The canonical injection of the Hardy-Orlicz space H^Ψ into the Bergman-Orlicz space $\mathfrak{B}^\Psi$

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May 13, 2010

Abstract. *We study the canonical injection from the Hardy-Orlicz space H^Ψ into the Bergman-Orlicz space $\mathfrak{B}^\Psi$.*

Mathematics Subject Classification. Primary: 46E30 – Secondary: 30D55; 30H05; 32A35; 32A36; 42B30

Key-words. absolutely summing operator – Bergman-Orlicz space – compactness – Dunford-Pettis operator – Hardy-Orlicz space – weak compactness

1 Introduction and notation

1.1 Introduction

There are two natural Orlicz spaces of analytic functions on the unit disk $\mathbb{D}$ of the complex plane: the Hardy-Orlicz space H^Ψ and the Bergman-Orlicz space $\mathfrak{B}^\Psi$. It is well-known that in the classical case $\Psi(x) = x^p$, $H^p \subseteq \mathfrak{B}^p$ and the canonical injection J_p from H^p to $\mathfrak{B}^p$ is bounded, and even compact. In fact, for any Orlicz function Ψ, one has $H^\Psi \subseteq \mathfrak{B}^\Psi$ and the canonical injection $J_\Psi \colon H^\Psi \to \mathfrak{B}^\Psi$ is bounded, but we shall see in this paper that its compactness requires that Ψ does not grow too fast. We actually characterize in Section 2 the compactness: J_Ψ is compact if and only if $\lim_{x\to+\infty} \Psi(Ax)/[\Psi(x)]^2 = 0$ for every $A > 1$, and the weak compactness: J_Ψ is weakly compact if and only if $\limsup_{x\to+\infty} \Psi(Ax)/[\Psi(x)]^2 < +\infty$ for every $A > 1$. We show that, if these two properties are "often" equivalent (this happens for example if $\Psi(x)/x$ is non-decreasing for x large enough), it is not always the case. We actually show a stronger result in Section 4: there is an Orlicz function Ψ such that J_Ψ is weakly compact and is Dunford-Pettis, but such that J_Ψ is not compact.

1.2 Notation

An Orlicz function is a non-decreasing convex function $\Psi \colon [0, +\infty[\to [0, +\infty[$ such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. One says that the Orlicz function Ψ has property Δ_2 ($\Psi \in \Delta_2$) if $\Psi(2x) \leq C\,\Psi(x)$ for some constant $C > 0$ and x large enough. It is equivalent to say that, for every $\beta > 1$, $\Psi(\beta x) \leq C_\beta \Psi(x)$. It is known that if $\Psi \in \Delta_2$, then $\Psi(x) = O\,(x^p)$ for some $1 \leq p < +\infty$. One says (see [6], [7]) that Ψ satisfies the condition Δ^0 if, for some $\beta > 1$, one has $\lim\limits_{x\to\infty} \Psi(\beta x)/\Psi(x) = +\infty$. If $\Psi \in \Delta^0$, then $\Psi(x)/x^p \underset{x\to\infty}{\longrightarrow} +\infty$ for every $1 \leq p < \infty$. Indeed, let $1 \leq p < \infty$. For every $\beta > 1$ one can find $x_0 > 0$ such that $\Psi(\beta x)/\Psi(x) \geq \beta^p$ for $x \geq x_0$; then $\Psi(\beta^n x_0) \geq \beta^{np}\Psi(x_0)$ for every $n \geq 1$. That implies that $\Psi(x) \geq C_p\, x^p$ for every $x > 0$ large enough. Since $p \geq 1$ is arbitrary, we get $x^p = o\,[\Psi(x)]$.

We say that $\Psi \in \nabla_0(1)$ if, for every $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing for x large enough. This is equivalent to say (see [7], Proposition 4.7) that $\log \Psi(\mathrm{e}^x)$ is convex. When $\Psi \in \nabla_0(1)$, one has either $\Psi \in \Delta_2$, or $\Psi \in \Delta^0$.

If $(S, \mathcal{S}, \mu)$ is a finite measure space, one defines the Orlicz space $L^\Psi(\mu)$ as the set of all (classes of) measurable functions $f\colon S \to \mathbb{C}$ for which there is a $C > 0$ such that $\int_S \Psi(|f|/C)\,d\mu$ is finite. The norm $\|f\|_\Psi$ is the infimum of all $C > 0$ for which the above integral is ≤ 1. The Morse-Transue space $M^\Psi(\mu)$ is the subspace of $f \in L^\Psi(\mu)$ for which $\int_S \Psi(|f|/C)\,d\mu$ is finite for all $C > 0$; it is the closure of $L^\infty(\mu)$ in $L^\Psi(\mu)$. One has $M^\Psi(\mu) = L^\Psi(\mu)$ if and only if $\Psi \in \Delta_2$.

If $\Psi(x)/x \underset{x\to+\infty}{\longrightarrow} +\infty$, the conjugate function Φ of Ψ is defined by $\Phi(y) = \sup_{x>0} \big(xy - \Psi(x)\big)$. It is an Orlicz function and $[M^\Psi(\mu)]^* = L^\Phi(\mu)$, isomorphically.

We may note that if $\Psi(x)/x$ does not converges to infinity, we must have $\Psi(x) \leq ax$ for some $a \geq 1$ and x large enough. Then $L^\Psi(\mu) = L^1(\mu)$ isomorphically and then $\Phi(y) = +\infty$ for $y > a$ (giving $L^\Phi(\mu) = L^\infty(\mu)$ isomorphically).

We denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$ and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle. The normalized area-measure on $\mathbb{D}$ is denoted by $\mathcal{A}$ and the normalized Lebesgue measure on $\mathbb{T}$ is denoted by m.

The Hardy-Orlicz space H^Ψ is defined as $\{f \in H^1\,;\ f^* \in L^\Psi(m)\}$, where f^* is the boundary values function of f, and $HM^\Psi = H^\Psi \cap M^\Psi(m)$ is the closure of H^∞ in H^Ψ. The Bergman-Orlicz space $\mathfrak{B}^\Psi$ is the subspace of analytic $f \in L^\Psi(\mathcal{A})$, and $\mathfrak{B}M^\Psi = \mathfrak{B}^\Psi \cap M^\Psi(\mathcal{A})$ is the closure of H^∞ in $\mathfrak{B}^\Psi$. Since, for $f \in H^\Psi$, $\|f\|_{H^\Psi} = \sup_{0<r<1} \|f_r\|_{H^\Psi}$ (see [7], Proposition 3.1), where $f_r(z) = f(rz)$, one has:

$$\int_0^{2\pi} \Psi\bigg(\frac{|f(r\mathrm{e}^{it})|}{\|f\|_{H^\Psi}}\bigg)\,\frac{dt}{2\pi} \leq \int_0^{2\pi} \Psi\bigg(\frac{|f(r\mathrm{e}^{it})|}{\|f_r\|_{H^\Psi}}\bigg)\,\frac{dt}{2\pi} \leq 1\,;$$

hence:

$$\int_{\mathbb{D}} \Psi\bigg(\frac{|f(r\mathrm{e}^{it})|}{\|f\|_{H^\Psi}}\bigg)\,d\mathcal{A} = \int_0^1 \bigg[\int_0^{2\pi} \Psi\bigg(\frac{|f(r\mathrm{e}^{it})|}{\|f\|_{H^\Psi}}\bigg)\,\frac{dt}{2\pi}\bigg]\,2r\,dr \leq 1\,,$$

so $f \in \mathfrak{B}^\Psi$ and $\|f\|_{\mathfrak{B}^\Psi} \leq \|f\|_{H^\Psi}$. It follows that $H^\Psi \subseteq \mathfrak{B}^\Psi$ and the canonical injection $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is bounded, and has norm 1. Let us point out that the boundedness also follows from [7], Theorem 4.10, 2), since J_Ψ is a Carleson embedding $J_\Psi \colon H^\Psi \to \mathfrak{B}^\Psi \subseteq L^\Psi(\mathcal{A})$.

This injection is not onto, since there are functions $f \in \mathfrak{B}^\Psi$ with no radial limit on a subset of $\mathbb{T}$ of positive measure (the proof is the same as in $\mathfrak{B}^p$: see [4], § 3.2, Lemma 2, page 81). Note that J_Ψ is not an into-isomorphism: take $f_n(z) = z^n$, for every $n \in \mathbb{N}$; it is easy to see that $\{f_n\}_n$ tends to 0 in $\mathfrak{B}^\Psi$, but not in H^Ψ.

Acknowledgment. This work is partially supported by a Spanish research project MTM 2009-08934. Part of this paper was made during an invitation of the second-named author by the *Departamento de Análisis Matemático* of the *Universidad de Sevilla*. It is a pleasure to thanks the members of this department for their warm hospitality.

2 Compactness and weak-compactness

In order to characterize the compactness and the weak-compactness of J_Ψ, we introduce the following quantity Q_A, $A > 1$:

$$Q_A = \limsup_{x \to +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2}\,, \tag{2.1}$$

which will turn out to be essential.

We are going to start with the compactness.

Theorem 2.1 *The canonical injection $J_\Psi \colon H^\Psi \to \mathfrak{B}^\Psi$ is compact if and only if*

$$\lim_{x \to +\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = 0 \quad \textit{for every } A > 1\,. \tag{2.2}$$

Remarks. 1) Condition (2.2) means that $Q_A = 0$ for every $A > 1$. It is equivalent to say that:

$$\sup_{A>1} Q_A < +\infty. \tag{2.3}$$

Indeed, assume that $M := \sup_{A>1} Q_A < +\infty$. Let $0 < \varepsilon \leq 1$ and $A > 1$; we can find $x_A = x_A(\varepsilon) > 0$ such that $\Psi(Ax/\varepsilon)/[\Psi(x)]^2 \leq 2M$ for $x \geq x_A$. By convexity, one has $\Psi(Ax) \leq \varepsilon\, \Psi(Ax/\varepsilon)$, and hence $\Psi(Ax)/[\Psi(x)]^2 \leq 2\varepsilon M$ for $x \geq x_A$. We get $Q_A = 0$.

