

The defining ideals of conjugacy classes of nilpotent matrices and a conjecture of Weyman

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Abstract

Tanisaki introduced generating sets for the defining ideals of the schematic intersections of the closure of conjugacy classes of nilpotent matrices with the set of diagonal matrices. These ideals are naturally labeled by integer partitions. Given such a partition λ , we define several methods to produce a reduced generating set for the associated ideal \mathcal{I}_λ . For particular shapes we find nice generating sets. By comparing our sets with some generating sets of \mathcal{I}_λ arising from a work of Weyman, we find a counterexample to a related conjecture of Weyman.

1 Introduction

Let X be the set of $n \times n$ matrices over a field k of characteristic 0. In his paper Kostant [K] showed that the ideal of polynomial functions vanishing on the set of nilpotent matrices in X , is given by the invariants of the action by conjugation of $GL(n)$ on X . Let C_λ be the conjugacy class of nilpotent matrices in X having Jordan block sizes $\lambda'_1, \dots, \lambda'_h$, with λ a partition of n and λ' its transpose. Let \overline{C}_λ be the nilpotent orbit variety defined as the Zariski closure of C_λ . De Concini and Procesi [DP] asked for a description of the ideal \mathcal{J}_λ of polynomial functions vanishing on C_λ , for a general partition λ . They were interested in a refinement of Kostant's result, which corresponds to the case $\lambda = (1^n)$. De Concini and Procesi described a set of elements of \mathcal{J}_λ that they conjectured to be a generating set. Later, Tanisaki [T] conjectured a simpler generating set, and Eisenbud and Saltman [ES] generalized Tanisaki's conjecture to rank varieties. Finally, in 1989 Weyman [W1] used geometric methods to show that the three conjectures hold, and conjectured a minimal generating set \mathcal{W}_λ for these ideals.

In the present paper we focus on a related family of ideals that we denote by \mathcal{I}_λ and call *De Concini-Procesi ideals*. These are the ideals of the scheme-theoretic intersection of nilpotent orbit varieties \overline{C}_λ with the set of diagonal matrices. De Concini and Procesi [DP] produced a set of generators for these ideals that was later simplified by Tanisaki [T]. In both cases, the sets of generators are highly nonminimal. In the case $\lambda = (1^n)$, Kostant's theorem implies that the elementary symmetric functions of the eigenvalues of the matrices give a minimal set of generators for $\mathcal{I}_{(1^n)}$.

Our work in this paper is motivated by the search for a minimal generating set for De Concini-Procesi ideals. To this end, we simplify the generating set described by Tanisaki using elementary facts of the theory of symmetric functions. We provide several reduction methods. The obtained sets are minimal in special cases, and are generally much smaller. The main tool we use is a special filling of the Young diagram of the partition λ which we call the *regular filling*.

Clearly, by adding the defining ideal of the diagonal matrices to any generating set for the ideal \mathcal{J}_λ , we obtain a generating set for \mathcal{I}_λ . The following question is natural: Is it true that, after adding these generators to Weyman's conjectured minimal generating set for \mathcal{J}_λ , a minimal generating set for \mathcal{I}_λ is obtained? We give a negative answer to this question and provide some infinite families of counterexamples. With the help of Macaulay 2 we verify that one of these counterexamples is also a counterexample to the original conjecture

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of Weyman on a minimal generating set of \mathcal{I}_λ . This has been a well studied problem that has been open for the past seventeen years. We hope that our methods together with those of Weyman will eventually lead to a complete solution of the problem of finding a minimal generating set for both ideals \mathcal{I}_λ and \mathcal{J}_λ .

Our paper is organized as follows. In Section 2 we introduce some basic tools from the theory of symmetric functions. In Section 3, we introduced Tanisaki's generating set for the De Concini-Procesi ideal, and derive a simple combinatorial description for it. This leads to a simple rule to read a set of generators of the ideal directly from a special filling of the Young diagram of the partition that call the *regular filling*. In Section 4 we show that only generators read from the top entries of the regular filling are necessary in order to construct a generating set for \mathcal{I}_λ . The resulting generating set is in a one-to-one correspondence with a generating set that arises from the work of Weyman [W1]. In the case where the partition λ is a hook, our result coincides with the minimal generating set we introduced in [BFR]. For a general shape though, this generating set could be far from minimal. In Section 5 we reduce the number of generators coming from each column of the Young diagram. Finally in Section 6, we provide many examples and counterexamples to the modified version of Weyman's conjecture, and discuss classes where our reductions work best. Inside those families we are able to find a counterexample to the original conjecture of Weyman on a minimal generating set for the ideal \mathcal{J}_λ . Throughout the paper, we raise new questions whose answers could help illuminate the problem of finding minimal generating sets for \mathcal{I}_λ and \mathcal{J}_λ .

2 Basic Tools

We will be working in the polynomial ring $R = k[x_1, \dots, x_n]$, where k may be an arbitrary field of characteristic 0.

We define a *partition* of $n \in \mathbb{N}$ to be a finite sequence $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, such that $\sum_{i=1}^k \lambda_i = n$ and $\lambda_1 \geq \dots \geq \lambda_k$. If λ is a partition of n we write $\lambda \vdash n$. The nonzero terms λ_i are called *parts* of λ . The number of parts of λ is called the *length* of λ , denoted by $\ell(\lambda)$, so $\lambda_i = 0$ if $i > \ell(\lambda)$.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n . The *Young diagram* of a partition λ is the left-justified array with λ_i squares in the i -th row, from bottom to top. We use the symbol λ for both a partition and its associated Young diagram. For example, the diagram of $\lambda = (4, 4, 2, 1)$ is illustrated in Figure 1 on the left.

For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we define its *conjugate* partition as $\lambda' = (\lambda'_1, \dots, \lambda'_h)$, where for each $i \geq 1$, λ'_i is the number of parts of λ that are bigger than or equal to i . The diagram of λ' is obtained by flipping the diagram of λ across the diagonal.



Figure 1: The partition $\lambda = (4, 4, 2, 1)$ and its conjugate $\lambda' = (4, 3, 2, 2)$.

We shall need some basic definitions from the theory of symmetric functions. First, we introduce the generating series for the elementary and the complete symmetric polynomials (denoted respectively by $E(S, z)$ and $H(S, z)$). These series are defined as:

$$E(S, z) = \sum_{i \geq 0} z^i e_i(S) = \prod_{a \in S} (1 + za), \quad \text{and} \quad H(S, z) = \sum_{i \geq 0} z^i h_i(S) = \prod_{a \in S} \frac{1}{1 - za}, \quad (1)$$

where S is a set of variables, and z is a formal variable. Therefore, the *elementary symmetric polynomial* $e_r(S)$ is the sum of all square free monomials of degree r in the variables of S , and the *complete symmetric polynomial* $h_r(S)$ is the sum of all monomials of degree r in the variables of S .

In order to introduce the monomial symmetric polynomials $m_\lambda(S)$, we say that a monomial $\mathbf{x}^s = x_1^{s_1} x_2^{s_2} \dots x_n^{s_n}$ has *type* λ , if the partition λ is obtained by rearranging the sequence (s_1, s_2, \dots, s_n) in weakly descending order. Given a partition λ , the *monomial symmetric polynomial* $m_\lambda = m_\lambda(S)$ is defined as

$$m_\lambda(S) = \sum \mathbf{x}^s$$

where the sum is taken over all different monomials \mathbf{x}^s of type λ and with all variables in S .

If $f \in k[x_1, \dots, x_n]$ is a symmetric polynomial, and $S \subseteq \{x_1, \dots, x_n\}$, we define $f(S)$ as the evaluation of f at the set S , by setting all variables $x \in \{x_1, \dots, x_n\} \setminus S$ to be equal to 0 in f . For instance, $e_2(x_1, x_3) = x_1x_3$. The polynomial $f(S)$ is called a *partially symmetric polynomial*. In general, it is no longer invariant under the action of the symmetric group on n letters.

For simplicity, given a symmetric polynomial $f \in k[x_1, \dots, x_n]$, for all $1 \leq k \leq n$, we will denote by $f(k)$ the following set of partially symmetric polynomials,

$$f(k) = \{f(S) \mid S \subseteq \{x_1, \dots, x_n\}, |S| = k\}.$$

For example, let $n = 4$, then $e_2(3) = \{x_1x_2 + x_1x_3 + x_2x_3, x_1x_2 + x_1x_4 + x_2x_4, x_1x_3 + x_1x_4 + x_3x_4, x_2x_3 + x_2x_4 + x_3x_4\}$. Note that if $r > k$ we have $e_r(k) = \emptyset$.

Notation. Let $S \subseteq \{x_1, \dots, x_n\}$. For $x \in S$, and $I = \{x_{i_1}, \dots, x_{i_k}\} \subseteq S$, we let

$$S_x = S \setminus \{x\} \text{ and } S_{i_1, \dots, i_k} = S \setminus I.$$

We shall be using the following elementary lemma later in the paper.

Lemma 2.1 (Basic Lemma). *Let $S \subseteq \{x_1, \dots, x_n\}$, $|S| = s$, and let $j \leq s$. Then*

1. $e_j(S) = e_j(S_x) + xe_{j-1}(S_x)$ for all $x \in S$;
2. $\sum_{x \in S} e_j(S_x) = (s - j)e_j(S)$;
3. $\sum_{x \in S} xe_{j-1}(S_x) = je_j(S)$.

