Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor *

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Introduction

In the present work we prove a structure theorem for operators of the 0-th term of the $\mathcal{V}_{\bullet}^{Y}$-filtration relative to a free divisor Y of a complex analytic variety X. As an application, we give a formula for the logarithmic de Rham complex in terms of $\mathcal{V}_0^Y$-modules, which generalizes the classical formula for the usual de Rham complex in terms of $\mathcal{D}_X$-modules, and the formula of Esnault-Viehweg in the case that Y is a normal crossing divisor. Using this, we give a sufficient condition for perversity of the logarithmic de Rham complex. Now we comment on the contents of each part of the paper:

In the first section, we recall the concepts of logarithmic derivation and logarithmic form, as well as free divisor, all of them due to Kyogi Saito [14], and the definition of the ring $\mathcal{V}_0^Y(\mathcal{D}_X)$ of logarithmic differential operators along Y.

In the second part, we study the logarithmic operators in the case that Y is free. We give a structure theorem in which we prove that the ring of logarithmic differential operators is the polynomial algebra generated by the logarithmic derivations over the sheaf $\mathcal{O}_X$ of holomorphic functions. As a consequence, $\mathcal{V}_0^Y(\mathcal{D}_X)$ is a coherent sheaf. Thanks to this theorem, we can prove the equivalence between $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules and $\mathcal{O}_X$-modules with logarithmic connections. Therefore, an $\mathcal{V}_0^Y(\mathcal{D}_X)$-module (or logarithmic $\mathcal{D}_X$-module) $\mathcal{M}$ defines a logarithmic de Rham complex $\Omega_X^{\bullet}(\log Y)(\mathcal{M})$.

In the third part, we prove that the logarithmic de Rham complex is canonically isomorphic to the complex $\mathbf{R}\mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{O}_X, \mathcal{M})$. To show this, we first construct a resolution of $\mathcal{O}_X$ as $\mathcal{V}_0^Y(\mathcal{D}_X)$-module, which we call the logarithmic Spencer complex and denote by $\mathcal{S}p^{\bullet}(\log Y)$.

*Supported by DGICYT PB94-1435

Finally, we give a sufficient condition for perversity of the logarithmic de Rham complex, which is a perverse sheaf if the symbols of a minimal generating set of logarithmic derivations form a regular sequence in the graded ring associated to the filtration by the order on $\mathcal{D}_X$. This condition always holds in dimension 2.

Some results of this paper have been announced in [4]. We give here the complete proofs of all of the results announced in that note and other new results.

Acknowledgements: I am grateful to David Mond for his interest and encouragement. I wish to thank my advisor Luis Narváez for introducing me to the subject of this work and for giving me suggestions for the proofs of some of the results.

1 Notations and Preliminaries

Let X be a complex analytic variety of dimension n, and Y a hypersurface of X defined by the ideal $\mathcal{I}$. We will denote by $\mathcal{D}_X$ the sheaf of linear differential operators over X, $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$ the sheaf of derivations of $\mathcal{O}_X$, and $\mathcal{D}_X[\star Y]$ the sheaf of meromorphic differential operators with poles along Y. Given a point x of Y, we will denote by $I = (f)$, $\mathcal{O}$, $\mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ and $\mathcal{D}$ the respective stalks at x. We will denote by $F^{\bullet}$ the filtration of $\mathcal{D}_X$ by the order of the operators and $\Omega_X^{\bullet}[\star Y]$ the meromorphic de Rham complex with poles along Y.

1.1 Logarithmic forms and logarithmic derivations. Free divisors

We are going to recall some notions of [14] that we will use repeatedly:

A section δ of $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$, defined over an open set U of X, is called a *logarithmic derivation* (or vector field) if for each point x in $Y \cap U$, $\delta_x(\mathcal{I}_x)$ is contained in the ideal $\mathcal{I}_x$ (if $I = \mathcal{I}_x = (f)$, it is sufficient that $\delta_x(f)$ belongs to $(f)\mathcal{O}$). The sheaf of logarithmic derivations is denoted by $\mathcal{D}\mathrm{er}(\log Y)$, and is a coherent $\mathcal{O}_X$-submodule of $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$ and a Lie subalgebra. We denote by $\mathrm{Der}(\log f)$, or $\mathrm{Der}(\log I)$, the stalks at x of $\mathcal{D}\mathrm{er}(\log Y)$:

$$\mathrm{Der}(\log f) = \{\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}) \ / \ \delta(f) \in (f)\}.$$

We say that a meromorphic q-form ω with poles along Y, defined in an open set U, is a logarithmic q-form along Y or, simply, a *logarithmic q-form*, if for every point x in U, $f\omega$ and $df \wedge \omega$ are holomorphic at x. The sheaf of logarithmic q-forms along Y in U is denoted by $\Omega_X^q(\log Y)(U)$. This definition gives rise to a coherent $\mathcal{O}_X$-module $\Omega_X^q(\log Y)$, whose stalks are:

$$\Omega^q(\log f) = \Omega_X^q(\log Y)_x = \{\omega \in \Omega_X^q[\star Y]_x \ / \ f\omega \in \Omega^q, \ df \wedge \omega \in \Omega^{q+1}\}.$$

The logarithmic q-forms along Y define a subcomplex of the meromorphic de Rham complex along Y, that we call the logarithmic de Rham complex and denote by $\Omega_X^\bullet(\log Y)$.

Contraction of forms by vector fields defines a perfect duality between the $\mathcal{O}_X$-modules $\Omega_X^1(\log Y)$ and $\mathcal{D}\text{er}(\log Y)$, that we denote by $\langle\ ,\ \rangle$. Thus, both of them are reflexive. In particular, when $n = \dim_{\mathbb{C}} X = 2$, $\Omega_X^1(\log Y)$ and $\mathcal{D}\text{er}(\log Y)$ are locally free $\mathcal{O}_X$-modules of rank 2.

We say that Y *is free at* x, or I is a free ideal of $\mathcal{O}$, if $\text{Der}(\log I)$ is free as $\mathcal{O}$-module (of rank n). If $f \in \mathcal{O}$, we say that f *is free* if the ideal $I = (f)$ is free. We say that Y is free if it is at every point x. In this case, $\mathcal{D}\text{er}(\log Y)$ is a locally free $\mathcal{O}_X$-module of rank n. We can use the following criterion to determine when an hypersurface Y is free at x:

Saito's Criterion: The $\mathcal{O}$-module $\text{Der}(\log f)$ is free if and only if there exist n elements $\delta_1, \delta_2, \cdots, \delta_n$ in $\text{Der}(\log f)$, with $\delta_i = \sum_{j=1}^n a_{ij}(z)\frac{\partial}{\partial z_j}$ $(i = 1, \ldots, n)$, where $z = (z_1, z_2, \cdots, z_n)$ is a system of coordinates of X centered in x, such that the determinant $\det(a_{ij})$ is equal to af, with $a \in \mathcal{O}$ a unit. Moreover, in this case, $\{\delta_1, \delta_2, \cdots, \delta_n\}$ is a basis of $\text{Der}(\log f)$.

When Y is free, we have the equality: $\Omega_X^p(\log Y) = \overset{p}{\wedge}\ \Omega_X^1(\log Y)$. Using the fact that $\Omega_X^1(\log Y) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}\text{er}(\log Y), \mathcal{O}_X)$, we can construct a natural isomorphism:

$$\Omega_X^p(\log Y) \overset{\gamma^p}{\cong} \mathcal{H}om_{\mathcal{O}_X}(\overset{p}{\wedge}\ \mathcal{D}\text{er}(\log Y), \mathcal{O}_X),$$

defined locally by $\gamma^p(\omega_1 \wedge \cdots \wedge \omega_p)(\delta_1 \wedge \cdots \wedge \delta_p) = \det(\langle \omega_i, \delta_j \rangle)_{1 \le i,j \le p}$.

1.2 $\mathcal{V}$-filtration

We define the $\mathcal{V}$-filtration relative to Y on $\mathcal{D}_X$ as in the smooth case ([10], [9]):

$$\mathcal{V}_k^Y(\mathcal{D}_X) = \{P \in \mathcal{D}_X \ /\ P(\mathcal{I}^j) \subset \mathcal{I}^{j-k}, \forall j \in \mathbb{Z}\}, \quad k \in \mathbb{Z},$$

where $\mathcal{I}^p = \mathcal{O}_X$ when p is negative. Similarly, $\mathcal{V}_k^I(\mathcal{D}) = \{P \in \mathcal{D} \ /\ P(I^j) \subset I^{j-k}, \forall j \in \mathbb{Z}\}$, with k an integer, and $I^p = \mathcal{O}$ when $p \geq 0$. In the case of $I = (f)$, we note $\mathcal{V}_k^f(\mathcal{D}) = \mathcal{V}_k^I(\mathcal{D})$.

Definition 1.2.1.– A logarithmic differential operator (or, simplify, a logarithmic operator) is a differential operator of degree 0 with respect to the $\mathcal{V}$-filtration.

We see that:

$$\mathcal{D}\text{er}(\log Y) = \mathcal{D}\text{er}_{\mathbb{C}}(\mathcal{O}_X) \cap \mathcal{V}_0^Y(\mathcal{D}_X) = \mathcal{G}r^1_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right),$$

$$F^1(\mathcal{V}_0^Y(\mathcal{D}_X)) = \mathcal{O}_X \oplus \mathcal{D}\text{er}(\log Y),$$

where the last expression is consequence of $F^1(\mathcal{D}_X) = \mathcal{O}_X \oplus \mathcal{D}\text{er}_{\mathbb{C}}(\mathcal{O}_X)$.

