)+ \\
&+\beta(1-\beta)|x-x_0|^{\beta-2}\sum_{i,j=1}^N a_{ij}D_i(|x-x_0|)D_j(|x-x_0|))+ \\
&+RR^{q/(1-q)}|x-x_0|^{\beta q} + g(x)R^{p/(1-q)}|x-x_0|^{\beta p} \geq \\
&\geq R^{1/(1-q)}(|x-x_0|^{\beta q} + \beta|x-x_0|^{\beta-1}L(|x-x_0|)+ \\
&\beta(1-\beta)|x-x_0|^{\beta-2}\sum_{i,j=1}^N a_{ij}D_i(|x-x_0|)D_j(|x-x_0|)) \geq 0,
\end{aligned}$$

by (14). In $B(x_0, \delta) \setminus B(x_0, \delta_0)$, we have that

$$L(\Phi_2) + R\Phi_2^q + g(x)\Phi_2^p \geq 0.$$

Finally, in $\partial B(x_0, \delta)$, Φ is bigger than γ provided that (15) holds.
Hence, we can apply Lemma I.1 in [4] and conclude that Φ is a supersolution of (13) in $B(x_0, \delta)$. This completes the proof.

For $R > 0$, we define the set

$$N(R) := \{x \in \Omega : f^-(x) \geq R\} = \{x \in \Omega : f(x) \leq -R\},$$

where $f^\pm(x) := \max\{\pm f(x), 0\}$. Assume that $f^\pm \not\equiv 0$. The main result of this section is the following one.

Theorem 2.4. *Assume that there exists $R > 0$ such that*

1.

$$\delta_R := \left(\frac{f_M e_M}{R}\right)^{1/(\beta(1-q))} \leq \delta_0,$$

2.

$$M(R) := \{x \in N(R) : dist(x, \partial N(R) \setminus \partial\Omega) \geq \delta_R\} \neq \emptyset.$$

Then, there exists a dead core for any nonnegative solution w of (8)*. Moreover, we have*

$$M(R) \subset \Omega_0 = \{x \in \Omega : w(x) = 0\}.$$

PROOF. Let $x_0 \in M(R)$, then

$$\mathcal{B}(x_0, \delta_R) := \{x \in \Omega : |x - x_0| < \delta_R\} \subset N(R). \qquad (16)$$

We call z the unique nonnegative solution of (13) in $\mathcal{B}(x_0, \delta_R)$ with $\gamma = (f_M e_M)^{1/(1-q)}$. Then, by (16) we have that

$$L\theta_{[L,q,p,f,g]} \leq -R\theta^q_{[L,q,p,f,g]} - g(x)\theta^p_{[L,q,p,f,g]} \quad \text{in } \mathcal{B}(x_0, \delta_R),$$

which implies that

$$L(z - \theta_{[L,q,p,f,g]}) \geq R(\theta^q_{[L,q,p,f,g]} - z^q) + g(x)(\theta^p_{[L,q,p,f,g]} - z^p) \quad \text{in } \mathcal{B}(x_0, \delta_R),$$

and by (9) and the choice of γ we get

$$z \geq \theta_{[L,q,p,f,g]} \quad \text{on } \partial\mathcal{B}(x_0, \delta_R).$$

Hence, if we denote by $\Omega_1 := \{x \in \mathcal{B}(x_0, \delta_R) : z(x) < \theta_{[L,q,p,f,g]}(x)\}$ then

$$\begin{aligned} L(z - \theta_{[L,q,p,f,g]}) &\geq 0 \quad \text{in } \Omega_1, \\ z - \theta_{[L,q,p,f,g]} &\geq 0 \quad \text{on } \partial\Omega_1 \cap \partial\mathcal{B}(x_0, \delta_R), \\ z - \theta_{[L,q,p,f,g]} &= 0 \quad \text{on } \partial\Omega_1 \cap \mathcal{B}(x_0, \delta_R). \end{aligned}$$

The maximum principle implies that $z \geq \theta_{[L,q,p,f,g]}$ in $\mathcal{B}(x_0, \delta_R)$. Finally, we can apply Lemma 2.3 because δ_R satisfies (15). This finishes the proof.

As consequence of the above result, we have

COROLLARY 2.5. *Any nonnegative solution of* (8) *has a dead core provided that f_M is sufficiently small.*

PROOF. It is sufficient to repeat the proof of Remark 2.13 in [14] and to take account that $\delta_R \to 0$ as $f_M \to 0$.

3. Existence of nonnegative solutions

Hereafter we write

$$\theta_{[L_1,f,g]} := \theta_{[L_1,1/m,2/m,f,g]}, \qquad \theta_{[L_2,f,g]} := \theta_{[L_2,1/n,2/n,f,g]}.$$

The following result gives us a necessary and sufficient condition to obtain semicoexistence states.

THEOREM 3.1. *Problem* (3) *has a semicoexistence state if, and only if,* $\lambda > 0$ *and* $\mu > 0$.

PROOF. By Theorem 2.1 3) it follows that

$$u \leq \theta_{[L_1,\lambda,a]}, \qquad v \leq \theta_{[L_2,\mu,d]}. \tag{17}$$

So, if $\lambda \leq 0$, again by Theorem 2.1 1) we obtain that $u \equiv 0$. Analogously, if $\mu \leq 0$, $v \equiv 0$.

Assume now that $\lambda > 0$ and $\mu > 0$. In this case, we have that

$$A(x) := \lambda - b(x)\theta^{1/n}_{[L_2,\mu,d]}(x) \qquad B(x) := \mu - c(x)\theta^{1/m}_{[L_1,\lambda,a]}(x) \tag{18}$$

satisfy $A_M = \lambda > 0$ and $B_M = \mu > 0$. We consider the pair

$$(\underline{u}, \overline{u}) = (\theta_{[L_1,A,a]}, \theta_{[L_1,\lambda,a]}), \quad (\underline{v}, \overline{v}) = (\theta_{[L_2,B,d]}, \theta_{[L_2,\mu,d]}).$$

By definition of A and B and Theorem 2.1 it follows that $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ and that $\underline{u}$ and $\underline{v}$ are nonnegative and nontrivial functions. Finally, it is not hard to prove that the pair $(\underline{u}, \overline{u}) - (\underline{v}, \overline{v})$ is a sub-supersolution of (3). This completes the proof.

The following result provides us with conditions which assure the existence of coexistence states as well as bounds of them.

THEOREM 3.2. *If* λ *and* μ *satisfy*

$$\lambda > (b(x)\theta^{1/n}_{[L_2,\mu,d]})_M, \qquad \mu > (c(x)\theta^{1/m}_{[L_1,\lambda,a]})_M, \tag{19}$$

then, (3) *possesses a coexistence state. Moreover, for any coexistence state* (u, v) *of* (3) *we have the following estimates: if* $\lambda > (b(x)\theta^{1/n}_{[L_2,\mu,d]})_M$ *then*

$$\varepsilon_1\varphi_1 \leq \theta_{[L_1,A,a]} \leq u \leq \theta_{[L_1,\lambda,a]} \leq \lambda^{m/(m-1)}(e_1)_M^{1/(m-1)}e_1, \tag{20}$$

and if $\mu > (c(x)\theta^{1/m}_{[L_1,\lambda,a]})_M$, *then*

$$\varepsilon_2\varphi_2 \leq \theta_{[L_2,B,d]} \leq v \leq \theta_{[L_2,\mu,d]} \leq \mu^{n/(n-1)}(e_2)_M^{1/(n-1)}e_2, \tag{21}$$

where φ_i and e_i, $i = 1, 2$ are the principal positive eigenfunctions of L_i and solutions of (10) *with L_i respectively, and ε_1 and ε_2 are the positive solutions of*

$$\begin{aligned} \varepsilon_1^{1-1/m}\sigma_1(L_1) + a_M\varepsilon_1^{1/m} &= \lambda - b_M\mu^{1/(n-1)}(e_2)_M^{1/(n-1)}, \\ \varepsilon_2^{1-1/n}\sigma_1(L_2) + d_M\varepsilon_1^{1/n} &= \mu - c_M\lambda^{1/(m-1)}(e_1)_M^{1/(m-1)}. \end{aligned} \tag{22}$$

Proof. Consider the same sub-supersolution that in the proof of Theorem 3.1. Observe that if (19) is satisfied, then $A_L > 0$ and $B_L > 0$. Hence, Theorem 2.1 5) completes the existence of coexistence state.