Proof. 1. Clear.

2. Fix a square-free monomial M of degree j appearing in $e_j(S)$. Without loss of generality, assume $M = x_1 \cdots x_j$ and $S = \{x_1, \dots, x_s\}$. Then each $e_j(S_{x_t})$ contains exactly one copy of M , for $t = j + 1, \dots, s$. There are exactly $s - j$ such indices t , so M appears $s - j$ times in the left-hand sum.
3. We use the equation in Part 1, and sum over all elements of S : $\sum_{x \in S} e_j(S) = \sum_{x \in S} e_j(S_x) + \sum_{x \in S} xe_{j-1}(S_x)$ so by Part 2 we have $se_j(S) = (s - j)e_j(S) + \sum_{x \in S} xe_{j-1}(S_x)$ and hence $je_j(S) = \sum_{x \in S} xe_{j-1}(S_x)$. □

Proposition 2.2 (Another presentation of the partially symmetric polynomials). *Let $S = \{x_1, \dots, x_n\}$, $i \leq n$, and define the ideal $\mathcal{E}_i(S) = (e_1(S), \dots, e_i(S))$ in the polynomial ring $k[x_1, \dots, x_n]$. Let $U \subseteq S$ be a subset of cardinality u . Then for $i \leq n - u$ we have*

$$e_i(S \setminus U) = (-1)^i h_i(U) \text{ mod } \mathcal{E}_i(S). \quad (2)$$

Proof. This result follows from a formal manipulation of the generating functions in (1). We have

$$E(S \setminus U, z) = \prod_{\substack{a \in S \\ a \notin U}} (1 + za) = \frac{\prod_{a \in S} (1 + za)}{\prod_{a \in U} (1 + za)} = E(S, z)H(U, -z).$$

Therefore, extracting the coefficient of z^i from both sides of the resulting equation $E(S \setminus U, z) = E(S, z)H(U, -z)$ we obtain

$$e_i(S \setminus U) = \sum_{j=0}^i e_j(S) (-1)^{i-j} h_{i-j}(U).$$

By hypothesis $e_j(S)$ is in the ideal for $j = 1, \dots, i$. Since $e_0(S) = 1$, the result follows. □

3 A new combinatorial description of Tanisaki's generating set for \mathcal{I}_λ

In this section, we define a family of ideals \mathcal{I}_λ in the polynomial ring $R = k[x_1, \dots, x_n]$ indexed by partitions λ of n . The ideal \mathcal{I}_λ was first introduced by De Concini and Procesi [DP] in order to describe the coordinate ring of the schematic intersection of the Zariski closure of the conjugacy class of nilpotent matrices of shape λ , with the set of diagonal matrices.

In order to manipulate De Concini-Procesi ideals, we use a generating set defined by Tanisaki [T]. A nice feature of Tanisaki's generating set is that its elements are elementary partially symmetric polynomials. Furthermore, Tanisaki's proof of the correctness of his generating set is both elegant and elementary, and it is based on standard linear algebra facts. Finally, Tanisaki's generating set has proven to be very fruitful in algebraic combinatorics, see for example [AB, BG, GP].

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n . For the purpose of the next formula, we add enough zeroes to the end of λ so that it has n terms: $\lambda = (\lambda_1, \dots, \lambda_n)$. For any $1 \leq k \leq n$, we define

$$\delta_k(\lambda) = \lambda'_n + \lambda'_{n-1} + \dots + \lambda'_{n-k+1}. \quad (3)$$

It is clear that $\delta_n(\lambda) \geq \delta_{n-1}(\lambda) \geq \dots \geq \delta_1(\lambda)$, and that $\delta_n(\lambda) = n$.

Theorem 3.1 (Tanisaki's generating set [T]). *The ideal \mathcal{I}_λ is generated by the following collection of elementary partially symmetric polynomials*

$$\mathcal{I}_\lambda = (e_r(k) \mid k = 1, \dots, n, \text{ and } k \geq r > k - \delta_k(\lambda)). \quad (4)$$

Definition 3.2 (De Concini-Procesi ideal). We call the ideal \mathcal{I}_λ defined in Theorem 3.1 the *De Concini-Procesi ideal* of the partition λ .

Since for any partition λ of n , $\delta_n(\lambda) = n$, when we set $k = n$ in (4) we conclude that \mathcal{I}_λ contains all the elementary symmetric polynomials in all the variables x_1, \dots, x_n .

Example 3.3. Let $\lambda = (4, 4, 2, 1, 0, 0, 0, 0, 0, 0, 0) \vdash 11$ be the partition appearing in Figure 1. Then $(\delta_1(\lambda), \dots, \delta_{11}(\lambda)) = (0, 0, 0, 0, 0, 0, 0, 2, 4, 7, 11)$. Hence

$$(1 - \delta_1(\lambda), \dots, 11 - \delta_{11}(\lambda)) = (1, 2, 3, 4, 5, 6, 7, 6, 5, 3, 0).$$

Here $n = 11$. For $k = 1, \dots, 7$ there is no admissible $e_r(k)$ in the generating set described in (4). So the generating set of $\mathcal{I}_{(4421)}$ consists of the following elements

| | Generators |
|----------|---------------------------------------|
| $k = 8$ | $e_7(8), e_8(8)$ |
| $k = 9$ | $e_6(9), e_7(9), e_8(9), e_9(9)$ |
| $k = 10$ | $e_4(10), e_5(10), \dots, e_{10}(10)$ |
| $k = 11$ | $e_1(11), e_2(11), \dots, e_{11}(11)$ |

We now give a simple combinatorial description of the set of generators for \mathcal{I}_λ described in Theorem 3.1, and then demonstrate how to shorten it so that one can read a reduced generating set for \mathcal{I}_λ directly from the diagram of the partition λ . In order to do so we introduce the notion of regular filling.

Definition 3.4 (The regular filling of a partition). Let λ be a partition of n . Draw its Young diagram and then fill its cells with the numbers $1, 2, \dots, n$ from top to bottom and from left to right, skipping the cells in the bottom row, which should be filled at the end from right to left. This is called the *regular filling* of λ , denoted rf .

Definition 3.5 (The reading process). We associate to any filling f of the Young diagram of λ a set of partial symmetric polynomials, denoted by $\mathcal{G}_f(\lambda)$. We read the elements of this set from the filling as follows. For a given column of λ we add to $\mathcal{G}_f(\lambda)$ all the elements of the sets $e_r(k)$, where k is the entry in the bottom cell of the column, and the degrees r 's are given by all the entries in that column.

| | | | |
|----|----|---|---|
| 1 | | | |
| 2 | 4 | | |
| 3 | 5 | 6 | 7 |
| 11 | 10 | 9 | 8 |

Figure 2: The regular filling of $(4, 4, 2, 1)$.

Notation. From now on, we enumerate columns and rows of a Young diagram from left to right by starting from zero. So the “first” column will be the 0-th column; similarly for rows.

Example 3.6. For the partition $\lambda = (4, 4, 2, 1)$, the regular filling rf is illustrated in Figure 2. The reading process of this filling gives the set $\mathcal{G}_{rf}(\lambda)$ consisting of: the elementary symmetric polynomials $e_1(x_1, \dots, x_{11})$, $e_2(x_1, \dots, x_{11})$, $e_3(x_1, \dots, x_{11})$, $e_{11}(x_1, \dots, x_{11})$, coming from the 0-th column; the partially symmetric polynomials of the sets $e_4(10)$, $e_5(10)$, $e_{10}(10)$ read from the first column, $e_6(9)$, $e_9(9)$ from the second column, and $e_7(8)$, $e_8(8)$ from the last column.

By using this reading process, we are going to read Tanisaki’s generators from a special filling.

Definition 3.7 (The antidiagonal filling). Let λ be a partition of n . Compute the partition $\delta(\lambda)$

$$\delta(\lambda) = \delta_n(\lambda) \geq \delta_{n-1}(\lambda) \geq \dots \geq \delta_1(\lambda),$$

where $\delta_k(\lambda)$ is defined as in (3), and draw the Young diagram of its conjugate $\delta'(\lambda)$. Now fill the 0-th column of $\delta'(\lambda)$ by $1, 2, \dots, n$ from top to bottom, and then fill the remainder of the diagram so that the filling is constant following each antidiagonal. We call this the *antidiagonal filling* of $\delta'(\lambda)$ and denote it by af .

For our running example $\lambda = (4, 4, 2, 1, 0^7)$, we have $\delta(\lambda) = (11, 7, 4, 2, 0^7)$; the antidiagonal filling of $\delta'(\lambda)$ is given in Figure 3. Note that the bottom entry of the k -th column of $\delta'(\lambda)$ is $n - k$.

| | | | |
|----|----|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | 4 | | |
| 6 | 5 | | |
| 7 | 6 | | |
| 8 | 7 | 6 | |
| 9 | 8 | 7 | |
| 10 | 9 | 8 | 7 |
| 11 | 10 | 9 | 8 |

Figure 3: The antidiagonal filling of $\delta'(\lambda)$.

Let λ be a partition of n . Compute the set $\mathcal{G}_{af}(\delta'(\lambda))$ by applying the reading process to the antidiagonal filling af of $\delta'(\lambda)$. We have the following lemma.