Remark 1.2.2.– The inclusion $\mathcal{D}\mathrm{er}(\log Y) \subset \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)$ gives rise to a canonical graded morphism of graded algebras:

$$\kappa: \ \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{D}\mathrm{er}(\log Y)) \ \longrightarrow \ \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right).$$

Similarly, we have a canonical graded morphism of graded $\mathcal{O}$-algebras: $\kappa_x: \ \mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}(\log I)) \ \longrightarrow \ \mathrm{Gr}_{F^\bullet}\left(\mathcal{V}_0^I(\mathcal{D})\right)$, which is the stalk of κ at x.

2 Logarithmic operators relative to a free divisor

2.1 The Structure Theorem

We denote by $\{\ ,\ \}$ the Poisson bracket defined in the graded ring $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ (cf. [12], [8]). Given two polynomials F, G in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{O}[\xi_1, \cdots, \xi_n]$:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial \xi_i}\frac{\partial G}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial \xi_i}.$$

Proposition 2.1.1.– Let f be free. Consider a minimal system of generators $\{\delta_1, \delta_2, \cdots, \delta_n\}$ of $\mathrm{Der}(\log f)$. Let R_0 be a polynomial in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, homogeneous of order d, and such that there exist other polynomials R_k in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, with $k = 1, \cdots, d$, homogeneous of order $d-k$ such that:

$$\{R_k, f\} = fR_{k+1}, \ (0 \leq k < d) \tag{1}$$

(we will say that R_0 verifies the property (1) for $R_1, R_2, \cdots, R_d$). Then there exist polynomials H_j^k in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, homogeneous of order $d-k-1$, with $j = 1, \cdots, n$ and $k = 1, \cdots, d-1$, such that:

a) $R_k = \sum_{j=1}^n H_j^k \sigma(\delta_j)$, where $\sigma(\delta_j)$ denotes the principal symbol of δ_j.

b) $\{H_j^k, f\} = fH_j^{k+1}$ $(1 \leq j \leq n, \ 0 \leq k < d-1)$. This is the same as saying: H_j^k verifies the property (1) for $H_j^{k+1}, \cdots, H_j^{d-1}$.

Proof: Let $A = (\alpha_i^j)$ be the square matrix whose rows are the coefficients of the basis $\{\delta_1, \delta_2, \cdots, \delta_n\}$ of $\mathrm{Der}(\log f)$ with respect to the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}$ of $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$:

$$\delta_j = \sum_{i=1}^{n} \alpha_i^j \frac{\partial}{\partial x_i} = \underline{\alpha}^j \bullet \underline{\partial}^t,$$

with $j = 1, \cdots, n$, where we write $\underline{\partial}$ instead of $\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$. We consider the ring $\mathcal{O}_{2n} = \mathbb{C}\{x_1, \cdots, x_2, \xi_1, \cdots, \xi_n\}$. Thanks to the Saito's Criterion, we know that the set

$$\{\delta_1, \cdots, \delta_n, \frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}\}$$

is a basis of the $\mathcal{O}_{2n}$-module $\mathrm{Der}_{\mathcal{O}_{2n}}(\log f)$. So, as we have, for $k = 1, \cdots, d$,

$$(f) \ni \{R_k, f\} = \sum_{i=1}^{n} (R_k)_{\xi_i} f_{x_i},$$

where f_{x_i} represents $\frac{\partial f}{\partial x_i}$ and $(R_k)_{\xi_i}$ represents $\frac{\partial R_k}{\partial \xi_i}$, then there exist homogeneous polynomials G_j^k in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, of degree $d-k-1$, or null, with $j = 1, \cdots, n$ and $k = 1, \cdots, d-1$, such that

$$((R_k)_{\xi_1}, (R_k)_{\xi_2}, \cdots, (R_k)_{\xi_n}) = \sum_{j=1}^{n} G_j^k \underline{\alpha}^j.$$

Using the Euler relation $R_k = \frac{1}{d}\sum_{i=1}^{n}(R_k)_{\xi_i}\xi_i$, and as $\sigma(\delta_i) = \underline{\alpha}^i \bullet \underline{\xi}^t$, we obtain

$$R_k = \frac{1}{d}\sum_{i=1}^{n}\sum_{j=1}^{n} G_j^k \alpha_i^j \xi_i = \frac{1}{d}\sum_{j=1}^{n} G_j^k \sigma(\delta_j).$$

By Saito's Criterion, the determinant of the matrix A is equal to uf, with $u \in \mathcal{O}$ invertible. Let $B = (b_{ij}) = Adj(A)^t$. We have:

$$((R_k)_{\xi_1}, (R_k)_{\xi_2}, \cdots, (R_k)_{\xi_n}) = \left(G_1^k, G_2^k, \cdots, G_n^k\right) A,$$

so

$$((R_k)_{\xi_1}, (R_k)_{\xi_2}, \cdots, (R_k)_{\xi_n})\, B = g\left(G_1^k, G_2^k, \cdots, G_n^k\right).$$

Now:

$$g\{G_j^k, f\} = \{gG_j^k, f\} = \sum_{i=1}^{n} f_{x_i} \frac{\partial(gG_j^k)}{\partial \xi_i} = \sum_{i=1}^{n} f_{x_i} \sum_{l=1}^{n} \frac{\partial (R_k)_{\xi_l}}{\partial \xi_i} b_{lj} =$$

$$\sum_{l=1}^{n} b_{lj} \sum_{i=1}^{n} \frac{\partial^2 R_k}{\partial \xi_l \partial \xi_i} f_{x_i} = \sum_{l=1}^{n} b_{lj} \frac{\partial(\{R_k, f\})}{\partial \xi_l} = f\sum_{l=1}^{n} b_{lj} \frac{\partial R_{k+1}}{\partial \xi_l} = f\sum_{l=1}^{n} b_{lj} (R_{k+1})_{\xi_l} =$$

$$f\sum_{l=1}^{n} b_{lj} \sum_{p=1}^{n} G_p^{k+1}\alpha_l^p = f\sum_{p=1}^{n} G_p^{k+1} \sum_{l=1}^{n} b_{lj}\alpha_l^p = fgG_j^{k+1}.$$

Therefore,

$$\{G_j^k, f\} = fG_j^{k+1},$$

with $k = 0, \cdots, d-2$ and $j = 0, \cdots, n$. We conclude by setting $H_j^k = \frac{1}{d}G_j^k$, for $j = 1, \cdots, n$ and $k = 0, \cdots, d-1$. □

Proposition 2.1.2.– Let be $\{\delta_1, \delta_2, \cdots, \delta_n\}$ a basis of $\mathrm{Der}(\log f)$. If a polynomial R_0 of $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ is homogeneous and verifies the property (1) of the last proposition, we can find a differential operator Q in $\mathcal{O}[\delta_1, \delta_2, \cdots, \delta_n]$ such that R_0 is the symbol of Q.

Proof: We will do the proof by induction on the order of R_0. If $R_0 \in \mathcal{O}$, it is obvious. We suppose that the result holds if the order of R_0 is less than d. Now let R_0 of order d verifying (1). By the last proposition there exist n homogeneous polynomials H_j^0 of order $d-1$ such that:

$$R_0 = \sum_{j=1}^{n} H_j^0 \sigma(\delta_j), \;\; H_j^0 \text{ verifies (1) } (j = 1, \ldots, n).$$

By induction hypothesis, there exist $Q_j \in \mathcal{O}[\delta_1, \delta_2, \cdots, \delta_n]$ such that $H_j^0 = \sigma(Q_j)$. So

$$R_0 = \sum_{i=1}^{n} \sigma(Q_i)\sigma(\delta_i) = \sum_{i=1}^{n} \sigma(Q_i \delta_i) = \sigma(\sum_{i=1}^{n} Q_i \delta_i) = \sigma(Q)$$

and $Q = \sum_{i=1}^{n} Q_i \delta_i \in \mathcal{O}[\delta_1, \delta_2, \cdots, \delta_n]$. □

Remark 2.1.3.– Really, the previous argument proves that if R_0 verifies (1), then R_0 is a polynomial in $\mathcal{O}[\sigma(\delta_1), \cdots, \sigma(\delta_n)]$.