The estimates (17) and (9) yield the upper bounds of (20) and (21). On the other hand, thanks to (17), u is a supersolution of (8) with $L = L_1$, $f \equiv A$ and $g \equiv a$. So, since $\lambda > (b(x)\theta_{[L_2,\mu,d]}^{1/n})_M$ then $A_L > 0$ and by Theorem 2.1 5) the lower bounds of (20) follow. Estimates (21) can be proved similarly.

Remark. 1. Using (20) and (21) we can obtain a sufficient condition for the existence of coexistence states involving the coefficients of the problem. Indeed, if λ and μ satisfy

$$\lambda > b_M(e_2)_M^{1/(n-1)}\mu^{1/(n-1)} \qquad \mu > c_M(e_1)_M^{1/(m-1)}\lambda^{1/(m-1)} \tag{23}$$

then λ and μ satisfy (19), so that (3) has a coexistence state.

2. Observe that when all coefficients are positive constants (see [9] and [14]) the conditions which assure the existence of coexistence states are independent of m and n. This is due to the fact that positive constants are supersolutions of (3).

In Figure 1 we have shown the different forms of the region defined in the (λ, μ)-plane by (23) when m and n vary. We have denoted by $f(\lambda) = c_M(e_1)_M^{1/(m-1)}\lambda^{1/(m-1)}$ and $g(\mu) = b_M(e_2)_M^{1/(n-1)}\mu^{1/(n-1)}$.

Figure 1 near here.

3.1. Existence of dead cores We will use the results of Section 2 to show the existence of dead cores for (3). The first result provides us conditions which assure the non-existence of dead cores, and it is a direct consequence of (20) and (21).

Proposition 3.3. *Assume that λ and μ satisfy* (19). *Then any nonnegative solution of* (3) *does not have a dead core.*

To state the main result of this section we need some notation. We fix $\lambda > 0$. It is not hard to prove that the map $\mu \mapsto \theta_{[L_2,\mu,d]}$ is strictly increasing, and so also is $\mu \mapsto (b(x)\theta_{[L_2,\mu,d]}^{1/n})_M$. Hence, there exists a unique

value $\mu^*(\lambda)$ such that $\lambda = (b(x)\theta^{1/n}_{[L_2,\mu^*(\lambda),d]})_M$. For such λ fixed, we write $\mu_*(\lambda) = (c(x)\theta^{1/m}_{[L_1,\lambda,a]})_M$.
Analogously, fixed $\mu > 0$, there exists a unique $\lambda^*(\mu) > 0$ such that $\mu = (c(x)\theta^{1/m}_{[L_1,\lambda^*(\mu),a]})_M$ and define $\lambda_*(\mu) = (b(x)\theta^{1/n}_{[L_2,\mu,d]})_M$.

THEOREM 3.4. *1. Assume $\lambda > 0$. Then there exist $0 < \mu_1(\lambda) < \mu_2(\lambda)$ with $\mu_1(\lambda) < \min\{\mu_*(\lambda), \mu^*(\lambda)\}$ and $\max\{\mu_*(\lambda), \mu^*(\lambda)\} < \mu_2(\lambda)$ such that if $0 < \mu < \mu_1(\lambda)$ or $\mu > \mu_2(\lambda)$ any nonnegative solution of (3) has a dead core.*

2. Assume $\mu > 0$. Then there exist $0 < \lambda_1(\mu) < \lambda_2(\mu)$ with $\lambda_1(\mu) < \min\{\lambda_*(\mu), \lambda^*(\mu)\}$ and $\max\{\lambda_*(\mu), \lambda^*(\mu)\} < \lambda_2(\mu)$ such that if $0 < \lambda < \lambda_1(\mu)$ or $\lambda > \lambda_2(\mu)$ any nonnegative solution of (3) has a dead core.

PROOF. We will prove 1). The second part follows analogously. Observe that if $\mu < \mu^*(\lambda)$ by (20) we get

$$u \geq \theta_{[L_1,A(x),a]}. \tag{24}$$

Now we define

$$F(x,\mu) := \mu - a(x,\mu) := \mu - c(x)\theta^{1/m}_{[L_1,A(x),a]}.$$

Now, using (24) it is not hard to prove that v is a subsolution of (8) with $L = L_2$, $f(x) = F(x,\mu)$ and $g(x) = d(x)$, and so by Theorem 2.1 3), it follows that

$$v \leq \theta_{[L_2,F(x,\mu),d]}. \tag{25}$$

Now, we are going to use Theorem 2.4 to prove that $\theta_{[L_2,F(x,\mu),d]}$ has a dead core, so that by (25) the result follows.

Observe that in this case

$$(F(x,\mu))_M = \mu, \quad \text{and so} \quad \delta_R = \left(\frac{\mu(e_2)_M}{R}\right)^{n/(\beta(n-1))}.$$

On the other hand, since $(a(x,0))_M > 0$ and $(a(x,\mu))_M$ is decreasing in μ, there exists a unique $\mu_0(\lambda) > 0$ such that $\mu_0 = (a(x,\mu_0))_M$. Observe that, by the definition of $\mu_*(\lambda)$, we have that $\mu_0(\lambda) < \mu_*(\lambda)$. Taking $\mu \leq \mu_0/2$, we have $0 < \mu \leq \mu_0/2 < \mu_0 = (a(x,\mu_0))_M < (a(x,\mu))_M$. Hence, there exists $R_0 > 0$ such that the following set is nonempty,

$$\{x \in \Omega : \mu_0(\lambda)/2 - a(x,\mu_0(\lambda)/2) \leq -R_0\} \neq \emptyset.$$

Now, we define

$$N(\mu) := N(R_0) = \{x \in \Omega : F(x,\mu) \leq -R_0\}.$$

Again by Theorem 2.1 we get that if $\mu_1 \leq \mu_2$ then $F(x,\mu_1) \leq F(x,\mu_2)$. Hence, if $\mu_1 \leq \mu_2 \leq \mu_0/2$ then

$$\emptyset \neq N(\mu_0/2) \subset N(\mu_2) \subset N(\mu_1).$$

Let $x_0 \in \Omega$ be the point where $F(x,\mu_0/2)$ attains its negative minimum. For that $x_0 \in \Omega$, there exists $r_0 > 0$ such that $B(x_0,r_0) \subset N(\mu_0/2)$.
Finally, since $\delta_{R_0} \to 0$ as $\mu \to 0$ there exists $\mu'(\lambda) > 0$ such that for $\mu < \mu'(\lambda)$, $\delta_{R_0} < \min\{r_0, \delta_0\}$. Define $\mu_1(\lambda) := \min\{\mu_*(\lambda), \mu^*(\lambda), \mu_0/2(\lambda), \mu'(\lambda)\}$, and therefore for $\mu < \mu_1(\lambda)$ we get

$$\mathrm{dist}(x_0, \partial N(\mu)\backslash\partial\Omega) \geq \mathrm{dist}(x_0, \partial N(\mu_0/2)\backslash\partial\Omega) \geq \mathrm{dist}(x_0, \partial B(x_0,r_0)) > \delta_{R_0}.$$

Therefore, $M(\mu) \neq \emptyset$ and Theorem 2.4 completes the first part of 1).

