Explicit models for perverse sheaves

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Abstract

We consider categories of generalized perverse sheaves, with relaxed constructibility conditions, by means of the process of gluing t-structures and we exhibit explicit abelian categories defined in terms of standard sheaves categories which are equivalent to the former ones. In particular, we are able to realize perverse sheaves categories as non full abelian subcategories of the usual bounded complexes of sheaves categories. Our methods use induction on perversities. In this paper, we restrict ourselves to the two-strata case, but our results extend to the general case.

KEYWORDS: perverse sheaf, derived category, t-structure, stratified space, abelian category.
MSC: 18E30, 32S60, 14F43.

Introduction

Perverse sheaves first appear in context of Complex Analytic Geometry by the coming together of the Riemann-Hilbert correspondence of Mebkhout-Kashiwara and the Intersection Cohomology of Goresky-MacPherson at the beginning of the 1980s. In the work [1] the notion of t-structure over a triangulated category was extracted and it was proved that the category of analytic constructible perverse sheaves, that we call "classical perverse sheaves", can be obtained by a general process of "gluing" t-structures, that makes sense in a much more general framework. In fact, the main contribution of *loc. cit.* is the use of that process to define ℓ-adic perverse sheaves over algebraic varieties in positive characteristics and to prove the theorem of purity of intersection complexes.

In this paper, we develop some ideas and complete some results in [13] and [5] on the core of the t-structure obtained by gluing standard t-structures shifted by "perversities" of strata, as in the classical case but without imposing necessarily any constructibility conditions. Objects in this core can be thought of as "generalized perverse sheaves".

In the complex analytic case, and when we consider the middle perversity, the category of classical perverse sheaves is a full abelian subcategory of that of generalized perverse

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sheaves. Furthermore, a classical perverse sheaf is the same as a generalized perverse sheaf which is complex analytic constructible.

The advantage of our point of view consists of being able to work simultaneously with different perversities and to establish some precise relations between perverse sheaves with respect to different perversities, which we do not know how to do if we are restricted to the classical case.

Our main result is theorem (3.2.2), from which we deduce (see 4.1) that any (generalized) perverse sheaf, and then any classical perverse sheaf, has a canonical model (16). As a consequence, the category of (generalized) perverse sheaves is equivalent to a non full (resp. full) abelian subcategory of the category of the usual bounded complexes (resp. up to homotopy).

The main idea consists of constructing a functor Φ relating d-perverse and $(d-1)$-perverse sheaves, and by iteration, d-perverse sheaves with 0-perverse sheaves, which are nothing but usual sheaves. In this way we develop the idea pointed out in [13], rem. (2.3.7), where we were restricted to the "conical" case.

Construction of functor Φ and many other results in this paper are inspired by the formalism of vanishing cycles [3] and the gluing of classical perverse sheaves of Deligne-Verdier [2, 14] and MacPherson-Vilonen [9], but our framework is more general.

In order to simplify, in this paper we restrict ourselves to the two-strata case, but our results extend to the general case.

Let us now comment on the content of this paper.

In section 1 we recall first the gluing process of t-structures and the notion of (generalized) perverse sheaf is introduced. Second, we recall some elementary constructions with adjoint functors that play a fundamental role in the proof of theorem (3.2.2) and in the manipulation of our explicit models for perverse sheaves.

Section 2 deals with the construction of functor Φ and the "induction on perversities" (2.3.1).

In section 3 we show the main result in this paper, namely that the category of (generalized) perverse sheaves Perv is equivalent to an explicit abelian category described in terms of abelian categories of usual sheaves.

In section 4 we give some applications of theorem (3.2.2). First, we associate to any perverse sheaf a canonical model. More precisely, we lift the inclusion functor of Perv into the derived category to a faithful exact functor into the category of usual bounded complexes.

Second, we lift the inclusion functor of Perv into the derived category to a fully faithful functor into the category of bounded complexes up to homotopy, K^b. In particular, Perv can be realized as a full abelian subcategory of K^b.

Third, we give quiver descriptions of conical perverse sheaves with respect to a $K(\pi, 1)$ basis.

Finally, we compute in terms of our listed canonical models the different perverse direct images and the intersection complex associated to a sheaf on the open stratum, and we announce further results.

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1 Preliminaries and notations

1.1 Perverse sheaves

Let X be a topological space stratified by $\Sigma = \{C, U\}$, where $i : C \to X$ is a closed immersion and $j : U = X - C \to X$ is its complementary dense open immersion. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_U = j^*\mathcal{O}_X, \mathcal{O}_C = i^*\mathcal{O}_X$. For $* = X, U, C$, let us denote by $\mathfrak{B}_*$ the abelian category of sheaves of $\mathcal{O}_*$-modules, and let $\mathfrak{A}_* \subset \mathfrak{B}_*$ be full abelian subcategories stable for kernels, cokernels and extensions. Let us denote by $\mathcal{D}_* := \mathrm{D}^+_{\mathfrak{A}_*}(\mathfrak{B}_*)$ the full triangulated subcategory of the derived category $\mathrm{D}(\mathfrak{B}_*)$ whose objects are bounded below complexes with cohomology in $\mathfrak{A}_*$. Let us suppose that the usual functors $i_* = i_!, i^*, \mathbb{R}i^!, \mathbb{R}j_*, j_!, j^* = j^!$ induce functors

$$\mathcal{D}_C \underset{i^*,\mathbb{R}i^!}{\overset{i_*=i_!}{\rightleftarrows}} \mathcal{D}_X \underset{\mathbb{R}j_*,j_!}{\overset{j^*=j^!}{\rightleftarrows}} \mathcal{D}_U$$

in such a way that we are in the conditions of gluing t-structures on $\mathcal{D}_U$ and on $\mathcal{D}_C$ [1].

(1.1.1) Example.
(1) If $\mathfrak{A}_* = \mathfrak{B}_*$, then $\mathcal{D}_* = \mathrm{D}^+(\mathfrak{B}_*)$.
(2) Let S be a compact topological space, X the cone of S, C its vertex, $\mathcal{O}_X$ the constant sheaf with fiber a ring (resp. a noetherian ring) k and $\mathfrak{A}_*$ the abelian categories of Σ-constructible sheaves of k-modules not necessarily finitely generated (resp. finitely generated).
(3) The space X is a pseudomanifold stratified by Σ, $\mathcal{O}_X$ is the constant sheaf with a field k as fiber, and the $\mathfrak{A}_*$ are the abelian categories of Σ-constructible sheaves of k-vector spaces of arbitrary (resp. of finite) rank. For instance, X can be a complex analytic space and $C \subset X$ a smooth closed analytic set satisfying the Whitney conditions.

(1.1.2) Definition. *For any integer $d \geq 0$, the category of d-*perverse sheaves *on X with respect to the stratification Σ,* $\mathrm{Perv}^d(X, \Sigma)$*, is the core of the t-structure on $\mathcal{D}_X$ obtained by gluing the natural t-structure on $\mathcal{D}_U$ and the image by $[-d]$ of the natural t-structure on $\mathcal{D}_C$ [1]. We will say that the* perversity *of the stratum C (resp. U) is d (resp. 0).*

(1.1.3) Remark. Observe that, if $d = 0$, then the category $\mathrm{Perv}^0(X, \Sigma)$ coincides with the category $\mathfrak{A}_X$.

(1.1.4) Proposition. (Characterization of d-perverse sheaves) *An object K of $\mathcal{D}_X$ is a d-perverse sheaf (with respect to Σ) if and only if the following properties hold:*

(a) K is concentrated in degrees $[0, d]$,

(b) j^*K *is concentrated in degree* 0*,*

(c) $h^n\mathbb{R}i^!K = 0$ *for* $n < d$*.*

PROOF. By definition of $\mathrm{Perv}^d(X,\Sigma)$, an object K of $\mathcal{D}_X$ is a d-perverse sheaf if and only if $h^n(j^*K) = 0$ for $n \neq 0$, $h^n i^*K = 0$ for $n > d$ and $h^n\mathbb{R}i^!K = 0$ for $n < d$, and it is clear that a K satisfying properties (a), (b), (c) is d-perverse.

Let us now take a d-perverse sheaf K. Properties (b) and (c) are clear. The long exact sequence associated with the triangle

$$j_!j^*K \longrightarrow K \longrightarrow i_*i^*K \xrightarrow{+1}$$

gives rise to isomorphisms $h^l(K) \simeq h^l(i_*i^*K)$ for any $l \geq 1$ and then $h^l(K) = 0$ for any $l > d$.

In a similar way, the long exact sequence associated with the triangle

$$i_*\mathbb{R}i^!K \longrightarrow K \longrightarrow \mathbb{R}j_*j^*K \xrightarrow{+1}$$

and the fact that $\mathbb{R}j_*j^*K$ is concentrated in non-negative degrees gives rise to isomorphisms $h^l(i_*\mathbb{R}i^!K) \simeq h^l(K)$ for $l < 0$ and then $h^l(K) = 0$ for any $l < 0$, and K is concentrated in degrees $[0,d]$. Q.E.D.

1.2 Functors acting on morphisms of functors

Let $\mathcal{B},\mathcal{C}$ be categories, $F,G : \mathcal{B} \to \mathcal{C}$ functors and $\tau : F \to G$ a morphism of functors (or natural transformation) which associates to any object B in $\mathcal{B}$ a morphism $\tau_B : FB \longrightarrow GB$ in $\mathcal{C}$ with the usual naturality properties.

For any functors $E : \mathcal{A} \to \mathcal{B}, H : \mathcal{C} \to \mathcal{D}$ we denote by $\tau E : FE \to GE$, $H\tau : HF \to HG$ the morphisms given by

$$(\tau E)_A = \tau_{EA}, \quad (H\tau)_B = H(\tau_B)$$

for any objects A in $\mathcal{A}$ and B in $\mathcal{B}$.

(1.2.1) We have the following rules:

(a) $H(\tau E) = (H\tau)E$, $H1_F = 1_{HF}$, $1_F E = 1_{FE}$.

(b) $(\tau \circ \varepsilon)E = (\tau E) \circ (\varepsilon E)$, $H(\tau \circ \varepsilon) = (H\tau) \circ (H\varepsilon)$ for any other morphism $\varepsilon : F' \to F$.

(c) $(\sigma G) \circ (K\tau) = (L\tau) \circ (\sigma F)$ for any other functors $K, L : \mathcal{C} \to \mathcal{D}$ and any other morphism $\sigma : K \to L$.

(d) $(\tau + \tau')E = (\tau E) + (\tau' E)$, $\quad H(\tau + \tau') = (H\tau) + (H\tau')$ in the case of additive functors between additive categories.

1.3 Adjoint functors

(1.3.1) In this section, we consider a couple of adjoint additive functors $G : \mathcal{A} \to \mathcal{A}'$, $F : \mathcal{A}' \to \mathcal{A}$ between abelian categories with adjunction morphisms $\alpha : \mathrm{Id}_{\mathcal{A}} \to FG$ and $\beta : GF \to \mathrm{Id}_{\mathcal{A}'}$ such that F is left exact, G is exact and α is injective. We denote $\mathbb{F} := FG$ and $(\mathbb{Q}, q) := \operatorname{coker} \alpha$. We have then a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Id}_{\mathcal{A}} \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \mathbb{F} \arrow[r, "q"] \arrow[d, "\alpha\mathbb{F}"'] & \mathbb{Q} \arrow[r] \arrow[d, "\alpha\mathbb{Q}"'] & 0 \\
0 \arrow[r] & \mathbb{F} \arrow[r, "\mathbb{F}\alpha"] \arrow[d, "q"'] & \mathbb{F}\mathbb{F} \arrow[r, "\mathbb{F}q"] \arrow[d, "q\mathbb{F}"'] & \mathbb{F}\mathbb{Q} \arrow[d, "q\mathbb{Q}"'] & \\
0 \arrow[r] & \mathbb{Q} \arrow[r, "\mathbb{Q}\alpha"] \arrow[d] & \mathbb{Q}\mathbb{F} \arrow[r, "\mathbb{Q}q"] \arrow[d] & \mathbb{Q}\mathbb{Q} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{1}$$

(1.3.2) Let us call $\gamma : \mathbb{Q} \longrightarrow \mathbb{F}\mathbb{F}$ the unique morphism satisfying $\gamma \circ q = \alpha\mathbb{F} - \mathbb{F}\alpha$. From (1) we deduce the relations

$$(\mathbb{F}q) \circ \gamma = \alpha\mathbb{Q}, \quad (q\mathbb{F}) \circ \gamma = -\mathbb{Q}\alpha.$$

(1.3.3) From the adjunction properties, the exact sequence

$$0 \longrightarrow G \xrightarrow{G\alpha} G\mathbb{F} \xrightarrow{Gq} G\mathbb{Q} \longrightarrow 0$$

splits, with retraction $\beta G : GFG \longrightarrow G$. Then the sequence

$$0 \longrightarrow \mathbb{F} \xrightarrow{\mathbb{F}\alpha} \mathbb{F}\mathbb{F} \xrightarrow{\mathbb{F}q} \mathbb{F}\mathbb{Q} \longrightarrow 0 \tag{2}$$

is exact and splits, with retraction $\nu := F\beta G$. Let us call $\mu : \mathbb{F}\mathbb{Q} \to \mathbb{F}\mathbb{F}$ the corresponding section, i.e.

$$\mu \circ (\mathbb{F}q) = 1_{\mathbb{F}^2} - (\mathbb{F}\alpha) \circ \nu, \quad (\mathbb{F}q) \circ \mu = 1_{\mathbb{F}\mathbb{Q}}.$$

(1.3.4) With the above notations, the relation $\nu \circ (\alpha\mathbb{F}) = 1_{\mathbb{F}}$ holds and the sequence

$$0 \longrightarrow \mathbb{F} \xrightarrow{\alpha\mathbb{F}} \mathbb{F}\mathbb{F} \xrightarrow{q\mathbb{F}} \mathbb{Q}\mathbb{F} \longrightarrow 0$$

also splits with the same retraction as in (2). Let us call $\mu' : \mathbb{Q}\mathbb{F} \to \mathbb{F}\mathbb{F}$ the corresponding section, i.e.

$$\mu' \circ (q\mathbb{F}) = 1_{\mathbb{F}^2} - (\alpha\mathbb{F}) \circ \nu, \quad (q\mathbb{F}) \circ \mu' = 1_{\mathbb{Q}\mathbb{F}}.$$

We have $\gamma = \mu \circ (\alpha\mathbb{Q})$.