Theorem 2.1.4.– If f is free and $\{\delta_1, \delta_2, \cdots, \delta_n\}$ is a basis of the $\mathcal{O}$-module $\mathrm{Der}(\log f)$, each logarithmic operator P can be written in a unique way as a polynomial

$$P = \sum \beta_{i_1 \cdots i_n} \delta_1^{i_1} \delta_2^{i_2} \cdots \delta_n^{i_n}, \quad \beta_{i_1 \cdots i_n} \in \mathcal{O}.$$

In other words, the ring of logarithmic operators is the $\mathcal{O}$-subalgebra of $\mathcal{D}$ generated by logarithmic derivations:

$$\mathcal{V}_0^I(\mathcal{D}) = \mathcal{O}[\delta_1, \delta_2, \cdots, \delta_n] = \mathcal{O}[\mathrm{Der}(\log f)].$$

Proof: The inclusion $\mathcal{O}[\delta_1, \delta_2, \cdots, \delta_n] \subseteq \mathcal{V}_0^I(\mathcal{D})$ is clear. We will prove the other inclusion by induction on the order of $P_0 \in \mathcal{V}_0^I(\mathcal{D})$. If the order of P_0 is zero, then it is a holomorphic function and the result is obvious. We suppose the result is true for every logarithmic operator Q whose order is strictly less than d. Let P_0 be a logarithmic operator of order d. We know that:

$$[P_0, f] = fP_1,$$

with $P_1 \in \mathcal{V}_0^I(\mathcal{D})$. So, there exist several P_k, with $k = 0, \cdots, d$, such that $[P_k, f] = fP_{k+1}$. If we set $R_k = \sigma(P_k)$, in the case that P_k has order $d-k$, and $R_k = 0$ otherwise, we obtain:

$$\{R_k, f\} = \{\sigma_{d-k}(P_k), f\} = \sigma_{d-k-1}([P_k, f]) = f\sigma_{d-k-1}(P_{k+1}) = fR_{k+1}.$$

By the previous proposition, there exists Q in $\mathcal{O}[\delta_1, \delta_2, \cdots, \delta_n]$ of order d and such that $\sigma(P_0) = \sigma(Q)$. As the order of $P_0 - Q \in \mathcal{V}_0^I(\mathcal{D})$ is strictly less than d, we apply the induction hypothesis to $P_0 - Q$ and obtain

$$P_0 = P_0 - Q + Q \in \mathcal{O}[\delta_1, \delta_2, \cdots, \delta_n],$$

as we wanted.
On the other hand, using the structure of Lie algebra it is clear that we can write a logarithmic operator as a $\mathcal{O}$-linear combination of the monomials $\{\delta_1^{i_1}, \cdots, \delta_n^{i_n}\}$. The uniqueness of this expression follows from the fact that these monomials are linearly independent over $\mathcal{O}$.

□

Remark 2.1.5.– As a immediate consequence of the theorem (see the previous remark), we obtain an isomorphism:

$$\mathrm{Gr}_{F^\bullet}\left(\mathcal{V}_0^I(\mathcal{D})\right) \overset{\alpha}{\cong} \mathcal{O}[\sigma(\delta_1), \cdots, \sigma(\delta_n)].$$

Corollary 2.1.6.– If Y is free at x, the morphism κ_x from the symmetric algebra $\mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}(\log f))$ to $\mathrm{Gr}_{F^\bullet}\left(\mathcal{V}_0^f(\mathcal{D})\right)$ (see remark 1.2.2) is an isomorphism of graded $\mathcal{O}$-algebras. As a consequence, if Y is a free divisor, the canonical morphism

$$\kappa: \quad \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{D}\mathrm{er}(\log Y)) \to \mathcal{G}r_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)$$

is an isomorphism.

Proof: Let x be in X and $f \in \mathcal{O}$ a local reduced equation of Y at a neighbourhood of x. Let $\{\delta_1, \cdots, \delta_n\}$ be a basis of $\mathrm{Der}(\log f)$.

$$\mathrm{Der}(\log f) = \oplus_{i=1}^n \mathcal{O}\delta_i \cong \oplus_{i=1}^n \mathcal{O}\sigma(\delta_i).$$

The symmetric algebra of the $\mathcal{O}$-module $\mathrm{Der}(\log f)$ is isomorphic to a polynomial ring:

$$\mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}(\log f)) \overset{\beta}{\cong} \mathcal{O}[\sigma(\delta_1), \cdots, \sigma(\delta_n)].$$

We also have the inclusion:

$$\oplus_{i=1}^n \mathcal{O}\sigma(\delta_i) = \mathrm{Gr}_{F^\bullet}^1\left(\mathcal{V}_0^I(\mathcal{D})\right) \subset \mathrm{Gr}_{F^\bullet}\left(\mathcal{V}_0^I(\mathcal{D})\right),$$

where $\sigma(\delta_i)$ is the image of δ_i by the morphism κ_x. Therefore we conclude that the morphism $\kappa_x = \alpha^{-1}\beta$ is an isomorphism (see remark 2.1.5). On the other hand, the inclusion

$$\mathcal{D}\mathrm{er}(\log Y) = \mathcal{G}r_{F^\bullet}^1\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right) \subset \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)$$

gives rise to a canonical graded morphism of graded $\mathcal{O}_X$-algebras (see remark 1.2.2): $\kappa: \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{D}\mathrm{er}(\log Y)) \longrightarrow \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)$, whose stalk at each point x of Y is the canonical graded isomorphism κ_x. So, κ is also an isomorphism. □

Corollary 2.1.7.– $\mathcal{V}_0^Y(\mathcal{D}_X)$ is a coherent sheaf of rings.

Proof: By theorem 9.16 of [1] (p. 83), we have only to prove that $\mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)$ is coherent, but this sheaf is locally isomorphic to the polynomial ring $\mathcal{O}_X[T_1, \cdots, T_n]$, which is coherent ([3, lemma 3.2, VI, pg. 205]). □

2.2 Equivalence between $\mathcal{O}_X$-modules with a logarithmic connection and left $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules.

Definition 2.2.1.– (cf. [6]) Let $\mathcal{M}$ be a $\mathcal{O}_X$-module. A connection on $\mathcal{M}$, with logarithmic poles along Y, (or logarithmic connection on $\mathcal{M}$), is a $\mathbb{C}$-homomorphism ∇,

$$\nabla : \ \mathcal{M} \ \rightarrow \ \Omega^1_X(\log Y) \otimes \mathcal{M},$$

that verifies Leibniz's identity: $\nabla(hm) = dh \cdot m + h \cdot \nabla(m)$, where d is the exterior derivative over $\mathcal{O}_X$. We will note $\Omega^q_X(\log Y)(\mathcal{M}) = \Omega^q_X(\log Y) \otimes \mathcal{M}$.

If δ is a logarithmic derivation along Y, it defines a $\mathbb{C}$-morphism:

$$\begin{array}{ccc} \mathcal{D}\mathrm{er}(\log Y) & \longrightarrow & \mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{M}), \\ \delta & \mapsto & \nabla_\delta \end{array}$$

where $\nabla_\delta(m) = \langle \delta, \nabla(m) \rangle$

Remark 2.2.2.– A logarithmic connection ∇ on $\mathcal{M}$ gives rise to a morphism of $\mathcal{O}_X$-modules

$$\nabla' : \ \mathcal{D}\mathrm{er}(\log Y) \ \rightarrow \ \mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$$

which verifies Leibniz's condition: $\nabla'_\delta(fm) = \delta(f) \cdot m + f \cdot \nabla'_\delta(m)$.
Conversely, given ∇' verifying this condition, we define

$$\nabla : \mathcal{M} \rightarrow \Omega^1_X(\log Y)(\mathcal{M}),$$

with $\nabla(m)$ the element of $\Omega^1_X(\log Y)(\mathcal{M}) = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{D}\mathrm{er}(\log Y), \mathcal{M})$ such that:

$$\nabla(m)(\delta) = \nabla'_\delta(m).$$

Definition 2.2.3.– A logarithmic connection ∇ is integrable if, for each pair δ and δ' of logarithmic derivations, it verifies:

$$\nabla_{[\delta,\delta']} = [\nabla_\delta, \nabla_{\delta'}],$$

where [,] represents the Lie bracket in $\mathcal{D}\mathrm{er}(\log Y)$ and the commutator in $\mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$.

Given a logarithmic connection ∇ and the exterior derivative d, we can construct a morphism:

$$\nabla^q : \Omega^q_X(\log Y)(\mathcal{M}) \rightarrow \Omega^{q+1}_X(\log Y)(\mathcal{M}),$$

for each $q = 1, \cdots, n$. If ω and m are sections of the sheaves $\Omega^p_X(\log Y)$ and $\mathcal{M}$:

$$\nabla^q(\omega \otimes m) = d\omega \otimes m + (-1)^q \omega \wedge \nabla(m).$$

The integrability condition is equivalent to $\nabla^q \circ \nabla^{q-1} = 0$, for every q (cf. [6]).

Definition 2.2.4.– Let $\mathcal{M}$ be a $\mathcal{O}_X$-module, and ∇ an integrable logarithmic connection along Y on $\mathcal{M}$. With the above notation, we call the logarithmic de Rham complex of $\mathcal{M}$, and we denote by $\Omega^\bullet_X(\log Y)(\mathcal{M})$, the complex (of sheaves of $\mathbb{C}$-vector spaces):

$$0 \to \mathcal{M} \xrightarrow{\nabla} \Omega^1_X(\log Y)(\mathcal{M}) \xrightarrow{\nabla^1} \cdots \xrightarrow{\nabla^{q-1}} \Omega^q_X(\log Y)(\mathcal{M}) \xrightarrow{\nabla^q}$$

$$\Omega^{q+1}_X(\log Y)(\mathcal{M}) \xrightarrow{\nabla^{q+1}} \cdots \xrightarrow{\nabla^{n-1}} \Omega^n_X(\log Y)(\mathcal{M}) \to 0.$$

In the particular case where the $\mathcal{O}_X$-module $\mathcal{M}$ is equal to $\mathcal{O}_X$ and the logarithmic connection ∇ is equal to the exterior derivative $d : \mathcal{O}_X \to \Omega^1_X(\log Y)$, the morphisms

$$\nabla^q : \Omega^q_X(\log Y) \longrightarrow \Omega^{q+1}_X(\log Y),$$

define the logarithmic de Rham complex of Saito.

