A dynamical characterization of sub-Arakelian subsets

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Abstract

We are going to characterize those sets which can be covered by an Arakelian set in terms of dynamical properties of entire functions via similarities. Moreover, if we consider the set of universal entire functions via similarities that are bounded on such a sub-Arakelian set, then it is shown that its algebraic size is as large as possible. As a consequence, we prove that the set of universal entire functions bounded on every (straight) line is algebraically large.

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1. Introduction and known results

In this paper we consider the concept of Arakelian sets which is well-known in Complex Approximation Theory. If F is a subset of the complex plane $\mathbb{C}$, then F is said to be an *Arakelian set* provided that F is a nonempty,

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closed subset of $\mathbb{C}$ whose complement with respect to the extended plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is connected and locally connected at ∞. The concept of an Arakelian set is essential in the Arakelian Approximation Theorem (see [20, pp. 153–154]) which has revealed as a very important tool in many results related to the existence of entire functions f that are universal in Birkhoff's sense.

In 1929 Birkhoff [13] proved the existence of an entire function f which is "universal" under translations, that is, whose family of translates $\{f(a + \cdot) : a \in \mathbb{C}\}$ is dense in $H(\mathbb{C})$ –the space of all entire functions–. Recall that $H(\mathbb{C})$, when endowed with the compact open topology, becomes a separable completely metrizable space [24]. In a more general setting, Birkhoff-universal functions are precisely the hypercyclic vectors of $H(\mathbb{C})$ with respect to the translation operators $T_a f := f(\cdot + a)$ (see [14], [22], [23]), but this generality will not be considered in the sequel. The harmonic analogue of Birkhoff's theorem –that is, $H(\mathbb{C})$ is replaced by the space $h(\mathbb{R}^N)$ of harmonic functions on $\mathbb{R}^N$– can be found in Armitage-Gauthier's paper [2].

Several analysts have recently focused their attention to find entire functions bearing such an "extremely wild" behavior and simultaneously "extremely tamed" behavior. To be more specific, the compatibility of Birkhoff universality with "opposite" properties –such as boundedness or even rapid decay to zero (as $z \to \infty$) on large sets– has been studied. A number of results have been recently produced in this vein.

Namely, the first author [16] (see also [21]) showed in 2002 the existence of a dense linear manifold of Birkhoff-universal entire functions f which are bounded on any domain lying between two parallel straight lines; and even many more additional properties hold. For similar results in the harmonic setting see [15].

Independently, Costakis and Sambarino [17] proved in 2004 that, given a compact set $K \subset \mathbb{C}$, there exists an entire function f whose translates $z \mapsto f(z + n)$ $(n \in \mathbb{N})$ are dense in $H(\mathbb{C})$ and such that f tends to zero on certain "translated sector" of K. Gharibyan, Luh and Niess [21, Theorem 1.1] demonstrated that, given a sector S there is a dense subset $M \subset H(\mathbb{C})$ such that every function $f \in M$ is bounded in S and Birkhoff-universal.

In 2006, Bernal and Bonilla [8] were able to establish that, for a prescribed subset $A \subset \mathbb{C}$, there exists a Birkhoff-universal entire function f that is bounded on A if and only if there exists an Arakelian subset F of $\mathbb{C}$ such that $A \subset F$ and $\rho(\mathbb{C} \setminus F) = +\infty$. Here $\rho(A) = \sup\{r > 0 :$ there exists a closed ball B of radius r with $B \subset A\}$ is the inscribed radius of a subset $A \subset \mathbb{C}$. In 2010, Bernal, Luh and the first author [9] stated the following: If $F \subset \mathbb{C}$ is an unbounded Arakelian set with $\rho(\mathbb{C} \setminus F) = +\infty$, then there is a dense linear manifold M of entire functions all of whose nonzero members are Birkhoff-universal and $\exp(|z|^\alpha) f(z) \to 0$ $(z \to 0,\ z \in A)$ for all $\alpha < 1/2$ and $f \in M$.

Recall that the set of automorphisms of $\mathbb{C}$ is the set of all similarities, that is, $\mathrm{Aut}(\mathbb{C}) = \{a + bz : \ a, b \in \mathbb{C},\ b \neq 0\}$. In general, we say that an entire function f is *universal under similarities* if $\{f \circ \varphi : \ \varphi \in \mathrm{Aut}(\mathbb{C})\}$ is dense in $H(\mathbb{C})$.

In this paper, we provide a rather general statement about the existence of large linear manifolds of universal entire functions under similarities that are bounded on a prescribed subset $A \subset \mathbb{C}$. In particular, such sets A are characterized in terms of its ability to be "filled as Arakelian set". We also deal with prescribed sequences of similarities.