For the second one, take $\mu > \max\{\mu_*(\lambda), \mu^*(\lambda)\}$, so $\mu > (c(x)\theta^{1/m}_{[L_1,\lambda,a]})_M$ and by (21) we get that

$$v \geq \theta_{[L_2,B(x),d]}.$$

Now we define

$$G(x,\mu) := \lambda - b(x,\mu) := \lambda - b(x)\theta^{1/n}_{[L_2,B(x),d]}.$$

Now, with a similar reasoning to that used in (25) we get that

$$u \leq \theta_{[L_1,G(x,\mu),a]}.$$

In this case, we take $R = \mu^r$ with $r > 0$ to be chosen later. So,

$$N(R) = N(\mu) := \{x \in \Omega : \lambda \leq b(x,\mu) - \mu^r\},$$

and

$$(G(x,\mu))_M = \lambda, \qquad \delta_R = \left(\frac{\lambda(e_1)_M}{\mu^r}\right)^{m/(\beta(m-1))}.$$

Using (21), we have that

$$b(x,\mu) \geq b(x)\varepsilon_2^{1/n}(\mu)\varphi_2^{1/n},$$

where ε_2 is defined in (22).
Let $\delta > 0$ be sufficiently small such that

$$B^+_\delta := \{x \in B^+ : \mathrm{dist}(x, \partial B^+) \geq \delta\} \neq \emptyset,$$

where $B^+ := \{x \in \Omega : b(x) > 0\}$. Define the set

$$T(\mu) := \{x \in B^+_\delta : \lambda \leq \varepsilon_2^{1/n}(\mu)b(x)\varphi_2^{1/n}(x) - \mu^r\}.$$

Clearly, $T(\mu) \subset N(\mu)$.
On the other hand, by Remark 2.1, $\varepsilon_2^{1/n}(\mu) = O(\mu^{1/(n-1)})$ if $n \geq 2$ and

$\varepsilon_2^{1/n}(\mu) = O(\mu)$ if $n < 2$ when μ is large. Take $r < 1/(n-1)$ if $n \geq 2$ and $r < 1$ if $n < 2$. So, there exists $\mu^0(\lambda) > 0$ such that for $\mu > \mu^0(\lambda)$

$$T(\mu^0(\lambda)) \neq \emptyset \quad \text{and} \quad T(\mu^0(\lambda)) \subset T(\mu). \tag{26}$$

Moreover, there exist $x_0 \in B_\delta^+$ and $r_0 > 0$ such that

$$B(x_0, r_0) \subset T(\mu_0(\lambda)). \tag{27}$$

Furthermore, since $\delta_R \to 0$ as $\mu \to \infty$, there exists $\mu''(\lambda)$ such that for $\mu > \mu''(\lambda)$ we get $\delta_R < \min\{r_0, \delta_0\}$. Hence, using (26) and (27), for $\mu > \mu_2(\lambda) := \max\{\mu_*(\lambda), \mu^*(\lambda), \mu^0(\lambda), \mu''(\lambda)\}$, we obtain

$$\begin{aligned}&\operatorname{dist}(x_0, \partial N(\mu) \setminus \partial\Omega) \geq \operatorname{dist}(x_0, \partial T(\mu)) \geq \\ &\operatorname{dist}(x_0, \partial T(\mu^0)) \geq \operatorname{dist}(x_0, \partial B(x_0, r_0)) = r_0 > \delta_R.\end{aligned}$$

Theorem 2.4 completes the proof.

Remark. 1. By the proof of Theorem 3.4, we can see that if $\mu < \mu_1(\lambda)$ (or $\lambda > \lambda_2(\mu)$) for any semicoexistence state (u, v) then v has dead core. Similarly, if $\mu > \mu_2(\lambda)$ (or $\lambda < \lambda_1(\mu)$) then u has dead core.

2. We can give a biological interpretation to Theorem 3.4. If we fix the growth rate of u, λ, then the other species does not live in all its habitat if its growth rate is small. But, if the growth rate of v is large, then u can not survive in all Ω.
On the other hand, when $m = n = 1$ it was shown in [22] (see also [28]) that if interaction rate (for example) b is large, then v drives u to extinction. This is in strong contrast with the case $m, n > 1$, because by Theorem 3.1 neither species drives the other to extinction when b or c is large.

4. Maximum principle for singular system

We define in X^2 the following order: given $(u_1, v_1), (u_2, v_2) \in X^2$,

$$(u_1, v_1) \preceq (u_2, v_2) \qquad \text{if, and only if,} \qquad u_1 \leq u_2 \quad \text{and} \quad v_1 \geq v_2.$$

Analogously, we write $(u_1, v_1) \prec (u_2, v_2)$ if $u_1 < u_2$ and $v_1 \geq v_2$ or $u_1 \leq u_2$ and $v_1 > v_2$.

Let $M(x) = (m_{ij}(x))$ be a 2×2 matrix whose elements belong to the Fréchet space $C^1(\Omega)$ and such that there exist $K > 0$ and $\alpha \in (0, 2]$ satisfying:

(HM) $m_{ij} \geq 0$, $m_{ij} \not\equiv 0$, $i \neq j$;

$$|m_{ij}(x)| \operatorname{dist}(x, \partial\Omega)^{2-\alpha} \leq K \qquad i, j = 1, 2. \tag{28}$$

The object of this section is to analyse the following singular eigenvalue problem:

$$\begin{cases} \mathcal{L}U + M(x)U = \sigma U & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{29}$$

where

$$\mathcal{L} = \operatorname{diag}(L_1, L_2), \quad U = (u, v)^t,$$

and L_i, $i = 1, 2$ are operators as in (2).
The next result characterizes the existence of a positive eigenvalue of (29) by means of the existence of a strict positive supersolution in the following sense. The proof of the result follows from Theorem 6.3 in [28] and Section 2 in [16].

Definition 4.1. *We say that $\Phi \in (C^2(\Omega) \cap C^{1,\delta}(\overline{\Omega}))^2$, $\delta \in (0,1)$, $\Phi \succ 0$ is a supersolution of $\mathcal{L} + M$ if $(\mathcal{L} + M)\Phi \succeq 0$ in Ω and $\Phi \succeq 0$ on $\partial\Omega$. If in addition, $(\mathcal{L} + M)\Phi \succ 0$ in Ω or $\Phi \succ 0$ on $\partial\Omega$, then it is said that Φ is a strict supersolution.*

Theorem 4.2. *Under the assumption (HM), the following conditions are equivalent:*

1. *$\mathcal{L} + M$ admits a positive strict supersolution;*
2. *The operator $[\mathcal{L} + M]^{-1} : X^2 \longmapsto X^2$ is well defined, compact and strongly positive;*
3. *The problem*

$$\begin{cases} \mathcal{L}U + M(x)U = F & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{30}$$

   where $F \in Y^2$, satisfies the strong maximum principle, i.e., if $F \succeq 0$ and $F \neq 0$, then $U \succ 0$;
4. *The operator $[\mathcal{L} + M] : X^2 \longmapsto Y^2$ possesses a strictly positive eigenvalue, denoted by $\sigma_1(\mathcal{L} + M)$. This eigenvalue is simple and it is the only eigenvalue of (29) possessing a positive eigenfunction $\Phi_1 \succ 0$.*

In the present work, we need to apply this result assuming less regularity for the strict supersolution.

Proposition 4.3. *Assume that M satisfies (HM). Then: $\sigma_1(\mathcal{L}+M) > 0$ if, and only if, there exists $\Phi \in (C^2(\Omega) \cap C_0^0(\overline{\Omega}))^2$ such that $\Phi \succ 0$ in Ω and $(\mathcal{L} + M)\Phi \succ 0$ in Ω.*

Remark. Since $\Phi = (\Phi_1, \Phi_2) \notin X^2$, when we write $\Phi \succ 0$ we mean that $\Phi_1(x) > 0$ and $\Phi_2(x) < 0$ for all $x \in \Omega$.