Lemma 3.8. *Let λ be a partition of n . Then Tanisaki’s set of generators is $\mathcal{G}_{af}(\delta'(\lambda))$. In particular,*

$$\mathcal{I}_\lambda = (\mathcal{G}_{af}(\delta'(\lambda))).$$

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_n)$. Compute $\delta'(\lambda)$ and fill its diagram with the antidiagonal filling. According to Theorem 3.1, to compute Tanisaki’s generating set, we need to find for which k the interval $[k - \delta_k(\lambda) + 1, \dots, k - 1, k]$ is nonempty; clearly this happens when $\delta_k(\lambda) > 0$.

From the definition of $\delta_k(\lambda)$, the only times $\delta_k(\lambda) > 0$ is when $k = n - \lambda_1 + 1, \dots, n$. So we are considering values $e_r(S)$ for sets S such that $n - \lambda_1 + 1 \leq |S| \leq n$. This is an interval of length λ_1 , and the numbers $k = |S|$ we are considering are exactly the entries in the first row of $\delta'(\lambda)$.

Now, fix a column t that has entry $n - t$ in its bottom cell. The generating set described in Theorem 3.1 has $e_r(S)$, where $|S| = n - t$ and $r = n - t - \delta_{n-t}(\lambda) + 1, \dots, n - t$. Note that there exactly $\delta_{n-t}(\lambda)$ values that r takes, and that is exactly the size of the t -th column of $\delta'(\lambda)$. The mentioned values of r are exactly the entries of the t -th column of the antidiagonal filling of $\delta'(\lambda)$. \square

One can easily check that this procedure applied to the antidiagonal filling in Figure 3 produces the generators given in the table of Example 3.3.

We are now able to show the main result of this section, namely, that \mathcal{I}_λ is the sum of three simpler ideals. In order to do so we will use the regular filling.

Theorem 3.9. *Let λ be a partition of n . Fill the diagram of λ with the regular filling, and compute the set $\mathcal{G}_{rf}(\lambda)$ by using the reading process described in Definition 3.5. Then*

$$\mathcal{I}_\lambda = (\mathcal{G}_{rf}(\lambda)).$$

Proof. Compute the partition $\delta'(\lambda)$, fill its diagram with the antidiagonal filling and read off all of Tanisaki's generators. By Part 2 of Lemma 2.1, if $e_r(x_1, \dots, x_j) \neq 0$ belongs to the ideal, so does $e_r(x_1, \dots, x_J)$ for any $J > j$. Therefore, for each entry $r = 1, \dots, n$, we only need to keep the generators coming from the rightmost occurrence of that r in the antidiagonal filling of $\delta'(\lambda)$. So we delete all other occurrences of r in that filling, and the corresponding cell. We obtain a filling that contains exactly one occurrence of each of the numbers from 1 to n . Now observe that the differences of heights between adjacent columns of $\delta'(\lambda)$ are given by the sequence $\lambda'_1, \dots, \lambda'_{\lambda_1}$. So after the deletion process, explained above, the remaining diagram will have columns of height $\lambda'_1, \dots, \lambda'_{\lambda_1}$. Hence it is the diagram of our partition λ . Moreover the resulting is the regular filling, and we are done. The case of the partition $\lambda = (4, 4, 2, 1)$ is displayed in Figure 4. \square

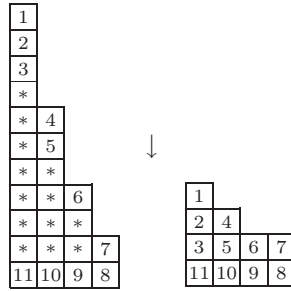


Figure 4: From the antidiagonal to the regular filling.

Remark 3.10. Observe that $e_j(S)$ for S of cardinality j is a square free monomial of degree j . So once we have all square-free monomials of degree $n - \lambda_1 + 1$ in our ideal, then we have the ones of higher degree. These monomials are obtained when we read the generators coming from the rightmost entry of the bottom row.

The following statement follows easily from the previous remark and Theorem 3.9.

Corollary 3.11 (First reduction of Tanisaki's generating set for \mathcal{I}_λ). *Let λ be a partition of n . Then \mathcal{I}_λ can be described as the sum of the following three ideals:*

$$\mathcal{I}_\lambda = \mathcal{M}_\lambda + \mathcal{E}_\lambda + \mathcal{K}_\lambda,$$

where

- \mathcal{M}_λ is generated by all square-free monomials of degree $n - \lambda_1 + 1$;
- \mathcal{E}_λ is generated by the elementary symmetric polynomials $e_1(x_1, \dots, x_n), \dots, e_{\ell(\lambda)-1}(x_1, \dots, x_n)$;
- \mathcal{K}_λ is generated by the partially symmetric polynomials in $e_r(k)$, where $n - 1 \geq k \geq n - \lambda_1 + 1$, and r in an entry of the regular filling of λ , in the same column as k , and strictly above it.

In the particular case where the indexing partition λ is a hook, we recover the minimal generating set for \mathcal{I}_λ described in [BFR, Proposition 3.4].

4 Second reduction of the generating set for \mathcal{I}_λ

Our goal in the rest of the paper is to shave off as many redundant generators as possible from the generating set given in Corollary 3.11. It turns out that only partially symmetric polynomials coming from the top value of each column are required in the generating set. This finding already gives a large reduction in the number of generator needed in the generating set of Tanisaki. Several other reductions will be obtained in the following sections.

Suppose we have a partition λ of an integer n , and fill the diagram of λ with the regular filling defined in Definition 3.4. For $k \geq 1$ we label the value in the top cell of the k -th column with b_k , as long as the height of the k -th column is ≥ 2 . If the right-most column of λ has height 1, then we label its entry b_s . This is reflected in the diagram in Figure 5. Note that with this notation we have

$$b_1 = \lambda'_1, b_2 = \lambda'_1 + \lambda'_2 - 1, \dots, b_k = \lambda'_1 + \dots + \lambda'_k - k + 1 \text{ for } k \leq t, b_s = n - s,$$

where we set

$$t = \lambda_2 - 1, \text{ and } s = \lambda_1 - 1. \quad (5)$$

Clearly if $\lambda_1 = \lambda_2$, then $t = s$ and b_s does not exist.

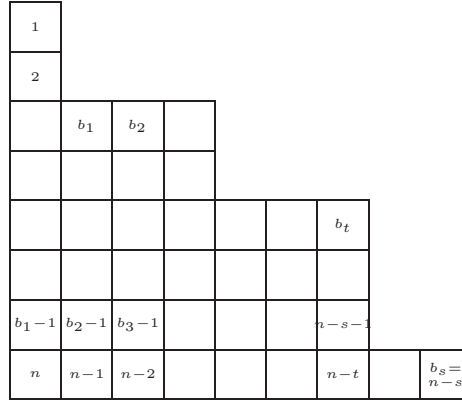


Figure 5: Diagram of a partition λ of n with the regular filling.

By Corollary 3.11) the reduced form of Tanisaki's generating set for \mathcal{I}_λ is the union of the following sets:

$$\begin{array}{ll} \text{Column 0} & e_1(n), \dots, e_{b_1-1}(n) \\ \text{Column 1} & e_{b_1}(n-1), \dots, e_{b_2-1}(n-1) \\ \text{Column 2} & e_{b_2}(n-2), \dots, e_{b_3-1}(n-2) \\ \vdots & \vdots \\ \text{Column } t & e_{b_t}(n-t), \dots, e_{n-s-1}(n-t) \\ \text{Column } s \text{ (if } s > t) & e_{n-s}(n-s), \text{ or all square-free monomials of degree } (n-s). \end{array} \quad (6)$$

Our goal here is to show that it is enough to pick only one set of generators in each column, other than the 0-th column; namely, the ones coming from the top values in each column.

Theorem 4.1 (Principal reduction of the generating set for \mathcal{I}_λ). *Let λ be a partition of n , and suppose that the diagram of λ has been filled as in Figure 5. Then a generating set for \mathcal{I}_λ is*

$$\begin{array}{ll} \text{Column 0} & e_1(n), \dots, e_{b_1-1}(n) \\ \text{Column 1} & e_{b_1}(n-1) \text{ (or } x_1^{b_1}, \dots, x_n^{b_1}) \\ \text{Column 2} & e_{b_2}(n-2) \\ \vdots & \vdots \\ \text{Column } t & e_{b_t}(n-t) \\ \text{Last column (if } s > t) & e_{n-s}(n-s), \text{ or all square-free monomials of degree } (n-s). \end{array} \quad (7)$$

If $\lambda = (1^n)$ is the one-column partition, then we also need to add the element $e_n(n) = x_1 \cdots x_n$ to this generating set. If $\lambda = (n)$ is the one-row partition, we only need generators from the last column, in other words $\mathcal{I}_{(n)} = (x_1, \dots, x_n)$.

Proof. We need to show that having in the ideal all generators read from the top index of each column implies that the other partially symmetric functions coming from the larger indices in that column also belong to the ideal. We go column by column, and build a new ideal I_λ by adding generators described in (7) for each column of λ . We show, each time, that I_λ contains all the other generators described in (6) (coming from the same column), and therefore $I_\lambda = \mathcal{I}_\lambda$.