2. Sub-Arakelian sets and boundedness universality. Algebraic genericity

Inspired by the results of Bernal and Bonilla [8] we introduce the following concept.

Definition 2.1. *We say that a subset $A \subset \mathbb{C}$ is a* sub-Arakelian set *whenever there exists an Arakelian set $F \subset \mathbb{C}$, $F \neq \mathbb{C}$, with $A \subset F$.*

Observe that any Arakelian set, except $\mathbb{C}$, is trivially sub-Arakelian, but the converse is not true. For instance, the set $A = \{z \in \mathbb{C} : \ |\Re z| \leq 2,\ |z| \geq 1\}$ is sub-Arakelian, just considering $F = \{z \in \mathbb{C} : \ |\Re z| \leq 2\}$, but A is not Arakelian because its complement is not connected. In a way, a set A is sub-Arakelian if it becomes Arakelian when their (bounded) "holes" are filled. Moreover, if we take an Arakelian set (not equal to $\mathbb{C}$) and we make "holes" inside it, the new set is sub-Arakelian.

Of course, there are many sets which are not sub-Arakelian. For example, the sets $\mathbb{C} \setminus \{0\}$, $\mathbb{C} \setminus \mathbb{N}$ and $\mathbb{C} \setminus \bigcup_{n=1}^{\infty} B\,(n, 1/2)$ are not sub-Arakelian sets.

The following technical Lemma shows us that it is always possible to "add" to an Arakelian set a sequence of compact sets tending to ∞ with no loss of the Arakelian condition. Given a subset $A \subset \mathbb{C}$ we denote by A° (resp., $\overline{A}$, and ∂A) the interior (resp., the closure, and the boundary) of A.

Lemma 2.1. *Given an Arakelian set $F \subset \mathbb{C}$, $F \neq \mathbb{C}$, and a point $z_0 \in \mathbb{C} \setminus F$, there exist two double sequences $\{a_{n,k}\}_{k \in \mathbb{N},\, n \leq k} \subset \mathbb{C}$, $\{r_{n,k}\}_{k \in \mathbb{N},\, n \leq k} \subset (0, +\infty)$ and an increasing sequence $\{R_k\}_{k \geq 0}$ of positive numbers tending to infinity, such that:*

(i) $B_0 \cap F = \varnothing$*, where* $B_0 := \overline{B}(z_0, R_0)$*.*

(ii) *For all $k \in \mathbb{N}$ and all $n \leq k$, if we define the sets $K_{n,k} := \overline{B}(a_{n,k}, r_{n,k})$ and $B_k := \overline{B}(0, R_k)$, then the sets $K_{n,k}$ are pairwise disjoints and $K_{n,k} \subset B_{k+1} \setminus (B_k \cup F)$.*

(iii) *The set $C_0 := F \cup B_0 \cup \bigcup_{\substack{k \in \mathbb{N} \\ j \leq k}} K_{j,k}$ is an Arakelian set.*

(iv) *For all $n \in \mathbb{N}$, the set $C_n := (F \setminus B_{n+1}^\circ) \cup B_n \cup \bigcup_{\substack{k \geq n \\ j \leq k}} K_{j,k}$ is an Arakelian set.*

Proof. As $z_0 \notin F$ and F is a proper closed subset of $\mathbb{C}$, there is $R_0 \in (0, 1)$ such that $\overline{B}(z_0, R_0) \cap F = \varnothing$.

There is $n_0 \in \mathbb{N}$ such that $B(0, n) \cap F \neq \varnothing$ for all $n \geq n_0$. Let $R_k := n_0 + k$ ($k \in \mathbb{N}$) and define $B_k := \overline{B}(0, R_k)$. For each $k \in \mathbb{N}$ the set $[(\mathbb{C} \setminus F) \setminus B_k] \cap B_{k+1}^\circ$ is nonempty and open, so we can select exactly k pairwise disjoint closed balls $K_{n,k} := \overline{B}(a_{n,k}, r_{n,k})$ ($n = 1, \cdots, k$) contained in $[(\mathbb{C} \setminus F) \setminus B_k] \cap B_{k+1}^\circ$.