2) It is clear that condition (2.2) is satisfied whenever $\Psi \in \Delta_2$, but $\Psi(x) = \mathrm{e}^{[\log(x+1)]^2} - 1$ satisfies (2.2) without being in Δ_2. However, condition (2.2) implies that Ψ cannot grow too fast. More precisely, we must have

$$\Psi(x) = o\left(\mathrm{e}^{x^\alpha}\right) \quad \text{for every } \alpha > 0\,.$$

Indeed, one has $\Psi(At) \leq [\Psi(t)]^2$ for $t \geq t_A$, and, by iteration, $\Psi(A^n t_A) \leq [\Psi(t_A)]^{2^n}$ for every $n \geq 1$. For every $x > 0$ large enough, taking $n \geq 1$ such that $A^n t_A \leq x < A^{n+1} t_A$, we get $\Psi(x) \leq C_1 \,\mathrm{e}^{C_2 x^\alpha}$, with $\alpha = \log 2 / \log A$. Since $A > 1$ is arbitrary, α may be any positive number. The little-oh condition follows from the fact that the inequality is true for all $\alpha > 0$.

Proof of Theorem 2.1. By definition, $\mathfrak{B}^\Psi$ is a subspace of $L^\Psi(\mathbb{D}, \mathcal{A})$; hence we can see J_Ψ as a Carleson embedding $J_\Psi \colon H^\Psi \to L^\Psi(\mathbb{D}, \mathcal{A})$. If $S(\xi, h) = \{z \in \mathbb{D}\,;\ |z - \xi| < h\}$, the compactness of J_Ψ implies, by [7], Theorem 4.11, that, for every $A > 1$, every $\varepsilon > 0$, and $h > 0$ small enough:

$$h^2 \leq 4\,\mathcal{A}[S(\xi, h)] \leq \frac{4\varepsilon}{\Psi[A\Psi^{-1}(1/h)]}\,,$$

that is, setting $x = \Psi^{-1}(1/h)$, $\Psi(Ax) \leq 4\varepsilon\,[\Psi(x)]^2$, and (2.2) is satisfied.

Conversely, one has:

$$\sup_{0<t\leq h} \sup_{|\xi|=1} \frac{\mathcal{A}[S(\xi, t)]}{t} \leq \sup_{0<t\leq h} \frac{t^2}{t} = h\,,$$

which is $o\big((1/h)/\Psi[A\Psi^{-1}(1/h)]\big)$ for every $A > 1$, if (2.2) holds; hence, by [7], Theorem 4.11, again, J_Ψ is compact. □

We now turn ourself to the weak compactness.

Theorem 2.2 *The following assertions are equivalent:*

(a) $J_\Psi \colon H^\Psi \to \mathfrak{B}^\Psi$ *is weakly compact;*
(b) J_Ψ *fixes no copy of* c_0*;*
(c) J_Ψ *fixes no copy of* ℓ_∞*;*
(d) $Q_A < +\infty$*, for every* $A > 1$*;*
(e) $H^\Psi \subseteq \mathfrak{B}M^\Psi$*;*
(f) J_Ψ *is strictly singular.*

Recall that an operator $T \colon X \to Y$ between two Banach spaces is said to be strictly singular if there is no infinite-dimensional subspace X_0 of X on which T is an into-isomorphism.

The proof will be somewhat long, and before beginning it, we shall remark that if $\Psi \in \Delta^0$, then condition

$$Q_A < +\infty \quad \text{for every } A > 1 \tag{2.4}$$

implies condition (2.2). Indeed, if $\lim_{x\to+\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty$, we get, for every $A > 1$:

$$\limsup_{x\to+\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = \limsup_{x\to+\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)}\,\frac{\Psi(\beta Ax)}{[\Psi(x)]^2} \leq \limsup_{x\to+\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)}\,Q_{\beta A} = 0\,.$$

Now, if, for some $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing for x large enough (in particular if $\Psi \in \nabla_0(1)$), one has the dichotomy: either $\Psi \in \Delta_2$, and then J_Ψ is compact; or $\Psi \in \Delta^0$ and hence the weak compactness of J_Ψ implies, by the two above theorems, its compactness. Hence:

Proposition 2.3 *If, for some $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing, for x large enough, then the weak compactness of J_Ψ is equivalent to its compactness.*

However, it is easy to construct an Orlicz function Ψ which satisfies condition (2.4), but not condition (2.2). We do not give an axample here because we have a stronger result in Section 4.

In order to prove Theorem 2.2, we shall need several lemmas.

Lemma 2.4 *Let Ψ be any Orlicz function. If we define $\Psi_1(t) = [\Psi(t)]^2$, $t \geq 0$, then Ψ_1 is an Orlicz function for which $H^\Psi \subseteq \mathfrak{B}^{\Psi_1}$ and the canonical injection of H^Ψ into $\mathfrak{B}^{\Psi_1}$ is continuous.*

Proof. It is enough to see that H^Ψ continuously embeds into $L^{\Psi_1}(\mathcal{A})$, and for this we can use Theorem 4.10 in [7]. Following the notation of that theorem for the measure $\mu = \mathcal{A}$, it is easy to see that, as $h \to 0^+$, $\rho_{\mathcal{A}}(h) \approx h^2$, and $K_{\mathcal{A}}(h) \approx h$. Observe that, for $t > 1$, we have $\Psi_1[\Psi^{-1}(t)] = t^2$, and so, for $h \in (0,1)$,

$$\frac{1/h}{\Psi_1[\Psi^{-1}(1/h)]} = \frac{1/h}{1/h^2} = h \succeq K_{\mathcal{A}}(h).$$

Using part 2) of Theorem 4.10 in [7], the lemma follows. □

Lemma 2.5 *Let $M > \delta > 0$ and $\{f_n\}_n$ be a sequence in $H^\Psi \cap \mathfrak{B}M^\Psi$ such that:*

(a) *$\{f_n\}_n$ tends to 0 uniformly on compact subsets of $\mathbb{D}$;*

(b) *$\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, for every $n \geq 1$;*

(c) *$\|f_n\|_{H^\Psi} \leq M$, for every $n \geq 1$.*

Then there exists a subsequence $\{f_{n_k}\}_k$ such that $\sum_k |f_{n_k}(z)| < +\infty$, for every $z \in \mathbb{D}$, and for every $\alpha = (\alpha_k)_k \in \ell_\infty$, one has, writing $T\alpha(z) = \sum_{k=1}^\infty \alpha_k f_{n_k}(z)$:

$$T\alpha \in \mathfrak{B}^\Psi \qquad \textit{and} \qquad (\delta/2)\|\alpha\|_\infty \leq \|T\alpha\|_{\mathfrak{B}^\Psi} \leq 2M\|\alpha\|_\infty. \tag{2.5}$$

Remark. It is clear that, by (2.5), we are defining an operator T from ℓ_∞ into $\mathfrak{B}^\Psi$ which is an isomorphism between ℓ_∞ and its image. In particular, the subsequence $\{f_{n_k}\}_k$ is equivalent, in $\mathfrak{B}^\Psi$, to the canonical basis of c_0.

Proof. First we are going to construct, inductively, a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$, and an increasing sequence $\{r_k\}_k$ in $(0,1)$, such that $\lim_{k\to\infty} r_k = 1$ and, setting

$$D_k = \{z \in \mathbb{D}\,;\ |z| \leq r_k\}\,, \quad \text{for } k \geq 1,$$

and

$$C_1 = D_1\,, \qquad C_k = D_k \setminus D_{k-1} = \{z \in \mathbb{D}\,;\ r_{k-1} < |z| \leq r_k\}, \quad k \geq 2,$$

we have:

$$|f_{n_k}(z)| \leq 2^{-k}, \quad \text{for every } z \in D_{k-1}, \text{ and every } k \geq 2\,; \tag{2.6}$$

and

$$\|f_{n_k}\mathbb{1}_{\mathbb{D}\setminus C_k}\|_{L^\Psi} < \delta 2^{-k-2}\,, \quad \text{for every } k \geq 1. \tag{2.7}$$

Start the construction by taking $n_1 = 1$. It is a known fact that, for every function f in the Morse-Transue space $M^\Psi(\mathcal{A})$, we have

$$\lim_{\mathcal{A}(A)\to 0} \|f\,\mathbb{1}_A\|_{L^\Psi} = 0. \tag{2.8}$$

Now, using (2.8), with $f = f_{n_1}$ and considering sets A of the form $A = \{z \in \mathbb{D}\,;\ r < |z| < 1\}$, we get $r_1 \in (0,1)$ so that, for $C_1 = D_1 = \{z \in \mathbb{D}\,;\ |z| \leq r_1\}$, we have

$$\|f_1\mathbb{1}_{\mathbb{D}\setminus C_1}\|_{L^\Psi} < \delta 2^{-3}\,.$$

By the uniform convergence of $\{f_n\}_n$ to 0 on D_1, we can find $n_2 > n_1$ such that

$$|f_{n_2}(z)| \leq 1/4, \text{ for every } z \in D_1, \qquad \text{and} \qquad \|f_{n_2}\mathbb{1}_{D_1}\|_{L^\Psi} < \delta 2^{-5}\,.$$

Using this last inequality and (2.8) again (for $f = f_{n_2}$), we get $r_2 \in (r_1, 1)$, $r_2 > 1 - 1/2$, such that, setting $C_2 = \{z \in \mathbb{D}\,;\ r_1 < |z| \leq r_2\}$, we have

$$\|f_{n_2}\mathbb{1}_{\mathbb{D}\setminus C_2}\|_{L^\Psi} < \delta 2^{-4}\,.$$

Now that we have (2.6) and (2.7) for $k = 1$ and $k = 2$, it is clear how we must iterate the inductive construction. At the time of choosing $r_k \in (r_{k-1}, 1)$, we also impose the condition $r_k > 1 - 1/k$ in order to get $\lim_{k\to\infty} r_k = 1$.