Col. 0. There is nothing to prove here, as we are keeping all the generators $e_1(n), \dots, e_{b_1-1}(n)$.

Col. 1. Assume that we have $e_{b_1}(S) \in I_\lambda$ for all S with $|S| = n - 1$. By Part 2 of Lemma 2.1, setting $j = b_1$, we see that we have $e_{b_1}(n) \in I_\lambda$.

For each $i > b_1$, we can assume by induction on i that

$$e_i(n), \dots, e_{i-1}(n) \in I_\lambda \text{ and } e_{b_1}(n-1), \dots, e_{i-1}(n-1) \in I_\lambda.$$

Apply Part 3 of Lemma 2.1 with $j = i$, to see that $e_i(n) \in I_\lambda$.

Fix a set S with $|S| = n - 1$ and $x \notin S$. Let $S^x = S \cup \{x\}$. Part 1 of Lemma 2.1 implies that

$$e_i(S) = e_i(S^x) - x e_{i-1}(S)$$

which demonstrates that $e_i(S) \in I_\lambda$. Hence $e_i(n-1) \in I_\lambda$.

The fact that the generators $e_{b_1}(n-1)$ can be replaced by the powers $x_1^{b_1}, \dots, x_n^{b_1}$ follows directly from Proposition 2.2. Note that, in particular, we have $e_i(n-1) \in I_\lambda$, for all $i \geq b_1$.

Col. j. Suppose I_λ contains all generators from the previous columns $0, \dots, j-1$ as described in (7). Let $|S| = n - j$, and suppose $x \notin S$, so that $|S^x| = n - j + 1$, ($S^x = S \cup \{x\}$). We know by induction that I_λ contains $e_h(S^x)$ for all $h \geq b_{j-1}$. Therefore, since $b_j > b_{j-1}$, for $i \geq b_j$ we have by Part 1 of Lemma 2.1

$$\begin{aligned} e_i(S) &= e_i(S^x) - x e_{i-1}(S) &&= -x e_{i-1}(S) \\ &= -x(e_{i-1}(S^x) - x e_{i-2}(S)) &&= x^2 e_{i-2}(S) \\ &= x^2(e_{i-2}(S^x) - x e_{i-3}(S)) &&= -x^3 e_{i-3}(S) \\ &\vdots \\ &= (-1)^{i-b_j} x^{i-b_j} e_{b_j}(S) && \pmod{\text{Col. } j-1} \end{aligned}$$

This means that once we include $e_{b_j}(S)$ in I_λ , we will have all $e_i(S) \in I_\lambda$ for $i \geq b_j$.

□

In the case where λ is a hook, the generating set described in Theorem 4.1 coincides with the minimal generating set for \mathcal{I}_λ introduced in our earlier work [BFR].

Example 4.2. Let $\lambda = (5, 4, 4, 3)$. Then, the regular filling of λ is

| | | | | |
|----|----|----|----|----|
| 1 | 4 | 7 | | |
| 2 | 5 | 8 | 10 | |
| 3 | 6 | 9 | 11 | |
| 16 | 15 | 14 | 13 | 12 |

So the generators of \mathcal{I}_λ are

| Column | Generators | Number of generators |
|--------|--|----------------------|
| 0 | $e_1(16), e_2(16), e_3(16)$ | 3 |
| 1 | x_1^4, \dots, x_{16}^4 | 16 |
| 2 | $e_7(14)$ | 120 |
| 3 | $e_{10}(13)$ | 560 |
| 4 | $e_{12}(12)$, or all square-free monomials of degree 12 | 1820 |
| | Total | 2519 |

Later in Example 6.4 we shall further reduce the generating set of this particular partition.

4.1 Remarks on a related work and conjecture of Weyman

We end this section by showing some relations between the generating set of Theorem 4.1 and two generating sets for \mathcal{I}_λ arising in the work of Weyman [W1].

In [W1] Weyman uses the representation theory of the general linear group to construct and study generating sets for the ideal \mathcal{J}_λ of polynomial functions vanishing on the conjugacy class \mathcal{C}_λ . The generators in the first family, denoted by V_λ , are expressed as sums of minors, and come from reducible representations of $GL(n)$. The second set of generators U_λ , on the other hand, arises from the irreducible representations of $GL(n)$. The set U_λ is smaller than V_λ , but how to compute its elements is not explicit in the paper.

The set V_λ (respectively U_λ) is given by the disjoint union of sets $V_{i,p}$ (respectively $U_{i,p}$), where the family of indices (i, p) can be read off from a special diagram introduced by Weyman; see [W1, Example (4.5)]. We call this diagram the *Weyman diagram* of λ . It is possible to construct the Weyman diagram of a partition starting from the antidiagonal filling (see Definition 3.7) as follows. First, consider the antidiagonal filling of $\delta'(\lambda)$, and justify its columns in such a way that equal entries are now in same rows. Then, replace any entry of this diagram by an X . The resulting picture is the Weyman diagram. In Figure 6 we illustrate the Weyman diagram corresponding to the partition $\lambda = (4, 4, 2, 1)$. Compare this diagram to the one in Figure 3. Note that if the top X in the i -th column of Weyman diagram of λ has coordinates (i, p) , then the top cell of the i -th column of the regular filling of λ is filled by p .

| | | | | |
|----------|----------|----------|----------|----------|
| $p = 1$ | <u>X</u> | | | |
| $p = 2$ | <u>X</u> | | | |
| $p = 3$ | <u>X</u> | | | |
| $p = 4$ | <u>X</u> | <u>X</u> | | |
| $p = 5$ | <u>X</u> | X | | |
| $p = 6$ | <u>X</u> | X | <u>X</u> | |
| $p = 7$ | <u>X</u> | X | X | <u>X</u> |
| $p = 8$ | <u>X</u> | X | X | X |
| $p = 9$ | <u>X</u> | X | X | |
| $p = 10$ | <u>X</u> | X | | |
| $p = 11$ | <u>X</u> | | | |
| $i =$ | 0 | 1 | 2 | 3 |

Figure 6: Weyman diagram for $\lambda = (4, 4, 2, 1)$.

We would like to remark that Weyman follows a convention opposite to ours when labelling the ideals \mathcal{I}_λ and \mathcal{J}_λ : he labels \mathcal{J}_λ the ideal of polynomial functions vanishing on all nilpotent matrices with Jordan blocks $\lambda_1, \dots, \lambda_n$, while we use the transpose. On the other hand, he associates to a partition λ what in our setting would be the Weyman diagram of λ' . These two facts cancel out, and we do not need to take any transpose when reading statements involving his diagrams.

Definition 4.3 (Weyman’s generating set for \mathcal{I}_λ). In [W1, Theorem (4.6)] Weyman shows that the ideal \mathcal{I}_λ is generated by the $U_{i,p}$, where the (i, p) ’s are the coordinates of the top cells of the columns ($i \geq 1$) of the Weyman diagram of λ , together with the invariants $U_{0,p}$ with $1 \leq p \leq n$. This result implies that the ideal \mathcal{I}_λ is also generated by the $V_{i,p}$ coming from the same set of indices (i, p) .

Example 4.4. For the partition $\lambda = (4, 4, 2, 1)$, whose Weyman diagram is in Figure 6, Weyman’s set U_λ consists of $U_{0,p}$, with $1 \leq p \leq 11$, $U_{1,4}$, $U_{2,6}$, and $U_{3,7}$ (and similarly for the set V_λ). The cells X whose coordinates label this generating set are underlined.

After adding the generators for the ideal defining the diagonal matrices to the two sets V_λ and U_λ , one gets two generating sets for \mathcal{I}_λ ; we denote these two generating sets by \tilde{V}_λ and \tilde{U}_λ .

Instead of going into the definitions of V_λ and U_λ that can be found in [W1, Section 4], we explicitly state the cardinalities of their components in order to compare them with our generating set. We emphasize the fact that Tanisaki’s generators (the ones we use) are easier to handle than Weyman’s generators. We have that

$$|V_{i,p}| = \binom{n}{i}^2 \quad \text{and} \quad |\tilde{V}_{i,p}| = \binom{n}{i},$$

and

$$|U_{i,p}| = \binom{n}{i}^2 - \binom{n}{i-1}^2 \quad \text{and} \quad |\tilde{U}_{i,p}| = \binom{n}{i} - \binom{n}{i-1}.$$

It turns out that the cardinalities of the generating set for \mathcal{I}_λ given by the $\tilde{V}_{i,p}$ ’s and the generating set given in Theorem 4.1 are the same. Moreover, it is not difficult to describe a one-to-one correspondence between the two generating sets. Under this correspondence Weyman’s $V_{i,p}$ generators correspond to our generators read from the top cell of the i -th column of the regular filling, as described in Theorem 4.1.

Weyman conjectured that a special subset of U_λ gives a minimal generating set of \mathcal{I}_λ ; see Conjecture 5.1 and Remark 5.3 of [W1].

Conjecture 4.5 (Weyman’s original conjecture). Let λ be a partition. The set consisting of $U_{0,p}$ for $1 \leq p \leq \ell(\lambda)$, and $U_{i,p}$, where (i, p) labels a top cell of the i -th row (in the Weyman diagram of λ), such that there are no X ’s to the right of or on the line segment joining (i, p) with $(0, 1)$, is a minimal set of generators \mathcal{W}_λ of \mathcal{I}_λ .