If now we define $C_0 = F \cup B_0 \cup \bigcup_{\substack{k \in \mathbb{N} \\ n \leq k}} K_{n,k}$, then C_0 is Arakelian because B_0, $K_{j,k}$ is a countable family of pairwise disjoint closed balls which does not cut F and such that it goes to ∞. It is clear that $C_n := (F \setminus B_{n+1}^\circ) \cup B_n \cup \bigcup_{\substack{k \geq n \\ j \leq k}} K_{j,k}$ ($n \in \mathbb{N}$) is also Arakelian, just taking into account that B_{n+1}° is a bounded connected open subset of $\mathbb{C}$ with $B_{n+1}^\circ \setminus F \neq \varnothing$. □

Let X be a topological vector space and $A \subset X$. We say that A is *spaceable* (see [4]) if there exists a closed infinite-dimensional linear subspace M such that $M \setminus \{0\} \subset A$. We say A is *dense-lineable* (resp. *maximal dense-lineable*, see [6]) if there exists a dense linear subspace $M \subset X$ such that $M \setminus \{0\} \subset A$ (resp. and M has maximal algebraic dimension, that is, $\dim(M) = \dim(X)$). Note that, since $H(\mathbb{C})$ is a separable complete metrizable infinite-dimensional vector space, we have $\dim(H(\mathbb{C})) = \mathfrak{c} :=$ the cardinality of the continuum. Thus $\mathfrak{c}$ is the maximal dimension allowed for any subspace of $H(\mathbb{C})$. Observe that there is no relationship between the two concepts: spaceability and dense-lineability. Of course, maximal dense-lineability implies dense-lineability but not the converse.

Let us see now how it is possible to establish the connection between the existence of "algebraically many" universal entire functions bounded on a set A and the mere topological condition of being sub-Arakelian. The following is our main result.

Theorem 2.2. *Assume that $A \subset \mathbb{C}$. The following statements are equivalent:*

(a) *A is a sub-Arakelian set.*

(b) *There exists an entire function universal with respect to similarities and bounded on A.*

(c) *The set of functions universal with respect to similarities and bounded on A is dense-lineable.*

(d) *The set of functions universal with respect to similarities and bounded on A is spaceable.*

(e) *The set of functions universal with respect to similarities and bounded on A is maximal dense-lineable.*

Proof. It is obvious that (c), (d) and (e) (separately) imply (b). If (b) is true, let f be the universal entire function bounded on A; by [18, Theorem 1] the set $F := \{z \in \mathbb{C} : \ |f(z)| \leq \sup_A |f|\}$ is an Arakelian set and it is clear that $A \subset F$. Moreover, F is not equal to $\mathbb{C}$; otherwise, f would be bounded in $\mathbb{C}$ and, by Liouville's theorem, should be constant. But f is universal, so f is not constant. Hence A is sub-Arakelian and we have (a). To have all the equivalences we are going to prove that (a) implies (c) and, in order to clarify the proof, that (a) implies (d) and finally that (a) implies (e).

Let us prove that (a) implies (c). Assume that A is sub-Arakelian, so there is an Arakelian set F with $A \subset F$. Let $K_{n,k} := \overline{B}(a_{n,k}, r_{n,k})$ and $B_k := \overline{B}(0, R_k)$ be the sequences of sets given by Lemma 2.1, when applied to F; and let $\{P_n(z)\}_n$ be a sequence of entire functions dense in $H(\mathbb{C})$.

For each $n \in \mathbb{N}$, we define the following function:

$$g_n(z) := \begin{cases} 0 & \text{if } \; z \in F \setminus B_{n+1}^\circ \\ P_n(z) & \text{if } \; z \in B_n \\ 0 & \text{if } \; z \in K_{j,k}, \; k \geq n, \; n \neq j \leq k \\ P_k\left(\frac{z-a_{n,k}}{r_{n,k}/R_k}\right) & \text{if } \; z \in K_{n,k}, \; k \geq n. \end{cases}$$

(If F is bounded, then there is $n_0 \in \mathbb{N}$ such that $F \subset B_{n_0+1}^\circ$. In this case we do not consider the part $F \setminus B_{n+1}^\circ (= \varnothing)$ in the definition of g_n ($n \geq n_0$), and the proof of the boundedness on F is trivial.)

The sets are pairwise disjoint by construction, so each function g_n is continuous in $C_n := (F \setminus B_{n+1}^\circ) \cup B_n \cup \bigcup_{\substack{k \geq n \\ j \leq k}} K_{j,k}$ and holomorphic in C_n°. According to Lemma 2.1, each C_n is an Arakelian set.

We consider now the functions $\varepsilon_n : [0, +\infty) \to (0, +\infty)$ given by $\varepsilon_n(t) = \frac{1}{n} e^{-t^{1/4}}$. By a variant for $\mathbb{C}$ of the Arakelian Theorem (see [1] or [20, pp. 160–162]), there are entire functions $\{f_n\}_n$ such that

$$|f_n(z) - g_n(z)| < \frac{1}{n} \exp\left(-|z|^{1/4}\right) \qquad \text{for each } z \in C_n. \tag{1}$$

Let $M := \operatorname{span}\{f_n : n \in \mathbb{N}\}$. This is the linear subspace we are looking for.

