Infinitesimal Carleson property for weighted measures induced by analytic self-maps of the unit disk

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Abstract. *We prove that, for every $\alpha > -1$, the pull-back measure $\varphi(\mathcal{A}_\alpha)$ of the measure $d\mathcal{A}_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, d\mathcal{A}(z)$, where $\mathcal{A}$ is the normalized area measure on the unit disk $\mathbb{D}$, by every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is not only an $(\alpha + 2)$-Carleson measure, but that the measure of the Carleson windows of size εh is controlled by $\varepsilon^{\alpha+2}$ times the measure of the corresponding window of size h. This means that the property of being an $(\alpha + 2)$-Carleson measure is true at all infinitesimal scales. We give an application by characterizing the compactness of composition operators on weighted Bergman-Orlicz spaces.*

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1 Introduction and notation

It is well-known that every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator $f \mapsto C_\varphi(f) = f \circ \varphi$ from the Bergman space $\mathfrak{B}^2$ into itself. By Hastings's version of the Carleson inclusion theorem ([4]), that means that the pull-back measure $\mathcal{A}_\varphi$ of the normalized area measure $\mathcal{A}$ by φ is a 2-Carleson measure, that is, for some constant $C > 0$,

$$\mathcal{A}\big(\{z \in \mathbb{D}\,;\ \varphi(z) \in W(\xi, \varepsilon)\}\big) \leq C\,\varepsilon^2$$

for every $\varepsilon \in (0, 1)$ and every $\xi \in \mathbb{T}$, where $W(\xi, \varepsilon)$ is the Carleson window centered at ξ and of size ε. It was proved in [6], Theorem 3.1, that one actually has an infinitesimal version of this property, namely, for some constant $C > 0$:

$$\mathcal{A}\big(\{z \in \mathbb{D}\,;\ \varphi(z) \in W(\xi, \varepsilon h)\}\big) \leq C\,\mathcal{A}\big(\{z \in \mathbb{D}\,;\ \varphi(z) \in W(\xi, h)\}\big)\,\varepsilon^2\,, \tag{1.1}$$

for every $\varepsilon \in (0,1)$ and $h > 0$ small enough.

Now, consider, for $\alpha > -1$, the *weighted Bergman space* $\mathfrak{B}_\alpha^2$. By Littlewood's subordination principle, every analytic self-map φ of $\mathbb{D}$ induces a bounded composition operator C_φ from $\mathfrak{B}_\alpha^2$ into itself (see [8], Proposition 3.4). By Stegenga's version of the Carleson theorem ([9], Theorem 1.2), that means that the pull-back measure of $\mathcal{A}_\alpha$ (see (1.3) below) by φ is an $(\alpha+2)$-Carleson measure. Our goal in this paper is to show the analog of (1.1) in the following form.

Theorem 1.1 *For each $\alpha > -1$, there exists a constant $C_\alpha > 0$ such that, for every analytic self-map of the unit disk $\varphi\colon \mathbb{D} \to \mathbb{D}$, every $\varepsilon \in (0,1)$ and every $h > 0$ small enough, one has, for every $\xi \in \mathbb{T}$:*

$$\begin{aligned} &\mathcal{A}_\alpha\big(\{z \in \mathbb{D}\,;\ \varphi(z) \in W(\xi, \varepsilon h)\}\big) \\ &\qquad\qquad \leq C_\alpha\,\varepsilon^{\alpha+2}\,\mathcal{A}_\alpha\big(\{z \in \mathbb{D}\,;\ \varphi(z) \in W(\xi, h)\}\big)\,. \end{aligned} \tag{1.2}$$

It should be stressed that the heart of the proof given in [6] in the case $\alpha = 0$ cannot be directly used for the other $\alpha > -1$, and we have to change it, justifying the current paper. Moreover, the present proof is simpler than that of [6]. We also pointed out that the result holds in the limiting case $\alpha = -1$, corresponding to the Hardy space H^2 ([5], Theorem 4.19), but the proof is different, due to the fact that one uses the normalized Lebesgue measure on $\mathbb{T}$ and the boundary values of φ instead of measures on $\mathbb{D}$ and the function φ itself.

We end the paper by an application to the compactness of composition operators on weighted Bergman-Orlicz spaces.

Another application of Theorem 1.1 is given in [7].

Notation. In this paper, $\mathbb{D} = \{z \in \mathbb{C}\,;\ |z| < 1\}$ denotes the open unit disk of the complex plane $\mathbb{C}$, and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. The normalized area measure $\frac{dx\,dy}{\pi}$ is denoted by $\mathcal{A}$.

For $\alpha > -1$, the *weighted Bergman space* $\mathfrak{B}_\alpha^2$ is the space of all analytic functions $f(z) = \sum_{n=0}^\infty a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|_\alpha^2 := \int_{\mathbb{D}} |f(z)|^2\,d\mathcal{A}_\alpha(z) < +\infty\,,$$

where $\mathcal{A}_\alpha$ is the weighted measure

$$d\mathcal{A}_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha d\mathcal{A}(z)\,. \tag{1.3}$$

The *Carleson window* centered at $\xi \in \mathbb{T}$ and of size h, $0 < h < 1$, is the set

$$W(\xi, h) = \{z \in \mathbb{D}\,;\ |z| \geq 1 - h \text{ and } |\arg(z\bar{\xi})| \leq h\}\,.$$

A measure μ on $\mathbb{D}$ is called an *α-Carleson measure* ($\alpha \geq -1$) if

$$\sup_{|\xi|=1} \mu[W(\xi,h)] = O_{h\to 0}\,(h^\alpha).$$

Actually, instead of the Carleson window $W(\xi,h)$, we shall merely use the sets

$$S(\xi,h) = \{z \in \mathbb{D}\,;\ |z-\xi| \leq h\}\,,$$

which have essentially the same size, so μ is an *α-Carleson measure* if and only if $\sup_{|\xi|=1} \mu[S(\xi,h)] = O_{h\to 0}\,(h^\alpha)$.