Functors $\mathbb{FQ}$ and $\mathbb{QF}$ are canonically isomorphic by means of $h := (q\mathbb{F}) \circ \mu : \mathbb{FQ} \to \mathbb{QF}$ and its inverse $h^{-1} = (\mathbb{F}q) \circ \mu'$.

(1.3.5) Lemma. *For any objects A, B in $\mathcal{A}$, the sequence*

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Q}A, \mathbb{F}B) \xrightarrow{q_A^*} \mathrm{Hom}(\mathbb{F}A, \mathbb{F}B) \xrightarrow{\alpha_A^*} \mathrm{Hom}(A, \mathbb{F}B) \longrightarrow 0$$

is exact and splits.

Proof. From (1.3.4), application

$$f \in \mathrm{Hom}(A, \mathbb{F}B) \mapsto \nu_B \circ (\mathbb{F}f) \in \mathrm{Hom}(\mathbb{F}A, \mathbb{F}B)$$

is a section of the above sequence. Q.E.D.

2 Construction of categories and functors

2.1 The functor Ω

(2.1.1) Let $\mathfrak{A}$ be a category. Let us denote by $\mathrm{Arr}(\mathfrak{A})$ the category of arrows of $\mathfrak{A}$, by $s, t : \mathrm{Arr}(\mathfrak{A}) \to \mathfrak{A}$ the functors defined by

$$s(A \xrightarrow{u} B) := A, \quad t(A \xrightarrow{u} B) := B$$

and by $\zeta : s \to t$ the morphism defined by $\zeta_{(A \xrightarrow{u} B)} := u$.

If $\mathfrak{A}$ is abelian, the category $\mathrm{Arr}(\mathfrak{A})$ is also abelian and functors s, t are exact and induce exact functors $\overline{s}, \overline{t} : \mathrm{C}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{C}(\mathfrak{A})$. They induce triangulated functors $\mathrm{K}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{K}(\mathfrak{A})$, $\mathrm{D}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{D}(\mathfrak{A})$, also denoted by $\overline{s}, \overline{t}$. Let us denote by $\overline{\zeta} : \overline{s} \to \overline{t}$ the morphism of functors induced by ζ.

(2.1.2) The functor $N : \mathrm{C}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{Arr}(\mathrm{C}(\mathfrak{A}))$ defined by $N = \overline{s} \xrightarrow{\overline{\zeta}} \overline{t}$ is an isomorphism of abelian categories. In a similar way we define functors $N : \mathrm{K}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{Arr}(\mathrm{K}(\mathfrak{A}))$, $N : \mathrm{D}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{Arr}(\mathrm{D}(\mathfrak{A}))$, which are no longer equivalence of categories. Nevertheless, a morphism in $\mathrm{K}(\mathrm{Arr}(\mathfrak{A}))$ is a quasi-isomorphism if and only if its images by $\overline{s}$ and $\overline{t}$ are quasi-isomorphisms.

For any abelian category $\mathfrak{A}$ and any object $(U \xrightarrow{\beta} V) \in \mathrm{C}(\mathrm{Arr}(\mathfrak{A})) \equiv \mathrm{Arr}(\mathrm{C}(\mathfrak{A}))$ we define

$$\Omega(U \xrightarrow{\beta} V) := (V \xrightarrow{q} \mathrm{cone}(\beta)) \in \mathrm{C}(\mathrm{Arr}(\mathfrak{A})),$$

where q is the canonical inclusion. One can easily define the action of Ω on morphisms and we obtain an exact functor $\Omega : \mathrm{C}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{C}(\mathrm{Arr}(\mathfrak{A}))$ which commutes (up to isomorphism) with the translation functor and satisfies $\overline{s}\Omega = \overline{t}$.

(2.1.3) PROPOSITION. *The functor Ω above induces a triangulated functor $\Omega_{\mathrm{K}} : \mathrm{K}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{K}(\mathrm{Arr}(\mathfrak{A}))$ such that $\overline{s}\Omega_{\mathrm{K}} = \overline{t}$.*

PROOF. It is an exercise we leave to the reader. Q.E.D.

The definition of distinguished triangles in $\mathrm{K}(\mathfrak{A})$ gives rise to a morphism $\vartheta : \overline{t}\Omega_{\mathrm{K}} \to \overline{s}[1]$ in such a way that the following triangle of functors

$$\overline{s} \xrightarrow{\overline{\zeta}} \overline{t} = \overline{s}\Omega_{\mathrm{K}} \xrightarrow{\overline{\zeta}\Omega_{\mathrm{K}}} \overline{t}\Omega_{\mathrm{K}} \xrightarrow{\vartheta} \overline{s}[1] \tag{3}$$

is distinguished, i.e. its evaluation on any object of $\mathrm{K}(\mathrm{Arr}(\mathfrak{A}))$ is a distinguished triangle of $\mathrm{K}(\mathfrak{A})$.

The following proposition is basically the same as the axiom (TR2) of triangulated categories for $\mathrm{K}(\mathfrak{A})$ ([15], chap. I, prop. 3.3.3).

(2.1.4) PROPOSITION. *Under the above hypothesis, there is an isomorphism of functors $\chi : \overline{s}[1] \xrightarrow{\simeq} \overline{t}\Omega^2_{\mathrm{K}}$ such that the following diagram is commutative:*

$$\begin{array}{ccccc}
\overline{t}\Omega_{\mathrm{K}} & \xrightarrow{\vartheta} & \overline{s}[1] & \xrightarrow{-\overline{\zeta}[1]} & \overline{t}[1] \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle \chi}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle =}\big\downarrow \\
\overline{s}\Omega^2_{\mathrm{K}} & \xrightarrow{\overline{\zeta}\Omega^2_{\mathrm{K}}} & \overline{t}\Omega^2_{\mathrm{K}} & \xrightarrow{\vartheta\Omega_{\mathrm{K}}} & \overline{s}[1]\Omega_{\mathrm{K}}.
\end{array}$$

(2.1.5) PROPOSITION. *The functor $\Omega_{\mathrm{K}} : \mathrm{K}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{K}(\mathrm{Arr}(\mathfrak{A}))$ transforms quasi-isomorphisms into quasi-isomorphisms and then it induces a triangulated functor $\Omega_{\mathrm{D}} : \mathrm{D}(\mathrm{Arr}(\mathfrak{A})) \to \mathrm{D}(\mathrm{Arr}(\mathfrak{A}))$ such that $\overline{s}\Omega_{\mathrm{D}} = \overline{t}$. Moreover, there is a morphism $\vartheta : \overline{t}\Omega_{\mathrm{D}} \to \overline{s}[1]$ and an isomorphism $\chi : \overline{s}[1] \xrightarrow{\simeq} \overline{t}\Omega^2_{\mathrm{D}}$ such that the following triangle of functors*

$$\overline{s} \xrightarrow{\overline{\zeta}} \overline{t} = \overline{s}\Omega_{\mathrm{D}} \xrightarrow{\overline{\zeta}\Omega_{\mathrm{D}}} \overline{t}\Omega_{\mathrm{D}} \xrightarrow{\vartheta} \overline{s}[1] \tag{4}$$

is distinguished and the following diagram is commutative:

$$\begin{array}{ccccc}
\overline{t}\Omega_{\mathrm{D}} & \xrightarrow{\vartheta} & \overline{s}[1] & \xrightarrow{-\overline{\zeta}[1]} & \overline{t}[1] \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle \chi}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle =}\big\downarrow \\
\overline{s}\Omega^2_{\mathrm{D}} & \xrightarrow{\overline{\zeta}\Omega^2_{\mathrm{D}}} & \overline{t}\Omega^2_{\mathrm{D}} & \xrightarrow{\vartheta\Omega_{\mathrm{D}}} & \overline{s}[1]\Omega_{\mathrm{D}}.
\end{array}$$

PROOF. The first part follows from the relation $\overline{s}\Omega_{\mathrm{K}} = \overline{t}$, from the fact that a morphism ξ in $\mathrm{K}(\mathrm{Arr}(\mathfrak{A}))$ is a quasi-isomorphism if and only if $\overline{s}(\xi), \overline{t}(\xi)$ are quasi-isomorphisms and from triangle (3).

The second part is basically the axiom (TR2) for the triangulated category $\mathrm{D}(\mathfrak{A})$ and follows from triangle (3) and from proposition (2.1.4). Q.E.D.

(2.1.6) REMARK. The functor $\Omega_{\mathfrak{A}}$ defined in [12, 13] is related to the functor Ω_{D} above by the equality $\Omega_{\mathfrak{A}} = N\Omega_{\mathrm{D}}$.

(2.1.7) Let us denote by $\mathfrak{Q}$ the full subcategory of $\mathrm{Arr}(\mathrm{D}(\mathfrak{A}))$ whose objects are the $A \xrightarrow{v} B$ such that A is concentrated in degree 0 and B is concentrated in degrees ≥ 0. For such objects the morphism v (in $\mathrm{D}(\mathfrak{A})$) is determined by its cohomology of degree 0. More precisely we have the following result:

(2.1.8) PROPOSITION. *Functor N defines an equivalence of (additive) categories between $N^{-1}\mathfrak{Q}$ and $\mathfrak{Q}$.*

PROOF. We sketch the definition of a quasi-inverse of $N : N^{-1}\mathfrak{Q} \to \mathfrak{Q}$ and leave the details to the reader.

Given an object $Y = (A \xrightarrow{v} B)$ in $\mathfrak{Q}$, let U, V be the complexes defined by $U^0 = h^0 A$, $U^n = 0$ for $n \neq 0$ and

$$V = \tau_{\geq 0} B = \cdots \longrightarrow 0 \longrightarrow \operatorname{coker} d_B^{-1} \xrightarrow{\overline{d_B^0}} B^1 \xrightarrow{d_B^1} \cdots ,$$

where $\operatorname{coker} d_B^{-1}$ is placed in degree 0, and let $\widetilde{v} : U \to V$ the morphism of complexes determined by $\widetilde{v}^0 = h^0 v : U^0 \to h^0 B \subset V^0$.

Correspondence $Y \mapsto (U \xrightarrow{\widetilde{v}} V)$ extends to a functor $\overline{N} : \mathfrak{Q} \to N^{-1}\mathfrak{Q}$. It is easy to see that $N\overline{N} \simeq \mathrm{Id}_{\mathfrak{Q}}$.

On the other hand, for any object $X = (U \xrightarrow{\beta} V)$ in $N^{-1}\mathfrak{Q}$, the following commutative diagram in $\mathrm{C}(\mathfrak{A})$

$$\begin{tikzcd}
U \arrow[r, "\beta"] \arrow[d, "\text{nat.}"'] & V \arrow[d, "\text{nat.}"] \\
\tau_{\geq 0} U \arrow[r, "\tau_{\geq 0}\beta"] & \tau_{\geq 0} V \\
h^0 U \arrow[r, "\widetilde{\beta}"] \arrow[u, "\text{nat.}"] & \tau_{\geq 0} V \arrow[u, "="']
\end{tikzcd}$$

defines a natural isomorphism

$$\kappa(X) : X \longrightarrow \overline{N}(N(X)). \tag{5}$$

Q.E.D.