A very interesting question is the following.

Question 4.6 (Diagonal version of Weyman’s conjecture). Is the generating set $\tilde{\mathcal{W}}_\lambda$ for \mathcal{I}_λ arising from Weyman’s conjecture minimal ?

In the following sections we show that the the answer to this question is negative. Indeed, we provide some infinite families of counterexamples. These observations, together with the help of Macaulay 2 led us to the discovery that even the original conjecture of Weyman (Conjecture 4.5) fails already for one of the smallest elements in these families.

5 Reducing generators of \mathcal{I}_λ of a fixed degree

The aim of this section is to consider the generating set of \mathcal{I}_λ described in Theorem 4.1, and eliminate as many redundant generators as possible from each column.

Proposition 5.1 (Columns of height > 1). *Let λ be a partition whose diagram is represented in Figure 5. For $k \geq 2$, if the height of the $(k-1)$ -st column is > 1 , then we can eliminate $\binom{n-1}{k-1} + 1$ generators of \mathcal{I}_λ (as described in (7)) that come from the k -th column. Indeed, if S denotes the set of variables x_1, \dots, x_n , we can eliminate the elements in the set $\{e_{b_k}(S_{1,i_2,\dots,i_k}) \mid 1 < i_2 < \dots < i_k \leq n\}$ and $e_{b_k}(S_{2,3,\dots,k+1})$.*

Proof. Let $k > 1$, by using Part 2 of Lemma 2.1 we write

$$\sum_{j \notin \{i_1, \dots, i_{k-1}\}} e_{b_k}(S_{i_1, \dots, i_{k-1}, j}) = (n - b_k - k + 1)e_{b_k}(S_{i_1, \dots, i_{k-1}}) \equiv 0 \pmod{\mathcal{I}_{k-1}} \quad (8)$$

| | | | | | | | | | | | | |
|---|----|----|----|----|----|----|--|-----|-----|-----|-----|---|
| | 12 | 13 | 14 | 23 | 24 | 34 | | 123 | 124 | 134 | 234 | |
| 2 | 1 | 0 | 0 | 1 | 1 | 0 | | 23 | 1 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 | 1 | 0 | 1 | | 24 | 0 | 1 | 0 | 1 |
| 4 | 0 | 0 | 1 | 0 | 1 | 1 | | 34 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | | 12 | 1 | 1 | 0 | 0 |
| | | | | | | | | 13 | 1 | 0 | 1 | 0 |
| | | | | | | | | 14 | 0 | 1 | 1 | 0 |

Figure 7: The non-singular submatrices for $n = 4, k = 2$, and $n = 4, k = 3$.

where \mathcal{I}_{k-1} is the ideal of generators coming from columns 0 to $k - 1$.

So we have a system of $\binom{n}{k-1}$ linear homogeneous equations, in $\binom{n}{k}$ variables. In fact we have one equation for each choice of a $(k - 1)$ -subset $\{i_1, \dots, i_{k-1}\}$, and one variable $e_{b_k}(S_{i_1, \dots, i_{k-1}, j})$ for each k -subset $\{i_1, \dots, i_{k-1}, j\}$.

The matrix associated to this system has columns J indexed by the k -subsets of $\{1, 2, \dots, n\}$, and rows I indexed by $k - 1$ -subsets of $\{1, 2, \dots, n\}$. Equation (8), says that at position (I, J) the entry will be 1 if $I \subseteq J$ and 0 if $I \not\subseteq J$.

We claim that we can drop from the generating set of Theorem 4.1 $e_{b_k}(S_J)$, for all J of cardinality k containing 1, and $e_{b_k}(S_{2, \dots, k+1})$. To prove this it suffices to show that the submatrix corresponding to these columns has full rank $\binom{n-1}{k-1} + 1$.

We order the columns of this submatrix in this way: we put first the the columns indexed by a J containing 1 in alphabetical order, and then column indexed by $\{2, \dots, k + 1\}$. Similarly, we order the rows starting with those indexed by subsets I that do not contain 1, in alphabetical order, and then the row indexed by $\{1, \dots, k - 1\}$, and then the other rows in any order. In Figure 7 two examples are displayed.

The square submatrix given by the first $\binom{n-1}{k-1} + 1$ rows consists of two blocks. An identity $\binom{n-1}{k-1}$ -matrix together with an additional row: $(1, \dots, 1, 0, \dots, 0)$, with $n - k + 1$ ones. In fact, this last row is indexed by $\{1, \dots, k - 1\}$, and the entries are 1 at columns indexed by $\{1, 2, \dots, k - 1, j\}$ for $j > k$, and zero otherwise. By Gauss elimination, it is easy to see that this submatrix has full rank. \square

Remark 5.2. The system (8) has $\binom{n}{k-1}$ linear equations and $\binom{n}{k}$ variables. If all the equations are independent, then $\binom{n}{k-1}$ variables are redundant. Hence only $\binom{n}{k} - \binom{n}{k-1}$ of them are necessary. Then using Gauss elimination we would obtain an explicit generating set of the same size as Weyman's $\tilde{U}_{k,p}$. We note that there is no explicit construction for the generators in U_λ in Weyman's paper [W1].

Remark 5.3. Let λ be a partition of n different than (n) . As a consequence of Proposition 5.1, the number of generators coming from the top cell of column k in our generating set for \mathcal{I}_λ is $\binom{n}{k} - \binom{n-1}{k-1} - 1$. On the other hand, and as discussed in Section 4.1 the corresponding $\tilde{U}_{k,p}$ in Weyman's generating set consists of $\binom{n}{k} - \binom{n}{k-1}$ elements. Since for all partitions other than (n) , we have that $n > k$, we conclude that the difference between the two sets is $\binom{n-1}{k-2} - 1$, for each $k > 2$. For columns 0, 1, and 2 their cardinalities coincide.

We now focus on eliminating generators from a column of height 1.

Proposition 5.4 (Columns of height 1). *Let λ be a diagram represented in Figure 5. If $s > t \geq 1$, then we can eliminate $\binom{n-s+t}{t}$ square-free monomial generators of \mathcal{I}_λ coming from the last column.*

Proof. Note that as $n - s > b_t$ (see Figure 5), from the proof of Theorem 4.1 we know that $e_{n-s}(n-t) \in \mathcal{I}_\lambda$. We now claim that we can drop monomial generators of the form

$$e_{n-s}(S_{1,2,\dots,s-t,i_1,\dots,i_t}), \quad s-t < i_1 < i_2 < \dots < i_t \leq n$$

from the generating set for \mathcal{I}_λ . Since there are $\binom{n-s+t}{t}$ such choices for sets $\{i_1, \dots, i_t\}$, this will settle the statement of the proposition. But this follows from the trivial identity

$$e_k(A) = \sum_{\substack{J \subseteq A \\ |J|=k}} e_k(J),$$

which implies

$$e_{n-s}(S_{1,2,\dots,s-t,i_1,\dots,i_t}) = e_{n-s}(S_{i_1,\dots,i_t}) - \sum_{\substack{\{j_1,\dots,j_{s-t}\} \neq \{1,\dots,s-t\} \\ \{j_1,\dots,j_{s-t}\} \cap \{i_1,\dots,i_t\} = \emptyset}} e_{n-s}(S_{j_1,\dots,j_{s-t},i_1,\dots,i_t}) \in \mathcal{I}_\lambda.$$

□

Therefore using Propositions 5.1 and 5.4, we have reduced our generating set to that in the table in Figure 8, using the Vandermonde identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

| Column | Generators | Number |
|-------------------|-------------------------------|--|
| 0 | $e_1(n), \dots, e_{b_1-1}(n)$ | $b_1 - 1 = \lambda'_1 - 1$ |
| 1 | $x_1^{b_1}, \dots, x_n^{b_1}$ | $\binom{n}{1} = \binom{n-1}{1} + 1$ |
| 2 | $e_{b_2}(n-2)$ | $\binom{n}{2} - \binom{n-1}{1} - 1 = \binom{n-1}{2} - 1$ |
| \vdots | \vdots | \vdots |
| t | $e_{b_t}(n-t)$ | $\binom{n}{t} - \binom{n-1}{t-1} - 1 = \binom{n-1}{t} - 1$ |
| s (if $s > t$) | $e_{n-s}(n-s)$ | $\binom{n}{s} - \binom{n-s+t}{t}$ |

Figure 8: Number of generators in each degree in the reduced generating set for \mathcal{I}_λ

Example 5.5. Consider the partition $\lambda = (4, 4, 2, 1)$ in Figure 2. Our formula gives 177 generators, but in fact, Macaulay2 verifies that 168 generators are enough. The extra generators are in degree 7 (see table in Figure 8):

| Degrees | Number of generators from Table 8 | Actual number of generators required |
|---------|-----------------------------------|--------------------------------------|
| 1, 2, 3 | 1 in each degree | 1 in each degree |
| 4 | 11 | 11 |
| 6 | 44 | 44 |
| 7 | 119 | 110 |

While in many examples such as the previous one, the predictions of the diagonal version of Weyman's conjecture are correct, this is not always the case.