The following boundary point result will be used in the proof.

LEMMA 4.4. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be such that $u \geq 0$ in Ω, $u \not\equiv 0$ and*

$$(L+q)u \geq 0 \quad \text{in } \Omega, \qquad u \geq 0 \quad \text{on } \partial\Omega,$$

where $q \in C^1(\Omega)$ satisfies (28). *Then $u(x) > 0$ for all $x \in \Omega$ and for all $x_0 \in \partial\Omega$ such that $u(x_0) = 0$, $(\partial u/\partial n)(x_0) < 0$.*

PROOF. It is an easy consequence of Lemma 3.6 in [5] with $\rho(r) = r^{\alpha-2}$.

PROOF (OF PROPOSITION 4.3). It is clear that if $\sigma_1(\mathcal{L}+M) > 0$, we can take $\Phi = \Phi_1$ the eigenfunction associated with $\sigma_1(\mathcal{L}+M)$.
Now, assume that there exists $\Phi \in (C^2(\Omega) \cap C_0^0(\overline{\Omega}))^2$ such that $\Phi \succ 0$ in Ω and $(\mathcal{L}+M)\Phi := G \succ 0$ in Ω. Let $F \succeq 0$ and $F \not\equiv 0$ and U be the solution of (30). We have to prove that $U \succ 0$ and then, by Theorem 4.2, the proof is concluded.
For each $\varepsilon > 0$ and $K > 0$, we define

$$W := U + (\varepsilon, -\varepsilon)^t + \varepsilon K \Phi \in (C^2(\Omega) \cap C^0(\overline{\Omega}))^2.$$

Since $\Phi \in (C_0^0(\overline{\Omega}))^2$, for any $\varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that $W \succ 0$ in $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \gamma(\varepsilon)\}$. Moreover

$$(\mathcal{L}+M)W \succeq \varepsilon((m_{11}-m_{12}, m_{21}-m_{22})^t + KG) \succ 0 \quad \text{in } \Omega\backslash\overline{\Omega}_\varepsilon, \qquad (31)$$

for K sufficiently large. Now in $\Omega\backslash\overline{\Omega}_\varepsilon$ the coefficients m_{ij} are bounded. So, since Φ is a strict supersolution in $\Omega\backslash\overline{\Omega}_\varepsilon$, we can apply Theorem 6.3 in [28] to get that $W \succ 0$ in $\Omega\backslash\overline{\Omega}_\varepsilon$. Thus, $W \succ 0$ in Ω for all $\varepsilon > 0$, and we obtain that $U \succeq 0$ in Ω. Let $U = (u_1, u_2)^t$ be. Since $U \neq (0,0)$ we can assume that $u_1 \geq 0$ and $u_1 \not\equiv 0$. Then, denoting $F = (f_1, f_2)^t$ and taking account that $m_{12} \geq 0$, we obtain

$$L_1 u_1 + m_{11} u_1 = f_1 - m_{12} u_2 \geq 0 \quad \text{in } \Omega, \quad u_1 = 0 \quad \text{on } \partial\Omega,$$

and so applying Lemma 4.4, we get $u_1 > 0$. For the second equation,

$$L_2(-u_2) + m_{22}(-u_2) = -f_2 + m_{21} u_1 > 0 \quad \text{in } \Omega, \quad u_2 = 0 \quad \text{on } \partial\Omega$$

and so $-u_2 > 0$. So, $U \succ 0$. This completes the proof.

Again, the next result is consequence of Theorem 6.5 in [28] and Theorem 4 in [16].

THEOREM 4.5. *Assume (HM). There exists one real eigenvalue of* (29), *denoted $\sigma_1(\mathcal{L}+M)$ associated with a positive eigenfunction $\Phi_1 \succ 0$. The eigenvalue is simple and there is no other eigenvalue associated with a positive eigenfunction.*

The following result will be used to compare principal eigenvalues of different matrices.

Lemma 4.6. *Let $A(x) = (a_{ij}(x))$ and $B(x) = (b_{ij}(x))$ be two matrices with a_{ij}, b_{ij} satisfying (HM), $b_{ii} \geq a_{ii}$ and $a_{ij} \geq b_{ji}$ for $i \neq j$ with some inequality strict. Then, $\sigma_1(\mathcal{L}+A) < \sigma_1(\mathcal{L}+B)$.*

Proof. Let $\Phi^A \succ 0$ be the eigenfunction associated with $\mathcal{L}+A$. Then, it is easy to show that

$$(\mathcal{L}+B-\sigma_1(\mathcal{L}+A)I)\Phi^A \succ 0,$$

and so, Φ^A is a strict supersolution of $\mathcal{L}+B-\sigma_1(\mathcal{L}+A)I$. Hence, by Theorem 4.2 we deduce that $\sigma_1(\mathcal{L}+B-\sigma_1(\mathcal{L}+A)I) > 0$, whence the conclusion follows.

5. Uniqueness result

Along this section we assume that λ and μ satisfy (19), and so the validity of the strong maximum principle is guaranteed. Indeed, by (21) we get

$$u^{1/m}(\lambda - b(x)v^{1/n}) - a(x)u^{2/m} \geq u^{1/m}(\lambda - b(x)\theta^{1/n}_{[L_2,\mu,d]}) - a(x)u^{2/m},$$

and so, by (19), there exists a positive constant M such that

$$u^{1/m}(\lambda - b(x)v^{1/n}) - a(x)u^{2/m} + Mu \geq 0, \tag{32}$$

whence it follows that if (u,v) is a non-negative solution of (3) with $u \not\equiv$, then $u(x) > 0$ for all $x \in \Omega$. Similarly we can reason with the second equation in (3).

In this section we obtain a uniqueness result for a coexistence state of (3). In order to get the result we use the fixed point index in cones.

Fixed $M > 0$ obtained in (32), consider the operator $\mathcal{K} : X^2 \mapsto X^2$ defined by

$$\mathcal{K}(u,v) := \begin{pmatrix} (L_1+M)^{-1}(u^{1/m}(\lambda - a(x)u^{1/m} - b(x)v^{1/n}) + Mu) \\ (L_2+M)^{-1}(v^{1/n}(\mu - d(x)v^{1/n} - c(x)u^{1/m}) + Mv) \end{pmatrix},$$

where $(L_i+M)^{-1}$, $i = 1,2$, stands for the inverse on the operator $L_i + M$ in Ω under homogeneous Dirichlet boundary conditions. Observe that by (7), $\sigma_1(L_i+M) > 0$ and so $(L_i+M)^{-1}$ is well-defined and it is a compact operator. Thanks to the choice of M, see (32), $\mathcal{K}$ is a positive operator whose fixed points are componentwise nonnegative solutions of (3).

On the other hand, by (20) and (21), there exist $R_i > 0$, $i = 1, 2$, such that for every (u, v) coexistence states of (3)

$$\|u\|_\infty \leq R_1 := (\lambda(e_1)_M)^{m/(m-1)}, \qquad \|v\|_\infty \leq R_2 := (\mu(e_2)_M)^{n/(n-1)}.$$

So, the fixed point index of $\mathcal{K}$ over $\mathcal{B}$ with respect to the cone $P \times P$ is well defined, where

$$\mathcal{B} := \{(u, v) \in P^2 : \|u\|_\infty \leq R_1 + 1,\ \|v\|_\infty \leq R_2 + 1\}.$$

Now, we are going to compute this index in some cases.

Proposition 5.1. *Assume that λ and μ satisfy* (19). *The following assertions are true:*

1. $i_{P\times P}(\mathcal{K}, \mathcal{B}) = 1$;
2. $i_{P\times P}(\mathcal{K}, (0, 0)) = 0$;
3. $i_{P\times P}(\mathcal{K}, (\theta_{[L_1,\lambda,a]}, 0)) = i_{P\times P}(\mathcal{K}, (0, \theta_{[L_2,\mu,d]})) = 0$.