Once the construction is achieved, let us see why the subsequence $\{f_{n_k}\}_k$ works. The condition (2.6) and the fact that $\lim_{k\to\infty} r_k = 1$ imply that, for every compact set K in $\mathbb{D}$ and $z \in \mathbb{D}$, there exists $l_K \in \mathbb{N}$ such that:

$$|f_{n_k}(z)| \leq 2^{-k}, \quad \text{for every } z \in K, \text{ and every } k \geq l_K\,.$$

This yields two facts. First, $\sum_k |f_{n_k}(z)| < +\infty$, for every $z \in \mathbb{D}$, and secondly: for every bounded complex sequence $\alpha = (\alpha_k)_k \in \ell_\infty$, the series $\sum_k \alpha_k f_{n_k}$ converges uniformly on compact subsets of $\mathbb{D}$, and its sum, the function $T\alpha$, is analytic on $\mathbb{D}$.

It remains to prove the estimates in (2.5) about the norm of $T\alpha$ in $L^\Psi(\mathcal{A})$. By homogeneity, we may assume that $\|\alpha\|_\infty = 1$. Let us write $g_k = f_{n_k}\mathbb{1}_{C_k}$ and $h_k = f_{n_k}\mathbb{1}_{\mathbb{D}\setminus C_k}$, for every $k \geq 1$,

$$g = \sum_{k=1}^{\infty} \alpha_k g_k \qquad \text{and} \qquad h = \sum_{k=1}^{\infty} \alpha_k h_k\,.$$

We have $T\alpha = g + h$. By (2.7) and the fact that $|\alpha_k| \leq 1$, we have that $h \in L^\Psi(\mathcal{A})$ and $\|h\|_{L^\Psi} \leq \delta/4$.

By the condition (c) in the statement and the definition of the norm in H^Ψ we have, for every n and every $r \in (0,1)$:

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi\big(|f_n(re^{it})|/M\big)\,dt \leq 1\,. \tag{2.9}$$

The function g_k is 0 outside of C_k, and the sequence $\{C_k\}_k$ is a partition of $\mathbb{D}$. Therefore:

$$\int_{\mathbb{D}} \Psi(|g|/M)\, d\mathcal{A} = \sum_{k=1}^{\infty} \int_{C_k} \Psi(|g|/M)\, d\mathcal{A} = \sum_{k=1}^{\infty} \int_{C_k} \Psi(|\alpha_k|\, |f_{n_k}|/M)\, d\mathcal{A}$$
$$\leq \sum_{k=1}^{\infty} \int_{C_k} \Psi(|f_{n_k}|/M)\, d\mathcal{A}\,.$$

Integrating in polar coordinates, setting $r_0 = 0$, and using (2.9), we get:

$$\int_{\mathbb{D}} \Psi(|g|/M)\, d\mathcal{A} \leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r\, \frac{1}{2\pi} \int_0^{2\pi} \Psi(|f_{n_k}(r\mathrm{e}^{it})|/M)\, dt\, dr$$
$$\leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r\, dr = 1\,,$$

and therefore $\|g\|_{L^\Psi} \leq M$, and $\|T\alpha\|_{L^\Psi} \leq \delta/4 + M \leq 2M$.

On the other hand, for every k, we have:

$$\|g\|_{L^\Psi} \geq \|g\, \mathbb{1}_{C_k}\|_{L^\Psi} = |\alpha_k|\|f_{n_k} - h_k\|_{L^\Psi} \geq |\alpha_k|\,(\delta - \delta/2^{2+k}) \geq \frac{3\delta}{4}|\alpha_k|\,.$$

Taking the supremum on k, we get $\|g\|_{L^\Psi} \geq (3\delta/4)\, \|\alpha\|_\infty = 3\delta/4$. Consequently,

$$\|T\alpha\|_{L^\Psi} \geq \|g\|_{L^\Psi} - \|h\|_{L^\Psi} \geq (3\delta/4) - \delta/4 \geq \delta/2\,,$$

and Lemma 2.5 is fully proved. □

In the following lemma we isolate the proof of the implication (c) $\Longrightarrow$ (d) in the statement of Theorem 2.2.

Lemma 2.6 *Assume that the Orlicz function Ψ is such that, for some $A > 1$,*

$$\limsup_{x\to+\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = +\infty \tag{2.10}$$

Then the injection $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ fixes a copy of ℓ_∞.

Proof. Let us take a sequence of positive numbers $\{d_n\}_n$, and a sequence $\{\xi_n\}_n$ in $\mathbb{T}$, such that the disks $\{D(\xi_n, d_n)\}_n$ are pairwise disjoint in $\mathbb{D}$. In particular, we should have $\lim_{n\to\infty} d_n = 0$.

The convexity of Ψ implies the existence of some $c > 0$ such that $\Psi(x) \geq cx$ for every $x \geq 1$. Given a sequence $\{\beta_n\}_n$ in $(4, +\infty)$ to be fixed later, we can find, thanks to (2.10), an increasing sequence $\{x_n\}$ satisfying:

$$x_n > 1, \quad \Psi(x_n) > 1, \qquad \Psi(Ax_n) > \beta_n[\Psi(x_n)]^2, \quad \text{for every } n \in \mathbb{N}\,. \tag{2.11}$$

Define y_n as the point in the interval (x_n, Ax_n) such that

$$[\Psi(y_n)]^2 = \Psi(Ax_n)\,. \tag{2.12}$$

Put now $h_n = 1/\Psi(y_n)$ and $r_n = 1 - h_n$. By (2.11) and (2.12), we have $[\Psi(y_n)]^2 > \beta_n > 4$, and therefore $h_n \in (0, 1/2)$. Define

$$u_n(z) = \Big(\frac{h_n}{1 - r_n\,\overline{\xi_n} z}\Big)^2, \qquad \text{and} \qquad f_n(z) = y_n\, u_n(z)\,.$$

It is easy to see that $\|u_n\|_\infty = 1$, and that $\|u_n\|_{H^1} \le h_n$.

The first condition imposed to β_n is $\beta_n > 16/d_n^2$. That gives $[\Psi(y_n)]^2 > 16/d_n^2$ and $h_n < d_n/4$. Let us write D_n for the disk $D(\xi_n, d_n)$. Observe that, for $z \in \overline{\mathbb{D}} \setminus D_n$, we have

$$|1 - r_n\,\overline{\xi_n} z| = |1 - r_n + r_n\,\xi_n\overline{\xi_n} - r_n\,\overline{\xi_n} z| \ge r_n|\xi_n - z| - h_n \ge (1/2)d_n - h_n \ge d_n/4\,,$$

and therefore, since $[\Psi(x_n)]^2 \ge \Psi(x_n) \ge c\,x_n$,

$$|f_n(z)| \le y_n\Big(\frac{4h_n}{d_n}\Big)^2 = \frac{16 y_n}{d_n^2[\Psi(y_n)]^2} \le \frac{16 A x_n}{d_n^2 \beta_n [\Psi(x_n)]^2} \le \frac{16A}{c\,d_n^2\beta_n}\,.$$

We also impose the condition $\beta_n > 16An^2/cd_n^2$, and so we have:

$$|f_n(z)| \le \frac{1}{n^2}\,, \qquad \text{for } z \in \overline{\mathbb{D}} \setminus D_n\,. \tag{2.13}$$

From (2.13) we deduce that $\{f_n\}_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Moreover (2.13) yields that, for every bounded sequence $\{\alpha_n\}_n$ of complex numbers, the series $\sum_{n\ge1} \alpha_n f_n$ is uniformly convergent on compact subsets of $\mathbb{D}$. Let us write f_n^* for the boundary value (on $\mathbb{T} = \partial\mathbb{D}$) of the function f_n. We claim that :

$$S = \sum_{n=1}^{\infty} |f_n^*| \in L^\Psi(\mathbb{T}, m). \tag{2.14}$$

From this, it is not difficult to deduce that, for every bounded sequence $\{\alpha_n\}_n$ of complex numbers, the function $\sum_{n=1}^{\infty} \alpha_n f_n$ is in H^Ψ and, for $M = \|S\|_{L^\Psi(\mathbb{T})}$,

$$\Big\|\sum_{n=1}^{\infty} \alpha_n f_n\Big\|_{H^\Psi} \le M \|\{\alpha_n\}_n\|_\infty\,. \tag{2.15}$$

On the other hand, taking $A_n = \{z \in \mathbb{D}\,;\ |z - \xi_n| \le h_n\}$, there exists a constant $\gamma \in (0,1)$ such that $\mathcal{A}(A_n) \ge \gamma h_n^2$, and, for every $z \in A_n$, we have:

$$|1 - r_n\,\overline{\xi_n} z| \le |1 - r_n| + |r_n\,\xi_n\overline{\xi_n} - r_n\,\overline{\xi_n} z| = h_n + r_n\,|z - \xi_n| \le 2h_n\,,$$

and consequently $|u_n(z)| \geq 1/4$. If $\delta = \gamma/4A$, we have, for every n,

$$\begin{aligned}\int_{\mathbb{D}} \Psi\Big(\frac{|f_n|}{\delta}\Big)\, d\mathcal{A} \geq \int_{A_n} \Psi\Big(\frac{y_n}{4\delta}\Big)\, d\mathcal{A} \geq \gamma h_n^2 \Psi\Big(\frac{1}{\gamma}\, A y_n\Big) \\ \geq h_n^2 \Psi(A y_n) > h_n^2 \Psi(A x_n) = 1\,.\end{aligned}$$

Thus $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, for every $n \in \mathbb{N}$. We can apply Lemma 2.5. Using this lemma and (2.15), we get a subsequence $\{f_{n_k}\}_k$ such that, for every $\alpha = (\alpha_k)_k \in \ell_\infty$, we have:

$$(\delta/2)\, \|\{\alpha_k\}_k\|_\infty \leq \Big\|\sum_{k=1}^\infty \alpha_k f_{n_k}\Big\|_{\mathfrak{B}^\Psi} \leq \Big\|\sum_{k=1}^\infty \alpha_k f_{n_k}\Big\|_{H^\Psi} \leq M \|\{\alpha_k\}_k\|_\infty\,.$$

This clearly says that J_Ψ fixes a copy of ℓ_∞.