Example 5.6. Consider the partition $\lambda = (5, 4, 1)$. We denote by $\mathcal{I}_{01} = (e_1(10), e_2(10), x_1^3, \dots, x_{10}^3)$ the

| | | | | |
|----|---|---|---|---|
| 1 | | | | |
| 2 | 3 | 4 | 5 | |
| 10 | 9 | 8 | 7 | 6 |

Figure 9: The partition $\lambda = (5, 4, 1)$

ideal generated by the elements of the 0-th and 1-st column. Now consider $e_4(8)$ coming from the second column. Let $A \subseteq \{1, \dots, n\}$ be a subset of of cardinality 8, and let B be its complement ($|B| = 2$). By Proposition 2.2, we have mod $\mathcal{E}_3(10)$

$$e_4(A) \equiv h_4(B) = m_{(4)}(B) + m_{(3,1)}(B) + m_{(2,2)}(B). \quad (9)$$

Among the monomial symmetric polynomials appearing in (9), $m_{(4)}$, and $m_{(3,1)}$ are already in the \mathcal{I}_{01} , since it contains x_1^3, \dots, x_n^3 . So from the second column we only need to add the set $m_{(2,2)}(2)$ to the generators of \mathcal{I}_{01} to obtain a bigger ideal denoted \mathcal{I}_{012} included in \mathcal{I}_λ . That is, we need to add all generators of the form $(x_i x_j)^2$ for $i < j$.

Now let us consider $e_5(A)$, where $|A| = 7$ and B is its complement. From the third column

$$- e_5(A) \equiv h_5(B) = m_{(5)}(B) + m_{(3,2)}(B) + m_{(4,1)}(B) + m_{(3,1,1)}(B) + m_{(2,2,1)}(B). \quad (10)$$

It is clear that each one of these monomial symmetric polynomials is already in the ideal \mathcal{I}_{012} . In fact, every monomial in the first four summands in (10) contains a power x_i^3 , and each element in $m_{(2,2,1)}(B)$ can be obtained as a combination of elements in $m_{(2,2)}(2)$. Hence the third column will not contribute any new generator. The same happens for the last column. Let $|A| = 6$ and B be its complement, $|B| = 4$. Then

$$\begin{aligned} e_6(A) = h_6(B) &= m_{(6)}(B) + m_{(5,1)}(B) + m_{(4,2)}(B) + m_{(3,3)}(B) \\ &+ m_{(4,1,1)}(B) + m_{(3,2,1)}(B) + m_{(2,2,2)}(B) \\ &+ m_{(3,1,1,1)}(B) + m_{(2,2,1,1)}(B), \end{aligned}$$

and all monomials in this sum are already in the ideal, since they contain either a power x_i^3 , or a monomial $(x_i x_j)^2$. So we have $\mathcal{I}_\lambda = \mathcal{I}_{012}$.

Counterexample 5.7 (Counterexample to the diagonal version of Weyman's conjecture). Example 5.6 proves that the generating set $\tilde{\mathcal{W}}_\lambda$ for \mathcal{I}_λ coming from the minimal generating set for \mathcal{J}_λ conjectured by Weyman is not in general minimal (see Question 4.6). More precisely, according to his diagram in Figure 10, some generators of degree 5 and 6 should be needed, while they are not, as we just showed. In Figure 10 the coordinates of the underlined X 's label the generators of \mathcal{I}_λ arising from the diagonal version of Weyman's conjecture. The generators coming from the shaded X 's are not needed. This is the convention that we shall use later as well.

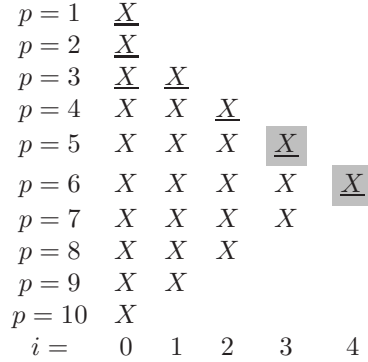


Figure 10: Weyman diagram for $\lambda = (5, 4, 1)$.

It might be possible to generalize the reasoning used in Example 5.6 with an algorithm, as explained below.

Algorithm 5.8. Consider the Young diagram of λ filled with the regular filling. Let b_1, \dots, b_s be the top-cell entries of λ as in Figure 5. Set $\mathcal{G}_0 = \{e_1(n), \dots, e_{b_1-1}(n)\}$, and create a list of partitions $L_0 = \emptyset$. For all $k \geq 1$, define

$$U_k = \{\mu \vdash b_k \mid \ell(\mu) \leq k \text{ and } \nu \not\subseteq \mu, \text{ for any } \nu \in L_{k-1}\},$$

where $\nu \subseteq \mu$ means that the Young diagram of ν is contained in that of μ .

- 1) If $|U_k| = 1$, say $U_k = \{\theta\}$, then $L_k = L_{k-1} \cup \{\theta\}$ and $\mathcal{G}_k = \mathcal{G}_{k-1} \cup m_\theta(k)$.
- 2) If $|U_k| = 0$, then $\mathcal{G}_k = \mathcal{G}_{k-1}$ and $L_k = L_{k-1}$.
- 3) If $|U_k| > 1$, then $\mathcal{G}_k = \mathcal{G}_{k-1} \cup (\bigcup_{l \geq k} h_{b_l}(l))$, and stop.

Denote by \mathcal{G} the set produced by the algorithm at the last step.

Question 5.9. Is the set \mathcal{G} a generating set for \mathcal{I}_λ ?

Clearly this algorithm produces a subset of the generating set given by the Theorem 4.1. All generators coming from cells labeled b_k satisfying condition 2) in the above algorithm would become redundant.

We used this algorithm to produce generating sets for all families of examples and counterexamples considered in the next section. Then, we proceeded to prove their correctness on a one by one basis. A proof of the correctness of the algorithm would be greatly welcomed.

6 Families of examples and a counterexample to Weyman's conjecture

We conclude the paper by producing simple generating sets for some particular families of shapes. In particular, this allows us to construct two infinite families of counterexamples to the diagonal version of Weyman's conjecture (Question 4.6), as well as a counterexample to the original conjecture of Weyman for a minimal generating set of the ideal \mathcal{J}_λ (see Conjecture 4.5).

Example 6.1 (The case of two-column partitions). As mentioned above a partition of n of the form $\lambda = (2^a, 1^c)$, where $a + c = \ell = \ell(\lambda)$ the length of the partition, \mathcal{I}_λ is generated by $e_1(n), \dots, e_{\ell-1}(n), x_1^\ell, \dots, x_n^\ell$.

Theorem 6.2 (The case of partially-rectangular partitions). *Let λ be a partition of n , and let $k > 2$ be any integer. If columns $0, 1, \dots, k - 1$ of the Young diagram have the same height, then in the generating set for the ideal \mathcal{I}_λ described in Theorem 4.1 generators coming from columns $2, \dots, k$ are redundant.*

Proof. The regular filling of the partition λ has the following form.

| | | | | | |
|-----|-------|--------|-----|--------|-----|
| 1 | $g+1$ | $2g+1$ | ... | | |
| 2 | $g+2$ | $2g+2$ | ... | $kg+1$ | |
| ... | ... | ... | ... | ... | ... |
| g | $2g$ | $3g$ | ... | ... | ... |
| n | ... | ... | ... | ... | ... |

By Theorem 4.1 and Proposition 2.2, modulo the previous columns, the generators coming from Column k are of the form

$$h_{kg+1} = \sum_{a_1 + \dots + a_k = kg+1} x_{j_1}^{a_1} \dots x_{j_k}^{a_k}$$

where $1 \leq j_1 \leq \dots \leq j_k \leq n$.

Consider a term $x_{j_1}^{a_1} \dots x_{j_k}^{a_k}$ in the sum above. We claim that for at least one power a_i , $a_i \geq g + 1$, making this monomial redundant in the presence of the second column generators, which are the $(g + 1)$ -st powers of the variables.

To see this, suppose $a_1 \leq g, \dots, a_k \leq g$. Then we should have that

$$kg + 1 = a_1 + \dots + a_k \leq kg$$

which is a contradiction. □

Remark 6.3. Drawing the Weyman diagram associated to partially rectangular partitions considered in Theorem 6.2, one can see that the points $(0, 1), (1, g + 1), (2, 2g + 1), \dots, (k, kg + 1)$ are collinear because they can successively be obtained by adding the vector $(1, g)$. Therefore, the diagonal version of Weyman's conjecture predicts that the generators coming from cells $(2, 2g + 1), \dots, (k, kg + 1)$ are redundant. This is true: in fact these are precisely the redundant cells according to Theorem 6.2.

| | | | | | |
|-------|----------|----------|----------|----------|----------|
| 1 | <u>X</u> | | | | |
| 2 | <u>X</u> | | | | |
| 3 | <u>X</u> | | | | |
| 4 | <u>X</u> | <u>X</u> | | | |
| 5 | X | X | | | |
| 6 | X | X | | | |
| 7 | X | X | <u>X</u> | | |
| 8 | X | X | X | | |
| 9 | X | X | X | | |
| 10 | X | X | X | <u>X</u> | |
| 11 | X | X | X | X | |
| 12 | X | X | X | X | <u>X</u> |
| 13 | X | X | X | X | |
| 14 | X | X | X | | |
| 15 | X | X | | | |
| 16 | X | | | | |
| $i =$ | 0 | 1 | 2 | 3 | 4 |

Figure 11: An example of a partially-rectangular partition $\lambda = (5, 4, 4, 3)$.