(2.1.9) Let us call $C = (\bar{t}\Omega_{\mathrm{D}}\overline{N})[-1] : \mathfrak{Q} \to \mathrm{D}(\mathfrak{A})$.

An object $(U \xrightarrow{\beta} V) \in \mathrm{D}(\mathrm{Arr}(\mathfrak{A}))$ is in $\mathfrak{P} := \Omega_{\mathrm{D}}^{-1} N^{-1}\mathfrak{Q}$ if and only if the complex U is concentrated in degrees ≥ 0, V is concentrated in degree 0 and $h^0\beta$ is injective.

From propositions (2.1.5) and (2.1.8) we obtain an isomorphism

$$\eta := (\bar{t}\Omega_{\mathrm{D}}\kappa\Omega_{\mathrm{D}})[-1] \circ \chi[-1] : \bar{s} \longrightarrow CN\Omega_{\mathrm{D}} \tag{6}$$

between functors from $\mathfrak{P}$ to $\mathrm{D}(\mathfrak{A})$.

(2.1.10) For any object $Y = (A \xrightarrow{v} B) \in \mathfrak{Q}$ such that $A^n = 0$ for all $n \neq 0$ and $B^n = 0$ for all $n < 0$ we can identify (by a canonical isomorphism)

$$C(Y) = \cdots \longrightarrow 0 \longrightarrow A^0 \xrightarrow{-v^0} B^0 \xrightarrow{-d_B^0} B^1 \xrightarrow{-d_B^1} \cdots , \tag{7}$$

where A^0 is placed in degree 0.

Consequently, for any object $X = (U \xrightarrow{\beta} V) \in \mathfrak{P}$ such that U and V are concentrated in degree 0 ($h^0\beta$ must be injective), we can identify

$$(CN\Omega_{\mathrm{D}})(X) = \cdots \to 0 \to h^0 V \xrightarrow{-\text{nat.}} \operatorname{coker} h^0\beta \to 0 \to \cdots$$

placed in degrees $0, 1$ and isomorphism $\eta_X : U \to (CN\Omega_{\mathrm{D}})(X)$ reduces to

$$\begin{array}{ccc} h^0 U & \xrightarrow{-h^0\beta} & h^0 V \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle -\text{nat.}} \\ h^1 U = 0 & \xrightarrow{0} & \operatorname{coker} h^0\beta, \end{array} \tag{8}$$

where minus sign in $-h^0\beta$ comes from the definition of χ in proposition (2.1.5).

2.2 The functor Φ

In this section we come back to the situation described in section 1.1.

(2.2.1) Let us choose an additive left exact functor $\mathbb{F} = FG : \mathcal{A} = \mathfrak{B}_U \to \mathcal{A} = \mathfrak{B}_U$ and an injective morphism $\alpha : 1 \to \mathbb{F}$ as in (1.3.1), such that $\mathbb{F}(\mathfrak{A}_U) \subset \mathfrak{A}_U$, $\mathbb{F}_{|\mathfrak{A}_U}$ is exact, $(\mathbb{R}^i j_*)(\mathbb{F}A) = 0$ and $j_*\mathbb{F}A \simeq \mathbb{R}(j_*\mathbb{F})A$, for $i > 0, A \in \mathfrak{A}_U$. The restriction to $\mathfrak{A}_U$ of the functor $\mathbb{Q} = \operatorname{coker}\alpha$ defined in (1.3.1) is also exact.

To simplify, let us write $\Omega : \mathrm{D}(\mathrm{Arr}(\mathfrak{B}_X)) \to \mathrm{D}(\mathrm{Arr}(\mathfrak{B}_X))$ instead of Ω_{D} in proposition (2.1.5).

(2.2.2) Let us first consider the additive left exact functor $\psi_{\mathbb{F}} : \mathfrak{B}_X \to \mathrm{Arr}(\mathfrak{B}_X)$ defined by

$$\psi_{\mathbb{F}} = (\mathrm{Id} \xrightarrow{\rho} j_*\mathbb{F}j^*), \quad \rho := (j_*\alpha j^*) \circ \mathrm{adj}, \tag{9}$$

where $\mathrm{adj} : \mathrm{Id} \to j_*j^*$ is the adjunction morphism, and second, functors

$$\Psi_{\mathbb{F}} := \Omega\mathbb{R}\psi_{\mathbb{F}} : \mathcal{D}_X \to \mathrm{D}(\mathrm{Arr}(\mathfrak{B}_X)), \quad \Phi_{\mathbb{F}} := \bar{t}\Psi_{\mathbb{F}} : \mathcal{D}_X \to \mathcal{D}_X.$$

Once the functor $\mathbb{F}$ is fixed, we omit subscripts and we will write ψ, Ψ, Φ instead of $\psi_{\mathbb{F}}, \Psi_{\mathbb{F}}, \Phi_{\mathbb{F}}$.

We have canonical isomorphisms

$$\bar{s}\mathbb{R}\psi \simeq \mathrm{Id}, \quad \bar{s}\Psi \simeq \bar{t}\mathbb{R}\psi \simeq \mathbb{R}(j_*\mathbb{F}j^*) = \mathbb{R}(j_*\mathbb{F})j^* \tag{10}$$

and triangle (4) gives rise to the triangle

$$\mathrm{Id} \xrightarrow{\rho} \mathbb{R}(j_*\mathbb{F})j^* \xrightarrow{u^1} \Phi \longrightarrow \mathrm{Id}[1]. \tag{11}$$

(2.2.3) Example.
(1) In example (1.1.1), (1) let us take U^{dis} as the discrete topological space with underlying set U, $\Delta : U^{\mathrm{dis}} \to U$ the identity map, $\mathcal{O}_{U^{\mathrm{dis}}} = \Delta^*\mathcal{O}_U$, $\mathcal{A}'$ the abelian category of $\mathcal{O}_{U^{\mathrm{dis}}}$-modules and $F = \Delta_*, G = \Delta^*$ (see [4], chap. II, §4.3).
(2) In example (1.1.1), (2), let us suppose that S is a $K(\pi, 1)$ space, $p : \widetilde{U} \to U$ the universal covering space of U, $\mathcal{O}_{\widetilde{U}} = p^*\mathcal{O}_U$, $\mathcal{A}'$ the abelian category of $\mathcal{O}_{\widetilde{U}}$-modules and $F = p_*, G = p^*$ (see [13]). If the fundamental group $\pi_1(U, x_0)$ is finite and k is noetherian, then we can also consider the categories $\mathfrak{A}_*$ as those of constructible sheaves of finitely generated modules.
(3) In example (1.1.1), (3), it is not possible in general to choose a functor $\mathbb{F}$ as above, but we will be able to apply the methods of this paper as explained in section 4.1.

2.3 Induction on perversities

The following theorem generalizes [13], prop. 2.3.3 and rem. 2.3.7.

(2.3.1) Theorem. *Let d be an integer ≥ 1 and let K be an object in $\mathcal{D}_X$. Then, $K \in \mathrm{Perv}^d(X, \Sigma)$ if and only if $j^*K \in \mathfrak{A}_U$ and $\Phi K \in \mathrm{Perv}^{d-1}(X, \Sigma)$.*

Proof. Let us consider the long exact sequence of cohomology associated with the triangle (11) evaluated on K:

$$K \xrightarrow{\rho_K} \mathbb{R}(j_*\mathbb{F})j^*K \xrightarrow{u^1_K} \Phi K \longrightarrow K[1]. \tag{12}$$

If K is d-perverse, j^*K belongs to $\mathfrak{A}_U$, $\mathbb{R}(j_*\mathbb{F})j^*K = j_*\mathbb{F}j^*K$ is concentrated in degree 0 and $h^i(\Phi K) \simeq h^{i+1}(K)$ for $i \neq 0, -1$. In particular ΦK is concentrated in degrees $[-1, d-1]$.

For $i = -1$ we have an exact sequence

$$0 \longrightarrow h^{-1}(\Phi K) \longrightarrow h^0 K \longrightarrow \mathbb{R}^0(j_*\mathbb{F})j^*K = j_*\mathbb{F}j^*(h^0K) \longrightarrow 0,$$

where the second arrow is nothing but ρ_{h^0K} (see (9)), which is injective because α is injective and $\mathbb{R}^0\Gamma_C(h^0K) = \mathbb{R}^0\Gamma_C K = i_*h^0\mathbb{R}i^!K = 0$. Then $h^{-1}(\Phi K) = 0$ and ΦK is concentrated in degrees $[0, d-1]$.

By applying j^* to (12) we obtain $j^*\Phi K \simeq \mathbb{Q}j^*K$, and then $j^*\Phi K$ is concentrated in degree 0.

On the other hand, by applying $\mathbb{R}i^!$ to (12) we deduce that $\mathbb{R}i^!\Phi K \simeq (\mathbb{R}i^!K)[1]$, hence $h^m\mathbb{R}i^!\Phi K = 0$ for any $m < d-1$.

By proposition (1.1.4), we conclude that ΦK is $(d-1)$-perverse.

Conversely, let us suppose that $j^*K \in \mathfrak{A}_U$ and that ΦK is $(d-1)$-perverse. By triangle (12) again we deduce that K is concentrated in degrees $[0, d]$, $h^m\mathbb{R}i^!K \simeq h^{m-1}\mathbb{R}i^!\Phi K = 0$ for any $m < d$ and then K is d-perverse. Q.E.D.

(2.3.2) REMARK. As pointed out in [13], rem. (2.3.7), theorem (2.3.1) suggests iterating the functor Φ in order to obtain, for any d-perverse sheaf K, an usual sheaf $\Phi^d K$. The main result of this paper (see theorem (3.2.2)) tells us how to reconstruct K from its restriction to the open set U and from $\Phi^d K$.

3 The equivalence of categories

3.1 Gluing data

We keep the notations in (1.3.1) and (2.2.1).

(3.1.1) For each integer $i \geq 0$ let us write $q^i := q\mathbb{Q}^i, \alpha^i := \alpha\mathbb{Q}^i, g^i := (\mathbb{F}q^i) \circ (\alpha\mathbb{F}\mathbb{Q}^i) = [(\mathbb{F}q) \circ (\alpha\mathbb{F})]\mathbb{Q}^i = [(\alpha\mathbb{Q}) \circ q]\mathbb{Q}^i = \alpha^{i+1} \circ q^i = g^0\mathbb{Q}^i$.

For $d \geq 1$, we have the complex

$$T_d := \mathbb{F} \xrightarrow{g^0} \cdots \xrightarrow{g^{d-2}} \mathbb{F}\mathbb{Q}^{d-1} \xrightarrow{q^{d-1}} \mathbb{Q}^d \tag{13}$$

placed in degrees $[0, d]$, which is a resolution of length d of the identity functor by means of the injection $\mathrm{Id} \xrightarrow{\alpha} \mathbb{F}$.

(3.1.2) Let $h_d : T_d\mathbb{Q} \to \mathbb{Q}T_{d+1}$ be the morphism of complexes given by

$$h_d^i = (-1)^{i+1} h\mathbb{Q}^i : \mathbb{F}\mathbb{Q}^{i+1} \to \mathbb{Q}\mathbb{F}\mathbb{Q}^i, \quad 0 \leq i \leq d-1$$

and $h_d^d = (-1)^d \mathbb{Q}\alpha\mathbb{Q}^d : \mathbb{Q}^{d+1} \to \mathbb{Q}\mathbb{F}\mathbb{Q}^d$, where h has been defined in (1.3.4). It is a quasi-isomorphism whose composition with $\alpha\mathbb{Q} : \mathbb{Q} \to T_d\mathbb{Q}$ is equal to $\mathbb{Q}\alpha : \mathbb{Q} \to \mathbb{Q}T_{d+1}$.

(3.1.3) DEFINITION. *For each integer $d \geq 1$, let $\mathfrak{B}^d_{\mathbb{F}}$ be the additive category whose objects are the $(\mathcal{L}, \mathcal{F}, u, \sigma)$, where $\mathcal{L} \in \mathfrak{A}_U, \mathcal{F} \in \mathfrak{A}_X$, $u : j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L} \to \mathcal{F}$ and $\sigma : \mathbb{Q}^d\mathcal{L} \xrightarrow{\sim} j^*\mathcal{F}$ such that $u \circ j_* g^{d-2}_{\mathcal{L}} = 0$ (if $d \geq 2$) and $\sigma \circ q^{d-1}_{\mathcal{L}} = j^*u$, and whose morphisms from $(\mathcal{L}, \mathcal{F}, u, \sigma)$ to $(\mathcal{L}', \mathcal{F}', u', \sigma')$ are defined as the pairs (f, g) where $f : \mathcal{L} \to \mathcal{L}', g : \mathcal{F} \to \mathcal{F}'$ are morphisms such that $u' \circ (j_*\mathbb{F}\mathbb{Q}^{d-1} f) = g \circ u$.*

Observe that in the above definition, the relation $(j^*g) \circ \sigma = \sigma' \circ (\mathbb{Q}^d f)$ holds.