It remains to prove (2.14). For obtaining this we impose the last condition to the sequence $\{\beta_n\}_n$. We shall need:

$$\sum_{n=1}^\infty 1/\sqrt{\beta_n} \leq 1\,. \tag{2.16}$$

Let us set $g_n = |f_n^*|\, 1\!\!1_{D_n}$. Thanks to (2.13), $S - \sum_{n=1}^\infty g_n$ is a bounded function. Thus we just need to prove that $G = \sum_{n=1}^\infty g_n$ is in $L^\Psi(\mathbb{T})$. We have $\|G\|_{L^\Psi(\mathbb{T})} \leq A$. Indeed, recalling that the D_n's are pairwise disjoint, and that each g_n is 0 out of D_n, we have:

$$\begin{aligned}\int_{\mathbb{T}} \Psi\Big(\frac{G}{A}\Big)\, dm &= \sum_{n=1}^\infty \int_{D_n\cap\mathbb{T}} \Psi\Big(\frac{G}{A}\Big)\, dm = \sum_{n=1}^\infty \int_{D_n\cap\mathbb{T}} \Psi\Big(\frac{|f_n^*|}{A}\Big)\, dm \\ &\leq \sum_{n=1}^\infty \int_{\mathbb{T}} \Psi\Big(\frac{y_n|u_n^*|}{A}\Big)\, dm\end{aligned}$$

and by the convexity of Ψ, and the fact that $|u_n| \leq 1$,

$$\begin{aligned}&\leq \sum_{n=1}^\infty \int_{\mathbb{T}} |u_n^*|\Psi\Big(\frac{y_n}{A}\Big)\, dm = \sum_{n=1}^\infty \|u_n\|_{H_1}\Psi\Big(\frac{y_n}{A}\Big) \\ &\leq \sum_{n=1}^\infty \frac{\Psi(y_n/A)}{\Psi(y_n)} \leq \sum_{n=1}^\infty \frac{\Psi(x_n)}{\Psi(y_n)} = \sum_{n=1}^\infty \frac{\Psi(x_n)}{\sqrt{\Psi(A x_n)}} \leq \sum_{n=1}^\infty \frac{1}{\sqrt{\beta_n}} \leq 1\,,\end{aligned}$$

by the required condition (2.16), and that ends the proof of Lemma 2.6. □

We are now in position to prove Theorem 2.2.

Proof of Theorem 2.2. We shall prove that:

$$\text{(a)} \implies \text{(b)} \implies \text{(c)} \implies \text{(d)} \implies \text{(e)} \implies \text{(a)}\,,$$

and that (b) $\iff$ (f).

The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) and (f) $\Longrightarrow$ (b) are trivial, and we have seen in Lemma 2.6 that (c) $\Longrightarrow$ (d).

(d) $\Longrightarrow$ (e). By Lemma 2.4, there exists a constant $C > 0$ such that, for every f in the unit ball of H^Ψ, we have:

$$\int_{\mathbb{D}} [\Psi(|f|/C)]^2 \, d\mathcal{A} \leq 1 \,. \tag{2.17}$$

For every $A > 0$, there exist x_A, such that $\Psi(Ax) \leq (Q_A + 1)[\Psi(x)]^2$, for every $x \geq x_A$. Thus for every $x \geq 0$ we have $\Psi(Ax) \leq (Q_A + 1)[\Psi(x)]^2 + \Psi(Ax_A)$. Then, by (2.17), we have

$$\int_{\mathbb{D}} \Psi(A|f|/C) \, d\mathcal{A} < +\infty \,, \qquad \text{for every } A > 0 \,.$$

Therefore $f \in \mathfrak{B}M^\Psi$, for every f in the unit ball of H^Ψ, and thus for every f in H^Ψ.

(e) $\Longrightarrow$ (a). Let $\{f_n\}_n$ be in the unit ball of H^Ψ. We have to prove that $\{f_n\}_n$ has a subsequence which converges in the weak topology of $\mathfrak{B}^\Psi$. By Montel's Theorem $\{f_n\}_n$ has a subsequence converging uniformly on compact subsets of $\mathbb{D}$, to a function g which, by Fatou's lemma, also belongs to the unit ball of H^Ψ. If this subsequence converges to g in the norm of $\mathfrak{B}^\Psi$ we are done. If not, after perhaps a new extraction of subsequence, there exist $\delta > 0$ and a subsequence $\{f_{n_k}\}_k$, such that

$$\|f_{n_k} - g\|_{\mathfrak{B}^\Psi} \geq \delta, \qquad \text{and} \qquad \|f_{n_k} - g\|_{H^\Psi} \leq 2 \,.$$

Since moreover $\{f_{n_k} - g\}_k$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ and, by condition (e), $f_{n_k} - g \in \mathfrak{B}M^\Psi$, we may apply Lemma 2.5 and we get that $\{f_{n_k} - g\}_k$ has a subsequence equivalent to the canonical basis of c_0 in $\mathfrak{B}^\Psi$, and is therefore weakly null. This yields that $\{f_n\}_n$ has a subsequence converging to g in the weak topology of $\mathfrak{B}^\Psi$.

(b) $\Longrightarrow$ (f). Suppose there exists an infinite-dimensional subspace X of H^Ψ on which the norms $\|\cdot\|_{\mathfrak{B}^\Psi}$ and $\|\cdot\|_{H^\Psi}$ are equivalent. We shall have finished if we prove that X contains a subspace isomorphic to c_0 because then J_Ψ will fix a copy of c_0.

We can assume that X is contained in $\mathfrak{B}M^\Psi$ because we already know that (b) implies (e). X being infinite-dimensional, there exists, for every $n \in \mathbb{N}$, $f_n \in X$, such that $\|f_n\|_{H^\Psi} = 1$, and $\widehat{f_n}(k) = 0$, for $k = 0, 1, \ldots, n$. By the equivalence of the norms in X, there exists $\delta > 0$ such that $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, for every n. The unit ball of H^Ψ is compact in the topology of $\mathcal{H}(\mathbb{D})$. Since

$$\lim_{n\to\infty} \widehat{f_n}(k) = 0 \,, \qquad \text{for every } k \geq 0 \,,$$

the only possible limit of a subsequence of $\{f_n\}_n$ is the function 0. So $\{f_n\}_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. As $f_n \in X \subseteq \mathfrak{B}M^\Psi$, for every n, we can apply Lemma 2.5, and we get that $\{f_n\}_n$ has a subsequence generating an space Y isomorphic to c_0 in $\mathfrak{B}^\Psi$. This space Y is contained in X, where the norms are equivalent, so Y is also isomorphic to c_0 for the norm of H^Ψ. □

3 Other properties

3.1 Dunford-Pettis

Recall that an operator $T\colon X \to Y$ between two Banach spaces X and Y is said to be *Dunford-Pettis* if $\{Tx_n\}_n$ converges in norm whenever $\{x_n\}_n$ converges weakly. Every compact operator is Dunford-Pettis. The next proposition shows that, in "most" of the cases, these two properties are equivalent for J_Ψ.

Proposition 3.1 *If the conjugate function of Ψ satisfies condition Δ_2, then $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is Dunford-Pettis if and only if it is compact.*

We shall see in Section 4 that without condition Δ_2 for the conjugate function, J_ψ may be Dunford-Pettis without being compact.

Proof. Remark first that speaking of the conjugate function of Ψ implicitly assume that $\Psi(x)/x$ tends to $+\infty$ as x goes to $+\infty$.

Assume that J_Ψ is not compact. By Theorem 2.1, there are some $A > 1$ and a sequence $\{x_j\}_j$ going to $+\infty$ such that $\Psi(Ax_j) \geq [\Psi(x_j)]^2$. Setting $r_j = 1 - 1/\Psi(x_j)$, this is equivalent to say that $A\Psi^{-1}\big(1/(1-r_j)\big) \geq \Psi^{-1}\big(1/(1-r_j)^2\big)$. Define:

$$f_j(z) = x_j \left(\frac{1-r_j}{1-r_j z}\right)^2 .$$

One has $f_j \in HM^\Psi$ and $\|f_j\|_{H^\Psi} \leq 1$ (see [7], Corollary 3.10). Since $\{f_j\}_j$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\{f_j\}_j$ converges to 0 in the weak-star topology of H^Ψ ([7], Proposition 3.7). But, since the conjugate function of Ψ satisfies condition Δ_2, H^Ψ is the bidual of HM^Ψ ([7], Corollary 3.3); hence $\{f_j\}_j$ converges weakly to 0 in HM^Ψ.