We denote by Π^+ the right-half plane

$$\Pi^+ = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0\}\,. \tag{1.4}$$

To avoid any misunderstanding, we denote by A the area measure on Π^+, and *not* this measure divided by π.

Let $T\colon \mathbb{D} \to \Pi^+$ be the conformal map defined by:

$$T(z) = \frac{1-z}{1+z}\,; \tag{1.5}$$

we denote by $\tau_\alpha = T(\mathcal{A}_\alpha)$ the pull-back measure defined by:

$$\tau_\alpha(B) = \mathcal{A}_\alpha[T^{-1}(B)] \tag{1.6}$$

for every Borel set B of Π^+. This is a probability measure on Π^+.

We also need another measure μ_α on Π^+, defined by:

$$d\mu_\alpha = x^\alpha\, dxdy\,. \tag{1.7}$$

Given two measures μ and ν, we shall write $\mu \sim \nu$ when the Radon-Nikodým derivative $\frac{d\mu}{d\nu}$ is bounded from above and from below.

The *pseudo-hyperbolic distance* ρ' on $\mathbb{D}$ is given by

$$\rho'(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|, \qquad z,w \in \mathbb{D}\,. \tag{1.8}$$

For every $z \in \mathbb{D}$ and $r \in (0,1)$,

$$\Delta'(z,r) = \{w \in \mathbb{D}\,;\ \rho'(w,z) < r\}$$

is called the *pseudo-hyperbolic disk* with center z and radius r. It is (see [1], [3], or [10], for example) the image of the Euclidean disk $D(0,r)$ by the automorphism

$$\varphi_z(\zeta) = \frac{z-\zeta}{1-\bar{z}\zeta}\,\cdot$$

The pseudo-hyperbolic distance ρ on Π^+ is deduced by transferring the pseudo-hyperbolic distance ρ' on $\mathbb{D}$ with the conformal map T:

$$\rho(a,b) = \rho'(T^{-1}a, T^{-1}b) = \left| \frac{a-b}{\bar{a}+b} \right|, \tag{1.9}$$

and, for every $w \in \Pi^+$ and $r \in (0,1)$,

$$\Delta(w,r) = \{z \in \Pi^+ \,;\ \rho(z,w) < r\}$$

is the *pseudo-hyperbolic disk* of Π^+ with center w and radius r.

Finally, we shall use the following notation:

$$\Omega = (0,2) \times (-1,1)\,. \tag{1.10}$$

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2 Transfer to the right half plane

As in [6], we only have to give the proof for $\xi = 1$ and, by considering $g = h/(1-\varphi)$, we are boiled down to prove:

Theorem 2.1 *Let $\alpha > -1$. There exist constants $K_0 > 0$, $c_0 > 0$ and $\lambda_0 > 1$ such that every analytic function $g\colon \mathbb{D} \to \Pi^+$ with $|g(0)| \leq c_0$ satisfies, for every $\lambda \geq \lambda_0$:*

$$\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \frac{K_0}{\lambda^{\alpha+2}}\, \mathcal{A}_\alpha(\{|g| > 1\})\,.$$

As said in the Introduction, this result is an infinitesimal version of the fact that the pull-back measure $\mathcal{A}_{\alpha,\varphi}$ of $\mathcal{A}_\alpha$ by any analytic self-map φ of $\mathbb{D}$ is an $(\alpha+2)$-Carleson measure. In fact, one has the following result.

Proposition 2.2 *There is some constant $C = C_\alpha > 0$ such that*

$$\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \frac{C}{\lambda^{\alpha+2}}\, |g(0)|^{\alpha+2} \tag{2.1}$$

for every analytic function $g\colon \mathbb{D} \to \Pi^+$ and every $\lambda > 0$.

The goal is hence to replace in the right-hand side the quantity $|g(0)|^{\alpha+2}$ by $\mathcal{A}_\alpha(\{|g| > 1\})$.

Proof of Proposition 2.2. We may assume that $|g(0)| = 1$. Hence we may assume that $\lambda > 2$, taking $C \geq 2^{\alpha+2}$, because $\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq 1$.

Set $\varphi(z) = [g(z) - g(0)]/[g(z) + \overline{g(0)}]$. Then $|g(z)| > \lambda$ implies that

$$|\varphi(z) - 1| = 2\,|\mathfrak{Re}\, g(0)|/|g(z) + \overline{g(0)}| \leq 2/(\lambda - 1) \leq 4/\lambda\,.$$

But φ maps $\mathbb{D}$ into itself, so the measure $\mathcal{A}_{\alpha,\varphi}$ is an $(\alpha+2)$-Carleson measure and (see the proof of [9], Theorem 1.2)

$$\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \mathcal{A}_{\alpha,\varphi}[S(1, 4/\lambda)] \leq C'_\alpha\, \|C_\varphi\|^2/(\lambda/4)^{\alpha+2}\,,$$

where $\|C_\varphi\|$ is the norm of the composition operator $C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$. But $\varphi(0) = 0$ and hence $\|C_\varphi\| = 1$, by using Littlewood's subordination principle and integrating. $\square$

For technical reasons, that we shall explain after Lemma 3.4, we need to work with functions defined on Π^+. Proposition 2.2 becomes:

Proposition 2.3 *There exists a constant $C = C_\alpha > 0$ such that, for every analytic function $f \colon \Pi^+ \to \Pi^+$, one has:*

$$\tau_\alpha(\{|f| > \lambda\}) \leq \frac{C}{\lambda^{\alpha+2}} |f(1)|^{\alpha+2}. \tag{2.2}$$

Proof. Set $E_f(\lambda) = \{|f| > \lambda\}$ and define similarly $E_g(\lambda) = \{|g| > \lambda\}$ where $g = f \circ T \colon \mathbb{D} \to \Pi^+$. We have $g(0) = f(1)$ as well as the simple but useful equation:

$$T^{-1}[E_f(\lambda)] = E_g(\lambda). \tag{2.3}$$

So that, by Proposition 2.2:

$$\begin{aligned}\tau_\alpha[E_f(\lambda)] &= \mathcal{A}_\alpha[T^{-1}(E_f(\lambda)] = \mathcal{A}_\alpha[E_g(\lambda)] \\ &\leq \frac{C}{\lambda^{\alpha+2}} |g(0)|^{\alpha+2} = \frac{C}{\lambda^{\alpha+2}} |f(1)|^{\alpha+2}\,,\end{aligned}$$

and Proposition 2.3 is proved. $\square$

Now, to prove Theorem 2.1, it suffices to prove that, when one localizes f on Ω, one may replace the quantity $|f(1)|$ in the right-hand side of (2.2) by $\tau_\alpha(\{|f| > 1\} \cap \Omega)$. This is what is claimed in the next result.