We consider the rings $R_0 = \mathcal{O}_X \subset R_1$ and $R = \mathcal{V}^Y_0(\mathcal{D}_X) = \bigcup_{k\geq 0} R_k$ ($1 \in R_0 \subset R$), with $R_k = F^k(\mathcal{V}^Y_0(\mathcal{D}_X))$. The ring $\mathcal{G}r(R)$ is commutative and verifies

(1) The canonical morphism $\alpha : Sym_{R_0}(\mathcal{G}r^1(R)) \to \mathcal{G}r(R)$, defined by $\alpha(s_1 \otimes \cdots \otimes s_t) = s_1 \cdots s_t$, is an isomorphism (see Corollary 2.1.6).

With these conditions, R_1 is an (R_0, R_0)-bimodule, and a Lie algebra ($[x, y] = xy - yx \in R_1$, because $\mathcal{G}r(R)$ is conmutative). Moreover, R_0 is a sub-(R_0, R_0)-bimodule of R_1 such that the two induced structures of R_0-module over the quotient R_1/R_0 are the same.

Let $\mathbf{T}_{R_0}(R_1) = R_0 \oplus R_1 \oplus (R_1 \otimes_{R_0} R_1) \oplus \cdots$ be the tensor algebra of the (R_0, R_0)-bimodule R_1, and let $\psi : \mathbf{T}_{R_0}(R_1) \to R$ be the canonical morphism defined by the inclusion $R_1 \subset R$. We prove a reciprocal theorem of one Poincaré-Birkhoff-Witt theorem [13, theorem 3.1,p.198] .

Proposition 2.2.5.– The morphism ψ induces an isomorphism:

$$\phi : \mathbf{S} = \frac{\mathbf{T}_{R_0}(R_1)}{J} \cong R, \quad \phi((i(x_1) \otimes \cdots \otimes i(x_t)) + J) = x_1 x_2 \cdots x_t,$$

where i the inclusion of R_1 in the tensor algebra, and J is the two sided ideal generated by the elements:

$$\text{a) } a - i(a),\ a \in R_0 \subset R_1, \quad \text{b) } i(x) \otimes i(y) - i(y) \otimes i(x) - i([x, y]),\ x, y \in R_1.$$

Proof: First, we check that the morphism $\phi : \mathbf{S} \to R$ is well defined:

$$\psi(a - i(a)) = a - a = 0,\ a \in R_0,$$
$$\psi(i(x) \otimes i(y) - i(y) \otimes i(x) - i([x, y])) = xy - yx - [x, y] = 0,\ x, y \in R_1.$$

The algebra $\mathbf{T}_{R_0}(R_1)$ is graded, so it is filtered, and induces a filtration on the quotient. The induced morphism $\phi : \mathbf{S} \to R$ is filtered:

$$\psi(a) = a \in R_0,\ \psi(i(x_1) \otimes \cdots \otimes i(x_t)) = x_1 x_2 \cdots x_t \in R_t.$$

So, we can define a graded morphism of R_0-rings.

$$\pi : \mathcal{G}r\,(\mathbf{S}) \to \mathcal{G}r(R),$$

$$\pi(\sigma_t(i(x_1) \otimes \cdots \otimes i(x_t) + J)) = \sigma'_t(x_1 \cdots x_t) = \overline{x_1} \cdots \overline{x_t},$$

where $x_i \in R_1$, $\overline{x_i} = \sigma'_1(x_1)$ is the class of x_i in R_1/R_0, $\sigma_t(P)$ is the class of $P \in \mathbf{S}$ in $\mathcal{G}r^t(\mathbf{S})$, and $\sigma'_t(Q)$ the class of $Q \in R_t$ in $\mathcal{G}r^t(R)$. Note that $\mathcal{G}r(\mathbf{S})$ is conmutative: it is generated by the elements $\sigma_0(a+J)$, $\sigma_1(i(x)+J)$, with $a \in R_0$, $x \in R_1$, and

$$[i(x) + J, i(y) + J] = i([x, y]) + J,$$
$$[a + J, i(x) + J] = i(ax - xa) + J = b + J,\ b = ax - xa \in R_0.$$

On the other hand, the image of $R_0 \subset R_1$ in $\mathbf{S}$ is exactly the part of degree zero of $\mathbf{S}$, and then we obtain a morphism of R_0-modules from $\mathcal{G}r^1(R) = R_1/R_0$ to $\mathcal{G}r^1(\mathbf{S})$ which induces a morphism of R_0-algebras:

$$\rho : \mathcal{S}ym_{R_0}\left(\frac{R_1}{R_0}\right) \to \mathcal{G}r\,(\mathbf{S})\,,$$

$$\rho(\overline{x_1} \otimes \cdots \otimes \overline{x_t}) = \sigma_t(i(x_1) \otimes \cdots \otimes i(x_t) + J),$$

which is obviously surjective. The composition $\pi\rho$ is equal to α, and, by property (1) of R, we deduce that ρ is injective. As ρ and $\pi\rho$ are isomorphisms, π is as well, as we wanted to prove. □

Corollary 2.2.6.– Let Y be a free divisor. Let $\mathcal{M}$ be a $\mathcal{O}_X$-module. An integrable logarithmic connection on $\mathcal{M}$ gives rise to a left $\mathcal{V}_0^Y(\mathcal{D}_X)$-structure on $\mathcal{M}$, and vice versa.

Proof: A $\mathcal{O}_X$-module $\mathcal{M}$ with an integrable logarithmic connection ∇ has a natural structure of left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module defined by its structure as $\mathcal{O}_X$-module. Let μ be the morphism of $(\mathcal{O}_X, \mathcal{O}_X)$-bimodules:

$$\mu : R_1 = \mathcal{O}_X \oplus \mathcal{D}\text{er}(\log Y) \to \mathcal{E}\text{nd}_{\mathbb{C}}(\mathcal{M}),\quad \mu(a)(m) = am,\quad \mu(\delta)(m) = \nabla_\delta(m).$$

μ induces a morphism $\nu : \mathbf{T}_{R_0}(R_1) \to \mathcal{E}\text{nd}_{\mathbb{C}}(\mathcal{M})$, and, as $\nu(J) = 0$, we have a morphism

$$\mathcal{V}_0^Y(\mathcal{D}_X) \simeq \frac{\mathbf{T}_{R_0}(R_1)}{J} \to \mathcal{E}\text{nd}_{\mathbb{C}}(\mathcal{M}),$$

which defines an structure of $\mathcal{V}_0^Y(\mathcal{D}_X)$-module on $\mathcal{M}$.

On the other hand, a left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module structure on $\mathcal{M}$ defines an integrable logarithmic connection ∇ on the $\mathcal{O}_X$-module $\mathcal{M}$:

$$\nabla : \mathcal{D}\mathrm{er}(\log Y) \to \mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{M}), \quad \nabla_\delta(m) = \delta \cdot m.$$

□

Remark 2.2.7.– A left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module structure on $\mathcal{M}$ defines a logarithmic de Rham complex. In local coordinates $(U; x_1, \cdots, x_n)$, with $\{\delta_1, \cdots, \delta_n\}$ a local basis of $\mathcal{D}\mathrm{er}(\log Y)$ and $\{\omega_1, \cdots, \omega_n\}$ its dual basis, the differential of the complex is defined by:

$$\nabla^p(U)(\omega \otimes m) = d\omega \otimes m + \sum_{i=1}^{n} \left((\omega_i \wedge \omega) \otimes \delta_i \cdot m\right),$$

for any sections $\omega \in \Omega_X^1(\log Y)$ and $m \in \mathcal{M}$. In the particular case of the left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module $\mathcal{O}_X$, defined as $\mathcal{V}_0^Y(\mathcal{D}_X)$-module in a natural way ($P \cdot g = P(g)$, with g a holomorphic function and P a logarithmic operator), this canonical structure of $\mathcal{O}_X$ as left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module is obviously equivalent to the integrable logarithmic connection over $\mathcal{O}_X$ defined naturally by the exterior derivative ($\nabla = d$):

$$\nabla_\delta(g) = \langle \delta, dg \rangle = \delta(g).$$

3 The Logarithmic de Rham Complex

In this section, Y will be a free divisor.

3.1 The Logarithmic Spencer Complex

Definition 3.1.1.– We call the logarithmic Spencer complex, and denote by $\mathcal{S}p^\bullet(\log Y)$, the complex:

$$0 \to \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \overset{n}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \overset{\varepsilon_{-n}}{\to} \cdots$$

$$\cdots \overset{\varepsilon_{-2}}{\to} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \overset{1}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \overset{\varepsilon_{-1}}{\to} \mathcal{V}_0^Y(\mathcal{D}_X),$$

where

$$\varepsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \delta_p) +$$

$$\sum_{1 \le i < j \le p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \widehat{\delta_j} \wedge \cdots \wedge \delta_p), \quad (2 \le p \le n).$$

$$\varepsilon_{-1}(P \otimes \delta) = P\delta.$$

We can augment this complex of left $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules by another morphism:

$$\varepsilon_0 : \mathcal{V}_0^Y(\mathcal{D}_X) \to \mathcal{O}_X, \quad \varepsilon_0(P) = P(1).$$

We call the new complex $\widetilde{\mathcal{S}p}^\bullet(\log Y)$.