On the other hand, if $S_j = D(1, 1-r_j) \cap \mathbb{D}$, one has $|1 - r_j z| \leq 2(1-r_j)$ for $z \in S_j$; hence, writing $K = \|f_j\|_{\mathfrak{B}^\Psi}$, one has:

$$1 = \int_{\mathbb{D}} \Psi(|f_j|/K)\, d\mathcal{A} \geq \int_{S_j} \Psi(|f_j|/K)\, d\mathcal{A} \geq \mathcal{A}(S_j)\Psi(x_j/4K) .$$

Since $\mathcal{A}(S_j) \geq \alpha(1-r_j)^2$, with $0 < \alpha < 1$, we get (since $\Psi(\alpha x_j/4K) \leq \alpha\Psi(x_j/4K)$, by convexity):

$$\|f_j\|_{\mathfrak{B}^\Psi} \geq (\alpha/4)\, \frac{x_j}{\Psi^{-1}\big(1/(1-r_j)^2\big)} = (\alpha/4)\, \frac{\Psi^{-1}\big(1/(1-r_j)\big)}{\Psi^{-1}\big(1/(1-r_j)^2\big)} \geq \frac{\alpha}{4A} .$$

Therefore J_Ψ is not Dunford-Pettis. □

On the other hand, one has:

Proposition 3.2 *If J_Ψ is Dunford-Pettis, then J_Ψ is weakly compact.*

Proof. By Theorem 2.2, if J_Ψ is not weakly compact, there is a subspace X_0 of H^Ψ isomorphic to c_0 on which J_Ψ is an into-isomorphism; hence J_Ψ cannot be Dunford-Pettis. □

We shall see in the next section that J_Ψ may be weakly compact without being Dunford-Pettis.

3.2 Absolutely summing

Every p-summing operator is weakly compact and Dunford-Pettis; so it may be expected that J_Ψ is p-summing for some $p < \infty$. The next results show that this is never the case as soon as Ψ grows faster than all the power functions.

Recall that an operator $T\colon X \to Y$ between two Banach spaces X and Y is called (p,q)*-summing* if there is a constant $C > 0$ such that

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^p\Big)^{1/p} \leq C \sup_{\|x^*\|_{X^*}\leq 1} \Big(\sum_{k=1}^{n} |x^*(x_k)|^q\Big)^{1/q},$$

for every finite sequence $(x_1, \ldots, x_n)$ in X. If $q = p$, it is said p-summing. Every p-summing operator is (p,q)-summing for $q \leq p$.

Theorem 3.3 *If $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is p-summing, then, for every $q > p$, $\Psi(x) = O\left(x^q\right)$ for x large enough. Moreover, if $p < 2$, then J_Ψ is compact.*

In order to prove this, we need two lemmas.

Lemma 3.4 *If the canonical injection $I_\Psi\colon A \to \mathfrak{B}^\Psi$ is $(p,1)$-summing, where $A = A(\mathbb{D})$ is the disk algebra, then $\Psi(x) = O\left(x^{2p}\right)$ for x large enough.*

In particular, $J_r\colon H^r \to \mathfrak{B}^r$ is $(p,1)$-summing for no $p < r/2$, and, if $\Psi \in \Delta^0$, then J_Ψ is $(p,1)$-summing for no $p < \infty$.

Recall that the disk algebra is the space of continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$.

We refer to [9] for a detailed study of r-summing Carleson embeddings $H^r \to L^r(\mu)$. In particular, it follows from these results that $J_r\colon H^r \to \mathfrak{B}^r$ is 1-summing for $1 \leq r < 2$. On the other hand, it is easy to see that $J_2\colon H^2 \to \mathfrak{B}^2$ is not Hilbert-Schmidt (*i.e.* not 2-summing): for the canonical orthonormal basis $\{z^n\}_n$ and $\{\sqrt{n+1}\, z^n\}_n$ of H^2 and $\mathfrak{B}^2$, J_2 is the diagonal operator of multiplication by $\{1/\sqrt{n+1}\}_n$. It also follows from [9] that, for $r \geq 2$, J_r is p-summing for no finite p.

Proof. Assume that we do not have $\Psi(x) = O\left(x^{2p}\right)$ for x large enough. Then $\limsup_{x\to+\infty} \Psi(x)/x^{2p} = +\infty$. Given any $K > 0$, take $y > 0$ such that $\Psi(y)/y^{2p} \geq K$ and such that $h = 1/\sqrt{\Psi(y)} \leq 1/2$. Let N be the integer part of $(1/h) + 1$. Writing $\xi_j = \mathrm{e}^{2\pi i j/N}$, we set:

$$u_j(z) = \frac{h^2}{[1 - (1-h)\,\overline{\xi_j} z]^2}\,.$$

We have $u_j \in A(\mathbb{D})$. By [7], Lemma 5.6, one has, since $h \geq 1/N$:

$$\sum_{j=0}^{N-1} |u_j(\mathrm{e}^{it})| \leq N\, h^2 \,\frac{1-(1-h)^{2N}}{[1-(1-h)^2][1-(1-h)^N]^2} \leq \frac{\mathrm{e}^2}{(1-\mathrm{e})^2} := C\,.$$

Hence:

$$\sup_{\|x^*\|_{A^*}\leq 1} \sum_{j=0}^{N-1} |x^*(u_j)| \leq C\,.$$

On the other hand, it is easy to see that $|u_j(z)| \geq 1/9$ when $|z-(1-h)\xi_j| < h$; hence, if $S_j = \{z \in \mathbb{D}\,;\; |z-(1-h)\xi_j| < h\}$, one has, since $\mathcal{A}(S_j) = h^2$:

$$1 = \int_{\mathbb{D}} \Psi\Big(\frac{|u_j(z)|}{\|u_j\|_{\mathfrak{B}^\Psi}}\Big)\, d\mathcal{A}(z) \geq \int_{S_j} \Psi\Big(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\Big)\, d\mathcal{A} \geq h^2\Psi\Big(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\Big)\,,$$

so $\|u_j\|_{\mathfrak{B}^\Psi} \geq 1/9\Psi^{-1}(1/h^2)$. Since $y = \Psi^{-1}(1/h^2)$, one gets:

$$\sum_{j=0}^{N-1} \|u_j\|_{\mathfrak{B}^\Psi}^p \geq (1/9)^p \frac{N}{y^p} \geq (1/9)^p \Big[\frac{\Psi(y)}{y^{2p}}\Big]^{1/2} \geq \frac{K^{1/2}}{9^p}\cdot$$

This yields that the $(p,1)$-summing norm of I_Ψ should be greater than $K^{1/2p}/9C$, and, as K is arbitrary, that I_Ψ is not $(p,1)$-summing. □

Remark. When $I_\Psi\colon A \hookrightarrow \mathfrak{B}^\Psi$ is p-summing, we have this shorter argument. By Pietsch's factorization theorem, this I_Ψ factors through H^p. It follows from [7], Theorem 4.10, that $\alpha\, h^2 \leq \rho_{\mathcal{A}}(h) \leq 1/\Psi^{-1}(A/h^{1/p})$, for some constants $0 < \alpha < 1$ and $A > 0$, and h small enough. That means that $\Psi(x) \leq C\, x^{2p}$ for x large enough.

Lemma 3.5 *If the canonical injection $I_\Psi\colon A \to \mathfrak{B}^\Psi$ is 1-summing, then J_Ψ is compact.*

Proof. The canonical injection $J_1\colon H^1 \to \mathfrak{B}^1$ (as well as J_Ψ whenever $\Psi \in \Delta_2$) is compact. Hence we may assume that H^Ψ is not H^1 and hence that $\Psi(x)/x$ tends to $+\infty$ as x tends to $+\infty$.

Assume that J_Ψ is not compact. Then, as in the proof of Proposition 3.1, there are some $A > 1$ and a sequence $\{x_k\}_k$ going to $+\infty$ such that $\Psi(Ax_k) \geq [\Psi(x_k)]^2$. Setting $h_k = 1/\Psi(x_k)$, we define, as in the proof of Proposition 3.4:

$$u_{k,j}(z) = \frac{h_k^2}{[1-(1-h_k)\overline{\xi_{k,j}}\,z]^2}\,,$$

where $\xi_{k,j} = \mathrm{e}^{2\pi ij/N_k}$, with N_k the integer part of $(1/h_k)+1$. One has $u_{k,j} \in A$ and (see the proofs of the two quoted propositions):

$$\sum_{j=0}^{N_k-1} |u_{k,j}(\mathrm{e}^{it})| \leq C \qquad \text{and} \qquad \|u_{k,j}\|_{\mathfrak{B}^\Psi} \geq \frac{\delta\alpha}{A}\,\frac{1}{\Psi^{-1}(1/h_k)}\,.$$

It follows that:

$$\sum_{j=0}^{N_k-1} \|u_{k,j}\|_{\mathfrak{B}^\Psi} \geq \frac{\delta\alpha}{A}\,\frac{N_k}{\Psi^{-1}(1/h_k)} \geq \frac{\delta\alpha}{A}\,\frac{1/h_k}{\Psi^{-1}(1/h_k)} = \frac{\delta\alpha}{A}\,\frac{\Psi(x_k)}{x_k} \underset{k\to\infty}{\longrightarrow} +\infty.$$

Hence I_Ψ is not 1-summing. □

Proof of Theorem 3.3. Since $J_\Psi \colon H^\Psi \to \mathfrak{B}^\Psi$ is p-summing and the canonical injection $I_\Psi \colon A \to \mathfrak{B}^\Psi$ factors as $I_\Psi \colon A \to H^\Psi \to \mathfrak{B}^\Psi$, this injection is p-summing. By Lemma 3.4, $\Psi(x) = O\left(x^{2p}\right)$ for x large enough. Hence we have the factorization $A \to H^{2p} \to H^\Psi \to \mathfrak{B}^\Psi$. Since the first injection is $2p$-summing and the last one is p-summing, the composition is $\max(1, p_1)$-summing, with $\frac{1}{p_1} = \frac{1}{2p} + \frac{1}{p}$ (see [2], Theorem 2.22), *i. e.* $p_1 = \frac{2}{3}\,p$. If $p_1 > 1$, we can use again Lemma 3.4 with p_1 instead of $2p$; we get that $\Psi(x) = O\left(x^{2p_1}\right)$, for x large enough, and that the factorization $I_\Psi \colon A \to H^{2p_1} \to H^\Psi \to \mathfrak{B}^\Psi$ is $\max(1, p_2)$-summing, with $\frac{1}{p_2} = \frac{1}{2p_1} + \frac{1}{p}$. Going on the same way, we get a decreasing sequence $\{p_n\}_n$ such that the canonical injection $A \to \mathfrak{B}^\Psi$ is $\max(1, p_n)$-summing and $\frac{1}{p_{n+1}} = \frac{1}{2p_n} + \frac{1}{p}\cdot$ Writing $p_n = \alpha_n p$, we get $\alpha_{n+1} = \frac{2\alpha_n}{2\alpha_n + 1}$; hence $p_n \underset{n\to\infty}{\longrightarrow} p/2$. In particular, $\Psi(x) = O\left(x^q\right)$ for every $q > p$.