Proof. 1.) Firstly, we define $\mathcal{G}_1 : X \mapsto X$ by

$$\mathcal{G}_1(u) := (L_1 + M)^{-1}(u^{1/m}(\lambda - a(x)u^{1/m}) + Mu)$$

By (9), taking $B_u := \{u \in P : \|u\|_\infty \leq R_1 + 1\}$ the fixed point index of $\mathcal{G}_1$ over B_u is well-defined. Applying Lemma 12.1 in [2] it can be proved that

$$i_P(\mathcal{G}_1, B_u) = 1. \tag{33}$$

Indeed, if there exist $t \geq 1$ and $u \in P$ such that $\|u\|_\infty = R_1 + 1$ and $\mathcal{G}_1(u) = tu$, then

$$L_1 u \leq u^{1/m}\left(\frac{\lambda}{t} - \frac{a(x)}{t}u^{1/m}\right),$$

and so,

$$\|u\|_\infty \leq \left(\frac{\lambda}{t}\right)^{m/(m-1)} (e_1)_M^{m/(m-1)} \leq R_1 < R_1 + 1.$$

Analogously,

$$i_P(\mathcal{G}_2, B_v) = 1 \tag{34}$$

with $\mathcal{G}_2(v) := (L_2 + M)^{-1}(v^{1/n}(\mu - d(x)v^{1/n}) + Mv)$ and $B_v := \{v \in P : \|v\|_\infty \leq R_2 + 1\}$.

Consider the operator $\mathcal{H}_1 : [0, 1] \times X^2 \mapsto X^2$ defined by

$$\mathcal{H}_1(t, u, v) := \begin{pmatrix} (L_1 + M)^{-1}(u^{1/m}(\lambda - a(x)u^{1/m} - tb(x)v^{1/n}) + Mu) \\ (L_2 + M)^{-1}(v^{1/n}(\mu - d(x)v^{1/n} - tc(x)u^{1/m}) + Mv) \end{pmatrix}.$$

Observe that by (20) and (21) any fixed point of $\mathcal{H}_1$ belongs to $\mathcal{B}$. So, it follows by homotopy invariance, (33) and (34) that

$$\begin{aligned} i_{P\times P}(\mathcal{K},\mathcal{B}) = & \; i_{P\times P}(\mathcal{H}_1(1,\cdot),\mathcal{B}) = i_{P\times P}(\mathcal{H}_1(0,\cdot),\mathcal{B}) \\ = & \; i_P(\mathcal{G}_1, B_u)\cdot i_P(\mathcal{G}_2, B_v) = 1. \end{aligned}$$

We now prove 2). Let $\psi_i \in Y$, $i = 1, 2$, be such that $\psi_i > 0$ in Ω. We define

$$\mathcal{H}_2(t,u,v) := \begin{pmatrix} (L_1+M)^{-1}(u^{1/m}(\lambda - a(x)u^{1/m} - b(x)v^{1/n}) + Mu + t\psi_1) \\ (L_2+M)^{-1}(v^{1/n}(\mu - d(x)v^{1/n} - c(x)u^{1/m}) + Mv + t\psi_2) \end{pmatrix}.$$

We claim that there exists $\delta > 0$ such that

$$(u,v) \neq \mathcal{H}_2(t,u,v), \quad \forall t\in[0,1],\ \forall (u,v)\in \mathcal{N}_\delta, \tag{35}$$

where $\mathcal{N}_\delta := \{(u,v)\in P^2 : \|u\|_\infty \leq \delta,\ \|v\|_\infty \leq \delta\}\setminus\{(0,0)\}$. Assume there exist sequences (u_r, v_r) of functions and $t_r \in [0,1]$ such that $(u_r,v_r)\to(0,0)$ as $r\to\infty$ and

$$(u_r, v_r) = \mathcal{H}_2(t_r, u_r, v_r).$$

Since $\lambda > 0$ and $\|v_r\|_\infty \to 0$, there exists $r_0 \in \mathbb{N}$ such that $(\lambda - b(x)v_r^{1/n})_L > 0$ for $r \geq r_0$. So, the strong maximum principle is satisfied in the first equation, and so $u_r > 0$. Let $K > 0$ be such that $K \geq \sigma_1(L_1)$. Since $\|u_r\|_\infty \to 0$, there exists $r_1 \in \mathbb{N}$ such that for $r \geq r_1$ we have

$$L_1 u_r = u_r^{1/m}(\lambda - b(x)v_r^{1/n}) - a(x)u_r^{2/m} + t_r\psi_1 > K u_r,$$

and hence $\sigma_1(L_1 - K) > 0$, a contradiction.
Thus, by (35) the homotopy is admissible and we get

$$\begin{aligned} i_{P\times P}(\mathcal{K},(0,0)) = & \; i_{P\times P}(\mathcal{K},\mathcal{N}_\delta) = i_{P\times P}(\mathcal{H}_2(0,\cdot),\mathcal{N}_\delta) \\ = & \; i_{P\times P}(\mathcal{H}_2(1,\cdot),\mathcal{N}_\delta) = 0, \end{aligned}$$

this last equality follows by (35).

It remains to prove 3). Let $\psi \in Y$ be such that $\psi > 0$ in Ω. We define another operator

$$\mathcal{H}_3(t,u,v) := \begin{pmatrix} (L_1+M)^{-1}(u^{1/m}(\lambda - a(x)u^{1/m} - b(x)v^{1/n}) + Mu) \\ (L_2+M)^{-1}(v^{1/n}(\mu - d(x)v^{1/n} - c(x)u^{1/m}) + Mv + t\psi) \end{pmatrix}.$$

We claim that there exists $\delta > 0$ such that

$$(u,v) \neq \mathcal{H}_3(t,u,v), \quad \forall t\in[0,1], \forall (u,v)\in \mathcal{M}_\delta, \tag{36}$$

where $\mathcal{M}_\delta := \{(u,v)\in P^2 : \|u - \theta_{[L_1,\lambda,a]}\|_\infty \leq \delta,\ \|v\|_\infty \leq \delta\}\setminus\{(\theta_{[L_1,\lambda,a]},0)\}$. Assume there exist sequences $(u_r,v_r)\to(\theta_{[L_1,\lambda,a]},0)$ as $r\to\infty$ and $t_r\in[0,1]$ such that

$$(u_r, v_r) = \mathcal{H}_3(t_r, u_r, v_r).$$

Since $u_r \leq \theta_{[L_1,\lambda,a]}$, and $\mu > (c(x)\theta^{1/m}_{[L_1,\lambda,a]})_M$ it follows that

$$\mu - c(x)u_r^{1/m}(x) \geq \mu - c(x)\theta^{1/m}_{[L_1,\lambda,a]} > 0. \tag{37}$$

Let $K > 0$ be such that $K \geq \sigma_1(L_2)$. Then, by (37) there exists $r_0 \in \mathbb{N}$ such that for $r \geq r_0$ we have

$$L_2 v_r = v_r^{1/n}(\mu - c(x)u_r^{1/m}) - d(x)v_r^{2/n} + t_r\psi > Kv_r, \quad \text{in } \Omega,$$

and hence $\sigma_1(L_2 - K) > 0$, a contradiction.
Thus, by (36) the homotopy is admissible and we get

$$\begin{aligned} i_{P\times P}(\mathcal{K}, (\theta_{[L_1,\lambda,a]}, 0)) = &\ i_{P\times P}(\mathcal{K}, \mathcal{M}_\delta) = i_{P\times P}(\mathcal{H}_3(0,\cdot), \mathcal{M}_\delta) \\ = &\ i_{P\times P}(\mathcal{H}_3(1,\cdot), \mathcal{M}_\delta) = 0. \end{aligned}$$

Analogously, it can be treated the solution $(0, \theta_{[L_2,\mu,d]})$.