Example 6.4. Let $\lambda = (5, 4, 4, 3)$ be the partition in Example 4.2. Theorem 6.2 implies that the generating set for \mathcal{I}_λ consists of the elements in the second column in the table below (compare with Example 4.2), and the reduced number from the table in Figure 8 is in the third column. No 7 and 10-degree generators are needed in the generating set. In this case the prediction of the diagonal version of Weyman’s conjecture was correct: cells (2, 7) and (3, 10) are redundant; see Figure 11.

| Column | Generators | Numbers from Figure 8 |
|--------------|-----------------------------|-----------------------|
| 0 | $e_1(16), e_2(16), e_3(16)$ | 3 |
| 1 | x_1^4, \dots, x_{16}^4 | 16 |
| 2 | redundant | – |
| 3 | redundant | – |
| 4 | $e_{12}(12)$ | 1365 |
| Total | | 1384 |

Corollary 6.5 (The case of rectangular partitions). *For a rectangular partition of n of the form $\lambda = (u^\ell)$, the generating set of \mathcal{I}_λ will simply be $e_1(n), \dots, e_{\ell-1}(n), x_1^\ell, \dots, x_n^\ell$, where $n = u\ell$.*

Corollary 6.6 (The case of two-row partitions). *For a two-row partition of n of the form $\lambda = (u, v)$, a generating set is given by $e_1(n), x_1^2, \dots, x_n^2$, and $e_u(u)$.*

Theorem 6.7. *Let λ be a partition of n .*

1. *If $\lambda = (u^a, (u-1)^c)$ with $g = a + c$, then a generating set of \mathcal{I}_λ is given by*

$$e_1(n), \dots, e_{g-1}(n), x_1^g, \dots, x_n^g.$$

2. *If $\lambda = (u^a, (u-1)^c, 1)$ with $u \geq 3$ and $g = a + c > 1$, then \mathcal{I}_λ is generated by*

$$e_1(n), \dots, e_g(n), x_1^{g+1}, \dots, x_n^{g+1}, (x_1x_2)^g, (x_1x_3)^g, \dots, (x_{n-1}x_n)^g.$$

3. If $\lambda = (u^a, (u-1)^c, 1, 1)$ with $u \geq 4$ and $g = a + c + 1 > 2$, then \mathcal{I}_λ is generated by

$$e_1(n), \dots, e_g(n), x_1^{g+1}, \dots, x_n^{g+1}, (x_i + x_j)(x_i x_j)^{g-1} \text{ for all } i \neq j, \text{ and } (x_i x_j x_k)^{g-1} \text{ for all } i < j < k.$$

Proof. 1. This is an easy consequence of Theorem 6.2.

2. The regular filling of $(u^a, (u-1)^c, 1)$ will be of the form:

| | | | | | | | |
|-----|-------|-------|--------|-----|----------------|-----|-----------------------|
| 1 | | | | | | | |
| 2 | $g+1$ | $2g$ | $3g-1$ | ... | $lg-l$ $+2$ | ... | |
| ... | ... | ... | ... | ... | ... | ... | |
| ... | ... | ... | ... | ... | ... | ... | $\frac{(u-1)g}{-u+3}$ |
| g | ... | ... | ... | ... | ... | ... | ... |
| n | $n-1$ | $n-2$ | $n-3$ | ... | $n-l$ | ... | $\frac{n-u}{+1}$ |

Columns 0 and 1 clearly provide the generators $e_1(n), \dots, e_g(n), x_1^{g+1}, \dots, x_n^{g+1}$. By Proposition 2.2, Column 2 provides generators of the form

$$h_{2g} = \sum_{a+b=2g} x_i^a x_j^b$$

for $1 \leq i < j \leq n$. Since we already have x_i^{g+1} and x_j^{g+1} in the ideal, this sum reduces to the monomial $x_i^g x_j^g$. Hence the third column provides the remaining generators $(x_1 x_2)^g, (x_1 x_3)^g, \dots, (x_{n-1} x_n)^g$.

It remains to show that the generators coming from Columns 3, $\dots, u-1$ are redundant. Let l be any integer such that $3 \leq l \leq u-1$. The generators from Column l , by Proposition 2.2 and the fact that we have all $(g+1)$ -st powers of the variables in the ideal, are of the form

$$h_{lg-l+2} = \sum_{\substack{a_1 + \dots + a_l = lg-l+2 \\ a_1, \dots, a_l \leq g}} x_{i_1}^{a_1} \dots x_{i_l}^{a_l}$$

where $1 \leq i_1 < i_2 < \dots < i_l \leq n$, and in each monomial $x_{i_1}^{a_1} \dots x_{i_l}^{a_l}$ at most one of the powers a_u is equal to g . For such a monomial in the sum, we therefore have

$$a_1 + \dots + a_l \leq (l-1)(g-1) + g = lg - l + 1 \implies lg - l + 2 \leq lg - l + 1$$

which is a contradiction. So there is no generator from Column l if $l \geq 3$.

3. The regular filling of $(u^a, (u-1)^c, 1, 1)$ will be of the following form.

| | | | | | | | |
|-----|-------|--------|--------|-----|-----------------|-----|------------------------|
| 1 | | | | | | | |
| 2 | | | | | | | |
| 3 | $g+1$ | $2g-1$ | $3g-3$ | ... | $lg-2l$ $+3$ | ... | |
| ... | ... | ... | ... | ... | ... | ... | |
| g | ... | ... | ... | ... | ... | ... | $\frac{(u-1)g}{-2u+3}$ |
| n | ... | ... | ... | ... | ... | ... | |

Again Columns 0 and 1 provide the generators $e_1(n), \dots, e_g(n), x_1^{g+1}, \dots, x_n^{g+1}$.

By Proposition 2.2, Column 2 provides generators of the form

$$h_{2g-1} = \sum_{a+b=2g-1} x_i^a x_j^b$$

for $1 \leq i < j \leq n$. Since we already have x_i^{g+1} and x_j^{g+1} in the ideal, we can additionally assume that $a, b \leq g$ for each monomial $x_i^a x_j^b$ in the sum, and so at least one of a or b would have to be $g-1$ and the other g . This produces a generator of the form $x_i^g x_j^{g-1} + x_i^{g-1} x_j^g = (x_i + x_j)(x_i x_j)^{g-1}$.

Similarly, Column 3 will produce generators of the form

$$h_{3g-3} = \sum_{a+b+c=3g-3} x_i^a x_j^b x_k^c$$

for $1 \leq i < j < k \leq n$. Once more, we can assume that $a, b, c \leq g$, which reduces the sum above to

$$\begin{aligned} & x_i^{g-1} x_j^{g-1} x_k^{g-1} + x_i^{g-2} (x_j^g x_k^{g-1} + x_j^{g-1} x_k^g) + x_j^{g-2} (x_i^g x_k^{g-1} + x_i^{g-1} x_k^g) + x_k^{g-2} (x_i^g x_j^{g-1} + x_i^{g-1} x_j^g) \\ &= x_i^{g-1} x_j^{g-1} x_k^{g-1} + x_i^{g-2} x_j^{g-1} x_k^{g-1} (x_j + x_k) + x_j^{g-2} x_i^{g-1} x_k^{g-1} (x_i + x_k) + x_k^{g-2} x_i^{g-1} x_j^{g-1} (x_i + x_j). \end{aligned}$$

The last three summands are in the ideal already (coming from Column 2), so the generators from Column 3 can all be written as $x_i^{g-1} x_j^{g-1} x_k^{g-1}$ for $1 \leq i < j < k \leq n$.

We now need to show that generators coming from Column l , where $4 \leq l \leq u-1$ are redundant. The generators from Column l , by Proposition 2.2 and the fact that we have all $(g+1)$ -st powers of the variables in the ideal, are of the form

$$h_{lg-2l+3} = \sum_{\substack{a_1+\dots+a_l=lg-2l+3 \\ a_1, \dots, a_l \leq g}} x_{i_1}^{a_1} \dots x_{i_l}^{a_l}$$

where $1 \leq i_1 < i_2 < \dots < i_l \leq n$.

Suppose that $M = x_{i_1}^{a_1} \dots x_{i_l}^{a_l}$ is a monomial in this sum.

If one of the powers, say a_1 , is equal to g , then we must have another power among a_2, \dots, a_l that is g or $g-1$. If not, all of a_2, \dots, a_l are $\leq g-2$, and we have

$$lg - 2l + 3 = a_1 + \dots + a_l \leq g + (l-1)(g-2) = lg - 2l + 2$$

which is a contradiction. So there is at least another power, say a_2 , such that $a_2 \geq g-1$.

- $a_1 = a_2 = g$. In this case, we can write

$$\begin{aligned} x_{i_1}^g x_{i_2}^g x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l} &= (x_{i_1} + x_{i_2})(x_{i_1} x_{i_2})^{g-1} [1/2 x_{i_2} x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l} + 1/2 x_{i_1} x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l}] \\ &\quad - 1/2 x_{i_1}^{g+1} x_{i_2}^{g-1} x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l} - 1/2 x_{i_1}^{g-1} x_{i_2}^{g+1} x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l} \end{aligned}$$

All the terms on the right-hand side are already in the ideal, and hence so is $x_{i_1}^g x_{i_2}^g x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l}$.