This definition is essentially the same as the definition of the usual Spencer complex $\mathcal{S}p^\bullet$ of $\mathcal{O}_X$ (cf. [11, 2.1]) and generalizes the definition given by Esnault and Viehweg [7, App. A] in the case of a normal crossing divisor. We denote by $\mathcal{S}p^\bullet[\star Y] = \mathcal{D}_X[\star Y] \otimes_{\mathcal{D}_X} \mathcal{S}p^\bullet$ the meromorphic Spencer complex of $\mathcal{O}_X[\star Y]$.

Theorem 3.1.2.– The complex $\mathcal{S}p^\bullet(\log Y)$ is a locally free resolution of $\mathcal{O}_X$ as left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module.

Proof: To see the exactness of $\widetilde{\mathcal{S}p}^\bullet(\log Y)$ we define a discrete filtration $G^\bullet$ such that it induces an exact graded complex (cf. [1, lemma 3.16]):

$$G^k\left(\mathcal{V}_0^Y(\mathcal{D}_X)\otimes \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y)\right) = F^{k-p}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right) \otimes \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y),$$

$$G^k(\mathcal{O}_X) = \mathcal{O}_X.$$

We have

$$\mathcal{G}\mathrm{r}_{G^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\otimes \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y)\right) = \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)[-p]\otimes \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y),$$

$$\mathcal{G}\mathrm{r}_{G^\bullet}(\mathcal{O}_X) = \mathcal{O}_X.$$

As the above filtrations are compatible with the differential of the complex $\widetilde{\mathcal{S}p}^\bullet(\log Y)$, we can consider the complex $\mathcal{G}\mathrm{r}_{G^\bullet}\left(\widetilde{\mathcal{S}p}^\bullet(\log Y)\right)$:

$$0 \to \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)[-n]\otimes_{\mathcal{O}_X} \overset{n}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \overset{\psi_{-n}}{\to} \cdots$$

$$\overset{\psi_{-2}}{\to} \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)[-1]\otimes_{\mathcal{O}_X} \overset{1}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \overset{\psi_{-1}}{\to} \mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right) \overset{\psi_0}{\to} \mathcal{O}_X \to 0,$$

where the local expression of the differential is defined by:

$$\psi_{-p}(G\otimes\delta_{j_1}\wedge\cdots\wedge\delta_{j_p}) = \sum_{i=1}^{p}(-1)^{i-1}G\sigma(\delta_{j_i})\otimes\delta_{j_1}\wedge\cdots\wedge\widehat{\delta_{j_i}}\wedge\cdots\wedge\delta_{j_p}, \quad (2 \le p \le n).$$

$$\psi_{-1}(G \otimes \delta_i) = G\sigma(\delta_i), \quad \psi_0(G) = G_0,$$

with $\{\delta_1, \cdots, \delta_n\}$ a (local) basis of $\mathcal{D}\mathrm{er}(\log Y)$. This complex is the Koszul complex of the ring

$$\mathcal{G}\mathrm{r}_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right) \cong \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}\left(\mathcal{D}\mathrm{er}(\log Y)\right)$$

with respect to the $\mathcal{G}r_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)$-regular sequence $\sigma(\delta_1),\cdots,\sigma(\delta_n)$ in the ring $\mathcal{G}r_{F^\bullet}\left(\mathcal{V}_0^Y(\mathcal{D}_X)\right)$. Consequently, it is exact. □

Lemma 3.1.3.– For every logarithmic operator $P \in \mathcal{V}_0^f(\mathcal{D})$, there exist, for each integer p, a logarithmic operator $Q \in \mathcal{V}_0^f(\mathcal{D})$ and an integer k such that $f^{-p}P = Qf^{-k}$.

Proof: We will prove the lemma by induction on the order of the logarithmic operator. If P has order 0, it is in $\mathcal{O}$, and it is clear that $f^{-p}P = Pf^{-p}$. Let P be of order d, and consider the logarithmic operator $[P, f^p]$, of order $d-1$. By induction hypothesis, there exists an integer m such that:

$$[P, f^{-p}]f^m \in \mathcal{V}_0^f(\mathcal{D}).$$

Let k be the greatest of the integers m and p. It is clear that:

$$f^{-p}Pf^k = Pf^{k-p} - [P, f^{-p}]f^k \in \mathcal{V}_0^f(\mathcal{D}).$$

This proves the result: $Q = Pf^{k-p} - [P, f^{-p}]f^k$. □

Remark 3.1.4.– For every operator Q in $\mathcal{D}_X[\star Y]_x$, we can always find a strictly positive integer m such that $f^mQ \in \mathcal{V}_0^f(\mathcal{D})$. Equivalently, for each meromorphic differential operator Q, there exists a positive integer p and a logarithmic operator Q' such that we can write:

$$Q = f^{-p}Q'.$$

Now we introduce several morphisms that we will use later.

Lemma 3.1.5.– We have the following isomorphisms:

1. $\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \xrightarrow{\sim} \mathcal{D}_X[\star Y] \xleftarrow{\sim} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X[\star Y]$.
2. $\alpha : \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{O}_X \cong \mathcal{O}_X[\star Y], \qquad \alpha(P \otimes g) = P(g)$.
3. $\rho : \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{D}_X[\star Y] \cong \mathcal{D}_X[\star Y], \qquad \rho(P \otimes Q) = PQ$.

Proof:

1. The inclusions $\mathcal{V}_0^Y(\mathcal{D}_X), \mathcal{O}_X[\star Y] \subset \mathcal{D}_X[\star Y]$ give rise to the previous isomorphisms of $(\mathcal{V}_0^Y(\mathcal{D}_X), \mathcal{O}_X[\star Y])$-modules. Locally:

$$af^{-k} \otimes P = af^{-k}P = aQ \otimes f^{-p},$$

with P and Q logarithmic operators such that $f^{-k}P = Qf^{-p}$. We have seen how to obtain Q from P (lemma 3.1.3), and we can obtain P from Q in the same way. On the other hand, we saw in the previous remark how to express a meromorphic differential operator as a product of a meromorphic function and a logarithmic operator.

2. We have to compose the following isomorphisms of left $\mathcal{D}_X[\star Y]$-modules:

$$\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{O}_X \cong \mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X[\star Y].$$

3. We obtain this isomorphism of $\mathcal{D}_X[\star Y]$-bimodules from the composition of the following isomorphisms:

$$\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{D}_X[\star Y] \cong \mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{D}_X[\star Y] \cong$$

$$\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \cong \mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \cong \mathcal{D}_X[\star Y],$$

where the isomorphism $\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{O}_X[\star Y] \cong \mathcal{O}_X[\star Y]$ sends (locally) the tensor product $g_1 \otimes g_2$ to the meromorphic function $g_1 g_2$. □

Proposition 3.1.6.– We have the following isomorphisms of complexes of $\mathcal{D}_X[\star Y]$-modules:

(a) $\mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet \cong \mathcal{S}p^\bullet[\star Y]$.

(b) $\mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log Y) \cong \mathcal{S}p^\bullet[\star Y]$.

Proof: (a) As $\mathcal{S}p^\bullet$ is a subcomplex of $\mathcal{D}_X$-modules of $\mathcal{S}p^\bullet[\star Y]$, and $\mathcal{D}_X[\star Y]$ is flat over ø $\mathcal{V}_0^Y(\mathcal{D}_X)$, the complex $\mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)}$ `ﬄø$\mathcal{S}p^\bullet$ is a subcomplex of $\mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet[\star Y]$, (see lemma 3.1.5, 1.). But, by the third isomorphism of lemma 3.1.5, this complex is the same as $\mathcal{S}p^\bullet[\star Y]$. Hence, we have an injective morphism of complexes:

$$\mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet \longrightarrow \mathcal{S}p^\bullet[\star Y],$$

defined locally in each degree by: $P \otimes Q \otimes \delta_1 \wedge \cdots \wedge \delta_p \mapsto PQ \otimes (\delta_1 \wedge \cdots \wedge \delta_p)$. This morphism is clearly surjective and, consequently, an isomorphism.

(b) We consider $\mathcal{V}_0^Y(\mathcal{D}_X)$ as a subsheaf of $\mathcal{O}$-modules of $\mathcal{D}_X$. Using the fact that $\overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y)$ is $\mathcal{O}_X$-free, we have an inclusion

$$\mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y).$$

On the other hand, as Y is free, we have a natural injective morphism from $\overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y)$ to $\overset{p}{\wedge} \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$ (cf. [2, AIII 88, Cor.]). As $\mathcal{D}_X$ is flat over $\mathcal{O}_X$, we have other inclusion:

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{p}{\wedge} \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X) \ (p \geq 0).$$