Theorem 2.4 *There exist constants $K = K_\alpha > 0$, $c_1 > 0$ and $\lambda_1 > 1$ such that every analytic function $f \colon \Pi^+ \to \Pi^+$ such that $|f(1)| \leq c_1$ satisfies, for every $\lambda \geq \lambda_1$:*

$$\tau_\alpha(\{|f| > \lambda\} \cap \Omega) \leq \frac{K}{\lambda^{\alpha+2}}\, \tau_\alpha(\{|f| > 1\} \cap \Omega).$$

We shall prove Theorem 2.4 in the next section, but before, let us see why it gives Theorem 2.1 and hence our main result, Theorem 1.1.

Proof of Theorem 2.1. Let $E\colon \Pi^+ \to \mathbb{D}$ be the exponential map defined by

$$E(z) = \mathrm{e}^{-\pi z}\,, \tag{2.4}$$

which (up to a radius) maps bijectively Ω onto the annulus

$$U = \{z \in \mathbb{D};\ |z| > \mathrm{e}^{-2\pi}\}. \tag{2.5}$$

For every $g\colon \mathbb{D} \to \Pi^+$ with $|g(0)| \leq (1-\beta)/(1+\beta)$ and $0 < \beta < 1$, one has, by Schwarz's lemma (see [6], eq. (3.9)):

$$|g(z)| > 1 \qquad \Longrightarrow \qquad |z| > \beta\,.$$

Therefore we only have to work on the annulus U, taking $c_0 \leq \tanh \pi$ in Theorem 2.1.

Let $L = E^{-1}$ be the inverse map of the restriction of E to Ω, and

$$\sigma_\alpha = L(\mathcal{A}_\alpha) \tag{2.6}$$

be the pull-back measure of $\mathcal{A}_\alpha$ by L. This measure is carried by Ω and we have:

Lemma 2.5 *On Ω, one has: $\sigma_\alpha \sim \mu_\alpha \sim \tau_\alpha$.*

Taking this lemma for granted for a while, let us finish the proof of Theorem 2.1 (the measure μ_α does not come into play here). Let $g\colon \mathbb{D} \to \Pi^+$ be an analytic function and $f = g \circ E\colon \Pi^+ \to \Pi^+$ (so that $g = f \circ L$ on $E(\Omega)$). We have $|f(1)| \leq c_1$ if $|g(0)| \leq c_0$, with $c_0 > 0$ small enough. In fact, the analytic function $h = T \circ g$ maps $\mathbb{D}$ into itself and hence, by the Schwarz-Pick inequality, h is a contraction for the pseudo-hyperbolic distance on $\mathbb{D}$ (see [1], eq. (3.3), page 18, for example); hence $\rho'[h(\mathrm{e}^{-\pi}), h(0)] \leq \rho'(\mathrm{e}^{-\pi}, 0) = \mathrm{e}^{-\pi}$, that is $\left|\frac{g(\mathrm{e}^{-\pi}) - g(0)}{g(\mathrm{e}^{-\pi}) + \overline{g(0)}}\right| \leq \mathrm{e}^{-\pi}$. It follows that $|g(\mathrm{e}^{-\pi})| - |g(0)| \leq \mathrm{e}^{-\pi}\big[|g(\mathrm{e}^{-\pi})| + |g(0)|\big]$, *i.e.* $|g(\mathrm{e}^{-\pi})| \leq \frac{1}{\tanh \pi}\,|g(0)|$. Therefore $|f(1)| = |g(\mathrm{e}^{-\pi})| \leq c_1$ if $|g(0)| \leq c_0$ with $c_0 \leq c_1 \tanh \pi$.

Set:

$$E_g(\lambda) = \{|g| > \lambda\} \cap U \qquad \text{and} \qquad E_f(\lambda) = \{|f| > \lambda\} \cap \Omega.$$

Observe that, as in (2.3),

$$L^{-1}[E_f(\lambda)] = E_g(\lambda) \qquad \text{and} \qquad E^{-1}[E_g(1)] = E_f(1).$$

Hence, in view of Theorem 2.4 and Lemma 2.5:

$$\begin{aligned}
\mathcal{A}_\alpha[E_g(\lambda)] &= \mathcal{A}_\alpha\big(L^{-1}[E_f(\lambda)]\big) = \sigma_\alpha[E_f(\lambda)] \\
&\leq \frac{K'_\alpha}{\lambda^{\alpha+2}}\,\sigma_\alpha[E_f(1)] = \frac{K'_\alpha}{\lambda^{\alpha+2}}\,\sigma_\alpha\big(E^{-1}[E_g(1)]\big) \\
&= \frac{K'_\alpha}{\lambda^{\alpha+2}}\,(E\sigma_\alpha)[E_g(1)] = \frac{K'_\alpha}{\lambda^{\alpha+2}}\,\mathcal{A}_\alpha[E_g(1)]\,,
\end{aligned}$$

which is exactly what we wanted to prove. $\square$

Proof of Lemma 2.5. Let us compute σ_α with the change of variable $w = E^{-1}(z)$. One has $z = E(w)$ and

$$d\mathcal{A}(z) = |E'(w)|^2 \, \frac{dA(w)}{\pi} = \frac{1}{\pi} \mathrm{e}^{-2\pi \Re\mathrm{e}\, w} \, dA(w).$$