Fix $n \in \mathbb{N}$. From (1), we have that $|f_n(z)| < 1$ for all $z \in F \setminus B_{n+1}^\circ$. But $B_{n+1} = \overline{B}(0, R_{n+1})$ is compact, hence f_n is bounded on F and, by linearity, each function $f \in M$ is bounded on F, so in A.

It follows from (1) and the definitions of f_n and g_n that

$$|f_n(z) - P_n(z)| < \frac{1}{n} \qquad \text{for each } z \in B_n, \tag{2}$$

$$|f_n(z)| < \exp\left(-|z|^{1/4}\right) \qquad \text{for each } z \in K_{j,k}, \; k \geq j \geq n+1 \tag{3}$$

and

$$\left| f_n(z) - P_k\left(\frac{z - a_{n,k}}{r_{n,k}/R_k}\right) \right| < \exp\left(-|z|^{1/4}\right) \qquad \text{for each } z \in K_{n,k}, \; k \geq n. \tag{4}$$

According to (2), we have that $\lim\limits_{n\to\infty} \sup\limits_{z \in B_n} |f_n(z) - P_n(z)| = 0$. Since $\{P_n\}_n$ is a dense sequence in $H(\mathbb{C})$, we can apply an elementary topological result (see, for instance, [9, Lemma 2.7]), to get the density of the sequence $\{f_n\}_n$. Hence the set M is a dense linear manifold of $H(\mathbb{C})$.

It only remains to prove the universality of each nonzero function of M. Fix a function $f = \sum_{j=1}^{N} \lambda_j f_j \in M \setminus \{0\}$. Without loss of generality (any nonzero scalar multiple of a universal function is also universal) we can assume that $\lambda_N = 1$.

Let us fix $k \geq N$. If $z \in B_k$, then $w := \frac{r_{N,k}}{R_k} z + a_{N,k} \in \overline{B}(a_{N,k}, r_{N,k}) = K_{N,k}$. Hence by (3) and (4) we have that

$$\begin{aligned}
\left| f\left(\frac{r_{N,k}}{R_k} z + a_{N,k}\right) - P_k(z) \right| &\leq \left| f_N(w) - P_k\left(\frac{w - a_{N,k}}{r_{N,k}/R_k}\right) \right| + \sum_{j=1}^{N-1} |\lambda_j| \cdot |f_j(w)| \\
&\leq \frac{1}{N} e^{-|w|^{1/4}} + \sum_{j=1}^{N-1} \frac{|\lambda_j|}{j} e^{-|w|^{1/4}} \\
&\leq \left(1 + \sum_{j=1}^{N-1} |\lambda_j|\right) e^{-|R_k|^{1/4}} \to 0 \quad (k \to \infty).
\end{aligned}$$

The last inequality is true because, by the construction of the sets $K_{n,k}$, we have $K_{N,k} \subset B_{k+1}^{\circ} \setminus B_k$. So $|w| > R_k$ if $w \in K_{N,k}$ and $R_k \to \infty$ $(k \to \infty)$. Hence,

$$\lim_{k \to \infty} \sup_{z \in B_k} \left| f\left(\frac{r_{N,k}}{R_k} z + a_{N,k}\right) - P_k(z) \right| = 0$$

and again [9, Lemma 2.7] gives us the denseness of $\{f \circ \varphi_k\}_k$, where $\varphi_k(z) := \frac{r_{N,k}}{R_k} z + a_{N,k} \in \mathrm{Aut}(\mathbb{C})$. Therefore f is universal under similarities. This finishes the proof of (a) implies (c).

In order to prove that (a) implies (d), we proceed in a similar way than the previous case. Without loss of generality, we may assume that the set B_0 given by Lemma 2.1 is, in fact the closed unit ball $\overline{\mathbb{D}}$.