Composing both of them, we obtain a new inclusion:

$$\mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \overset{p}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{p}{\wedge} \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X),$$

for $p = 0, \cdots, n$. These inclusions give rise to an injective morphism of complexes of $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules

$$\mathcal{S}p^\bullet(\log Y) \hookrightarrow \mathcal{S}p^\bullet.$$

As $\mathcal{D}_X[\star Y]$ is flat over $\mathcal{V}_0^Y(\mathcal{D}_X)$ (see lemma 3.1.5, 1.) we have an injective morphism of complexes of $\mathcal{D}_X[\star Y]$-modules:

$$\theta' : \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log Y) \hookrightarrow \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet,$$

defined by: $\theta'(P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p)$. This morphism is surjective, given P local section of $\mathcal{D}_X[\star Y]$, Q in $\mathcal{D}$ and $\delta_1, \cdots, \delta_n$ in $\mathrm{Der}_{\mathbb{C}}(\mathcal{O})$, we have:

$$P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p) = \theta'\left((Pf^{-k}) \otimes Q' \otimes (f\delta_1 \wedge \cdots \wedge f\delta_p)\right),$$

with $k > 0$ and Q' a local section of $\mathcal{V}_0^Y(\mathcal{D}_X)$ verifying $f^k Q = Q' f^p$ (see lemma 3.1.3). Composing θ' with the isomorphism of (a), we obtain the isomorphism:

$$\theta : \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log Y) \xrightarrow{\sim} \mathcal{S}p^\bullet[\star Y],$$

with local expression: $\theta(P \otimes Q \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = PQ \otimes (\delta_1 \wedge \cdots \wedge \delta_p)$. □

3.2 The Logarithmic de Rham Complex

For each divisor Y, we have a standard canonical isomorphism:

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\overset{p}{\wedge}\mathcal{D}\mathrm{er}(\log Y), \mathcal{O}_X\right) \overset{\lambda^p}{\cong} \mathcal{H}\mathrm{om}_{\mathcal{V}_0^Y(\mathcal{D}_X)}\left(\mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \overset{p}{\wedge}\mathcal{D}\mathrm{er}(\log Y), \mathcal{O}_X\right),$$

defined by: $\lambda^p(\alpha)(P \otimes \delta_1 \wedge \cdots \wedge \delta_p) = P\left(\alpha(\delta_1 \wedge \cdots \wedge \delta_p)\right).$

Composing this isomorphism with the isomorphism γ^p defined in section 1.1, we can construct a natural morphism $\psi^p = \lambda^p \circ \gamma^p$:

$$\Omega_X^p(\log Y) \overset{\psi^p}{\cong} \mathcal{H}\mathrm{om}_{\mathcal{V}_0^Y(\mathcal{D}_X)}\left(\mathcal{V}_0^Y(\mathcal{D}_X) \otimes \overset{p}{\wedge}\mathcal{D}\mathrm{er}(\log Y), \mathcal{O}_X\right),$$

for $p = 0, \cdots, n$. Locally:

$$\psi^p(\omega_1 \wedge \cdots \wedge \omega_p)(P \otimes \delta_1 \wedge \cdots \wedge \delta_p) = P\left(\det(\langle \omega_i, \delta_j \rangle)_{1 \le i,j \le p}\right).$$

with ω_i $(i = 1, \cdots, n)$ local sections of $\Omega_X^1(\log Y)$ and P a logarithmic operator.

Similarly, if $\mathcal{M}$ is a left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module, given an integer $p \in \{1, \cdots, n\}$, there exist the following canonical isomorphisms:

$$\gamma_{\mathcal{M}}^p : \Omega_X^p(\log Y) \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\sim} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\overset{p}{\wedge}\mathcal{D}\mathrm{er}(\log Y), \mathcal{M}_X\right),$$

$$\lambda^p_{\mathcal{M}} : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\overset{p}{\wedge}\,\mathcal{D}\mathrm{er}(\log Y), \mathcal{M}\right) \xrightarrow{\sim} \mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0}\left(\mathcal{V}^Y_0(\mathcal{D}_X)\otimes_{\mathcal{O}_X}\overset{p}{\wedge}\,\mathcal{D}\mathrm{er}(\log Y), \mathcal{M}\right),$$

$$\psi^p_{\mathcal{M}} = \lambda^p_{\mathcal{M}}\circ\gamma^p_{\mathcal{M}} :\ \Omega^p_X(\log Y)(\mathcal{M}) \xrightarrow{\sim} \mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}\left(\mathcal{V}^Y_0(\mathcal{D}_X)\otimes\overset{p}{\wedge}\,\mathcal{D}\mathrm{er}(\log Y), \mathcal{M}\right).$$

Locally:

$$\psi^p_{\mathcal{M}}(\omega_1\wedge\cdots\wedge\omega_p\otimes m)(P\otimes\delta_1\wedge\cdots\wedge\delta_p) = P\cdot\det(\langle\omega_i,\delta_j\rangle)_{1\le i,j\le p}\cdot m.$$

Theorem 3.2.1.– If $\mathcal{M}$ is a left $\mathcal{V}^Y_0(\mathcal{D}_X)$-module (or, equivalently, is a $\mathcal{O}_X$-module with an integrable logarithmic connection), the complexes of sheaves of $\mathbb{C}$-vector spaces $\Omega^\bullet_X(\log Y)(\mathcal{M})$ and $\mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}(\mathcal{S}p^\bullet(\log Y), \mathcal{M})$ are canonically isomorphic.

Proof: The general case is solved if we prove the case $\mathcal{M} = \mathcal{V}^Y_0(\mathcal{D}_X)$, using the isomorphisms:

$$\Omega^\bullet_X(\log Y)(\mathcal{M}) \cong \Omega^\bullet_X(\log Y)(\mathcal{V}^Y_0(\mathcal{D}_X))\otimes_{\mathcal{V}^Y_0(\mathcal{D}_X)}\mathcal{M},$$

$$\mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}(\mathcal{S}p^\bullet(\log Y), \mathcal{M}) \cong \mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}\left(\mathcal{S}p^\bullet(\log Y), \mathcal{V}^Y_0(\mathcal{D}_X)\right)\otimes_{\mathcal{V}^Y_0(\mathcal{D}_X)}\mathcal{M}.$$

For $\mathcal{M} = \mathcal{V}^Y_0(\mathcal{D}_X)$, we obtain the right $\mathcal{V}^Y_0(\mathcal{D}_X)$-isomorphisms

$$\phi^p = \psi^p_{\mathcal{V}^Y_0(\mathcal{D}_X)} :\ \Omega^p_X(\log Y)(\mathcal{V}^Y_0(\mathcal{D}_X)) \to \mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}\left(\mathcal{S}p^{-p}(\log Y), \mathcal{V}^Y_0(\mathcal{D}_X)\right),$$

whose local expression are:

$$\phi^p\left((\omega_1\wedge\cdots\wedge\omega_p)\otimes Q\right)\left(P\otimes(\delta_1\wedge\cdots\wedge\delta_p)\right) = P\cdot\det\left(\langle\omega_i,\delta_j\rangle\right)\cdot Q.$$

To prove that these isomorphisms produce a isomorphism of complexes we have to check that they commute with the differential of the complex. Thanks to the isomorphism (b) of the proposition 3.1.6,

$$\mathcal{D}_X[\star Y]\otimes_{\mathcal{V}^Y_0(\mathcal{D}_X)}\mathcal{S}p^\bullet(\log Y) \simeq \mathcal{S}p^\bullet[\star Y],$$

we obtain a natural morphism of complexes of sheaves of right $\mathcal{V}^Y_0(\mathcal{D}_X)$-modules:

$$\tau^\bullet : \mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}\left(\mathcal{S}p^\bullet(\log Y), \mathcal{V}^Y_0(\mathcal{D}_X)\right) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{D}_X[\star Y]}\left(\mathcal{S}p^\bullet[\star Y], \mathcal{D}_X[\star Y]\right),$$

locally defined by:

$$\tau^p(\alpha)\left(R\otimes(\delta_1\wedge\cdots\wedge\delta_p)\right) = f^{-k}\alpha\left(P\otimes(f\delta_1\wedge\cdots\wedge f\delta_p)\right)$$

(for any local sections α of $\mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}\left(\mathcal{S}p^\bullet(\log Y), \mathcal{V}^Y_0(\mathcal{D}_X)\right)$, R of $\mathcal{D}_X[\star Y]$ and $\delta_1,\cdots,\delta_p$ of $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$), where P is a local section of $\mathcal{V}^Y_0(\mathcal{D}_X)$ such that $Rf^{-p} = f^{-k}P$ (see lemma 3.1.3). The morphisms τ^i are injective, because:

$$\alpha\left(P\otimes(\delta_1\wedge\cdots\wedge\delta_p)\right) = \tau^i(\alpha)\left(P\otimes(\delta_1\wedge\cdots\wedge\delta_p)\right).$$

Let us see the following diagram commutes:

$$\begin{array}{ccc} \Omega^p_X(\log Y)(\mathcal{V}^Y_0(\mathcal{D}_X)) & \stackrel{j^p}{\longrightarrow} & \Omega^p_X[\star Y]\,(\mathcal{D}_X[\star Y]) \\ \downarrow \phi^p & \# & \downarrow \Phi^p \\ \mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}(\mathcal{S}p^p\left(\log Y),\mathcal{V}^Y_0(\mathcal{D}_X)\right) & \stackrel{\tau^p}{\longrightarrow} & \mathcal{H}\mathrm{om}_{\mathcal{D}_X[\star Y]}(\mathcal{S}p^p[\star Y],\mathcal{D}_X[\star Y]) \end{array}$$

for each $p \geq 0$, where the Φ^p are the isomorphisms:

$$\Phi^p:\ \Omega^p_X[\star Y]\,(\mathcal{D}_X[\star Y]) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{D}_X[\star Y]}\left(\mathcal{D}_X[\star Y]\otimes \stackrel{p}{\wedge}\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X),\mathcal{D}_X[\star Y]\right),$$

$$\Phi^p((\omega_1\wedge\cdots\wedge\omega_p)\otimes Q)\,(P\otimes(\delta_1\wedge\cdots\wedge\delta_p)) = P\cdot\det\left(\langle\omega_i\cdot\delta_j\rangle_{1\leq i,j\leq p}\right)\cdot Q.$$