We get:

$$\begin{aligned} \int_\Omega h(w)\, d\sigma_\alpha(w) &= \int_U h(Lz)\, d\mathcal{A}_\alpha(z) = (\alpha+1) \int_U h(E^{-1}z)(1-|z|^2)^\alpha d\mathcal{A}(z) \\ &= \frac{\alpha+1}{\pi} \int_\Omega h(w)\, \mathrm{e}^{-2\pi\Re\mathrm{e}\, w}(1-\mathrm{e}^{-2\pi\Re\mathrm{e}\, w})^\alpha \, dA(w), \end{aligned}$$

so that

$$d\sigma_\alpha(w) = \frac{\alpha+1}{\pi} \, \mathrm{e}^{-2\pi\Re\mathrm{e}\, w}(1-\mathrm{e}^{-2\pi\Re\mathrm{e}\, w})^\alpha \; \mathbb{1}_\Omega(w)\, dA(w). \tag{2.7}$$

Thus, on Ω, we have $\sigma_\alpha \sim \mu_\alpha$. Indeed, the factor $\mathrm{e}^{-2\Re\mathrm{e}\, w}$ is bounded from below and from above, and $(1-\mathrm{e}^{-2\Re\mathrm{e}\, w})^\alpha \sim (\Re\mathrm{e}\, w)^\alpha$ as $\Re\mathrm{e}\, w$ goes to 0. This proves the first equivalence of Lemma 2.5.

To prove the second equivalence, we use the change of variable formula $z = Tw$ in

$$\int_\Omega h(u)\, d\tau_\alpha(u) = \int_U h(Tz)\, d\mathcal{A}_\alpha(z);$$

it gives $d\tau_\alpha(w) = |T'(w)|^2(1-|T(w)|^2)^\alpha(\alpha+1)\, dA(w)/\pi$, i.e.:

$$d\tau_\alpha(w) = \frac{4^{\alpha+1}(\alpha+1)}{\pi} \, \frac{(\Re\mathrm{e}\, w)^\alpha}{|1+w|^{2(\alpha+2)}} \; \mathbb{1}_\Omega(w)\, dA(w), \tag{2.8}$$

showing that $\mu_\alpha \sim \tau_\alpha$ on Ω. □

3 Proof of Theorem 2.4

Let us split, up to a set of measure 0, the square Ω into dyadic sub-squares

$$Q_l = \left(\frac{2j}{2^n}, \frac{2(j+1)}{2^n}\right) \times \left(\frac{2k}{2^n} - 1, \frac{2(k+1)}{2^n} - 1\right) \tag{3.1}$$

of center

$$c_l = \frac{2j+1}{2^n} + i\left(\frac{2k+1}{2^n} - 1\right), \tag{3.2}$$

with $n \geq 0$, $0 \leq j, k \leq 2^n - 1$ and where $l = (n, j, k)$.

Note that $\Omega = Q_{(0,0,0)}$. We are going to use the special form of the measure τ_α, taken in (2.8), to get a localized version of Proposition 2.3 as follows.

Proposition 3.1 *There is a constant $C_\alpha > 0$ such that, for any analytic function $f \colon \Pi^+ \to \Pi^+$ and any dyadic sub-square Q_l of Ω, one has, for any $\lambda > 0$:*

$$\tau_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}}\, \tau_\alpha(Q_l)\, |f(c_l)|^{\alpha+2}. \tag{3.3}$$

Proof. Using Lemma 2.5, we may replace the measure τ_α by $d\mu_\alpha = x^\alpha\, dxdy$. This measure is no longer a probability measure, but it has the advantage of being invariant under vertical translations, and, especially, to react to a dilation of positive ratio λ by multiplying the result by the factor $\lambda^{\alpha+2}$.

We first need a simple lemma.

Lemma 3.2 *For every $0 \leq s < 1$, there exists a constant $M_s > 0$ such that, for any analytic function $f \colon \Pi^+ \to \Pi^+$ and any pseudo-hyperbolic disk $\Delta(c,s)$ in Π^+, we have, for every $z \in \Delta(c,s)$:*

$$1/M_s \leq |f(z)|/|f(c)| \leq M_s. \tag{3.4}$$

Proof. By the classical Schwarz-Pick inequality, any analytic map $f \colon \Pi^+ \to \Pi^+$ contracts the pseudo-hyperbolic distance ρ of Π^+ (see [1], Section 6), so that if $z \in \Delta(c,s)$, one has:

$$|u| := \left|\frac{f(z) - f(c)}{f(z) + \overline{f(c)}}\right| \leq \left|\frac{z-c}{z+\overline{c}}\right| \leq s.$$

Inverting that relation, we get $f(z) = \frac{u\overline{f(c)} + f(c)}{1-u}$, whence

$$|f(z)| \leq |f(c)|\, \frac{1+|u|}{1-|u|} \leq |f(c)|\, \frac{1+s}{1-s}$$

and, similarly, $|f(z)| \geq |f(c)|\, \frac{1-s}{1+s}$. The lemma follows, with $M_s = \frac{1+s}{1-s}\cdot$ □

Let us now continue the proof of Proposition 3.1.

Lemma 3.3 *Inequality* (3.3) *holds when the square Q_l, of the n-th generation, does not touch the boundary of Π^+, namely when $l = (n,j,k)$ with $j \geq 1$. More precisely, we have $Q_l \subseteq \Delta(c_l, s)$ where $s < 1$ is a numerical constant.*

Proof. Recall that c_l is the center of Q_l. We claim that we can find some numerical $s < 1$ such that $Q_l \subset \Delta(c_l, s)$. To show that claim, let $l = (n,j,k)$ and $z, w \in Q_l$. We have:

$$1 - \rho(z,w)^2 = 1 - \left|\frac{z-w}{z+\bar{w}}\right|^2 = 4\, \frac{\mathfrak{Re}\, z\, \mathfrak{Re}\, w}{|z+\bar{w}|^2}\cdot$$

But one has $2j/2^n \leq \mathfrak{Re}\, z, \mathfrak{Re}\, w \leq 2(j+1)/2^n$ whereas $|\mathfrak{Im}\,(z+\bar{w})| \leq 2^{-n+1}$; hence $\mathfrak{Re}\, z\, \mathfrak{Re}\, w \geq 4j^2 4^{-n}$ and $|z+\bar{w}|^2 = (\mathfrak{Re}\, z + \mathfrak{Re}\, w)^2 + [\mathfrak{Im}\,(z+\bar{w})]^2 \leq 16(j+1)^2 4^{-n} + 4.4^{-n} \leq 80 j^2 4^{-n}$, because $j \geq 1$. Therefore

$$1 - \rho(z,w)^2 \geq 4\, \frac{4j^2 4^{-n}}{80 j^2 4^{-n}} = \frac{1}{5}\,,$$

so that $\rho(z,w) \leq s = \sqrt{4/5}$. In particular, we have $Q_l \subseteq \Delta(c_l, s)$.