For $d \geq 2$, let us define the additive functor

$$\mathbb{G}_{d-1} := \operatorname{coker} j_* g^{d-2} : \mathfrak{A}_U \to \mathfrak{A}_X,$$

which is right exact.

In the above definition, we can replace condition $u \circ j_* g^{d-2}_{\mathcal{L}} = 0$ by taking objects $(\mathcal{L}, \mathcal{F}, \overline{u}, \sigma)$ with $\overline{u} : \mathbb{G}_{d-1}\mathcal{L} \to \mathcal{F}$.

The proof of the following proposition is an exercise left up to the reader.

(3.1.4) PROPOSITION. *The category $\mathfrak{B}^d_{\mathbb{F}}$ is abelian.*

(3.1.5) Remark. By using the fact that sheaves on X are determined by their restrictions to U and C and by the gluing morphism $i^* \to i^* j_* j^*$, category $\mathfrak{B}^d_{\mathbb{F}}$ fits into the construction of abelian categories in [9], 1. Namely, category $\mathfrak{B}^d_{\mathbb{F}}$ is equivalent to the category $\mathcal{C}(F, G; T)$ in *loc. cit.*, where $F = i^* \mathbb{G}_{d-1}, G = i^* j_* \mathbb{Q}^d : \mathfrak{B}_U \to \mathfrak{B}_C$ (F is right exact and G is left exact) and $T = i^* \overline{j_* q^{d-1}} : F \to G$, where $\overline{j_* q^{d-1}} : \mathbb{G}_{d-1} \to j_* \mathbb{Q}^d$ is the morphism induced by $j_* q^{d-1}$.

As in [9], any other choice of the functor $\mathbb{F}$ in (2.2.1) gives rise to a category equivalent to $\mathfrak{B}^d_{\mathbb{F}}$.

(3.1.6) By theorem (2.3.1), functors j^* and Φ can be considered as functors $j^* : \mathrm{Perv}^d(X, \Sigma) \to \mathfrak{A}_U, \quad \Phi : \mathrm{Perv}^d(X, \Sigma) \to \mathrm{Perv}^{d-1}(X, \Sigma)$.

From the properties of $\mathbb{F}$ in (2.2.1), we have

$$(j_* \mathbb{F})(\mathfrak{A}_U) \subset \bigcap_{m \geq 0} \mathrm{Perv}^m(X, \Sigma).$$

For any $K \in \mathrm{Perv}^d(X, \Sigma)$, we have $\mathbb{R}(j_* \mathbb{F}) j^* K = j_* \mathbb{F} j^* K$ and the morphism u^1 in (11) gives rise to a morphism

$$u^1 : j_* \mathbb{F} j^* \to \Phi$$

between functors $j_* \mathbb{F} j^*, \Phi : \mathrm{Perv}^d(X, \Sigma) \to \mathrm{Perv}^{d-1}(X, \Sigma)$.

As pointed out in the proof of theorem (2.3.1), by applying the functor j^* to (11) we deduce an isomorphism

$$\xi^1 : \mathbb{Q} j^* \xrightarrow{\sim} j^* \Phi \quad \text{such that} \quad \xi^1 \circ (q j^*) = j^* u^1. \tag{14}$$

(3.1.7) We define inductively

$$\begin{aligned} u^i &= (u^{i-1} \Phi) \circ (j_* \mathbb{F} \mathbb{Q}^{i-2} \xi^1) : j_* \mathbb{F} \mathbb{Q}^{i-1} j^* \to \Phi^i, \quad i \geq 2 \\ \xi^i &= (\xi^1 \Phi^{i-1}) \circ (\mathbb{Q} \xi^{i-1}) : \mathbb{Q}^i j^* \xrightarrow{\sim} j^* \Phi^i, \quad i \geq 1. \end{aligned}$$

The relations

$$\xi^i \circ (q \mathbb{Q}^{i-1} j^*) = j^* u^i, \quad u^i \circ (j_* g^{i-2} j^*) = 0, \quad (\xi^{i-1} \Phi) \circ (\mathbb{Q}^{i-1} \xi^1) = \xi^i \tag{15}$$

hold for every $i \geq 2$.

3.2 The theorem

With the notations introduced in (3.1.7), we do the following:

(3.2.1) Definition. *For each integer $d \geq 1$, let us define the additive functors*

$$D_d := (j^*, \Phi^d, u^d, \xi^d) : \mathrm{Perv}^d(X, \Sigma) \to \mathfrak{B}^d_{\mathbb{F}}$$

and $B_d : \mathfrak{B}^d_{\mathbb{F}} \to \mathrm{Perv}^d(X,\Sigma)$ *by*

$$B_d(\mathcal{L},\mathcal{F},u,\sigma) := j_*\mathbb{F}\mathcal{L} \xrightarrow{j_*g^0_{\mathcal{L}}} j_*\mathbb{F}\mathbb{Q}\mathcal{L} \xrightarrow{j_*g^1_{\mathcal{L}}} \cdots \xrightarrow{j_*g^{d-2}_{\mathcal{L}}} j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L} \xrightarrow{u} \mathcal{F}, \tag{16}$$

where the complex is placed in degrees $[0,d]$*, the action of* B_d *on morphisms being obvious.*

In the above definition we can identify

$$j^*B_d(\mathcal{L},\mathcal{F},u,\sigma) = T_d\mathcal{L} \tag{17}$$

by means of σ (see (13)). Furthermore, the acyclicity properties in (2.2.1) show that $j_*\mathbb{F}\mathbb{Q}^i\mathcal{L} = \mathbb{R}j_*\mathbb{F}\mathbb{Q}^i\mathcal{L}$. Then $\mathbb{R}i^!B_d(\mathcal{L},\mathcal{F},u,\sigma) = \mathbb{R}i^!\mathcal{F}[-d]$, and we deduce the perversity of $B_d(\mathcal{L},\mathcal{F},u,\sigma)$ from proposition (1.1.4).

The main result of this paper is the following:

(3.2.2) Theorem. *For any integer* $d \geq 1$*, functors* B_d *and* D_d *defined above are the quasi-inverse of each other and they define, thus, an equivalence of categories between* $\mathrm{Perv}^d(X,\Sigma)$ *and* $\mathfrak{B}^d_{\mathbb{F}}$*.*

As suggested by rem. (2.3.7) of [13] and theorem (2.3.1), the proof of theorem (3.2.2) can be approached by induction on perversity d.

(3.2.3) Remark. In case $d=1$ our proof of the isomorphism $\mathrm{Id} \simeq D_1B_1$ is essentially the same as in [13], th. 2.3.4, but it should be noticed that in *loc. cit.* there is a mistake in the proof of the faithfulness of D_1. Our proof of theorem (3.2.2) completes the one given in [5].

3.3 The proof

First Part: We are going to construct a natural isomorphism $\mathcal{O} \simeq D_dB_d\mathcal{O}$ for any $\mathcal{O}$ in $\mathfrak{B}^d_{\mathbb{F}}$.

For $d=0$ let us call $\mathfrak{B}^0_{\mathbb{F}} = \mathfrak{A}_X$ and $B_0 = \mathrm{Id} : \mathfrak{B}^0_{\mathbb{F}} \to \mathrm{Perv}^0(X,\Sigma)$.

For any $d \geq 2$ we consider functors $\mathfrak{T} : \mathfrak{B}^d_{\mathbb{F}} \to \mathfrak{B}^{d-1}_{\mathbb{F}}$ whose action on objects (resp. on morphisms) is given by $\mathfrak{T}(\mathcal{L},\mathcal{F},u,\sigma) := (\mathbb{Q}\mathcal{L},\mathcal{F},u,\sigma)$ (resp. $\mathfrak{T}(f,g) := (\mathbb{Q}f,g)$). For $d=1$, functor $\mathfrak{T} : \mathfrak{B}^1_{\mathbb{F}} \to \mathfrak{B}^0_{\mathbb{F}}$ is simply defined by $\mathfrak{T}(\mathcal{L},\mathcal{F},u,\sigma) = \mathcal{F}$.

For any $d \geq 1$ we also consider functors $s : \mathfrak{B}^d_{\mathbb{F}} \to \mathfrak{A}_U$, $t : \mathfrak{B}^d_{\mathbb{F}} \to \mathfrak{A}_X$ and morphism $\upsilon : j_*\mathbb{F}\mathbb{Q}^{d-1}s \to t$ defined by $s(\mathcal{L},\mathcal{F},u,\sigma) = \mathcal{L}$, $t(\mathcal{L},\mathcal{F},u,\sigma) = \mathcal{F}$, $\upsilon_{(\mathcal{L},\mathcal{F},u,\sigma)} = u$. We obviously have $s\mathfrak{T} = \mathbb{Q}s$ and $j^*t \overset{\sigma}{=} \mathbb{Q}^ds$.

From (17) we can identify $j^*B_d = T_ds$ for $d \geq 1$, and from the acyclicity properties of $\mathbb{F}$ with respect to j_* in (2.2.1), we deduce

$$\Phi B_d = \bar{t}\Omega\mathbb{R}\psi B_d = \bar{t}\Omega\psi B_d = \mathrm{cone}(B_d \xrightarrow{\rho B_d} j_*\mathbb{F}j^*B_d = j_*\mathbb{F}T_ds),$$

i.e.

$$(\Phi B_d)^{-1} = 0 \oplus (j_*\mathbb{F}s), \quad (\Phi B_d)^{d-1} = (j_*\mathbb{F}^2\mathbb{Q}^{d-1}s) \oplus t,$$
$$(\Phi B_d)^d = (j_*\mathbb{F}\mathbb{Q}^d s) \oplus 0,$$
$$(\Phi B_d)^i = (j_*\mathbb{F}^2\mathbb{Q}^i s) \oplus (j_*\mathbb{F}\mathbb{Q}^{i+1}s), \quad 0 \le i \le d-2,$$

and

$$d^{-1}_{\Phi B_d} = \begin{pmatrix} 0 & j_*\alpha\mathbb{F}s \\ 0 & -j_*g^0 s \end{pmatrix}, \quad d^i_{\Phi B_d} = \begin{pmatrix} j_*\mathbb{F}g^i s & j_*\alpha\mathbb{F}\mathbb{Q}^{i+1}s \\ 0 & -j_*g^{i+1}s \end{pmatrix}, \quad 0 \le i \le d-3,$$

$$d^{d-2}_{\Phi B_d} = \begin{pmatrix} j_*\mathbb{F}g^{d-2}s & j_*\alpha\mathbb{F}\mathbb{Q}^{d-1}s \\ 0 & -\upsilon \end{pmatrix}, \quad d^{d-1}_{\Phi B_d} = \begin{pmatrix} j_*\mathbb{F}q\mathbb{Q}^{d-1}s & (j_*\alpha\mathbb{Q}^d s)\circ \mathrm{adj} \\ 0 & 0 \end{pmatrix},$$

where $\mathrm{adj} : t \to j_*j^*t = j_*\mathbb{Q}^d s$ is the adjunction morphism.

By the same reason, morphism $u^1 B_d : j_*\mathbb{F}j^*B_d = j_*\mathbb{F}T_d s \to \Phi B_d$ becomes the natural inclusion.

Let Q_d be the complex of functors from $\mathfrak{B}^d_{\mathbb{F}}$ to $\mathfrak{A}_U$ obtained by plumbing $\mathbb{F}s$ in degree -1 and $\mathbb{F}T_d s$ in degrees ≥ 0 by means of $\mathbb{F}\alpha s$. From (13) and (2.2.1) we deduce that complexes Q_d and j_*Q_d are exact.