Now, let (u_0, v_0) be a coexistence state of (3). We consider the matrix $M_{(u_0,v_0)} := (m_{ij})$, $i, j = 1, 2$, which is related to the linearization of (3) about (u_0, v_0), where

$$\begin{aligned} m_{11} &= -\frac{1}{m}u_0^{1/m-1}(\lambda - 2a(x)u_0^{1/m} - b(x)v_0^{1/n}), \\ m_{12} &= \frac{1}{n}b(x)u_0^{1/m}v_0^{1/n-1}, \\ m_{21} &= \frac{1}{m}c(x)v_0^{1/n}u_0^{1/m-1}, \\ m_{22} &= -\frac{1}{n}v_0^{1/n-1}(\mu - 2d(x)v_0^{1/n} - c(x)u_0^{1/m}). \end{aligned} \tag{38}$$

Observe that since (u_0, v_0) is a coexistence state, by (20) and (21) there exists $k_0 > 0$ such that

$$k_0 \text{dist}(x, \partial\Omega) \leq u_0, \qquad k_0 \text{dist}(x, \partial\Omega) \leq v_0,$$

then $M_{(u_0,v_0)}$ satisfies (HM), so that $\sigma_1(\mathcal{L} + M_{(u_0,v_0)})$ makes sense.
The general uniqueness result reads

THEOREM 5.2. *Assume that λ and μ satisfy* (19) *and $\sigma_1(\mathcal{L}+M_{(u_0,v_0)}) > 0$ for any (u_0, v_0) coexistence state of* (3). *Then,* (3) *possesses a unique coexistence state.*

PROOF. Recall that by Proposition 3.3, if λ and μ satisfy (19) then any nonnegative solution of (3) is a coexistence state. We claim that if (u_0, v_0) is a coexistence state of (3), then

$$i_{P\times P}(\mathcal{K}, (u_0, v_0)) = 1. \tag{39}$$

Assume that we have proved (39), then since $\mathcal{K}$ is a compact operator, it possesses a finite number of coexistence states, say (u_i, v_i), $i = 1, \dots, r$. Then,

$$\begin{aligned} i_{P\times P}(\mathcal{K}, \mathcal{B}) = \ & i_{P\times P}(\mathcal{K}, (0,0)) + i_{P\times P}(\mathcal{K}, (\theta_{[L_1,\lambda,a]}, 0)) \\ & + i_{P\times P}(\mathcal{K}, (0, \theta_{[L_2,\mu,d]})) + \sum_{i=1}^{r} i_{P\times P}(\mathcal{K}, (u_i, v_i)) \end{aligned}$$

and so, by Proposition 5.1 and (39),

$$1 = 0 + 0 + 0 + r,$$

whence the conclusion now easily follows.

It remains to prove (39). Let $h \in C^1(\Omega)$ be such that h verifies that $|h(x)|\text{dist}(x, \partial\Omega)^{2-\alpha} \le K$ for some $\alpha \in (0, 2]$, $K > 0$ and

$$h \ge \max\{0, m_{11}, m_{22}\}, \tag{40}$$

where m_{11} and m_{22} are defined in (38). We define the operator

$$\mathcal{T}(u, v) := \begin{pmatrix} (L_1 + h)^{-1}(u^{1/m}(\lambda - a(x)u^{1/m} - b(x)v^{1/n}) + hu) \\ (L_2 + h)^{-1}(v^{1/n}(\mu - d(x)v^{1/n} - c(x)u^{1/m}) + hv) \end{pmatrix}.$$

Observe that $(L_i + h)^{-1}$ exists because $h \ge 0$ and so $\sigma_1(L_i + h) > 0$.
By the Leray-Schauder formula, $i_{P\times P}(\mathcal{T}, (u_0, v_0)) = (-1)^{\xi}$, where ξ is the sum of the multiplicities of the eigenvalues of $D_{(u,v)}\mathcal{T}(u_0, v_0)$ larger than one, being $D_{(u,v)}\mathcal{T}(u_0, v_0)$ the linearization of $\mathcal{T}$ about (u_0, v_0). It is clear that

$$D_{(u,v)}\mathcal{T}(u_0, v_0) = \text{diag}((L_1 + h)^{-1}, (L_2 + h)^{-1})(-M_{(u_0,v_0)} + \text{diag}(h, h)),$$

where $M_{(u_0,v_0)}$ is defined by (38). It is not hard to prove that if $r > 1$ is an eigenvalue of $D_{(u,v)}\mathcal{T}(u_0, v_0)$, then

$$\sigma_1(\mathcal{L} + M_{(u_0,v_0)} + B) = 0, \tag{41}$$

where

$$B = \begin{pmatrix} (m_{11} - h)(\frac{1}{r} - 1) & m_{12}(\frac{1}{r} - 1) \\ m_{21}(\frac{1}{r} - 1) & (m_{22} - h)(\frac{1}{r} - 1) \end{pmatrix}$$

Since $r > 1$, by (40) and Lemma 4.6 we get

$$\sigma_1(\mathcal{L} + M_{(u_0,v_0)} + B) > \sigma_1(\mathcal{L} + M_{(u_0,v_0)}) > 0,$$

contradicting (41).

The following result provides us with a sufficient condition for $\sigma_1(\mathcal{L} + M_{(u_0,v_0)}) > 0$ to be hold.

Proposition 5.3. *Assume that $m = n$, $a(x), d(x) > 0$ for all $x \in \overline{\Omega}$, λ and μ satisfy* (19) *and that for any* (u_0, v_0) *coexistence state of* (3)

$$\left(\frac{b}{a}\right)_M \left(\frac{c}{d}\right)_M \left(\frac{u_0}{v_0}\right)_M^{(2-m)/m} \left(\frac{v_0}{u_0}\right)_M^{(2-m)/m} < 1. \tag{42}$$

Then, (3) *possesses a unique coexistence state.*

Proof. Let

$$\Phi := (\alpha u_0^{1/m}, -\beta v_0^{1/m}) \in (C^2(\Omega) \cap C_0^0(\overline{\Omega}))^2,$$

with $\alpha, \beta > 0$ to be chosen. We will show that Φ is a supersolution in the sense of Definition 4.1 of $\mathcal{L} + M_{(u_0,v_0)}$ if (42) holds. Proposition 4.3 and Theorem 5.2 will complete the proof.

Firstly, observe that $\Phi \succ 0$. In order to show that Φ is a supersolution of $\mathcal{L} + M_{(u_0,v_0)}$ we have to prove that (for the first equation)

$$L_1(\alpha u_0^{1/m}) + m_{11}(x)\alpha u_0^{1/m} + m_{12}(x)(-\beta v_0^{1/m}) > 0, \tag{43}$$

where m_{11} and m_{12} are defined in (38). Taking into account the fact that

$$L_1(u_0^{1/m}) = \frac{1}{m} u_0^{1/m-1}[(1 - \frac{1}{m})u_0^{-1} \sum_{i,j=1}^{N} a_{ij}^1 D_i u_0 D_j u_0 + L_1 u_0],$$

to prove (43) it suffices that

$$a(x) u_0^{2/m-1} > b(x) v_0^{2/m-1} \cdot \frac{\beta}{\alpha}, \quad \text{for all } x \in \Omega.$$

Analogously, for the second equation it is sufficient that

$$d(x) v_0^{2/m-1} > c(x) u_0^{2/m-1} \cdot \frac{\alpha}{\beta}, \quad \text{for all } x \in \Omega.$$

Now, by (42) it is easy to show that there exist α and β satisfying the above inequalities.

The following result provides us another sufficient condition to obtain a uniqueness result.