If $p < 2$, one has $\max(1, p_n) = 1$ for n large enough, and Lemma 3.4 implies that J_Ψ is compact. □

Remark 1. It is not clear whether J_Ψ p-summing, with $p \geq 2$, implies that J_Ψ is compact. However, when $r \geq 2$, $J_r \colon H^r \to \mathfrak{B}^r$ is p-summing for no $p < \infty$ (see [9]).

Remark 2. An operator $T \colon X \to Y$ between two Banach spaces is said to be *finitely strictly singular* (or *superstrictly singular*) if for every $\varepsilon > 0$, there is an integer $N_\varepsilon \geq 1$ such that, for every subspace X_0 of X of dimension $\geq N_\varepsilon$, there is an $x \in X_0$ such that $\|Tx\| \leq \varepsilon\,\|x\|$. Every finitely strictly singular operator is strictly singular. It is not difficult to see that every compact operator is finitely strictly singular and it is shown in [10] (see also [5], Corollary 2.3) that every p-summing operator is finitely strictly singular. We do not know when J_Ψ is finitely strictly singular.

3.3 Order boundedness

Recall that an operator $T \colon X \to Y$ from a Banach space X into a Banach lattice Y is said to be *order bounded* if there is $y \in Y_+$ such that $|Tx| \leq y$ for every x in the unit ball of X. Since the Bergman-Orlicz space $\mathfrak{B}^\Psi$ is a subspace of the Banach lattice $L^\Psi(\mathbb{D}, \mathcal{A})$, we may study the order boundedness of J_Ψ. Actually, we are going to see that the natural space for the order boundedness of J_Ψ is not $L^\Psi(\mathbb{D}, \mathcal{A})$, but the *weak Orlicz space* $L^{\Psi,\infty}(\mathbb{D}, \mathcal{A})$, the definition of which we are recalling below (see [7], Definition 3.16).

Definition 3.6 *Let* $(S, \mathcal{S}, \mu)$ *be a measure space; the* weak-L^Ψ space $L^{\Psi,\infty}$ *is the set of the (classes of) measurable functions* $f\colon S \to \mathbb{C}$ *such that, for some constant* $c > 0$*, one has, for every* $t > 0$*:*

$$\mu(|f| > t) \leq \frac{1}{\Psi(ct)} \cdot$$

One has $L^\Psi \subseteq L^{\Psi,\infty}$ and ([7], Proposition 3.18) the equality $L^\Psi = L^{\Psi,\infty}$ implies that $\Psi \in \Delta^0$. On the other hand, this equality holds when Ψ grows sufficiently; for example, if Ψ satisfies the condition Δ^1: $x\Psi(x) \leq \Psi(\alpha x)$, for some constant $\alpha > 1$ and x large enough.

Proposition 3.7 $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ *is always order bounded into* $L^{\Psi,\infty}(\mathbb{D}, \mathcal{A})$*.*

Proof. Since (see [7], Lemma 3.11):

$$\frac{1}{4}\Psi^{-1}\Big(\frac{1}{1-|z|}\Big) \leq \sup_{\|f\|_{H^\Psi} \leq 1} |f(z)| \leq 4\Psi^{-1}\Big(\frac{1}{1-|z|}\Big)\,, \tag{3.1}$$

one has, denoting by $S(z)$ the supremum in (3.1), for t large enough:

$$\mathcal{A}(|S| > t) \leq \mathcal{A}\big(\{z \in \mathbb{D}\,;\ |z| > 1 - 1/\Psi(t/4)\}\big) \leq \frac{2}{\Psi(t/4)} \leq \frac{1}{\Psi(t/8)}\,,$$

and the result follows. □

Since we also have, for t large enough:

$$\mathcal{A}(|S| > t) \geq \mathcal{A}\big(\{z \in \mathbb{D}\,;\ |z| > 1 - 1/\Psi(4t)\}\big) \geq \frac{1}{\Psi(4t)}\,,$$

we get:

Corollary 3.8 J_Ψ *is order bounded into* $L^\Psi(\mathbb{D}, \mathcal{A})$ *if and only if* $L^\Psi = L^{\Psi,\infty}$*. This is the case if* $\Psi \in \Delta^1$*.*

Remark. Contrary to the compactness, or the weak compactness, which requires that Ψ does not grow too fast, the order boundedness of J_Ψ into $L^\Psi(\mathbb{D}, \mathcal{A})$ holds when Ψ grows fast enough. Nevertheless, for $\Psi(x) = \exp[\big(\log(x+1)\big)^2] - 1$, J_Ψ is compact and order bounded into $L^\Psi(\mathbb{D}, \mathcal{A})$.

When J_Ψ is weakly compact, J_Ψ maps H^Ψ into $\mathfrak{B}M^\Psi$ (Theorem 2.2); hence, we may ask whether J_Ψ may be order bounded into $M^\Psi(\mathbb{D}, \mathcal{A})$; however, we have:

Proposition 3.9 J_Ψ *is never order bounded into* $M^\Psi(\mathbb{D}, \mathcal{A})$*.*

Proof. If it were the case, we should have $S \in M^\Psi(\mathbb{D}, \mathcal{A})$, and hence

$$\int_{\mathbb{D}} \Psi\Big[4 \times \frac{1}{4}\Psi^{-1}\Big(\frac{1}{1-|z|}\Big)\Big]\, d\mathcal{A}(z) < +\infty\,,$$

which is false. □

4 An example

Theorem 4.1 *There exists an Orlicz function Ψ such that J_Ψ is weakly compact and Dunford-Pettis, but which is not compact.*

Note that such an Orlicz function is very irregular: $\Psi \notin \Delta_2$, $\Psi \notin \Delta^0$, so, for every $A > 1$, $\Psi(Ax)/\Psi(x)$ is not non-decreasing for x large enough, and the conjugate function of Ψ does not satisfies condition Δ_2.

The following lemma is undoubtedly well-known, but we have found no reference, so we shall give a proof. Recall that a sublattice X of $L^0(\mu)$ is solid if $|f| \leq |g|$ and $g \in X$ implies $f \in X$ and $\|f\| \leq \|g\|$.

Lemma 4.2 *Let $(S, \mathcal{S}, \mu)$ be a measure space, and let X be a solid Banach sublattice of $L^0(\mu)$, the space of all measurable functions. Then, for every weakly null sequence $\{f_n\}_n$ in X and every sequence $\{A_n\}_n$ of disjoint measurables sets, the sequence $\{f_n \mathbb{1}_{A_n}\}_n$ converges weakly to 0 in X.*

Proof. If the conclusion does not hold, there are a continuous linear functional $\sigma \colon X \to \mathbb{C}$ and some $\delta > 0$ such that, up to taking a subsequence, $|\sigma(f_n \mathbb{1}_{A_n})| \geq \delta$. Set, for every measurable set $A \in \mathcal{S}$:

$$\mu_n(A) = \sigma(f_n \mathbb{1}_A)\,.$$

Then μ_n is a finitely additive measure with bounded variation. By Rosenthal's lemma (see [3], Lemma I.4.1, page 18, or [1], Chapter VII, page 82), there is an increasing sequence of integers $\{n_k\}_k$ such that:

$$\Big|\mu_{n_k}\Big(\bigcup_{l\neq k} A_{n_l}\Big)\Big| \leq |\mu_{n_k}|\Big(\bigcup_{l\neq k} A_{n_l}\Big) \leq \delta/2\,.$$

Now, if $A = \bigcup_{l\geq 1} A_{n_l}$, $\{f_{n_k}\mathbb{1}_A\}_k$ is weakly null, but:

$$|\sigma(f_{n_k}\mathbb{1}_A)| \geq |\sigma(f_{n_k}\mathbb{1}_{A_{n_k}})| - |\mu_{n_k}|\Big(\bigcup_{l\neq k} A_{n_l}\Big) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}\,,$$

so we get a contradiction. □

Proof of Theorem 4.1. We begin by defining a sequence $\{x_n\}_n$ of positive numbers in the following way: set $x_1 = 4$ and, for every $n \geq 1$, $x_{n+1} > 2x_n$ is the abscissa of the second intersection point of the parabola $y = x^2$ with the straight line containing (x_n, x_n^2) and $(2x_n, x_n^4)$; we have $x_{n+1} = x_n^3 - 2x_n$ (for example, $x_2 = 56$). Define $\Psi \colon [0, +\infty) \to [0, +\infty)$ by $\Psi(x) = 4x$ for $0 \leq x \leq 4$, and, for $n \geq 1$:

$$\Psi(x_n) = x_n^2\,, \qquad \Psi(2x_n) = x_n^4\,, \qquad \Psi \text{ affine between } x_n \text{ and } x_{n+1}\,. \tag{4.1}$$

Then Ψ is an Orlicz function and

$$x^2 \leq \Psi(x) \leq x^4 \quad \text{for} \quad x \geq 4. \tag{4.2}$$

For this Orlicz function Ψ, J_Ψ is not compact, since $\Psi(2x)/[\Psi(x)]^2$ does not tend to 0. However, J_Ψ is weakly compact, because one has the factorization $H^\Psi \hookrightarrow H^2 \hookrightarrow \mathfrak{B}^4 \hookrightarrow \mathfrak{B}^\Psi$ (by (4.2) and Lemma 2.4).