Now, to prove (3.3), we may assume, by homogeneity (replace f by $f/|f(c_l)|$ and λ by $\lambda/|f(c_l)|$), that $|f(c_l)| = 1$. We then have, by Lemma 3.2, $|f(z)| \leq M_s|f(c_l)| = M_s$ for every $z \in Q_l$. Hence (3.3) trivially holds when $\lambda > M_s$, since then the set in the left-hand side is empty. So we assume $\lambda \leq M_s$. In that case, setting $C_\alpha = M_s^{\alpha+2}$, we have :

$$\tau_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \tau_\alpha(Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}}\, \tau_\alpha(Q_l).$$

This is the desired inequality, since we have supposed that $|f(c_l)| = 1$. $\square$

Lemma 3.4 *Inequality* (3.3) *holds when the square Q_l, of the n-th generation, touches the boundary of Π^+, namely when $l = (n, j, k)$ with $j = 0$.*

Proof. This case uses the specific properties of the measure μ_α. In view of Lemma 2.5, we have to prove that:

$$\mu_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}}\, \mu_\alpha(Q_l)\, |f(c_l)|^{\alpha+2}, \tag{3.5}$$

when the square $Q_l \subseteq \Omega$ is supported by the imaginary axis. We may again assume that $|f(c_l)| = 1$, and we proceed in three steps.

1) First, (3.5) holds if $Q_l = Q_{(0,0,0)} = \Omega$: this is just what we have proved in Proposition 2.3 with (2.2).

2) For $h > 0$, (3.5) holds when $Q_l = h\Omega = (0, 2h) \times (-h, h)$ is a square meeting the imaginary axis in an interval $(-h, h)$ centered at 0. Indeed, setting $E_f(\lambda) = \{|f| > \lambda\}$ as well as $f_h(z) = f(hz)$, we easily check that

$$E_f(\lambda) \cap h\,\Omega = h\,[E_{f_h}(\lambda) \cap \Omega]. \tag{3.6}$$

For example, if $v \in E_{f_h}(\lambda) \cap \Omega$, one has $|f(hv)| > \lambda$ and hence $w = hv \in E_f(\lambda) \cap h\Omega$, giving one inclusion in (3.6); the other is proved similarly. Using the already mentioned $(\alpha+2)$-homogeneity of the measure μ_α, we obtain, using (2.2) for f_h:

$$\begin{aligned} \mu_\alpha[E_f(\lambda) \cap h\Omega] = \mu_\alpha[h\big(E_{f_h}(\lambda) \cap \Omega\big)] &= h^{\alpha+2}\mu_\alpha[E_{f_h}(\lambda) \cap \Omega] \\ &\leq h^{\alpha+2}\frac{C_\alpha}{\lambda^{\alpha+2}}\, |f_h(1)|^{\alpha+2} = \mu_\alpha(Q_l)\, \frac{C'_\alpha}{\lambda^{\alpha+2}}\, |f(c_l)|^{\alpha+2}, \end{aligned}$$

with $C'_\alpha = 4^{-(\alpha+2)}(\alpha+1)C_\alpha$, since the center c_l of $Q_l = h\Omega$ is $c_l = h$.

3) Finally, (3.5) holds if Q_l is any square supported by the imaginary axis. Indeed, this Q_l is a vertical translate of the second case, and the measure μ_α is invariant under vertical translations, which exchange centers.

This ends the proof of the crucial Lemma 3.4 and thereby that of Proposition 3.1. $\square$

Remark. We see here why it is better to work with functions $f\colon \Pi^+ \to \Pi^+$ instead of functions $g\colon \mathbb{D} \to \Pi^+$; if the invariance of μ_α under vertical translations corresponds to the rotation invariance of $\mathcal{A}_\alpha$, the homogeneity of μ_α, used in part 2) of the proof, corresponds to an invariance by the automorphisms φ_a of $\mathbb{D}$, with real $a \in \mathbb{D}$, which is not shared by $\mathcal{A}_\alpha$, and writing a measure equivalent to $\mathcal{A}_\alpha$ having these properties is not so simple.

In order to exploit this proposition, we need the following precisions.

Lemma 3.5 *There exist constants $c > 0$ and $\delta_0 > 0$, depending only on α, such that for every l, there exists $R_l \subseteq Q_l$ with $\tau_\alpha(R_l) \geq c\,\tau_\alpha(Q_l)$ and, for every analytic map $f\colon \Pi^+ \to \Pi^+$,*

$$|f(z)| > \delta_0\,|f(c_l)| \quad \textit{for every } z \in R_l. \tag{3.7}$$

Proof. By Lemma 2.5, it suffices to prove this lemma with μ_α instead of τ_α. Let us consider two cases.

1) If $l = (n, j, k)$ with $j \geq 1$, we can simply take $R_l = Q_l$, in view of Lemma 3.2 and Lemma 3.3.