Given $\omega_1,\cdots,\omega_p$ local sections of $\Omega^1_X(\log Y)$, Q and R local sections of $\mathcal{D}_X[\star Y]$ and $\delta_1,\cdots,\delta_p$ local sections of $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$, we have

$$(\tau^p\circ\phi^p)((\omega_1\wedge\cdots\wedge\omega_p)\otimes Q)[R\otimes(\delta_1\cdots\wedge\delta_p)] =$$

$$f^{-k}\phi_p((\omega_1\wedge\cdots\wedge\omega_p)\otimes Q)[P\otimes(f\delta_1\wedge\cdots\wedge f\delta_p)] =$$

$$f^{-k}P\cdot\det(\langle\omega_i f\delta_j\rangle)\cdot Q = R\cdot f^{-p}\det(\langle\omega_i f\delta_j\rangle)\cdot Q = R\cdot\det(\langle\omega_i\delta_j\rangle)\cdot Q =$$

$$\Phi^p\circ j^p((\omega_1\wedge\cdots\wedge\omega_p)\otimes Q)[R\otimes(\delta_1\wedge\cdots\wedge\delta_p)],$$

with P a local section of $\mathcal{V}^Y_0(\mathcal{D}_X)$ such that $Rf^{-p} = f^{-k}P$.

But $\Phi^\bullet$, $j^\bullet$ and $\tau^\bullet$ are morphisms of complexes, and $\tau^\bullet$ is injective, hence we deduce that the ϕ^p commute with the differential and so define a isomorphism of complexes:

$$\phi^\bullet:\Omega^\bullet_X(\log Y)\left(\mathcal{V}^Y_0(\mathcal{D}_X)\right)\longrightarrow\mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}\left(\mathcal{S}p^\bullet(\log Y),\mathcal{V}^Y_0(\mathcal{D}_X)\right),$$

as we wanted to prove.

□

Corollary 3.2.2.– There exists a canonical isomorphism in the derived category:

$$\Omega^\bullet_X(\log Y)(\mathcal{M})\cong\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}(\mathcal{O}_X,\mathcal{M}).$$

Proof: By theorem 3.1.2, the complex $\mathcal{S}p^\bullet(\log Y)$ is a locally free resolution of $\mathcal{O}_X$ as left $\mathcal{V}^Y_0(\mathcal{D}_X)$-module. So, we have only to apply the theorem 3.2.1. □

Remark 3.2.3.– In the specific case that $\mathcal{M}=\mathcal{O}_X$, we have that the complexes $\Omega^\bullet_X(\log Y)$ and $\mathcal{H}om_{\mathcal{V}^Y_0(\mathcal{D}_X)}(\mathcal{S}p^\bullet(\log Y),\mathcal{O}_X)$ are canonically isomorphic and so, there exists a canonical isomorphism:

$$\Omega^\bullet_X(\log Y)\cong\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}(\mathcal{O}_X,\mathcal{O}_X).$$

Remark 3.2.4.– A classical problem is the comparison between the logarithmic de Rham complex and the meromorphic de Rham complex relative to a divisor Y,

$$\Omega_X^\bullet[\star Y] \cong \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X[\star Y]) \cong \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{O}_X, \mathcal{O}_X[\star Y])\,.$$

If Y is a normal crossing divisor, an easy calculation shows that they are quasi-isomorph (cf. [6]). The same result is true if Y is a strongly weighted homogeneous free divisor [5]. As a consequence of theorem 2.1.4, if Y is an arbitrary free divisor, the meromorphic de Rham complex and the logarithmic de Rham complex are quasi-isomorphic if and only if:

$$0 = \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\mathcal{D}_X \otimes^{\mathbf{L}}_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{O}_X, \frac{\mathcal{O}_X[\star Y]}{\mathcal{O}_X}\right)\left(= \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{V}_0^Y(\mathcal{D}_X)}\left(\mathcal{O}_X, \frac{\mathcal{O}_X[\star Y]}{\mathcal{O}_X}\right)\right).$$

4 Perversity of the logarithmic complex

Now we consider the complex $\mathcal{D}_X \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log Y)$:

$$0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{n}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \overset{\varepsilon_{-n}}{\to} \cdots\cdots \overset{\varepsilon_{-2}}{\to} \mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{1}{\wedge} \mathcal{D}\mathrm{er}(\log Y) \overset{\varepsilon_{-1}}{\to} \mathcal{D}_X,$$

where the local expressions of the morphisms are defined by:

$$\varepsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \delta_p) +$$

$$\sum_{1 \le i < j \le p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \wedge \widehat{\delta_i} \wedge \cdots \wedge \widehat{\delta_j} \wedge \cdots \wedge \delta_p), \quad (2 \le p \le n).$$

$$\varepsilon_{-1}(P \otimes \delta) = P\delta.$$

In the case that Y is a free divisor, we can work at each point x of Y with a basis $\{\delta_1, \cdots, \delta_n\}$ of $\mathrm{Der}(\log f)$, with f a local reduced equation of Y at x.

Proposition 4.0.5.– If $\{\delta_1, \cdots, \delta_n\}$ is a basis of $\mathrm{Der}(\log f)$, and the sequence $\{\sigma(\delta_1), \cdots, \sigma(\delta_n)\}$ is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular, it verifies

$$\sigma(\mathcal{D}(\delta_1, \cdots, \delta_n)) = \mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma(\delta_1), \cdots, \sigma(\delta_n)).$$

Proof: The inclusion $\mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma(\delta_1), \cdots, \sigma(\delta_n)) \subset \sigma(\mathcal{D}(\delta_1, \cdots, \delta_n))$ is clair. Let F be the symbol of an operator P of order d, with

$$P = \sum_{i=1}^{n} P_i \delta_i \in \mathcal{D}(\delta_1, \cdots, \delta_n).$$

We will prove by induction that $F = \sigma(P)$ belongs to $\mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma_1, \cdots, \sigma_n)$, with $\sigma_i = \sigma(\delta_i)$. We will do the induction on the maximum order of the P_i ($i = 1, \cdots, n$), order that we will denote by k_0. As P has order d, k_0 is greater or equal to $d-1$. If $k_0 = d-1$, we have:

$$\sigma(P) = \sum_{i\in K} \sigma(P_i)\sigma_i,$$

with K the set of subindexes j such that P_j has order k_0 in $\mathcal{D}$. We suppose that the result holds when $d-1 \leq k_0 < m$. Let $F = \sigma(P)$, with $P = \sum_{i=1}^n P_i\delta_i$ and $k_0 = m$. There are two possibilities:

1. $F = \sigma(P) = \sum_{i\in K} \sigma(P_i)\sigma_i \in \mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma_1, \cdots, \sigma_n)$, as we wanted to prove.

2. $\sum_{i\in K} \sigma(P_i)\sigma_i = 0$.

In this last case, as $\{\sigma_1, \cdots, \sigma_n\}$ is a $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular sequence, if we call F_i the symbol $\sigma(P_i)$ in the case that $i \in K$ and 0 otherwise, we have:

$$(F_1, \cdots, F_n) = \sum_{i<j} F_{ij}(0, \cdots, 0, \overset{i}{\smile}{\sigma_j}, 0, \cdots, 0, \overset{j}{\smile}{-\sigma_i}, 0, \cdots, 0),$$

with $F_{ij} \in \mathrm{Gr}_{F^\bullet}(\mathcal{D})$ homogeneous polynomials of order $m-1$. We choose, for $1 \leq i < j \leq n$, operators Q_{ij}, of order $m-1$ in $\mathcal{D}$, such that $\sigma(Q_{ij}) = F_{ij}$, and define:

$$(Q_1, \cdots, Q_n) = (P_1, \cdots, P_n) - \sum_{i<j} Q_{ij} \left((0, \cdots, 0, \overset{i}{\smile}{\delta_j}, 0, \cdots, 0, \overset{j}{\smile}{-\delta_i}, 0, \cdots, 0) - \underline{\alpha}_{ij} \right),$$

where $\underline{\alpha}_{ij}$ are the vectors with n coordinates in $\mathcal{O}$ defined by the relations:

$$[\delta_i, \delta_j] = \sum_{k=1}^n a_{ij}^k \delta_k = \underline{\alpha}_{ij} \bullet \underline{\delta},$$

with $\underline{\delta} = (\delta_1, \cdots, \delta_n)$. These Q_i, of order m in $\mathcal{D}$, verify

$$(\sigma_m(Q_1), \cdots, \sigma_m(Q_n)) =$$

$$(F_1, \cdots, F_n) - \sum_{i<j} F_{ij}(0, \cdots, 0, \overset{i}{\smile}{\sigma_j}, 0, \cdots, 0, \overset{j}{\smile}{-\sigma_i}, 0, \cdots, 0) = 0.$$

So, Q_i has order $m-1$ in $\mathcal{D}$. Moreover,

$$\sum_{i=1}^n Q_i\delta_i = \sum_{i=1}^n P_i\delta_i - \sum_{i<j} Q_{ij} \left(\delta_i\delta_j - \delta_j\delta_i - [\delta_i, \delta_j]\right) = \sum_{i=1}^n P_i\delta_i = P.$$

We apply the induction hypothesis to $F = \sigma(P)$, with $P = \sum_{i=1}^{n} Q_i \delta_i$, and obtain:

$$\sigma(P) \in \mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma_1, \cdots, \sigma_n).$$

$\square$

Proposition 4.0.6.– Let $\{\delta_1, \cdots, \delta_n\}$ be a basis of $\mathrm{Der}(\log\ f)$. If the sequence $\sigma(\delta_1), \cdots, \sigma(\delta_n)$ is a $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular sequence in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, the complex $\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} \mathcal{S}p^\bullet(\log\ f)$ is a resolution of the quotient module $\frac{\mathcal{D}}{\mathcal{D}(\delta_1, \cdots, \delta_n)}$.