Assume that J_Ψ is not Dunford-Pettis: there exists a weakly null sequence $\{f_n\}_n$ in the unit ball of H^Ψ which does not converges for the norm in $\mathfrak{B}^\Psi$. Then $\{f_n\}_n$ converges uniformly to 0 on the compact subsets of $\mathbb{D}$ (since it is weakly null) and we may assume that $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ for some $\delta > 0$. We may also assume that $\|f_n\|_\infty \underset{n\to\infty}{\longrightarrow} +\infty$ because if $\{f_n\}_n$ were uniformly bounded, we should have $\|f_n\|_{\mathfrak{B}^\Psi} \underset{n\to\infty}{\longrightarrow} 0$, by dominated convergence.

We are going to show that there exist a subsequence $\{f_{n_k}\}_k$ and pairwise disjoint measurable sets $A_k \subseteq \mathbb{T}$ such that the sequence $\{f_{n_k} \mathbb{I}_{A_k}\}_k \subseteq L^\Psi(\mathbb{T}, m)$ is equivalent to the canonical basis of ℓ_1, whence a contradiction with Lemma 4.2.

It is worth to note from now that the Poisson integral $\mathcal{P}$ maps boundedly $L^2(\mathbb{T})$ into $L^4(\mathbb{D})$. Indeed, $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$ and the canonical injection is bounded from H^2 into $\mathfrak{B}^4$, by Lemma 2.4.

We have seen in the proof of Lemma 2.5 that there exist a subsequence $\{f_{n_k}\}_k$ and disjoint measurable annuli $C_1 = \{z \in \mathbb{D}\,;\ |z| \leq r_1\}$ and $C_k = \{z \in \mathbb{D}\,;\ r_{k-1} < |z| \leq r_k\}$, $k \geq 2$, with $0 < r_1 < r_2 < \cdots < r_n < \cdots < 1$, such that $\|f_{n_k} \mathbb{I}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/2$. The assumptions of that lemma are satisfied here: $\|f_n\|_{H^\Psi} \leq 1$, $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, $\{f_n\}_n$ converges uniformly to 0 on the compact subsets of $\mathbb{D}$, and $f_n \in \mathfrak{B}M^\Psi$ because $H^\Psi \subseteq \mathfrak{B}M^\Psi$, since J_Ψ is weakly compact. Then:

Fact 1. *There exist two sequences $\{\alpha_k\}_k$ and $\{\beta_k\}_k$, with $\beta_n > \alpha_n \underset{n\to\infty}{\longrightarrow} +\infty$ such that, if $g_k = f_{n_k}^* \mathbb{I}_{\{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}}$, then:*

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \geq \delta/3\,,$$

where $f_{n_k}^*$ is the boundary value of f_{n_k} on $\mathbb{T}$.

Proof. 1) Let $\alpha_k = \frac{\delta}{12}\,\Psi^{-1}\big(1/\mathcal{A}(C_k)\big)$ and $v_k = \mathcal{P}\big(f_{n_k}^*\,\mathbb{I}_{\{|f_{n_k}^*| < \alpha_k\}}\big)\,\mathbb{I}_{C_k}$. One has:

$$\int_{\mathbb{D}} \Psi\big(|v_k|/(\delta/12)\big)\,d\mathcal{A} = \int_{C_k} \Psi\big(|v_k|/(\delta/12)\big)\,d\mathcal{A} \leq \Psi\big(\alpha_k/(\delta/12)\big)\,\mathcal{A}(C_k) = 1\,,$$

so $\|v_k\|_{L^\Psi(\mathbb{D})} \leq \delta/12$. Since $\mathcal{P}(f_{n_k}^*) = f_{n_k}$, we have $\|\mathcal{P}(f_{n_k}^*)\,\mathbb{I}_{C_k}\|_{L^\Psi(\mathbb{D})} = \|f_{n_k}\,\mathbb{I}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/2$, and we get:

$$\|\mathcal{P}(f_{n_k}^*\,\mathbb{I}_{\{|f_{n_k}^*| \geq \alpha_k\}})\,\mathbb{I}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \|f_{n_k}\,\mathbb{I}_{C_k}\|_{L^\Psi(\mathbb{D})} - \|v_k\|_{L^\Psi(\mathbb{D})} \geq \frac{\delta}{2} - \frac{\delta}{12} = \frac{5\delta}{12}\cdot$$

2) Let $w_k = f_{n_k}^*\,\mathbb{I}_{\{|f_{n_k}^*| \geq \alpha_k\}}$. Since $\mathcal{P}(w_k\,\mathbb{I}_{\{|w_k| > \beta\}})$ tends to 0 uniformly on C_k when β goes to infinity, Lebesgue's dominated convergence theorem gives:

$$\|\mathcal{P}(w_k\,\mathbb{I}_{\{|w_k|>\beta\}})\,\mathbb{I}_{C_k}\|_{L^\Psi(\mathbb{D})} \leq \|\mathcal{P}(w_k\,\mathbb{I}_{\{|w_k|>\beta\}})\,\mathbb{I}_{C_k}\|_{L^4(\mathbb{D})} \underset{\beta\to+\infty}{\longrightarrow} 0\,,$$

so there is some $\beta_k > \alpha_k$ such that $\|\mathcal{P}(w_k\, \mathbb{1}_{\{|w_k|>\beta\}})\, \mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \leq \delta/12$.

We then have, with $g_k = f^*_{n_k}\, \mathbb{1}_{\{\alpha_k \leq |f^*_{n_k}| \leq \beta_k\}}$:

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \geq \|\mathcal{P}(g_k)\, \mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \frac{5\delta}{12} - \frac{\delta}{12} = \frac{\delta}{3}\,,$$

and that ends the proof of Fact 1. □

Fact 2. *There are a further subsequence, denoted yet by $\{f_{n_k}\}_k$, and pairwise disjoint measurable subsets $E_k \subseteq \{\alpha_k \leq |f^*_{n_k}| \leq \beta_k\}$, such that, if $h_k = f^*_{n_k}\, \mathbb{1}_{E_k}$, then:*

$$\|\mathcal{P}(h_k)\|_{L^\Psi(\mathbb{D})} \geq \delta/4\,.$$

Proof. First, since $g_k \in L^\infty(\mathbb{T}) \subseteq M^\Psi(\mathbb{T})$, there exists $\varepsilon_k > 0$ such that $m(A) \leq \varepsilon_k$ implies $\|g_k\, \mathbb{1}_A\|_{L^\Psi(\mathbb{T})} \leq \delta/(12\,\|\mathcal{P}\|)$ (where $\|\mathcal{P}\|$ stands for the norm of $\mathcal{P}\colon L^2(\mathbb{T}) \to L^4(\mathbb{D})$). Now, $\mathcal{P}\colon L^\Psi(\mathbb{T}) \to L^\Psi(\mathbb{D})$ is bounded and its norm is $\leq \|\mathcal{P}\|$, thanks to the factorization $L^\Psi(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \hookrightarrow L^4(\mathbb{D}) \hookrightarrow L^\Psi(\mathbb{D})$. Hence $\|\mathcal{P}(g_k\, \mathbb{1}_A)\|_{L^\Psi(\mathbb{D})} \leq \delta/12$ for $m(A) \leq \varepsilon_k$.

Let $B_k = \{\alpha_k \leq |f^*_{n_k}| \leq \beta_k\}$. Up to taking a subsequence, we may assume that $\sum_{l>k} m(B_l) \leq \varepsilon_k$. Let

$$E_k = B_k \setminus \bigcup_{l>k} B_l\,.$$

The sets E_k, $k \geq 1$, are pairwise disjoint, and

$$\|\mathcal{P}(g_k\, \mathbb{1}_{E_k})\|_{L^\Psi(\mathbb{D})} \geq \|\mathcal{P}(g_k\, \mathbb{1}_{B_k})\|_{L^\Psi(\mathbb{D})} - \|\mathcal{P}\big(g_k\, \mathbb{1}_{\bigcup_{l>k} B_l}\big)\|_{L^\Psi(\mathbb{D})} \geq \frac{\delta}{3} - \frac{\delta}{12} = \frac{\delta}{4}\,;$$

so we get the Fact 2 with $h_k = g_k\, \mathbb{1}_{E_k} = f^*_{n_k}\, \mathbb{1}_{E_k}$. □

Set

$$F_k = \{z \in E_k\,;\ \Psi\big(|f^*_{n_k}(z)\big)| \leq M\, |f^*_{n_k}(z)|^2\}\,.$$

For $z \in E_k \setminus F_k$, one has:

$$\int_{E_k \setminus F_k} |f^*_{n_k}|^2\, dm \leq \frac{1}{M} \int_{\mathbb{T}} \Psi(|f^*_{n_k})|\, dm \leq \frac{1}{M}\,,$$

so $\|f^*_{n_k}\, \mathbb{1}_{E_k \setminus F_k}\|_{L^2(\mathbb{T})} \leq 1/\sqrt{M}$ and:

$$\begin{aligned}\|\mathcal{P}(f^*_{n_k}\, \mathbb{1}_{E_k \setminus F_k})\|_{L^\Psi(\mathbb{D})} &\leq \|\mathcal{P}(f^*_{n_k}\, \mathbb{1}_{E_k \setminus F_k})\|_{L^4(\mathbb{D})} \\ &\leq \|\mathcal{P}\|\, \|(f^*_{n_k}\, \mathbb{1}_{E_k \setminus F_k})\|_{L^2(\mathbb{T})} \leq \frac{\|\mathcal{P}\|}{\sqrt{M}} \leq \frac{\delta}{8}\,,\end{aligned}$$

for M large enough. It follows that, for M large enough, $\|\mathcal{P}(f^*_{n_k}\, \mathbb{1}_{F_k})\|_{L^\Psi(\mathbb{D})} \geq \delta/8$ and

$$\|f^*_{n_k}\, \mathbb{1}_{F_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/(8\,\|\mathcal{P}\|)\,. \tag{4.3}$$

Now, we may assume that, for some $\alpha > 0$,

$$\int_{\mathbb{T}} |f_{n_k}^*|^2 \, \mathbb{I}_{F_k} \, dm \geq \alpha \,,$$

because, if not, there would be a subsequence $\{f_{n_{k_j}}^* \, \mathbb{I}_{F_{k_j}}\}_j$ converging to 0 in $L^2(\mathbb{T})$; but then $\{\mathcal{P}\big(f_{n_{k_j}} \, \mathbb{I}_{F_{k_j}}\big)\}_j$ would converge to 0 in $\mathfrak{B}^4$, and hence in $\mathfrak{B}^\Psi$, contrary to (4.3). It follows, using (4.2), that:

$$\int_{F_k} \Psi(|f_{n_k}^*|) \, dm \geq \alpha \,. \tag{4.4}$$

The following lemma is now the key of the proof.