2) If $l = (n, j, k)$ with $j = 0$, we may assume that $Q_l = \Omega = (0, 2) \times (-1, 1)$, since either vertical translations or dilations of positive ratio are isometries for the pseudo-hyperbolic distance on Π^+ and, on the other hand, multiply the μ_α-measure by 1 or $h^{\alpha+2}$ respectively. It follows that $c_l = 1$. We are going to check that $\Delta(1, 1/4) \subseteq \Omega = Q_l$, so that we can take $R_l = \Delta(c_l, 1/4)$. Indeed, set $t = 1/4$; if $|u| := \left|\frac{z-1}{z+1}\right| \leq t$, we have $z = \frac{1+u}{1-u}$ and

$$0 < \mathfrak{Re}\, z = \frac{1 - |u|^2}{|1 - u|^2} \leq \frac{1 + t}{1 - t} < 2\,;$$
$$|\mathfrak{Im}\, z| = \frac{2\,|\mathfrak{Im}\, u|}{|1 - u|^2} \leq \frac{2t}{(1 - t)^2} = \frac{8}{9} < 1.$$

Moreover, in view of Lemma 3.2, (3.7) holds with $\delta_0 = M_t^{-1} = 3/5$.

Finally, the claim on the measures holds with $c = \mu_\alpha[\Delta(1, 1/4)]/\mu_\alpha(\Omega)$. $\square$

Now, we want to control mean values of f on some of the Q_l's. In order to get that, we have to do a Calderón-Zygmund decomposition.

To that end, we need to know that the mean of $|f|$ on Ω is small, namely less than 1, if $|f(1)|$ is small enough. This is the aim of the next proposition.

Proposition 3.6 *There exists a constant $C > 0$ such that, for every analytic function $f\colon \Pi^+ \to \Pi^+$, one has:*

$$|f(1)| \leq \iint_\Omega |f(x + iy)|\,\frac{dxdy}{\pi} \leq C\,|f(1)|\,. \tag{3.8}$$

Moreover, if c is the center of an open square Q contained in Π^+, then:

$$\frac{\pi}{4}\,|f(c)| \leq \frac{1}{A(Q)}\int_Q |f(z)|\,dA(z) \leq C\,\frac{\pi}{4}\,|f(c)|\,. \tag{3.9}$$

Proof. Let us see first that (3.9) follows from (3.8). Let $c = a + ib$ ($a > 0$ and $b \in \mathbb{R}$) be the center of the square $Q = (a-h, a+h) \times (b-h, b+h)$, with $0 < h \le a$. Consider the function f_1 defined by:

$$f_1(z) = f[\phi(z)]\,, \quad \text{where} \quad \phi(z) = hz - h + a + ib\,.$$

Observe that $\phi\colon \Pi^+ \to \Pi^+$ is an affine transformation sending 1 onto c and that $\phi(\Omega) = Q$. Applying (3.8) to f_1 gives:

$$\frac{\pi}{4}\,|f_1(1)| \le \frac{1}{A(\Omega)} \int_\Omega |f_1(z)|\, dA(z) \le C\,\frac{\pi}{4}\,|f_1(1)|\,.$$

This yields (3.9) using an obvious change of variable and $f_1(1) = f(c)$.

The left-hand side inequality in (3.8) is due to subharmonicity: consider the open disk D of center 1 and radius 1; then $D \subseteq \Omega$ and, $|f|$ being subharmonic, we have:

$$|f(1)| \le \frac{1}{\pi} \iint_D |f(x+iy)|\, dxdy \le \frac{1}{\pi} \iint_\Omega |f(x+iy)|\, dxdy\,.$$

We now prove the right-hand side inequality. Using Lemma 2.3 and the fact that $\mu_0 \sim \tau_0$ on Ω (note that μ_0 is just the area measure A on Π^+), we have the existence of a constant $\kappa > 0$ such that, for all $\lambda > 0$:

$$\mu_0(\{|f| > \lambda\} \cap \Omega) \le \frac{\kappa}{\lambda^2}\,|f(1)|^2\,. \tag{3.10}$$

From this estimate (3.10), we can control the integral of $|f|$ over Ω (recall that $\mu_0(\Omega) = 4$):

$$\begin{aligned} \int_\Omega |f|\, d\mu_0 &= \int_0^{+\infty} \mu_0(\{|f| > \lambda\} \cap \Omega)\, d\lambda \\ &\le 4\,|f(1)| + \int_{|f(1)|}^{+\infty} \frac{\kappa\,|f(1)|^2}{\lambda^2}\, d\lambda = (4+\kappa)\,|f(1)|. \end{aligned}$$

The proposition follows. □

Remark. We do not know if the constant $\pi/4$ in the left-hand side of (3.9) can be replaced by a better constant; however, it is not possible to replace this factor $\pi/4$ by 1. Let us see an example.

Let us define $f(z) = \exp\big((Tz)^4)\big)$ where $Tz = (1-z)/(1+z)$. Recall that T sends Π^+ to the unit disk $\mathbb{D}$, and therefore $f(z) \in \Pi^+$, for every $z \in \Pi^+$ because $|\arg(\exp w)| < 1 < \pi/2$, for all $w \in \mathbb{D}$.

Now let Q be the unit square $Q = (-1,1) \times (-1,1)$. For $0 < t \le 1/2$, let Q_t be the square, centered in 1, $Q_t = (1-t, 1+t) \times (-t, t)$, which is contained in Π^+, and define

$$\sigma(t) = \frac{1}{A(Q_t)} \int_{Q_t} |f(z)|\, dA(z) = \frac{1}{4t^2} \int_{-t}^{t} \left[\int_{1-t}^{1+t} |f(x+iy)|\, dx \right] dy\,.$$

Using a change of variable we have:

$$\sigma(t) = \frac{1}{4} \iint_Q |f(1 + tx + ity)|\, dx\, dy\,.$$

We are going to prove that there exists t such that $\sigma(t) < 1 = |f(1)|$ and the average of $|f|$ in the cube Q_t is smaller than $|f(1)|$. Now observe that

$$f(z) = \frac{1}{16}(z-1)^4 + O\big((z-1)^5\big), \qquad z \to 1.$$

Consequently, there exists a constant $C > 0$ such that, for $z \in Q_{1/2}$,

$$\Re\mathrm{e}\,\big(f(z)\big) \le \frac{1}{16}\,\Re\mathrm{e}\,\big((z-1)^4\big) + C|z-1|^5$$

and then, there exists $C_1 > 0$, such that for every $x + iy \in Q$ and $t \in (0, 1/2)$,

$$\begin{aligned} |f(1 + tx + ity)| &\le \exp\Big[\frac{1}{16}\,\Re\mathrm{e}\,\big(t^4(x+iy)^4\big) + C_1 t^5\Big] \\ &= \exp\Big[\frac{t^4}{16}\,(x^4 + y^4 - 6x^2y^2) + C_1 t^5\Big]\,. \end{aligned}$$