Proposition 5.4. *Assume that λ and μ satisfy* (19) *and that for any coexistence state* (u_0, v_0) *of* (3) *the following inequalities hold for all $x \in \Omega$,*

$$\begin{aligned} &\lambda(1 - \frac{1}{m}) + a(x) u_0^{1/m}(x)(\frac{2}{m} - 1) > b(x)\left(1 + \frac{1}{n} - \frac{1}{m}\right) v_0^{1/n}(x), \\ &\mu(1 - \frac{1}{n}) + d(x) v_0^{1/n}(x)(\frac{2}{n} - 1) > c(x)\left(1 + \frac{1}{m} - \frac{1}{n}\right) u_0^{1/m}(x). \end{aligned} \tag{44}$$

Then, (3) *possesses a unique coexistence state.*

Proof. Taking

$$\Phi := (u_0, -v_0),$$

it suffices to prove that $\Phi \succ 0$ is a supersolution of $\mathcal{L} + M_{(u_0,v_0)}$ provided that (44) and apply again Proposition 4.3 and Theorem 5.2. For the second equation, Φ is a supersolution if

$$L_2(-v_0) + m_{21}(x)(u_0) + m_{22}(x)(-v_0) < 0,$$

where m_{21} and m_{22} are defined in (38). For observe that

$$\begin{aligned} & L_2(-v_0) + m_{21}(x)(u_0) + m_{22}(x)(-v_0) \\ = {} & v_0^{1/n}\left(\mu\left(\tfrac{1}{n}-1\right) + d(x)v_0^{1/n}\left(1-\tfrac{2}{n}\right) + c(x)u_0^{1/m}\left(1+\tfrac{1}{m}-\tfrac{1}{n}\right)\right) < 0, \end{aligned}$$

provided that (44) holds. Similarly we can reason with the first equation.

Now, we will use the upper estimates of (20) and (21) giving sufficient conditions for the uniqueness of coexistence state in terms of several coefficients involved in the model setting.

Corollary 5.5. *Assume that $m = n$, $a(x), d(x) > 0$ for $x \in \overline{\Omega}$, λ and μ satisfy* (19) *and*

$$\left(\frac{(e_1)_M^{1/(m-1)}}{\varepsilon_1}\frac{(e_2)_M^{1/(m-1)}}{\varepsilon_2}\right)^{\frac{2-m}{m}} \left(\frac{e_1}{\varphi_2}\right)_M^{\frac{2-m}{m}} \left(\frac{e_2}{\varphi_1}\right)_M^{\frac{2-m}{m}} (\lambda\mu)^{\frac{2-m}{m-1}} < \frac{a_L d_L}{b_M c_M}, \tag{45}$$

where ε_1 and ε_2 are defined in (22)*. Then,* (3) *possesses a unique coexistence state.*

Proof. By (20) and (21) we have that

$$\left(\frac{u_0}{v_0}\right)_M \le \frac{\lambda^{m/(m-1)}(e_1)_M^{1/(m-1)}}{\varepsilon_2}\left(\frac{e_1}{\varphi_2}\right)_M,$$

and

$$\left(\frac{v_0}{u_0}\right)_M \le \frac{\mu^{m/(m-1)}(e_2)_M^{1/(m-1)}}{\varepsilon_1}\left(\frac{e_2}{\varphi_1}\right)_M,$$

and so, (42) is satisfied if (45) holds. It suffices to apply Proposition 5.3.

Corollary 5.6. *Assume that some of the following sets of inequality, 1 to 4, holds:*

1. *If $1 < m, n \le 2$,*

$$b_M\left(1+\frac{1}{n}-\frac{1}{m}\right)\mu^{1/(n-1)}(e_2)_M^{1/(n-1)} < \lambda(1-\frac{1}{m}),$$

$$c_M\left(1+\frac{1}{m}-\frac{1}{n}\right)\lambda^{1/(m-1)}(e_1)_M^{1/(m-1)} < \mu(1-\frac{1}{n}),$$

2. *If* $1 < n \leq 2$ *and* $m > 2$,

$$b_M\left(1+\tfrac{1}{n}-\tfrac{1}{m}\right)\mu^{1/(n-1)}(e_2)_M^{1/(n-1)} + (1-\tfrac{2}{m})a_M\lambda^{1/(m-1)}(e_1)_M^{1/(m-1)} < \lambda(1-\tfrac{1}{m}),$$

$$c_M\left(1+\tfrac{1}{m}-\tfrac{1}{n}\right)\lambda^{1/(m-1)}(e_1)_M^{1/(m-1)} < \mu(1-\tfrac{1}{n}),$$

3. *If* $1 < m \leq 2$ *and* $n > 2$,

$$c_M\left(1+\tfrac{1}{m}-\tfrac{1}{n}\right)\lambda^{1/(m-1)}(e_1)_M^{1/(m-1)} + (1-\tfrac{2}{n})d_M\mu^{1/(n-1)}(e_2)_M^{1/(n-1)} < \mu(1-\tfrac{1}{n}),$$

$$b_M\left(1+\tfrac{1}{n}-\tfrac{1}{m}\right)\mu^{1/(n-1)}(e_2)_M^{1/(n-1)} < \lambda(1-\tfrac{1}{m}),$$

4. *If* $m > 2$ *and* $n > 2$,

$$b_M\left(1+\tfrac{1}{n}-\tfrac{1}{m}\right)\mu^{1/(n-1)}(e_2)_M^{1/(n-1)} + (1-\tfrac{2}{m})a_M\lambda^{1/(m-1)}(e_1)_M^{1/(m-1)} < \lambda(1-\tfrac{1}{m}),$$

$$c_M\left(1+\tfrac{1}{m}-\tfrac{1}{n}\right)\lambda^{1/(m-1)}(e_1)_M^{1/(m-1)} + (1-\tfrac{2}{n})d_M\mu^{1/(n-1)}(e_2)_M^{1/(n-1)} < \mu(1-\tfrac{1}{n}),$$

then, (3) *possesses a unique coexistence state.*

Proof. Reasoning as in the proof of Corollary 5.5, it is sufficient to apply (20), (21) and Proposition 5.4.

Remark. 1. Observe that when $m = 1$, (42) is the condition obtained in Theorem 4.2 in [28] and Theorem 4.8 in [22]. Moreover, when $m = n$, and a and d are positive, we obtain uniqueness provided that b_M or c_M is small.

2. The (λ, μ)-regions defined in Corollary 5.6 are subsets of the coexistence region obtained in Theorem 3.2. Similar conditions to those imposed in Figure 1 assure the existence of these subregions.

6. Conclusions

We have studied the set of non-negative solutions of a spatially heterogeneous Lotka-Volterra competition model with degenerate diffusion. Basically, we have found three differences with the respect to the non-degenerate (linear) case:

1. In the degenerate case all the non-negative solutions are bounded, unlike the linear case in which a-priori bounds are lost for some values of the data of the problem.

2. In the degenerate case a new kind of non-negative solutions appears: non-negative and nontrivial solutions that vanish in a region of the habitat of the species. We obtain sufficient conditions in terms of some parameters involved in the setting of the model ensuring the existence or non-existence of such kind of solutions.

3. Unlike the non-degenerate case, in our model when the competition between the species is “strong” neither species drives to the other to extinction.

Finally, we have obtained uniqueness of positive solution of the problem under some conditions on the data of the problem.

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References

[1] H. Amann, “On the existence of positive solution of nonlinear elliptic boundary value problems”, *Indiana Univ. Math. J.*, **21** (1971) 125-146.

[2] H. Amann, “Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces”, *SIAM Rev.*, **18** (1976) 620-709.

[3] C. Bandle, M. A. Pozio and A. Tesei, “The asymptotic behaviour of the solutions of degenerate parabolic equations”, *Trans. Amer. Math. Soc.*, **303** (1987) 487-501.

[4] H. Berestycki and P. L. Lions, “Some applications of the method of sub and supersolutions”, in *Bifurcation and nonlinear eigenvalue problem*, Springer Lectures Notes, **782** (1980) 16-41.

[5] M. Bertsch and R. Rostamian, “The principle of linearized stability for a class of degenerate diffusion equations”, *J. Differential Equations*, **57** (1985) 373-405.