Lemma 4.3 *Let $\delta_n = 2x_{n-1}/x_n = 2/(x_{n-1}^2 - 2)$. If $\Psi(x) \leq Mx^2$ and $x \geq x_n$, then, for n large enough ($n \geq N$), one has $\Psi(\varepsilon x) \geq C_M \, \varepsilon \, \Psi(x)$ for $\delta_n \leq \varepsilon \leq 1$.*

Proof. We may assume that $x_n \leq x < x_{n+1}$, because if $x_k \leq x < x_{k+1}$ with $k \geq n$, then $\varepsilon \geq \delta_n$ implies $\varepsilon \geq \delta_k$.

Now, remark that:

$$\frac{\Psi(y)}{\Psi(x)} \leq 4 \, \frac{y}{x} \,, \qquad \text{for} \quad 2x_n \leq x \leq y \leq x_{n+1} \,. \tag{4.5}$$

Indeed, on the one hand, $\frac{\Psi(y)-\Psi(x_n)}{\Psi(x)-\Psi(x_n)} = \frac{y-x_n}{x-x_n} \leq \frac{y}{x/2} = 2 \, \frac{y}{x}$; and, on the other hand, $\Psi(y) - \Psi(x_n) \geq \Psi(y) - \Psi(y/2) \geq \Psi(y) - \frac{1}{2} \Psi(y) = \frac{1}{2} \Psi(y)$, so $\frac{\Psi(y)}{\Psi(x)} \leq \frac{\Psi(y)}{\Psi(x)-\Psi(x_n)} \leq 2 \, \frac{\Psi(y)-\Psi(x_n)}{\Psi(x)-\Psi(x_n)} \leq 4 \, \frac{y}{x} \cdot$

We shall separate three cases:

1) $\varepsilon x \leq x_n \leq x \leq 2x_n$. Then $\varepsilon x \geq \varepsilon x_n$ and hence $\Psi(\varepsilon x) \geq \Psi(\varepsilon x_n)$. But $2x_{n-1} \leq \varepsilon x_n \leq x_n$, since $\varepsilon \geq \delta_n$; hence (4.5) implies that $\Psi(\varepsilon x) \geq (\varepsilon/4) \, \Psi(x_n) = (\varepsilon/4) \, x_n^2$. On the other hand, one has, by hypothesis, $\Psi(x) \leq Mx^2 \leq M(2x_n)^2$, so we get $\Psi(\varepsilon x) \geq (\varepsilon/16M)\Psi(x)$.

2) $x_n \leq \varepsilon x \leq x \leq 2x_n$. Then, since $1 \leq 1/\varepsilon$:

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \leq \frac{Mx^2}{\Psi(x_n)} \leq \frac{M(2x_n)^2}{x_n^2} = 4M \leq \frac{4M}{\varepsilon} \cdot$$

3) For $x \geq 2x_n$, remark that the conditions $\Psi(x) \leq Mx^2$ and $x \geq 2x_n$ imply that $x \geq x_n^2/\sqrt{M}$. Indeed, if $x \geq 2x_n$, then $\Psi(x) \geq \Psi(2x_n) = x_n^4$, and the condition $\Psi(x) \leq Mx^2$ implies $x_n^4 \leq Mx^2$, *i.e.* $x \geq x_n^2/\sqrt{M}$.

In this case, one has $\varepsilon x \geq \varepsilon x_n^2/\sqrt{M} \geq \delta_n x_n^2/\sqrt{M} = 2(x_{n-1}/x_n)x_n^2/\sqrt{M} = 2x_{n-1}x_n/\sqrt{M} \geq 2x_n$, if $x_{n-1} \geq \sqrt{M}$. Hence (4.5) gives, for $2x_n \leq x < x_{n+1}$ (since then $2x_n \leq \varepsilon x \leq x < x_{n+1}$):

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \leq 4 \, \frac{x}{\varepsilon x} = \frac{4}{\varepsilon} \cdot$$

That ends the proof of Lemma 4.3. □

Extract now a further subsequence of $\{f_{n_k}\}$, yet denoted by $\{f_{n_k}\}$, in order that (see Fact 1) $\alpha_k \geq x_{N+k}$. Lemma 4.3 holds, with $x = \Psi(|f_{n_k}^*(z)|)$, $z \in F_k$, for every $k \geq 1$; one has (since, by definition, $\Psi(|f_{n_k}|) \leq M\,|f_{n_k}|^2$ on F_k):

$$\int_{F_k} \Psi(\varepsilon\,|f_{n_k}^*|)\,dm \geq \varepsilon\,C/\alpha := c\,\varepsilon\,, \qquad \text{for } \delta_{N+k} \leq \varepsilon \leq 1\,.$$

The proof of Theorem 4.1 reaches now its end: put $u_k = f_{n_k}^*\,\mathbb{1}_{F_k}$, and take an arbitrary sequence of complex numbers such that $\sum_{k\geq 1} |\lambda_k| = 1$. Let $\delta_0 = \sum_{k\geq N} \delta_k$. One has $\delta_0 < 1$, because we may assume that N had been taken large enough. One gets:

$$\begin{aligned}
\int_{\mathbb{T}} \Psi\Big(\Big|\sum_{k\geq 1} \lambda_k u_k\Big|\Big)\,dm &= \sum_{k\geq 1} \int_{F_k} \Psi(|\lambda_k f_{n_k}|)\,dm \\
&\geq \sum_{|\lambda_k|\geq \delta_{N+k}} c\,|\lambda_k| + \sum_{|\lambda_k|<\delta_{N+k}} \int_{F_k} \Psi(|\lambda_k f_{n_k}|)\,dm \\
&\geq \sum_{|\lambda_k|\geq \delta_{N+k}} c\,|\lambda_k| = c\,\Big(1 - \sum_{|\lambda_k|<\delta_{N+k}} |\lambda_k|\Big) \\
&\geq c\,\Big(1 - \sum_{k\geq N} \delta_k\Big) = c\,(1-\delta_0) := c_0\,.
\end{aligned}$$

Since $c_0 < 1$, this implies, by convexity, that

$$\Big\|\sum_{k\geq 1} \lambda_k u_k\Big\|_{L^\Psi(\mathbb{T})} \geq c_0\,.$$

Hence $\{u_k\}_k$ is equivalent to the canonical basis of ℓ_1, and that achieves the proof of Theorem 4.1. □

Remarks. 1) It follows from Theorem 3.3 that, for this Ψ, J_Ψ is not p-summing for $p < 4$. By modifying the definition of Ψ (taking $\Psi(x_n) = x_n^{r/2}$ and $\Psi(2x_n) = x_n^r$), we get, for every $4 \leq r < \infty$, an Orlicz function Ψ such that J_Ψ is Dunford-Pettis and weakly compact, without being p-summing for $p < r$, and without being compact. We do not know whether it is possible to have J_Ψ p-summing for no finite p.

2) Let us point out that the fact that J_Ψ is Dunford-Pettis does not trivially follows from its weak compactness: H^Ψ does not have the Dunford-Pettis property. In fact, if it were the case, the weakly compact injection $H^\Psi \hookrightarrow H^2$ would be Dunford-Pettis, and hence also $H^4 \hookrightarrow H^2$ (since $H^4 \hookrightarrow H^\Psi \hookrightarrow H^2$). But it is not the case: the sequence $\{z^n\}_n$ converges weakly to 0 in H^4, whereas it does not converges in norm to 0 in H^2.

Proposition 4.4 *There is an Orlicz function Ψ for which J_Ψ is weakly compact, but not Dunford-Pettis.*

Proof. Let us call Ψ_0 the Orlicz function constructed in Theorem 4.1, and let $\Psi(x) = \Psi_0(x^2)$. Then, with $\beta = 2$, $\Psi(\beta x) = \Psi_0(4x^2) \geq 4\Psi_0(x^2) = (2\beta)\Psi(x)$; that means that the conjugate function of Ψ satisfies Δ_2.

J_Ψ is weakly compact (since J_Ψ factors as $H^\Psi \hookrightarrow H^4 \hookrightarrow \mathfrak{B}^8 \hookrightarrow \mathfrak{B}^\Psi$), but is not compact, since $[\Psi(\sqrt{x_n})]^2 = \Psi(\sqrt{2}\sqrt{x_n})$. Since the conjugate function satisfies Δ_2, J_Ψ is not Dunford-Pettis, by Proposition 3.1. □

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