(3.3.1) For any $d \ge 1$ let us call $L_d = B_{d-1}\mathfrak{T}$, which can be considered as a complex of functors from $\mathfrak{B}^d_{\mathbb{F}}$ to $\mathfrak{A}_X$, and let us define the following exact sequences:

$$0 \to L^{-1}_d = 0 \xrightarrow{\lambda^{-1}_d = 0} (\Phi B_d)^{-1} \xrightarrow{\pi^{-1}_d = (0\ 1)} j_*Q^{-1}_d \to 0,$$

$$0 \to L^i_d \xrightarrow{\lambda^i_d = (-1)^{i+1}\begin{pmatrix} j_*\mu\mathbb{Q}^i s \\ -1 \end{pmatrix}} (\Phi B_d)^i \xrightarrow{\pi^i_d = (1\ \ j_*\mu\mathbb{Q}^i s)} j_*Q^i_d \to 0, \quad 0 \le i \le d-2,$$

$$0 \to L^{d-1}_d \xrightarrow{\lambda^{d-1}_d = (-1)^d\begin{pmatrix} (j_*\gamma\mathbb{Q}^{d-1}s)\circ(\mathrm{adj}) \\ -1 \end{pmatrix}} (\Phi B_d)^{d-1} \xrightarrow{\pi^{d-1}_d = (1\ \ (j_*\gamma\mathbb{Q}^{d-1}s)\circ(\mathrm{adj}))} j_*Q^{d-1}_d \to 0,$$

$$0 \to L^d_d = 0 \xrightarrow{\lambda^d_d = 0} (\Phi B_d)^d \xrightarrow{\pi^d_d = (1\ 0)} j_*Q^d_d \to 0,$$

where $\gamma : \mathbb{Q} \to \mathbb{F}^2$, $\mu : \mathbb{F}\mathbb{Q} \to \mathbb{F}^2$ have been defined in (1.3.2) and (1.3.3) respectively.

From (1.3.2), (1.3.3), (3.1.1) and (1.3.4) we deduce, first:

$$(\mathbb{F}g^i)\circ(\mu\mathbb{Q}^i) = [(\mathbb{F}\alpha\mathbb{Q})\circ(\mathbb{F}q)\circ\mu]\mathbb{Q}^i = \mathbb{F}\alpha\mathbb{Q}^{i+1},$$

$$(\mu\mathbb{Q}^i)\circ g^i = [\mu\circ(\mathbb{F}q)\circ(\alpha\mathbb{F})]\mathbb{Q}^i = [\alpha\mathbb{F} - \mathbb{F}\alpha]\mathbb{Q}^i,$$

$$(j_*\gamma\mathbb{Q}^{d-1}s)\circ(\mathrm{adj})\circ\upsilon = (j_*\gamma\mathbb{Q}^{d-1}s)\circ(j_*q\mathbb{Q}^{d-1}s) = [j_*(\alpha\mathbb{F} - \mathbb{F}\alpha)\mathbb{Q}^{d-1}]s,$$

$$(\mathbb{F}q\mathbb{Q}^{d-1}s)\circ(\gamma\mathbb{Q}^{d-1}s) - \alpha\mathbb{Q}^d s = [(\mathbb{F}q)\circ\gamma - \alpha\mathbb{Q}]\mathbb{Q}^{d-1}s = 0,$$

and second:

$$d^i_{\Phi B_d} \circ \lambda^i_d = \lambda^{i+1}_d \circ (j_* g^{i+1} s), \quad d^i_{Q_d} \circ \pi^i_d = \pi^{i+1}_d \circ d^{i+1}_{\Phi B_d} \quad \text{for any } i.$$

In particular we obtain an exact sequence of complexes

$$0 \longrightarrow B_{d-1}\mathfrak{T} \xrightarrow{\lambda_d} \Phi B_d \xrightarrow{\pi_d} j_* Q_d \longrightarrow 0,$$

which shows that $\lambda_d : B_{d-1}\mathfrak{T} \to \Phi B_d$ is a quasi-isomorphism and then an isomorphism between functors from $\mathfrak{B}^d_{\mathbb{F}}$ to $\mathrm{Perv}^{d-1}(X, \Sigma)$.

(3.3.2) For any $d \geq 1$ we consider the morphism of functors $\theta_d : j_*\mathbb{F}s \to B_{d-1}\mathfrak{T}$ given by $\theta_d = j_* g^0 s$ if $d \geq 2$ and $\theta_1 = \upsilon : j_*\mathbb{F}s \to t = B_0\mathfrak{T}$. Diagram

$$\begin{array}{ccc} j_*\mathbb{F}s & \xrightarrow{\theta_d} & B_{d-1}\mathfrak{T} \\ {\scriptstyle j_*\mathbb{F}\alpha s}\big\downarrow & & {\scriptstyle \lambda_d}\big\downarrow \\ j_*\mathbb{F}j^*B_d = j_*\mathbb{F}T_d s & \xrightarrow{u^1 B_d} & \Phi B_d \end{array} \tag{18}$$

commutes in the homotopy category of complexes and then in the derived category.

(3.3.3) For each $i \geq 1$ let us call

$$\phi_i := (j_*\mathbb{F}\mathbb{Q}^{i-1}\xi^{-1}B_{i+1}) \circ (j_*\mathbb{F}\mathbb{Q}^{i-1}j^*\lambda_{i+1}) : j_*\mathbb{F}\mathbb{Q}^{i-1}j^*B_i\mathfrak{T} \longrightarrow j_*\mathbb{F}\mathbb{Q}^i j^* B_{i+1}.$$

From rule (1.2.1), (c), and (3.1.7) we deduce that the following diagram of functors from $\mathfrak{B}^{i+1}_{\mathbb{F}}$ to $\mathrm{Perv}^0(X, \Sigma) = \mathfrak{A}_X$

$$\begin{array}{ccc} j_*\mathbb{F}\mathbb{Q}^{i-1}j^*B_i\mathfrak{T} & \xrightarrow{u^i B_i \mathfrak{T}} & \Phi^i B_i \mathfrak{T} \\ {\scriptstyle \phi_i}\big\downarrow & & {\scriptstyle \Phi^i\lambda_{i+1}}\big\downarrow \\ j_*\mathbb{F}\mathbb{Q}^i j^* B_{i+1} & \xrightarrow{u^{i+1}B_{i+1}} & \Phi^{i+1}B_{i+1} \end{array}$$

commutes, where the vertical arrows are isomorphisms.

(3.3.4) With identifications

$$j_*\mathbb{F}\mathbb{Q}^{i-1}j^*B_i\mathfrak{T} = j_*\mathbb{F}\mathbb{Q}^{i-1}T_i s\mathfrak{T} = j_*\mathbb{F}\mathbb{Q}^{i-1}T_i\mathbb{Q}s$$

and

$$j_*\mathbb{F}\mathbb{Q}^i j^* B_{i+1} = j_*\mathbb{F}\mathbb{Q}^{i-1}\mathbb{Q}T_{i+1}s$$

one can prove that $\phi_i = j_*\mathbb{F}\mathbb{Q}^{i-1}h_i s$, where h_i has been defined in (3.1.2), but we will not need that result in the rest of this paper.

Summing up (3.3.2) and (3.3.3), for any $d \geq 1$ we obtain a commutative diagram of functors from $\mathfrak{B}^d_{\mathbb{F}}$ to $\mathrm{Perv}^0(X,\Sigma) = \mathfrak{A}_X$

$$\begin{array}{ccc}
j_*\mathbb{F}s\mathfrak{T}^{d-1} & \xrightarrow{\upsilon\mathfrak{T}^{d-1}} & B_0\mathfrak{T}\mathfrak{T}^{d-1} = e\mathfrak{T}^{d-1} \\
{\scriptstyle j_*\mathbb{F}\alpha s\mathfrak{T}^{d-1}}\downarrow & & \downarrow{\scriptstyle \lambda_1\mathfrak{T}^{d-1}} \\
j_*\mathbb{F}j^*B_1\mathfrak{T}^{d-1} & \xrightarrow{u^1B_1\mathfrak{T}^{d-1}} & \Phi B_1\mathfrak{T}^{d-1} \\
{\scriptstyle \phi_1\mathfrak{T}^{d-2}}\downarrow & & \downarrow{\scriptstyle \Phi\lambda_2\mathfrak{T}^{d-2}} \\
\vdots & & \vdots \\
{\scriptstyle \phi_{d-2}\mathfrak{T}}\downarrow & & \downarrow{\scriptstyle \Phi\lambda_{d-1}\mathfrak{T}} \\
j_*\mathbb{F}\mathbb{Q}^{d-2}j^*B_{d-1}\mathfrak{T} & \xrightarrow{u^{d-1}B_{d-1}\mathfrak{T}} & \Phi^{d-1}B_{d-1}\mathfrak{T} \\
{\scriptstyle \phi_{d-1}}\downarrow & & \downarrow{\scriptstyle \Phi\lambda_d} \\
j_*\mathbb{F}\mathbb{Q}^{d-1}j^*B_d & \xrightarrow{u^dB_d} & \Phi^dB_d.
\end{array}$$

Compositions of vertical arrows give rise to the natural isomorphism $\mathrm{Id}_{\mathfrak{B}^d_{\mathbb{F}}} \xrightarrow{\sim} D_dB_d$ we wanted and the first part of the proof of theorem (3.2.2) is finished.

Second part:

In this part we prove that for any d-perverse sheaf K, there exists a natural isomorphism $K \simeq B_dD_dK$. We are using notations of (2.1.9). We proceed by induction on $d \geq 1$.

For any d-perverse sheaf we know (theorem (2.3.1)) that $\mathbb{R}\psi K \in \mathfrak{P}$ and

$$N\Psi K = j_*\mathbb{F}j^*K \xrightarrow{u^1_K} \Phi K.$$

Let us call

$$\omega_K : K \xrightarrow{\sim} C(j_*\mathbb{F}j^*K \xrightarrow{u^1_K} \Phi K)$$

the composition of isomorphism

$$\eta_{\mathbb{R}\psi K} : \overline{s}\mathbb{R}\psi K \to CN\Omega\mathbb{R}\psi K = CN\Psi K = C(j_*\mathbb{F}j^*K \xrightarrow{u^1_K} \Phi K)$$

defined in (2.1.9) and isomorphism $K \xrightarrow{\sim} \overline{s}\mathbb{R}\psi K$ of (10).

Functors ψ, Ω, N commute with j^* and we can identify

$$j^*C(j_*\mathbb{F}j^*K \xrightarrow{u^1_K} \Phi K) \overset{\xi_1}{=} C(\mathbb{F}j^*K \xrightarrow{q_{j^*K}} \mathbb{Q}j^*K). \tag{19}$$

Then, by using (8) we obtain

$$j^*\omega_K = -\alpha_{j^*K} : j^*K \longrightarrow C(\mathbb{F}j^*K \xrightarrow{q_{j^*K}} \mathbb{Q}j^*K). \tag{20}$$

For $d = 1$ we have $B_1 D_1 K = j_* \mathbb{F} j^* K \xrightarrow{u_K^1} \Phi K$ which is isomorphic to

$$C(j_* \mathbb{F} j^* K \xrightarrow{u_K^1} \Phi K) = j_* \mathbb{F} j^* K \xrightarrow{-u_K^1} \Phi K$$

by means of $(1, -1)$. The composition of this last isomorphism with ω_K gives rise to an isomorphism

$$\delta_K^1 : K \longrightarrow B_1 D_1 K$$

natural with respect to $K \in \mathrm{Perv}^1(X, \Sigma)$ such that $j^* \delta_K^1 = -\alpha_{j^* K}$.

Now let d be an integer ≥ 2 and suppose there exists $\delta^{d-1} : \mathrm{Id}_{\mathrm{Perv}^{d-1}(X,\Sigma)} \xrightarrow{\sim} B_{d-1} D_{d-1}$ such that

$$j^* \delta^{d-1} = (-1)^{d-1} \alpha j^* : j^* \longrightarrow j^* B_{d-1} D_{d-1} = T_{d-1} s D_{d-1} = T_{d-1} j^*. \tag{21}$$

Isomorphism

$$(\xi^1, 1) : \mathfrak{T} D_d = (\mathbb{Q} j^*, \Phi^d, u^d, \xi^d) \rightarrow D_{d-1} \Phi = (j^* \Phi, \Phi^d, u^{d-1} \Phi, \xi^{d-1} \Phi)$$

allows us to identify both functors and, by (21) we obtain

$$(j^* \delta^{d-1} \Phi) \circ (j^* u^1) = (-1)^{d-1} g^0 j^* : \mathbb{F} j^* \longrightarrow j^* B_{d-1} D_{d-1} \Phi = j^* B_{d-1} \mathfrak{T} D_d = T_{d-1} \mathbb{Q} j^*.$$

Then $(\delta^{d-1} \Phi) \circ u^1 = (-1)^{d-1} j_* g^0 j^*$ and

$$C(1, \delta^{d-1} \Phi) : C(j_* \mathbb{F} j^* \xrightarrow{u^1} \Phi) \xrightarrow{\sim} C(j_* \mathbb{F} j^* \xrightarrow{(-1)^{d-1} j_* g^0 j^*} B_{d-1} \mathfrak{T} D_d),$$

but

$$B_{d-1} \mathfrak{T} D_d = j_* \mathbb{F} \mathbb{Q} j^* \xrightarrow{j_* g^0 \mathbb{Q} j^*} \cdots \xrightarrow{j_* g^{d-3} \mathbb{Q} j^*} j_* \mathbb{F} \mathbb{Q}^{d-2} \mathbb{Q} \xrightarrow{u^d} \Phi^{d-1} \Phi$$

and $j_* g^{i-1} \mathbb{Q} j^* = j_* g^i j^*$. In particular, by using (7) we deduce an isomorphism

$$C(j_* \mathbb{F} j^* \xrightarrow{u^1} \Phi) \simeq$$
$$j_* \mathbb{F} \xrightarrow{(-1)^d j_* g^0 j^*} j_* \mathbb{F} \mathbb{Q} j^* \xrightarrow{-j_* g^1 j^*} \cdots \xrightarrow{-j_* g^{d-2} j^*} j_* \mathbb{F} \mathbb{Q}^{d-1} \xrightarrow{-u^d} \Phi^d, \tag{22}$$

and the complex (22) is isomorphic to

$$B_d D_d = j_* \mathbb{F} \xrightarrow{j_* g^0 j^*} j_* \mathbb{F} \mathbb{Q} j^* \xrightarrow{j_* g^1 j^*} \cdots \xrightarrow{j_* g^{d-2} j^*} j_* \mathbb{F} \mathbb{Q}^{d-1} \xrightarrow{u^d} \Phi^d$$

by means of

$$((-1)^{d-1}, -1, 1, -1, \ldots, (-1)^{d-1}, (-1)^d).$$

By composing isomorphisms above with ω we obtain an isomorphism

$$\delta^d : \mathrm{Id}_{\mathrm{Perv}^d(X,\Sigma)} \xrightarrow{\sim} B_d D_d$$

such that

$$j^* \delta^d = (-1)^{d-1}(-\alpha j^*) = (-1)^d \alpha j^*$$

and the proof of theorem (3.2.2) is finished.