Integrating over Q, putting:

$$\tau(s) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \exp\big((s/16)(x^4 + y^4 - 6x^2y^2) + C_1 s^{5/4}\big)\, dx\, dy\,,$$

we get that $\sigma(t) \le \tau(t^4)$, for $t \in (0, 1/2]$. We just need to prove that, for $s > 0$ close enough to 0, we have $\tau(s) < 1$. But this is easy because $\tau(0) = 1$, and

$$\tau'(0) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{1}{16}\,(x^4 + y^4 - 6x^2y^2)\, dx\, dy = \frac{1}{64}\Big(\frac{4}{5} + \frac{4}{5} - \frac{8}{3}\Big) = -\frac{1}{60} < 0\,.$$

Return now to the proof of Theorem 2.4.

Consider, for every $n \ge 0$, the conditional expectation of the restriction to Ω of $|f|$ with respect to the algebra $\mathcal{Q}_n$ generated by the squares $Q_{(n,j,k)}$, $0 \le j, k \le 2^n - 1$ (note that $\mathcal{Q}_n \subseteq \mathcal{Q}_{n+1}$):

$$(\mathbb{E}_n|f|)(z) = \sum_{j,k=0}^{2^n-1} \bigg(\frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f|\, dA\bigg)\, \mathbb{1}_{Q_{(n,j,k)}}(z)\,, \tag{3.11}$$

and the maximal function Mf is defined by:

$$Mf(z) = \sup_n\, (\mathbb{E}_n|f|)(z)\,. \tag{3.12}$$

One has

$$M(f)(z) = \sup_{z \in Q_{(n,j,k)}} \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f|\, dA\,. \tag{3.13}$$

Since f is continuous on Ω, one has $\lim_{n\to\infty} \mathbb{E}_n|f|(z) = |f(z)|$ for every $z \in \Omega$, and it follows that:

$$\{|f| > 1\} \subseteq \{Mf > 1\}\,. \tag{3.14}$$

Now, the set $\{Mf > 1\} \cap \Omega$ can be split into a disjoint union

$$\{Mf > 1\} \cap \Omega = \bigsqcup_{n\geq 1} Z_n\,,$$

where

$$Z_n = \{z \in \Omega\,;\; (\mathbb{E}_n|f|)(z) > 1 \text{ and } (\mathbb{E}_j|f|)(z) \leq 1 \text{ if } j < n\}\,.$$

(note that, by Proposition 3.6, $\mathbb{E}_0|f| \leq 1$ if $|f(1)|$ is small enough).

Since $\mathbb{E}_n|f|$ is constant on the sets $Q \in \mathcal{Q}_n$, each Z_n can be in its turn decomposed, up to a set of measure 0, into a disjoint union $E_n = \bigsqcup_{(j,k)\in J_n} Q_{(n,j,k)}$.

By definition, for $z \in Z_n$, one has $(\mathbb{E}_n|f|)(z) \geq 1$ and hence, for $(j,k) \in J_n$,

$$\frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f|\, dA \geq 1 \qquad \text{for } z \in Q_{(n,j,k)}\,.$$

But, on the other hand, $(\mathbb{E}_{n-1}|f|)(z) \leq 1$ for $z \in Z_n$, and we have, if $z \in Q_{(n,j,k)}$:

$$\begin{aligned}(\mathbb{E}_n|f|)(z) &= \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f|\, dA \leq \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n-1,j',k')}} |f|\, dA \\ &\leq 4\, \frac{1}{A(Q_{(n-1,j',k')})} \int_{Q_{(n-1,j',k')}} |f|\, dA \leq 4\,,\end{aligned}$$

where $Q_{(n-1,j',k')}$ is the square of rank $(n-1)$ containing $Q_{(n,j,k)}$.

Finally, we can write $\{Mf > 1\} \cap \Omega$ as a disjoint union, up to a set of measure 0,

$$\{Mf > 1\} \cap \Omega = \bigsqcup_{l\in L} Q_l\,, \tag{3.15}$$

where L is a subset of all the indices (n,j,k), for which:

$$1 \leq \frac{1}{A(Q_l)} \int_{Q_l} |f|\, dA \leq 4\,. \tag{3.16}$$

Equations (3.14), (3.15) and (3.16) define the Calderón-Zygmund decomposition of the function f.

We are now ready to end the proof of Theorem 2.4.

For $\lambda \geq 1$, set $E_\lambda = \{|f| > \lambda\}$; one has, by (3.15), Proposition 3.1 and (3.9):

$$\begin{aligned}
\tau_\alpha(E_\lambda \cap \Omega) &= \tau_\alpha(E_\lambda \cap \{Mf > 1\} \cap \Omega) = \sum_{l \in L} \tau_\alpha(E_\lambda \cap Q_l) \\
&\leq \frac{K_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l)\, |f(c_l)|^{\alpha+2} \\
&\leq \frac{K_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) \Big(\frac{16}{\pi}\Big)^{\alpha+2} = \frac{C_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) \\
&= \frac{C_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(\{Mf > 1\} \cap \Omega).
\end{aligned}$$

But, on the other hand, the sets R_l of Lemma 3.5 are disjoint, since $R_l \subseteq Q_l$ and we have $|f| > \delta_0\, |f(c_l)| > (4/\pi C)\, \delta_0 := \delta_1$ on R_l, in view of Lemma 3.5 and Proposition 3.6. Therefore:

$$\begin{aligned}
\tau_\alpha(|f| > \delta_1) &\geq \tau_\alpha\Big(\bigsqcup_l R_l\Big) = \sum_{l \in L} \tau_\alpha(R_l) \geq c \sum_{l \in L} \tau_\alpha(Q_l) = c\, \tau_\alpha\Big(\bigsqcup_{l \in L} Q_l\Big) \\
&= c\, \tau_\alpha(\{Mf > 1\} \cap \Omega).
\end{aligned}$$

We get hence $\tau_\alpha(E_\lambda \cap \Omega) \leq \frac{C'_\alpha}{\lambda^{\alpha+2}}\, \tau_\alpha(\{|f| > \delta_1\})$ for $\lambda \geq 1$, with $C'_\alpha = C_\alpha/c$. Applying this to f/δ_1 instead of f, we get:

$$\tau_\alpha(\{|f| > \lambda\} \cap \Omega) \leq \frac{C''_\alpha}{\lambda^{\alpha+2}}\, \tau_\alpha(\{|f| > 1\})$$

for $\lambda > \lambda_1 := 1/\delta_1$, and that finishes the proof of Theorem 2.4.