- $a_1 = g$ and $a_2 = g-1$. In this case, there is another monomial $M' = x_{i_1}^{g-1} x_{i_2}^g x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l}$ in the sum as well, and there is exactly one copy of M and one copy of M' in the sum. Now we have

$$M + M' = (x_{i_1} + x_{i_2}) x_{i_1}^{g-1} x_{i_2}^g x_{i_3}^{a_3} x_{i_4}^{a_4} \dots x_{i_l}^{a_l}.$$

So each such monomial M is paired with a unique monomial M' in the sum, and their sum is already in the ideal.

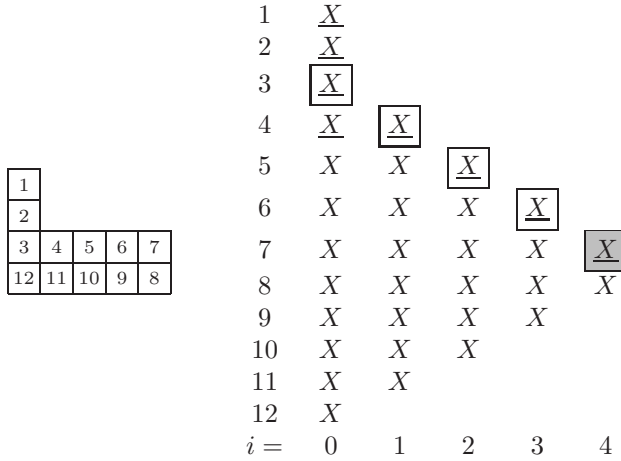


Figure 12: The regular filling and the Weyman diagram of $\lambda = (5, 5, 1, 1)$.

Now assume that all the powers a_1, \dots, a_l are $\leq g - 1$. If $l - 2$ of the powers a_1, \dots, a_l are $\leq g - 2$, then we have

$$lg - 2l + 3 = a_1 + \dots + a_l \leq (l - 2)(g - 2) + 2(g - 1) = lg - 2l + 2$$

which is a contradiction. So there are at least 3 powers among a_1, \dots, a_l that are equal to $g - 1$. But then the monomial $x_{i_1}^{a_1} \dots x_{i_l}^{a_l}$ is already in \mathcal{I}_λ , because it is a multiple of a generator coming from Column 3.

□

Corollary 6.8. *Suppose that the first $l + 1$ columns of a partition λ belong to one of the three families of shapes described in Theorem 6.7. Then*

- a) *In cases 1 and 2, the generators coming from Columns $3, \dots, l$ are redundant. For Columns 0, 1, 2 we can use the generators described in Theorem 6.7.*
- b) *In Case 3, the generators coming from columns $4, \dots, l$ are redundant. For Columns 0, 1, 2, 3 we can use the generators described in Theorem 6.7.*

Counterexample 6.9 (Counterexamples to the diagonal version of Weyman’s conjecture). The two infinite families of partitions described in parts 2 and 3 of Theorem 6.7 are counterexamples to the diagonal version of Weyman’s conjecture. Indeed, according to it, all generators coming from each of the top cells of their diagrams should be necessary because for $k > 0$, the top cells are collinear (for the first family we can move from one top cell to the next one by adding the vector $(1, g - 1)$, and for the second family, by adding the vector $(1, g - 2)$). But the line containing those points does not pass through $(0, 1)$. Instead it passes through $(0, 2)$ for the first family, and through $(0, 3)$ for the second family.

Let λ be a partition such that its first l columns belong to one of the two families of shapes described above, with $l > 2$ for the first family and $l > 3$ for the second one. The preceding corollary shows that the generators coming from Column k , with $3 < k \leq l$ are redundant. We conclude that each such λ is a counterexample to the diagonal version of Weyman’s conjecture. A first counterexample was shown in Counterexample 5.7.

Example 6.10. Consider the partition $(5, 5, 1, 1)$ that fits inside one of the families in Theorem 6.7. As proved in that theorem, the cell containing 7 is redundant. Translated into the Weyman diagram, this means that the X in position $(4, 7)$ is redundant (see Figure 12).

The following table, computed with Macaulay2, confirms our prediction that the 275 degree 7 generators that should be in the generating set according to the diagonal version of the conjecture, are not needed.

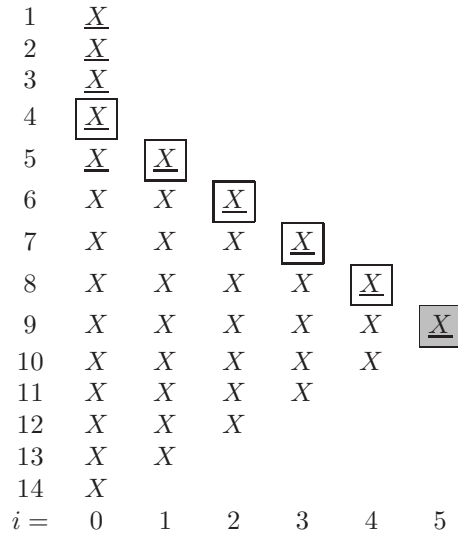


Figure 13: An evidence regarding the statement in Question 6.11 for $\lambda = (6, 5, 1, 1, 1)$

| Degrees | Minimal number of generators |
|---------|------------------------------|
| 1, 2, 3 | 1 in each degree |
| 4 | 12 |
| 5 | 54 |
| 6 | 154 |
| 7 | redundant |

Theorems 6.2 and 6.7 can be reformulated in a suggestive geometrical way as special instances of the following statement.

Question 6.11. Let λ be a partition and draw the Weyman diagram of λ . If the X 's at the top of columns $1, 2, \dots, r$ are collinear, and the line containing them passes through the point $(0, k)$, then are the generators coming from columns $k + 1, \dots, r$ redundant?

We have evidence that suggests that this statement is true: it was proven to be true when $k = 1$ in Theorem 6.2, for $k = 2$ in Theorem 6.7 Part 1, and for $k = 3$ in Theorem 6.7 Part 2, (see Figure 12: the collinear X 's have been surrounded). For $k = 4$, we used Macaulay2 to verify whether the statement is still true for the smallest possible member of this family, the partition $(6, 5, 1, 1, 1)$ (see Figure 13). As predicted, all degree 9 generators are redundant.

| Degrees | Minimal number of generators |
|------------|------------------------------|
| 1, 2, 3, 4 | 1 in each degree |
| 5 | 14 |
| 6 | 77 |
| 7 | 273 |
| 8 | 637 |
| 9 | redundant |

6.1 Weyman's original conjecture

To finish our work, we focus our attention at the original conjecture of Weyman. It seems plausible that those partitions that give counterexamples to the diagonal version of Weyman's conjecture are also counterexamples

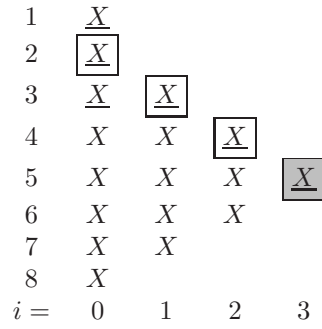


Figure 14: A counterexample to Weyman’s original conjecture: $(4, 3, 1)$.

to Weyman’s original conjecture. We used Macaulay2 to verify if this was the case for the smallest shape in the families described in Counterexample 6.9.

Counterexample 6.12 (Counterexample to Weyman’s original conjecture). Consider the partition $(4, 3, 1)$ whose Weyman diagram is represented in Figure 14. The points $(1, 3)$, $(2, 4)$ and $(3, 5)$ are collinear, but the line that contains them does not pass through $(0, 1)$. So according to Weyman’s conjecture, all these cells contribute generators to a minimal generating set of $\mathcal{J}_{(4,3,1)}$. However, Theorem 6.7 suggests that the generators coming from cell $(3, 5)$ may be redundant.

Using Macaulay 2, we computed the minimal generating set for $\mathcal{J}_{(4,3,1)}$ and verified that this is indeed the case. We conclude that $(4, 3, 1)$ is a counterexample to Weyman’s original conjecture.

| Degrees | Weyman’s conjecture | Minimal number of generators |
|--------------|---------------------|------------------------------|
| 1 | 1 | 1 |
| 2 | 1 | 1 |
| 3 | 64 | 64 |
| 4 | 720 | 720 |
| 5 | 2352 | redundant |
| Total | 3138 | 786 |

To summarize, in this particular case, Weyman’s conjecture predicts that we need 3138 generators, but only 786 of them are really necessary.

Unfortunately, even large servers were not able to handle slightly larger examples, so at this point we do not know if other partitions in the families described earlier are counterexamples to Weyman’s original conjecture.

We end the paper with a natural question.

Question 6.13. Does the statement of Question 6.11 hold for \mathcal{J}_λ ?

Acknowledgments

We wish to thank Jerzy Weyman for many interesting conversations and suggestions, as well as his interest in our project. At the same time that we were working on this project, he showed independently that $(4, 3, 1)$ is indeed a counterexample to his Conjecture 5.1 using geometric reasoning [W3]. Moreover, he has shown that Conjecture 5.1 holds for all the other partitions of $n \leq 9$ except for the three partitions of 9 belonging to the shapes described in Counterexample 6.9. We also wish to Mark Shimozono pointing out some references and Emmanuel Briand for his help during this project.

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