4 Applications

4.1 Explicit models for perverse sheaves

Theorem (3.2.2) provides explicit models (16) for d-perverse sheaves. Actually, functor B_d factorizes through the category of bounded complexes $\mathrm{C}^b(\mathfrak{B}_X)$ and it defines a faithful exact functor $B_d : \mathfrak{B}^d_{\mathbb{F}} \to \mathrm{C}^b(\mathfrak{B}_X)$ establishing an equivalence of categories between $\mathfrak{B}^d_{\mathbb{F}}$ and a non full abelian subcategory of $\mathrm{C}^b(\mathfrak{B}_X)$, whose objects are precisely complexes of the form (16). In particular, inclusion functor $\mathrm{Perv}^d(X,\Sigma) \subset \mathcal{D}_X$ can be lifted to an exact faithful functor $\mathrm{Perv}^d(X,\Sigma) \to \mathrm{C}^b(\mathfrak{B}_X)$.

The lifting above allows us to describe in a concrete way the realization functor (see [1], 3.1.9)

$$\mathrm{real} : \mathrm{D}(\mathrm{Perv}^d(X,\Sigma)) \longrightarrow \mathcal{D}_X$$

by taking single complexes associated with double complexes.

When no functor $\mathbb{F}$ is available for the given subcategories $\mathfrak{A}_* \subset \mathfrak{B}_*$, we can always work at the level of the full derived categories $\mathrm{D}^+(\mathfrak{B}_*)$ by using, for instance, Godement functor $\mathbb{F} = \Delta_*\Delta^*$, as shown in examples (1.1.1), (1) and (2.2.3), (1). The corresponding category of perverse sheaves $\mathrm{Perv}^d(X,\Sigma)$ (without any constructibility conditions, i.e. $\mathfrak{A}_* = \mathfrak{B}_*$) is, by theorem (3.2.2), equivalent to $\mathfrak{B}^d_{\mathbb{F}}$, whose objects are (3.1.3) the $(\mathcal{L},\mathcal{F},u,\sigma)$ where $\mathcal{L} \in \mathfrak{B}_U, \mathcal{F} \in \mathfrak{B}_X$, $u : j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L} \to \mathcal{F}$ and $\sigma : \mathbb{Q}^d\mathcal{L} \xrightarrow{\sim} j^*\mathcal{F}$ such that $u \circ j_* g^{d-2}_{\mathcal{L}} = 0$ (if $d \geq 2$) and $\sigma \circ q^{d-1}_{\mathcal{L}} = j^*u$, and whose morphisms from $(\mathcal{L},\mathcal{F},u,\sigma)$ to $(\mathcal{L}',\mathcal{F}',u',\sigma')$ are defined as the pairs (f,g) where $f : \mathcal{L} \to \mathcal{L}', g : \mathcal{F} \to \mathcal{F}'$ are morphisms such that $u' \circ (j_*\mathbb{F}\mathbb{Q}^{d-1}f) = g \circ u$.

Let us call $\mathrm{Perv}^d_c(X,\Sigma)$ the category of perverse sheaves "constructible" with respect to $\mathfrak{A}_* \subset \mathfrak{B}_*$. It is a full (abelian) subcategory of $\mathrm{Perv}^d(X,\Sigma)$ and then it is equivalent to the full subcategory $\mathfrak{B}^d_{\mathbb{F},c}$ of $\mathfrak{B}^d_{\mathbb{F}}$ whose objects are the $(\mathcal{L},\mathcal{F},u,\sigma)$ such that $\mathcal{L} \in \mathfrak{A}_U$ and morphism $\overline{u} : \mathbb{G}_{d-1}\mathcal{L} \to \mathcal{F}$ has kernel and cokernel in $\mathfrak{A}_X$.

So, even when no functor $\mathbb{F}$ is available for the given subcategories $\mathfrak{A}_* \subset \mathfrak{B}_*$, explicit models and liftings as above also exist.

(4.1.1) Example. *(Perverse sheaves categories which split)* In example (1.1.1), (2), let $d \geq 2$ be an integer and let us suppose S a "good" compact, connected and simply connected topological space, and k a field such that

$$\mathrm{H}^i(S,k) = 0 \quad \forall i = 1,\ldots,d. \tag{23}$$

For example, S can be the $(n-1)$-dimensional sphere and X the n-dimensional disk, stratified by the origin and its complement, for $n \geq d$, or in singularity theory, $(X,0) \subset \mathbb{C}^{d+1}$ is an isolated hypersurface singularity with complex link S a topological (exotic) sphere [11], §8.

Let us consider the category of Σ-constructible complexes of sheaves of k-vector spaces of arbitrary[1] rank on each stratum which are d-perverse sheaves, denoted by $\mathrm{Perv}^d_c(X,\Sigma)$. It is

[1] We may also consider only sheaves of finite rank.

a full subcategory of the category of d-perverse sheaves (without constructibility conditions) $\mathrm{Perv}^d(X,\Sigma)$, which is equivalent by theorem (3.2.2) to category $\mathfrak{B}^d_{\mathbb{F}}$, with $\mathbb{F}$ a functor satisfying the conditions (2.2.1) (see (2.2.3), (1)).

Since S is simply connected, any locally-constant sheaf $\mathcal{L}$ of k-vector spaces on U is constant with fiber $E = \Gamma(U,\mathcal{L}) \simeq k^r$ and

$$(\mathbb{R}^i j_*\mathcal{L})_C = \lim_{\epsilon\to 0} \mathrm{H}^i(]0,\epsilon[\times S,\mathcal{L}) = \mathrm{H}^i(S,E) = 0, \quad 1 \le i \le d.$$

In particular, the sequence

$$0 \to j_*\mathcal{L} \xrightarrow{j_*\alpha_{\mathcal{L}}} j_*\mathbb{F}\mathcal{L} \xrightarrow{j_*q_{\mathcal{L}}} j_*\mathbb{Q}\mathcal{L} \to 0$$

is exact and $\mathbb{R}^i j_*\mathbb{Q}\mathcal{L} \simeq \mathbb{R}^{i+1} j_*\mathcal{L}$ for all $i \ge 1$. Reasoning inductively we obtain that the sequences

$$0 \to j_*\mathbb{Q}^{i-1}\mathcal{L} \xrightarrow{j_*\alpha^{i-1}_{\mathcal{L}}} j_*\mathbb{F}\mathbb{Q}^{i-1}\mathcal{L} \xrightarrow{j_*q^{i-1}_{\mathcal{L}}} j_*\mathbb{Q}^{i}\mathcal{L} \to 0, \quad i = 1,\dots,d \tag{24}$$

are exact.

Given a constructible d-perverse sheaf $K \in \mathrm{Perv}^d_c(X,\Sigma)$, let us denote $(\mathcal{L},\mathcal{F},u,\sigma) = D_d K$ its corresponding object of $\mathfrak{B}^d_{\mathbb{F}}$ by means of theorem (3.2.2). Now K is naturally isomorphic to

$$j_*\mathbb{F}\mathcal{L} \xrightarrow{j_*g^0_{\mathcal{L}}} j_*\mathbb{F}\mathbb{Q}\mathcal{L} \xrightarrow{j_*g^1_{\mathcal{L}}} \cdots \xrightarrow{j_*g^{d-2}_{\mathcal{L}}} j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L} \xrightarrow{u} \mathcal{F}. \tag{25}$$

The exactness of (24) for $i = d-1, d$ implies that $\operatorname{coker} j_*g^{d-2}_{\mathcal{L}} = j_*\mathbb{Q}^d\mathcal{L}$. Let $s : j_*\mathbb{Q}^d\mathcal{L} \to \mathcal{F}$ be the morphism induced by u, whose restriction to U coincides with σ. Now, σ being an isomorphism, the adjunction properties for (j^*, j_*) give us a morphism $t : \mathcal{F} \to j_*\mathbb{Q}^d\mathcal{L}$ verifying $ts = 1$. Then, complex (25) is the direct sum of $j_*T_d\mathcal{L}$ and $(\ker t)[-d]$. On the other hand, the exactness of (24) implies the $j_*T_d\mathcal{L}$ is concentrated in degree 0, its 0-cohomology being equal to $j_*\mathcal{L}$ and, thus, a constant sheaf. Finally, we obtain a natural isomorphism

$$K \simeq (h^0 K) \oplus (h^d K)[-d]$$

expressing the category $\mathrm{Perv}^d_c(X,\Sigma)$ as a direct sum of the category of constant sheaves of k-vector spaces in X and the category of k-vector spaces, considered (this last category) as the category of complexes of sheaves on X concentrated in degree $-d$ and supported by the vertex C.

This is a purely topological result related to a well-known result of Kashiwara-Kawai [6] (see [9], 6.5, p. 427). It can be also directly deduced by using functors $j^p_!, j^p_*$ instead of our models. Namely[2], our hypothesis imply that $j^p_! j^* K \xrightarrow{\sim} j^p_* j^* K \xrightarrow{\sim} j_*\mathcal{L} \simeq h^0 K$ and then, from the canonical morphisms

$$j^p_! j^* K \to K \to j^p_* j^* K$$

we deduce that $h^0 K$ is a direct factor of K.

[2]We owe this remark to P. Deligne.

4.2 Perverse sheaves categories as full abelian subcategories of $\mathrm{K}^b(\mathfrak{B}_X)$

In this section we show that functor $B_d : \mathfrak{B}^d_{\mathbb{F}} \to \mathrm{K}^b_{\mathfrak{A}_X}(\mathfrak{B}_X)$ is fully faithful and then the inclusion functor $\mathrm{Perv}^d(X,\Sigma) \subset \mathcal{D}_X$ lifts to a fully faithful functor $\mathrm{Perv}^d(X,\Sigma) \to \mathrm{K}^b_{\mathfrak{A}_X}(\mathfrak{B}_X)$. In particular, category $\mathrm{Perv}^d(X,\Sigma)$ is realized as a full abelian subcategory of $\mathrm{K}^b_{\mathfrak{A}_X}(\mathfrak{B}_X)$.

(4.2.1) THEOREM. *Functor $B_d : \mathfrak{B}^d_{\mathbb{F}} \to \mathrm{K}^b_{\mathfrak{A}_X}(\mathfrak{B}_X)$ is fully faithful.*

PROOF. Let $\mathcal{O}_i = (\mathcal{L}_i, \mathcal{F}_i, u_i, \sigma_i)$, $i = 1, 2$ be two objects in $\mathfrak{B}^d_{\mathbb{F}}$. We have to prove that

$$B_d : \mathrm{Hom}_{\mathfrak{B}^d_{\mathbb{F}}}(\mathcal{O}_1, \mathcal{O}_2) \longrightarrow \mathrm{Hom}_{\mathrm{K}^b(\mathfrak{B}_X)}(B_d\mathcal{O}_1, B_d\mathcal{O}_2)$$

is bijective.

INJECTIVITY: Although the injectivity of B_d is a consequence of theorem (3.2.2) (the morphism $B_d : \mathrm{Hom}_{\mathfrak{B}^d_{\mathbb{F}}}(\mathcal{O}_1, \mathcal{O}_2) \to \mathrm{Hom}_{\mathcal{D}_X}(B_d(\mathcal{O}_1), B_d(\mathcal{O}_2))$ is bijective), we give here a direct independent proof.