4 An application to composition operators

In this section, we give an application of our main result to composition operators on weighted Bergman-Orlicz spaces.

Recall that an Orlicz function $\Psi \colon [0, \infty) \to \mathbb{R}_+$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(x)/x \to \infty$ as x goes to ∞. The *weighted Bergman-Orlicz space* $\mathfrak{B}_\alpha^\Psi$ is the space of all analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ such that

$$\int_{\mathbb{D}} \Psi(|f|/C)\, d\mathcal{A}_\alpha < +\infty$$

for some constant $C > 0$. The norm of f in $\mathfrak{B}_\alpha^\Psi$ is the infimum of the constants C for which the above integral is ≤ 1. With this norm, $\mathfrak{B}_\alpha^\Psi$ is a Banach space.

Now, every analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$ defines a bounded linear operator $C_\varphi \colon \mathfrak{B}_\alpha^\Psi \to \mathfrak{B}_\alpha^\Psi$ by $C_\varphi(f) = f \circ \varphi$, called the *composition operator of symbol* φ. This is a consequence of the classical Littlewood's subordination principle, using the facts that the measure $\mathcal{A}_\alpha$ is radial and the function $\Psi(|f|/C)$ is subharmonic for every analytic function $f \colon \mathbb{D} \to \mathbb{C}$. Such an operator may be seen as a Carleson embedding $J_\mu\colon \mathfrak{B}_\alpha^\Psi \to L^\Psi(\mu)$ for the pull-back measure $\mu = \varphi(\mathcal{A}_\alpha)$. S. Charpentier ([2]), following [6], has characterized the compactness of such embeddings (actually in the more general setting of the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$ instead of the unit disk $\mathbb{D}$ of $\mathbb{C}$):

Theorem 4.1 (S. Charpentier) *For every finite positive measure μ on $\mathbb{D}$ and for every $\alpha > -1$, one has:*

1) If $\mathfrak{B}_\alpha^\Psi$ is compactly contained in $L^\Psi(\mu)$, then

$$\lim_{h\to 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/\rho_\mu(h))} = 0\,. \tag{4.1}$$

2) Conversely, if

$$\lim_{h\to 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/h^{\alpha+2}K_\mu(h))} = 0\,, \tag{4.2}$$

then $\mathfrak{B}_\alpha^\Psi$ is compactly contained in $L^\Psi(\mu)$.

Here ρ_μ is the *Carleson function* of μ, defined as:

$$\rho_\mu(h) = \sup_{|\xi|=1} \mu[W(\xi,h)] \tag{4.3}$$

and

$$K_\mu(h) = \sup_{0<t\leq h} \frac{\rho_\mu(t)}{t^{\alpha+2}}\,. \tag{4.4}$$

When $\mu = \varphi(\mathcal{A}_\alpha)$ is the pull-back measure of $\mathcal{A}_\alpha$ by an analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, we denote them by $\rho_{\varphi,\alpha+2}$ and $K_{\varphi,\alpha+2}$ respectively.

We gave in [6], in the non-weighted case, examples showing that conditions (4.1) and (4.2) are not equivalent for general measures μ. However, Theorem 1.1 implies that $K_{\varphi,\alpha+2}(h) \lesssim \rho_{\varphi,\alpha+2}(h)/h^{\alpha+2}$ and so conditions (4.1) and (4.2) are equivalent in this case. Therefore, we get:

Theorem 4.2 *For every $\alpha > -1$, every Orlicz function Ψ, and every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, the composition operator $C_\varphi\colon \mathfrak{B}_\alpha^\Psi \to \mathfrak{B}_\alpha^\Psi$ is compact if and only if:*

$$\lim_{h\to 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/\rho_{\varphi,\alpha+2}(h))} = 0\,. \tag{4.5}$$

References

[1] A. F. Beardon and D. Minda, The hyperbolic metric and geometric function theory, Quasiconformal mappings and their applications, 9–56, Narosa, New Delhi (2007).

[2] S. Charpentier, Composition operators on weighted Bergman-Orlicz spaces on the ball, *to appear* in Complex Analysis and Operator Theory [DOI: 10.1007/s11785-011-0140-7].

[3] P. Duren and A. Schuster, Bergman Spaces, Math. Surveys and Monographs, No. 100, A.M.S. (2004).

[4] W. H. Hastings, A Carleson measure theorem for Bergman spaces, Proc. Amer. Math. Soc. 52 (1975), 237–241.

[5] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, Memoirs Amer. Math. Soc. 207 (2010) No. 974.

[6] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Compact composition operators on Bergman-Orlicz spaces, *preprint* (2009): hal-00426831 – arXiv : 0910.5368.

[7] D. Li, H. Queffélec, and L. Rodríguez-Piazza, On approximation numbers of composition operators, Journal Approx. Theory 164 (4) (2012), 431–459.

[8] B. McCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canad. J. Math. 38, no. 4 (1986), 878–906.

[9] D. A. Stegenga, Multipliers of the Dirichlet space, Illinois J. Math. 24 (1980), no. 1, 113–139.

[10] K. Zhu, Operator Theory in Function Spaces, Monographs and Textbooks in Pure and Applied Mathematics, 139, Marcel Dekker, Inc. (1990).