Proof: We consider the complex $\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} \mathcal{S}p^\bullet(\log\ f)$. We can augment this complex of $\mathcal{D}$-modules by another morphism:

$$\varepsilon_0 : \mathcal{D} \to \frac{\mathcal{D}}{\mathcal{D}(\delta_1, \cdots, \delta_n)}, \quad \varepsilon_0(P) = P + \mathcal{D}(\delta_1, \cdots, \delta_n).$$

We denote by $\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} \widetilde{\mathcal{S}p}^\bullet(\log f)$ the new complex. To prove that this new complex is exact, we define a discrete filtration $G^\bullet$ such that the graded complex be exact (cf. [1, lemma 3.16]):

$$G^k\left(\mathcal{D} \otimes_{\mathcal{O}} \overset{p}{\wedge} \mathrm{Der}(\log\ f)\right) = F^{k-p}(\mathcal{D}) \otimes_{\mathcal{O}} \overset{p}{\wedge} \mathrm{Der}(\log\ f),$$

$$G^k\left(\frac{\mathcal{D}}{\mathcal{D}(\delta_1, \cdots, \delta_n)}\right) = \frac{F^k(\mathcal{D}) + \mathcal{D} \cdot (\delta_1, \cdots, \delta_n)}{\mathcal{D}(\delta_1, \cdots, \delta_n)}.$$

Clairly the filtration is compatible with the differential of the complex. Moreover:

$$\mathrm{Gr}_{G^\bullet}\left(\mathcal{D} \otimes \overset{p}{\wedge} \mathrm{Der}(\log\ f)\right) = \mathrm{Gr}_{F^\bullet}(\mathcal{D})[-p] \otimes \overset{p}{\wedge} \mathrm{Der}(\log\ f),$$

and, by the previous proposition,

$$\mathrm{Gr}_{G^\bullet}\left(\frac{\mathcal{D}}{\mathcal{D}(\delta_1, \cdots, \delta_n)}\right) = \frac{\mathrm{Gr}_{F^\bullet}(\mathcal{D})}{\sigma\left(\mathcal{D} \cdot (\delta_1, \cdots, \delta_n)\right)} = \frac{\mathrm{Gr}_{F^\bullet}(\mathcal{D})}{\mathrm{Gr}_{F^\bullet}(\mathcal{D}) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n))}.$$

We consider the complex $\mathrm{Gr}_{G^\bullet}\left(\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} \widetilde{\mathcal{S}p}^\bullet(\log\ f)\right)$:

$$0 \to \mathrm{Gr}_{F^\bullet}(\mathcal{D})[-n] \otimes_{\mathcal{O}} \overset{n}{\wedge} \mathrm{Der}(\log\ f) \overset{\psi_{-n}}{\to} \cdots \overset{\psi_{-2}}{\to} \mathrm{Gr}_{F^\bullet}(\mathcal{D})[-1] \otimes_{\mathcal{O}} \overset{1}{\wedge} \mathrm{Der}(\log\ f)$$

$$\overset{\psi_{-1}}{\to} \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \overset{\psi_0}{\to} \frac{\mathrm{Gr}_{F^\bullet}(\mathcal{D})}{\mathrm{Gr}_{F^\bullet}(\mathcal{D}) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n))} \to 0,$$

where the local expression of the differential is defined by:

$$\psi_{-p}(G \otimes \delta_{j_1} \wedge \cdots \wedge \delta_{j_p}) = \sum_{i=1}^{p} (-1)^{i-1} G\sigma(\delta_{j_i}) \otimes \delta_{j_1} \wedge \cdots \wedge \widehat{\delta_{j_i}} \wedge \cdots \wedge \delta_{j_p}, \quad (2 \leq p \leq n),$$

$$\psi_{-1}(G \otimes \delta_i) = G\sigma(\delta_i),$$
$$\psi_0(G) = G + \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n)).$$

This complex is the Koszul complex of the ring $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ with respect to the sequence $\sigma(\delta_1), \cdots, \sigma(\delta_n)$. So we deduce that, if the sequence $\sigma(\delta_1), \cdots, \sigma(\delta_n)$ is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, the complex

$$\mathrm{Gr}_{G^\bullet}\left(\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} \widetilde{\mathcal{S}p}^\bullet(\log f)\right)$$

is exact. So, the complex $\mathcal{D}\otimes_{\mathcal{V}_0^f(\mathcal{D})}\widetilde{\mathcal{S}p}^\bullet(\log f)$ is exact too, and $\mathcal{D}\otimes_{\mathcal{V}_0^f(\mathcal{D})}\mathcal{S}p^\bullet(\log f)$ is a resolution of $\frac{\mathcal{D}}{\mathcal{D}(\delta_1,\cdots,\delta_n)}$. □

Corollary 4.0.7.– Let Y be a free divisor. With the conditions of the previous proposition (for each point x of Y, there exists a basis $\{\delta_1, \cdots, \delta_n\}$ of $\mathrm{Der}(\log f)$ such that the sequence $\sigma(\delta_1), \cdots, \sigma(\delta_n)$ is a $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular sequence), the sheaf $\Omega_X^\bullet(\log Y)$ is a perverse sheaf.

Proof: With the same conditions of the previous proposition, the homology of the complex $\mathcal{D}_X \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log\, Y)$ is concentrated in degree 0. All its homology groups are zero except the group in degree 0, which verifies:

$$h^0\left(\mathcal{D}_X \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log\, Y)\right) = \frac{\mathcal{D}_X}{\mathcal{D}_X \cdot \mathcal{D}\mathrm{er}(\log Y)} = \frac{\mathcal{D}_X}{\mathcal{D}_X \cdot (\delta_1, \cdots, \delta_n)} = \mathcal{E},$$

where $\{\delta_1, \cdots, \delta_n\}$ is a local basis of $\mathcal{D}\mathrm{er}(\log Y)$. But $\mathcal{E}$ is a holonomic $\mathcal{D}_X$-module because:

$$\mathcal{G}\mathrm{r}_F(\mathcal{E}) = \frac{\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)}{(\sigma(\delta_1), \cdots, \sigma(\delta_n))}$$

has dimension n (using the fact that $\sigma(\delta_1), \cdots, \sigma(\delta_n)$ is a $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$-regular sequence). So (using remark 3.2.3 for the first equality and teorema 3.1.2 for the last equality)):

$$\Omega_X^\bullet(\log\, Y) = \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\mathcal{D}_X \otimes^{\mathbf{L}}_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{O}_X, \mathcal{O}_X\right) =$$

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\mathcal{D}_X \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log\, Y), \mathcal{O}_X\right) = \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\frac{\mathcal{D}_X}{\mathcal{D}_X(\delta_1, \cdots, \delta_n)}, \mathcal{O}_X\right)$$

is a perverse sheaf (as solution of a holonomic $\mathcal{D}_X$-module, cf. [11]).

□

Corollary 4.0.8.– Let Y be any divisor in X, with $\dim_{\mathbb{C}} X = 2$. Then $\Omega_X^\bullet(\log\, Y)$ is a perverse sheaf.

Proof: We know that, if $\dim_{\mathbb{C}} X = 2$, any divisor Y in X is free [14]. So, we have only to check that the other hypothesis of the previous corollary holds. We consider the symbols $\{\sigma_1, \sigma_2\}$ of a basis $\{\delta_1, \delta_2\}$ of Der(log f), where f is a reduced equation of Y. We have to see that they form a $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular sequence. If they do not, they have a common factor $g \in \mathcal{O}$, because they are symbols of operators of order 1. If g is a unit, we divide one of them by g and eliminate the common factor. If g is not a unit, it would be in contradiction with Saito's Criterion, because the determinant of the coefficients of the basis $\{\delta_1, \delta_2\}$ would have as factor g^2, with g not invertible, and this determinant has to be equal to f multiplied by a unit. □

Remark 4.0.9.– The regularity of the sequence of the symbols of a basis of Der(log f) in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ is not necessary for the perversity of the logarithmic de Rham complex. For example, if $X = \mathbb{C}^3$ and $Y \equiv \{f = 0\}$, with $f = xy(x+y)(y+tx)$, f is a free divisor such that the graded complex

$$\mathcal{G}\mathrm{r}_{G^\bullet}(\mathcal{D}_X \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log Y)) = K(\sigma(\delta_1), \sigma(\delta_1), \sigma(\delta_3); \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X))$$

is not concentrated in degree 0, but the complex

$$\mathcal{D}_X \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{S}p^\bullet(\log Y)$$

is. Moreover, in this case the dimension of $\frac{\mathcal{D}_X}{\mathcal{D}_X \cdot (\delta_1, \delta_2, \delta_3)}$ is 3 and so, $\Omega_X^\bullet(\log Y)$ is a perverse sheaf.

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