Let $(f, g) : \mathcal{O}_1 \to \mathcal{O}_2$ be a morphism such that $B_d(f, g)$ is null-homotopic. We obviously have $f = j^* h^0 B_d(f, g) = 0$ and $B_d(0, g)$ is null-homotopic.

There exist $s^i : j_*\mathbb{F}\mathbb{Q}^i\mathcal{L}_1 \to j_*\mathbb{F}\mathbb{Q}^{i-1}\mathcal{L}_2$, $i = 1, \ldots, d-1$, $s^d : \mathcal{F}_1 \to j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L}_2$ such that $s^1 \circ j_* g^0_{\mathcal{L}_1} = 0$, $s^2 \circ j_* g^1_{\mathcal{L}_1} + j_* g^0_{\mathcal{L}_2} \circ s^1 = 0$, $\ldots$, $s^d \circ u_1 + j_* g^{d-2}_{\mathcal{L}_2} \circ s^{d-1} = 0$, $g = u_2 \circ s^d$,

$$\begin{tikzcd}
j_*\mathbb{F}\mathcal{L}_1 \arrow[r, "j_*g^0_{\mathcal{L}_1}"] \arrow[d, "0"'] & j_*\mathbb{F}\mathbb{Q}\mathcal{L}_1 \arrow[r, "j_*g^1_{\mathcal{L}_1}"] \arrow[dl, "s^1"'] & \cdots & & j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L}_1 \arrow[r, "u_1"] \arrow[dl, "s^{d-1}"'] \arrow[d, "0"'] & \mathcal{F}_1 \arrow[d, "g"] \arrow[dl, "s^d"'] \\
j_*\mathbb{F}\mathcal{L}_2 \arrow[r, "j_*g^0_{\mathcal{L}_2}"'] & \cdots & & j_*\mathbb{F}\mathbb{Q}^{d-2}\mathcal{L}_2 \arrow[r, "j_*g^{d-2}_{\mathcal{L}_1}"'] & j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L}_2 \arrow[r, "u_2"'] & \mathcal{F}_2.
\end{tikzcd}$$

In degree 0, from $0 = j^* s^1 \circ g^0_{\mathcal{L}_1} = j^* s^1 \circ \alpha^1_{\mathcal{L}_1} \circ q_{\mathcal{L}_1}$ we deduce $0 = j^* s^1 \circ \alpha^1_{\mathcal{L}_1}$ and then, there exists $t^1 : \mathbb{Q}^2\mathcal{L}_1 \to \mathbb{F}\mathcal{L}_2$ s.t. $t^1 \circ q^1_{\mathcal{L}_1} = j^* s^1$. From lemma (1.3.5), there exists $\tau^1 : \mathbb{F}\mathbb{Q}^2\mathcal{L}_1 \to \mathbb{F}\mathcal{L}_2$ s.t. $\tau^1 \circ \alpha^2_{\mathcal{L}_1} = t^1$.

In degree 1, from

$$0 = j^* s^2 \circ g^1_{\mathcal{L}_1} + g^0_{\mathcal{L}_2} \circ j^* s^1 = (j^* s^2 + g^0_{\mathcal{L}_2} \circ \tau^1) \circ \alpha^2_{\mathcal{L}_1} \circ q^1_{\mathcal{L}_1}$$

we deduce

$$0 = (j^* s^2 + g^0_{\mathcal{L}_2} \circ \tau^1) \circ \alpha^2_{\mathcal{L}_1}$$

and then, there exists $t^2 : \mathbb{Q}^3\mathcal{L}_1 \to \mathbb{F}\mathbb{Q}\mathcal{L}_2$ s.t.

$$j^* s^2 + g^0_{\mathcal{L}_2} \circ \tau^1 = t^2 \circ q^2_{\mathcal{L}_1}.$$

From lemma (1.3.5) again, there exists $\tau^2 : \mathbb{F}\mathbb{Q}^3\mathcal{L}_1 \to \mathbb{F}\mathbb{Q}\mathcal{L}_2$ s.t. $\tau^2 \circ \alpha^3_{\mathcal{L}_1} = t^2$.

We inductively construct

$$t^i : \mathbb{Q}^{i+1}\mathcal{L}_1 \to \mathbb{F}\mathbb{Q}^{i-1}\mathcal{L}_2, \quad 2 \le i \le d-1,$$

$$\tau^i : \mathbb{F}\mathbb{Q}^{i+1}\mathcal{L}_1 \to \mathbb{F}\mathbb{Q}^{i-1}\mathcal{L}_2, \quad 2 \le i \le d-2$$

such that

$$g_{\mathcal{L}_2}^{i-2} \circ \tau^{i-1} + j^* s^i = t^i \circ q_{\mathcal{L}_1}^i, \quad 2 \le i \le d-1,$$

$$\tau^i \circ \alpha_{\mathcal{L}_1}^{i+1} = t^i, \quad 2 \le i \le d-2.$$

Let us identify $j^*\mathcal{F}_1 = \mathbb{Q}^d\mathcal{L}_1, j^*u_1 = q_{\mathcal{L}_1}^{d-1}$ by means of σ_1.

In degree $d-1$, from $0 = s^d \circ u_1 + j_* g_{\mathcal{L}_2}^{d-2} \circ s^{d-1}$ we deduce first

$$0 = j^* s^d \circ q_{\mathcal{L}_1}^{d-1} + g_{\mathcal{L}_2}^{d-2} \circ j^* s^{d-1} = j^* s^d \circ q_{\mathcal{L}_1}^{d-1} + g_{\mathcal{L}_2}^{d-2} \circ (t^{d-1} \circ q_{\mathcal{L}_1}^{d-1} - g_{\mathcal{L}_2}^{d-3} \circ \tau^{d-2}) =$$
$$= (j^* s^d + g_{\mathcal{L}_2}^{d-2} \circ t^{d-1}) \circ q_{\mathcal{L}_1}^{d-1}$$

and second

$$0 = j^* s^d + g_{\mathcal{L}_2}^{d-2} \circ t^{d-1}.$$

But s^d is determined by its restriction $j^* s^d$

$$s^d = -(j_* g_{\mathcal{L}_2}^{d-2}) \circ (j_* t^{d-1}) \circ (\mathrm{adj}),$$

where $\mathrm{adj} : \mathcal{F}_1 \to j_* j^* \mathcal{F}_1 = j_* \mathbb{Q}^d \mathcal{L}_1$ is the adjunction morphism. Then

$$g = u_2 \circ s^d = -u_2 \circ (j_* g_{\mathcal{L}_2}^{d-2}) \circ (j_* t^{d-1}) \circ (\mathrm{adj}) = 0$$

and injectivity is proven.

Surjectivity: We need to prove that for any morphism of complexes $F^\bullet : B_d\mathcal{O}_1 \to B_d\mathcal{O}_2$, there exists $(f,g) : \mathcal{O}_1 \to \mathcal{O}_2$ s.t. $B_d(f,g)$ is homotopic to $F^\bullet$.

Obviously, morphism $f : \mathcal{L}_1 \to \mathcal{L}_2$ must be equal to $j^* h^0 F^\bullet$.

Let us consider the following commutative diagram with exact arrows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{L}_1 & \xrightarrow{\alpha_{\mathcal{L}_1}} & \mathbb{F}\mathcal{L}_1 & \xrightarrow{g_{\mathcal{L}_1}^0} & \mathbb{F}\mathbb{Q}\mathcal{L}_1 \\
 & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle j^*F^0 - \mathbb{F}f} & & \Big\downarrow{\scriptstyle j^*F^1 - \mathbb{F}\mathbb{Q}f} \\
0 & \longrightarrow & \mathcal{L}_2 & \xrightarrow{\alpha_{\mathcal{L}_2}} & \mathbb{F}\mathcal{L}_2 & \xrightarrow{g_{\mathcal{L}_2}^0} & \mathbb{F}\mathbb{Q}\mathcal{L}_2.
\end{array}$$

There exists $\sigma_0^1 : \mathbb{Q}\mathcal{L}_1 \to \mathbb{F}\mathcal{L}_2$ s.t. $\sigma_0^1 \circ q_{\mathcal{L}_1} = j^*F^0 - \mathbb{F}f$. From lemma (1.3.5), there exists $\sigma^1 : \mathbb{F}\mathbb{Q}\mathcal{L}_1 \to \mathbb{F}\mathcal{L}_2$ s.t. $\sigma^1 \circ \alpha_{\mathcal{L}_1}^1 = \sigma_0^1$, and then $\sigma^1 \circ g_{\mathcal{L}_1}^0 = j^*F^0 - \mathbb{F}f$. Writing $s^1 := j_*\sigma^1$, we have $s^1 \circ j_* g_{\mathcal{L}_1}^0 = F^0 - j_*\mathbb{F}f$.

In a similar way, we inductively construct $s^i : j_*\mathbb{F}\mathbb{Q}^i\mathcal{L}_1 \to j_*\mathbb{F}\mathbb{Q}^{i-1}\mathcal{L}_2$, $i = 2,\dots,d-1$, s.t. $s^i \circ j_* g^{i-1}_{\mathcal{L}_1} + j_* g^{i-2}_{\mathcal{L}_2} \circ s^{i-1} = F^{i-1} - j_*\mathbb{F}\mathbb{Q}^{i-1} f$. Let us write $\sigma^i = j^* s^i$.

In degree $d-1$ we have

$$\begin{aligned}(j^*F^{d-1} - \mathbb{F}\mathbb{Q}^{d-1}f - g^{d-2}_{\mathcal{L}_2} \circ \sigma^{d-1}) \circ \alpha^{d-1}_{\mathcal{L}_1} \circ q^{d-2}_{\mathcal{L}_1} = \\ = (j^*F^{d-1} - \mathbb{F}\mathbb{Q}^{d-1}f - g^{d-2}_{\mathcal{L}_2} \circ \sigma^{d-1}) \circ g^{d-2}_{\mathcal{L}_1} = \\ = g^{d-2}_{\mathcal{L}_2} \circ (j^*F^{d-2} - \mathbb{F}\mathbb{Q}^{d-2}f - j^*F^{d-2} + \mathbb{F}\mathbb{Q}^{d-2}f - g^{d-3}_{\mathcal{L}_2} \circ \sigma^{d-2}) = 0\end{aligned}$$

and then

$$0 = (j^*F^{d-1} - \mathbb{F}\mathbb{Q}^{d-1}f - g^{d-2}_{\mathcal{L}_2} \circ \sigma^{d-1}) \circ \alpha^{d-1}_{\mathcal{L}_1}.$$

There exists $\sigma^d_0 : \mathbb{Q}^d\mathcal{L}_1 \to \mathbb{F}\mathbb{Q}^{d-1}\mathcal{L}_2$ s.t.

$$\sigma^d_0 \circ q^{d-1}_{\mathcal{L}_1} = j^*F^{d-1} - \mathbb{F}\mathbb{Q}^{d-1}f - g^{d-2}_{\mathcal{L}_2} \circ \sigma^{d-1}.$$

Since $j^*\mathcal{F}_1 = \mathbb{Q}^d\mathcal{L}_1$, morphism σ^d_0 determines another morphism $s^d : \mathcal{F}_1 \to j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L}_2$ s.t.

$$s^d \circ u_1 + j_* g^{d-2}_{\mathcal{L}_2} \circ s^{d-1} = F^{d-1} - j_*\mathbb{F}\mathbb{Q}^{d-1}f.$$

To finish, we take

$$g := F^d - u_2 \circ s^d : \mathcal{F}_1 \to \mathcal{F}_2.$$

An straightforward computation shows that $(f,g) : \mathcal{O}_1 \to \mathcal{O}_2$ is a morphism in $\mathfrak{B}^d_{\mathbb{F}}$, and clearly the s^i, $i = 1,\dots,d$, give an homotopy between $F^\bullet$ and $B_d(f,g)$. Q.E.D.

(4.2.2) Corollary. *The inclusion functor* $\mathrm{Perv}^d(X,\Sigma) \subset \mathcal{D}_X = \mathrm{D}^+_{\mathfrak{A}_X}(\mathfrak{B}_X)$ *lifts to a fully faithful functor* $\mathrm{Perv}^d(X,\Sigma) \to \mathrm{K}^b_{\mathfrak{A}_X}(\mathfrak{B}_X)$. *In particular, category* $\mathrm{Perv}^d(X,\Sigma)$ *is realized as a full abelian subcategory of* $\mathrm{K}^b_{\mathfrak{A}_X}(\mathfrak{B}_X)$.

Proof. It is a direct consequence of theorems (3.2.2) and (4.2.1). Q.E.D.