[6] J. Blat and K. J. Brown, “Bifurcation of steady-state solutions in predator-prey and competition systems”, *Proc. Roy. Soc. Edinburgh Sect. A*, **97** (1984) 21-34.

[7] R. S. Cantrell and C. Cosner, “On the steady-state problem for the Volterra-Lotka competition model with diffusion”, *Houston J. Math.*, **13** (1987) 337-352.

[8] R. S. Cantrell and C. Cosner, “Should a park be an island?”, *SIAM J. Math. Anal.*, **53** (1993) 219-252.

[9] A. Cañada and J. L. Gámez, “Elliptic systems with nonlinear diffusion in population dynamics”, *Differential Equations Dynam. Systems*, **3** (1995) 189-204.

[10] C. Cosner and A. C. Lazer, “Stable coexistence states in the Volterra-Lotka competition model with diffusion”, *SIAM J. Appl. Math.*, **44** (1984) 1112-1132.

[11] E. N. Dancer, "On the indices of fixed points of mappings in cones and applications", *J. Math. Anal. Appl.* **91** (1983) 131-151.

[12] E. N. Dancer, "On positive solutions of some pairs of differential equations, part I", *Trans. Amer. Math. Soc.*, **284** (1984) 729-743.

[13] E. N. Dancer, "On the existence and uniqueness of positive solutions for competing species models with diffusion", *Trans. Amer. Math. Soc.* **326** (1991) 828-859.

[14] M. Delgado and A. Suárez, "On the existence of dead cores for degenerate Lotka-Volterra models", *Proc. Roy. Soc. Edinburgh Sect. A*, **130** (2000) 743-766.

[15] M. Delgado and A. Suárez, "On the structure of the positive solutions of the logistic equation with nonlinear diffusion" , *J. Math. Anal. Appl.*, **268** (2002) 200-216.

[16] M. Delgado and A. Suárez, "Stability and uniqueness for cooperative degenerate Lotka-Volterra model", *Nonlinear Anal.*, **49** (2002) 757-778.

[17] J. I. Díaz and J. Hernández, "On the existence of a free boundary for a class of reaction-diffusion systems", *SIAM. J. Math. Anal.*, **15** (1984) 670-685.

[18] Y. Du, "Effects of a degeneracy in the competition model: parts I and II", *J. Differential Equations*, **181** (2002) 92-132 and 133-164.

[19] Y. Du and K. J. Brown, "Bifurcation and monotonicity in competetion reaction-diffusion systems", *Nonlinear Anal.*, **23** (1994) 1-13.

[20] J. C. Eilbeck, J. E. Furter and J. López-Gómez, "Coexistence in the competition model with diffusion", *J. Differential Equations*, **107** (1994), 96-139.

[21] J. E. Furter and J. López-Gómez, "On the existence and uniqueness of coexistence states for the Lotka-Volterra competition model with diffusion and spatially dependent coefficients", *Nonlinear Anal.*, **25** (1995) 363-398.

[22] J. E. Furter and J. López-Gómez, "Diffusion-mediated permanence problem for an heterogeneus Lotka-Volterra competition model", *Proc. Roy. Soc. Edinburgh Sect. A*, **127** (1997) 281-336.

[23] M. E. Gurtin and R. C. MacCamy, "On the diffusion of biological populations", *Math. Biosci.*, **33** (1977) 35-49.

[24] A. W. Leung and G. Fan, "Existence of positive solutions for elliptic systems—degenerate and nondegenerate ecological models", *J. Math. Anal. Appl.* **151** (1990) 512-531.

[25] L. Li and R. Logan, "Positive solutions to general elliptic competition models", *Differential Integral Equations*, **4** (1991) 817-834.

[26] J. López-Gómez, "On the structure of the permanence region for the competing species models with general diffusivities and transports effects", *Discrete Contin. Dynam. Systems*, **2** (1996) 525-542.

[27] J. López-Gómez, "Coexistence and metacoexistence states in competing species model", *Houston J. Math.*, **29** (2003) 483-536.

[28] J. López-Gómez and J. C. Sabina de Lis, "Coexistence states and global attractivity for some convective diffusive competing species models", *Trans. Amer. Math. Soc.*, **347** (1995) 3797-3833.

[29] T. Namba, "Density-dependent dispersal and spatial distribution of a population", *J. Theor. Biol.*, **86** (1980) 351-363.

[30] M. A. Pozio and A. Tesei, "Support properties of solutions for a class of degenerate parabolic problems", *Commun. Partial Differential Equations*, **12** (1987) 47-75.

[31] M. A. Pozio and A. Tesei, "Degenerate parabolic problems in population dynamics", *Japan J. Appl. Math.*, **2** (1985) 351-380.

[32] W. H. Ruan and C. V. Pao, "Positive steady-state solutions of a competing reaction-diffusion system", *J. Differential Equations*, **117** (1995) 401-427.

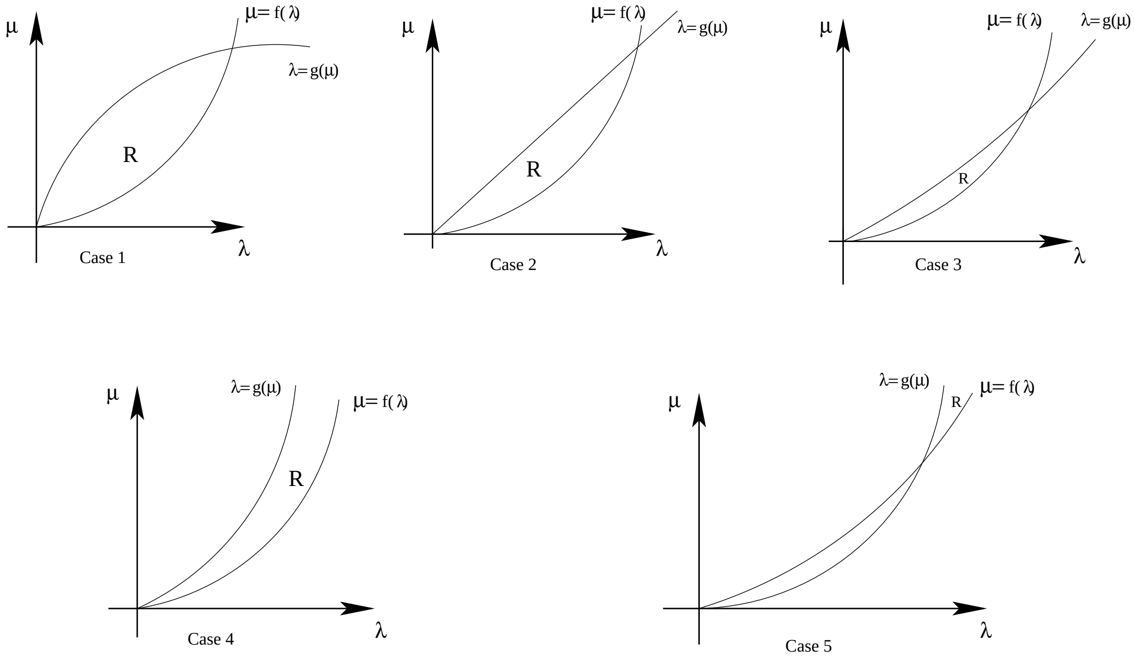

FIGURE 1. R stands for the semicoexistence region in the following cases: Case 1: $1 < m, n < 2$; Case 2: $1 < m < 2 = n$; Case 3: $1 < m < 2 < n$, $n-1 < 1/(m-1)$; Case 4: $1 < m < 2 < n$, $n-1 = 1/(m-1)$, $c_M b_M^{n-1}(e_1)_M^{n-1}(e_2)_M^{n-1} < 1$; Case 5: $1 < m < 2 < n$, $n-1 > 1/(m-1)$.