4.3 Conical perverse sheaves with respect to a $K(\pi,1)$ basis

In case of examples (1.1.1), (2) and (2.2.3), (2), we suppose that S is connected and its universal covering space is contractible. Let us choose a base point $x_0 \in S$ and let us denote $H = \pi_1(S,x_0) = \pi_1(U,x_0)$. Let $\mathfrak{A}_U$ be (resp. $\mathfrak{A}_X$) the abelian category of locally constant sheaves of k-modules (not necessarily finitely generated) on U (resp. of Σ-constructible sheaves of k-modules on X). We can take $\mathbb{F} = p_*p^*$, where p is the universal covering space of (U,x_0).

Objects of category $\mathrm{Perv}^d(X,\Sigma) \subset \mathrm{D}^+_{\mathfrak{A}_X}(k_X)$ are called "conical perverse sheaves" in [13], def. (2.1.1) and rem. (2.3.7).

(4.3.1) The standard equivalence of categories between $\mathfrak{A}_U$ and $\mathrm{Mod}(k[H])$ allows us to translate the exact sequence of functors of $\mathfrak{A}_U$

$$0 \to \mathrm{Id} \xrightarrow{\alpha} \mathbb{F} \xrightarrow{q} \mathbb{Q} \to 0$$

in the following way. For each $k[H]$-module E we have:
1) $\mathbb{F}E = E^H = \{f : H \to E\}$, where the action of H is given by $(hf)(\sigma) = f(\sigma h), \quad f \in E^H, h, \sigma \in H$.
2) Adjunction morphism $\alpha_E : E \to \mathbb{F}E$ is given by $(\alpha_E e)(\sigma) = \sigma e$, $e \in E, \sigma \in H$.
3) $\mathbb{Q}E = \{\psi : H \to E \mid \psi(1) = 0\}$, where the action of H is

$$(h\psi)(\sigma) = \psi(\sigma h) - \sigma\psi(h), \qquad \psi \in \mathbb{Q}E, \sigma, h \in H.$$

4) Morphism $q_E : \mathbb{F}E \to \mathbb{Q}E$ is given by

$$(q_E f)(\sigma) = f(\sigma) - \sigma f(1), \quad f \in \mathbb{F}E = E^H, \sigma \in H.$$

5) The application $c : e \in E \mapsto c(e) \in E^H$, where $c(e)(\sigma) = e$ for any $\sigma \in H$, gives rise to a natural identification $E = (\mathbb{F}E)^{inv}$.
6) For any $r \geq 1$ we have a natural identification

$$\mathbb{Q}^r E = \{\psi : H^r \to E \mid \psi(h_1, \dots, h_r) = 0 \ \text{ if } \ \exists j, h_j = 1\}$$

where the action of H is given by

$$(h_{r+1}\psi)(h_1, \dots, h_r) = \sum_{i=1}^{r} (-1)^{r-i} \psi(h_1, \dots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \dots, h_{r+1}) + \\ + (-1)^r h_1 \psi(h_2, \dots, h_{r+1}).$$

7) Morphisms $q_E^r : \mathbb{F}\mathbb{Q}^r E \to \mathbb{Q}^{r+1} E$, $g_E^r : \mathbb{F}\mathbb{Q}^r E \to \mathbb{F}\mathbb{Q}^{r+1} E$ (see (3.1.1)) are given by

$$(q_E^r f)(h_1, \dots, h_r, h_{r+1}) = f(h_{r+1})(h_1, \dots, h_r) - [h_{r+1} f(1)](h_1, \dots, h_r),$$
$$(g_E^r f)(\sigma) = \sigma(q_E^r f), \quad f \in \mathbb{F}\mathbb{Q}^r E = (\mathbb{Q}^r E)^H, \ h_1, \dots, h_{r+1}, \sigma \in H.$$

8) By 5), morphism $\varrho_E^r := (g_E^r)^{inv} : (\mathbb{F}\mathbb{Q}^r E)^{inv} = \mathbb{Q}^r E \to (\mathbb{F}\mathbb{Q}^{r+1} E)^{inv} = \mathbb{Q}^{r+1}$ is

$$(\varrho_E^r \psi)(h_1, \dots, h_{r+1}) = \psi(h_1, \dots, h_r) - (h_{r+1}\psi)(h_1, \dots, h_r)$$

for $r \geq 1$, $\psi \in \mathbb{Q}^r E$, $h_i \in H$. For $r = 0$, morphism $\varrho_E^0 := (g_E^0)^{inv} : (\mathbb{F}\mathbb{Q}^0 E)^{inv} = E \to (\mathbb{F}\mathbb{Q}^1 E)^{inv} = \mathbb{Q}^{r+1}$ is

$$(\varrho_E^0 e)(h_1) = e - h_1 e, \quad e \in E, h_1 \in H.$$

(4.3.2) Remark. The complex $(\mathbb{Q}^r E, \varrho_E^r)_{r \geq 0}$ is the usual complex of E-valued cochains obtained from the normalized bar resolution [7], chap. IV, §5.

(4.3.3) Category $\mathfrak{A}_X$ is equivalent to the category
-) whose objects are triplets (V, W, ς) where V is a $k[H]$-module (representing the restriction j^* of a constructible sheaf), W is a k-module (representing the fiber i^* at F of a constructible sheaf) and $\varsigma : W \to V^{inv}$ is a k-linear morphism (representing the adjunction morphism $i^* \to i^* j_* j^*$).
-) whose morphisms are defined in the obvious way.

By (4.3.1), (4.3.3) and the fact that sheaves on X are determined by their restrictions j^*, i^* and the adjunction morphism $i^* \to i^* j_* j^*$, we deduce that category $\mathfrak{B}^d_{\mathbb{F}}$ is equivalent to the category $\mathfrak{C}^d(k,H)$:
-) whose objects are 4-uples (E, M, u, v) where E is a $k[H]$-module, M is a k-module and u, v appear in a commutative diagram

$$\begin{tikzcd} \mathbb{Q}^{d-2}E \arrow[r, "\varrho_E^{d-2}"] & \mathbb{Q}^{d-1}E \arrow[rr, "\varrho_E^{d-1}"] \arrow[dr, "u"'] & & (\mathbb{Q}^d E)^{inv} \\ & & M \arrow[ur, "v"'] & \end{tikzcd}$$

such that $u \circ \varrho_E^{d-2} = 0$, if $d \geq 2$.
-) whose morphisms are defined in the obvious way.

By theorem (3.2.2) we conclude that the category of d-conical perverse sheaves is equivalent to $\mathfrak{C}^d(k,H)$.

In case $d = 1$, by defining $v_\sigma(y) = -v(y)(\sigma)$, $\sigma \in H, y \in M$, we obtain an equivalence between $\mathfrak{C}^1(k,H)$ and the category of k-module diagrams

$$\begin{tikzcd} E \arrow[r, shift left, "u"] & M \arrow[l, shift left, "\{v_\sigma\}_{\sigma \in H}"] \end{tikzcd}$$

such that
(1) $v_{\tau\sigma} = v_\tau \circ u \circ v_\sigma + v_\tau + v_\sigma$ for all $\sigma, \tau \in H$.
(2) $1_E + v_\sigma \circ u$ is an automorphism of E for any $\sigma \in H$.

Property (1) comes from the fact that $v(y) \in (\mathbb{Q}E)^{inv}$ for every $y \in M$. In property (2), automorphism $1_E + v_\sigma \circ u$ coincides with the action of σ on E.

In this way we find a new proof of theorem (2.3.4) in [13]. This theorem is a natural generalization of the first known case [2] on explicit description of perverse sheaves, namely $S = S^1, H = \mathbb{Z}$. (see also [10], [12]).

4.4 Explicit description of perverse direct images and intersection complexes

In this section we give models (16) for $j^p_*\mathcal{L}$, $j^p_!\mathcal{L}$ and $j_{!*}\mathcal{L}$, where $\mathcal{L}$ is an object of $\mathfrak{A}_U$. The computations consist of interpreting the proof of theorem 1.4.10 in [1] in terms of our $(\mathbb{F},\mathbb{Q})$-resolutions (13).

(4.4.1) For each $\mathcal{L} \in \mathfrak{A}_U$ we have a natural isomorphism

$$D_d(j^p_*\mathcal{L}) \simeq (\mathcal{L}, j_*\mathbb{Q}^d\mathcal{L}, j_*q^d_{\mathcal{L}}, 1).$$

In particular the complex

$$j_*\mathbb{F}\mathcal{L} \xrightarrow{j_*g^0_{\mathcal{L}}} j_*\mathbb{F}\mathbb{Q}\mathcal{L} \xrightarrow{j_*g^1_{\mathcal{L}}} \cdots \xrightarrow{j_*g^{d-2}_{\mathcal{L}}} j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L} \xrightarrow{j_*q^d_{\mathcal{L}}} j_*\mathbb{Q}^d\mathcal{L}$$

(in degrees $[0,d]$) is an explicit model for $j_*^p\mathcal{L}$, which coincides with $\tau_{\leq d}\mathbb{R}j_*\mathcal{L}$ [1], prop. 1.4.23.

(4.4.2) For $d=1$ we have a natural isomorphism

$$D_1(j_!^p\mathcal{L}) \simeq (\mathcal{L}, j_*\mathbb{F}\mathcal{L}/j_!\mathcal{L}, \mathrm{can}, 1).$$

In particular the complex

$$j_*\mathbb{F}\mathcal{L} \xrightarrow{\mathrm{can}} j_*\mathbb{F}\mathcal{L}/j_!\mathcal{L}$$

(in degrees $0,1$) is an explicit model for $j_!^p\mathcal{L}$. It is quasi-isomorphic to $j_!\mathcal{L}$ since $j_!\mathcal{L}$ is 1-perverse.

For $d \geq 2$ we have a natural isomorphism

$$D_d(j_!^p\mathcal{L}) \simeq (\mathcal{L}, \mathrm{coker}\, j_*g_{\mathcal{L}}^{d-2}, \mathrm{can}, 1).$$

In particular the complex

$$j_*\mathbb{F}\mathcal{L} \xrightarrow{j_*g_{\mathcal{L}}^0} \cdots \xrightarrow{j_*g_{\mathcal{L}}^{d-2}} j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L} \xrightarrow{\mathrm{can}} \mathrm{coker}\, j_*g_{\mathcal{L}}^{d-2},$$

(in degrees $[0,d]$) is an explicit model for $j_!^p\mathcal{L}$, which coincides with $\tau_{\leq d-2}\mathbb{R}j_*\mathcal{L}$ [1], prop. 1.4.23.

(4.4.3) By interpreting natural morphisms $j_!^p\mathcal{L} \longrightarrow j_*^p\mathcal{L}$ on models above, we have a natural isomorphism

$$D_d(j_{!*}\mathcal{L}) \simeq (\mathcal{L}, \mathrm{Img}\, j_*q_{\mathbb{Q}^{d-1}\mathcal{L}}, j_*q_{\mathbb{Q}^{d-1}\mathcal{L}}, 1).$$

In particular the complex

$$j_*\mathbb{F}\mathcal{L} \xrightarrow{j_*g_{\mathcal{L}}^0} \cdots \xrightarrow{j_*g_{\mathcal{L}}^{d-2}} j_*\mathbb{F}\mathbb{Q}^{d-1}\mathcal{L} \xrightarrow{j_*q_{\mathbb{Q}^{d-1}\mathcal{L}}} \mathrm{Img}\, j_*q_{\mathbb{Q}^{d-1}\mathcal{L}}$$

(in degrees $[0,d]$) is an explicit model for the intersection complex $IC(\mathcal{L}) = j_{!*}\mathcal{L}$, which coincides with $\tau_{\leq d-1}\mathbb{R}j_*\mathcal{L}$ [1], prop. 1.4.23.

4.5 Further results

Following a suggestion of Deligne, explicit models of perverse sheaves can be constructed by using other functorial resolutions instead of (13). For instance, given $\mathbb{F} = FG : \mathcal{A} = \mathfrak{B}_U \to \mathcal{A} = \mathfrak{B}_U$, $\alpha : 1 \to \mathbb{F}$ under the conditions of (1.3.1), with $\mathbb{F}^k\mathcal{L}$ j_*-acyclic for $k \geq 1$ and $\mathcal{L} \in \mathfrak{A}_U$, and not requiring $\mathbb{F}(\mathfrak{A}_U) \subset \mathfrak{A}_U$, we can use the "simplicial" resolution

$$\mathbb{F} \xrightarrow{\partial^0} \mathbb{F}^2 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{d-2}} \mathbb{F}^d \xrightarrow{\partial^{d-1}} \cdots$$

where

$$\partial^i = \alpha\mathbb{F}^{i+1} - \mathbb{F}\alpha\mathbb{F}^i + \cdots + (-1)^{i+1}\mathbb{F}^{i+1}\alpha$$

(cf. [4], Appendice, 5 and [8], VII, 6). This is the aim of an article in preparation.

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