For each $n \in N$, we define the following function:

$$G_n(z) := \begin{cases} z^n & \text{if } \ z \in \overline{\mathbb{D}} \\ 0 & \text{if } \ z \in F \\ 0 & \text{if } \ z \in K_{j,k}, \ j \neq n, \ k \geq j \\ P_k\left(\frac{z - a_{n,k}}{r_{n,k}/R_k}\right) & \text{if } \ z \in K_{n,k}, \ k \geq n. \end{cases}$$

Again from Lemma 2.1, the set $C_0 := F \cup \overline{\mathbb{D}} \cup \bigcup_{\substack{k\in\mathbb{N}\\ j\le k}} K_{j,k}$ is an Arakelian set and the functions G_n are continuous in C_0 and holomorphic in C_0°. Hence, by considering $\varepsilon_n(t) = \frac{1}{3^n} e^{-1/t^4}$, there are functions $F_n \in H(\mathbb{C})$ $(n \in \mathbb{N})$ such that

$$|F_n(z) - G_n(z)| < \frac{1}{3^n} \exp\left(-|z|^{1/4}\right) \qquad \text{for all } z \in C_0. \tag{5}$$

In particular, we have that

$$|F_n(z) - z^n| < \frac{1}{3^n} \quad \text{for all } z \in \overline{\mathbb{D}}.$$

Therefore, by the basis perturbation theorem (see [19, p. 50]), $\{F_n\}_n$ is a basic sequence in $L^2(\partial\mathbb{D})$ (the Hilbert space of measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$ with finite quadratic norm $\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\, d\theta)$ equivalent to $\{z^n\}_n$, and the functions F_n $(n \ge 1)$ are linearly independent.

Let $S_0 := \text{span}\{F_n : n \in \mathbb{N}\}$ and $S := \overline{S_0} = \overline{\text{span}}\{F_n : n \in \mathbb{N}\}$. It is obvious that S is a closed linear subspace with infinite dimension. It only remains to prove that each non-zero function in S is bounded on A and universal under similarities.

Let $f \in S\backslash\{0\}$ and let $f = \sum_{j=1}^\infty \alpha_j F_j$ be its representation on $L^2(\partial\mathbb{D})$. As $f \neq 0$ there is some nonzero coefficient, say α_1. Without loss of generality, we can assume that $\alpha_1 = 1$ (because of the invariance under scalar multiplication of boundedness and universality).

Since $S = \overline{S_0}$, there is a sequence $\left\{h_l := \sum_{j=1}^{N(l)} \alpha_j^{(l)} F_j\right\}_l \subset S_0$ converging to f uniformly in compacta of $\mathbb{C}$. But convergence in $H(\mathbb{C})$ is stronger than convergence in $L^2(\partial\mathbb{D})$, so $h_l \to f$ in $L^2(\partial\mathbb{D})$. Now the continuity of each projection guarantees that $\alpha_1^{(l)} \to 1$ $(l \to \infty)$. We are going to assume that $\alpha_1^{(l)} = 1$ for all l (if it were not the case, we would take $H_l := h_l + (1-\alpha_1^{(l)})F_1$ which enjoy all the above properties). Finally, by [10, Lemma 2.3], there is a constant $H > 0$ such that $\sum_{j=1}^{N(l)} |\alpha_j^{(l)}|^2 < H$ for all $l \in \mathbb{N}$.

Let us prove first that f is bounded on A. Fix $n \in \mathbb{N}$. The set $F \cap B_n$ is compact, so there is $l_n \in \mathbb{N}$ such that $|h_{l_n}(z) - f(z)| < 1$ for all $z \in F \cap B_n$.

Hence, for $z \in F \cap B_n$, we have that

$$|f(z)| \leq 1 + |h_{l_n}(z)| \leq 1 + \left(\sum_{j=1}^{N(l_n)} |\alpha_j^{(l_n)}|^2 \right)^{1/2} \left(\sum_{j=1}^{N(l_n)} |F_j(z)|^2 \right)^{1/2}$$

$$\leq 1 + \sqrt{H} \left(\sum_{j=1}^{\infty} \left(\frac{1}{3^j} \right)^2 \right)^{1/2} \leq 1 + \sqrt{H},$$

where we have used the Cauchy-Schwarz inequality and the fact that if $z \in F$, then, by (5), $|F_j(z)| < \frac{1}{3^j}$ $(j \in \mathbb{N})$. But $F = \bigcup_{n \in \mathbb{N}} (F \cap B_n)$ and the constant $1 + \sqrt{H}$ does not depend on n, so we have that f is bounded on F and, consequently on A.

In order to prove the universality part, fix $k \in \mathbb{N}$. Since $h_l \to f$ $(l \to \infty)$ in $H(\mathbb{C})$, there is $l_0 \in \mathbb{N}$ such that $|f(z) - h_{l_0}(z)| < \frac{1}{k}$ for all $z \in K_{1,k}$. Then, for any $z \in B_k$ we have that

$$\left| f\left(\frac{r_{1,k}}{R_k} z + a_{1,k} \right) - P_k(z) \right| \leq \left| f\left(\frac{r_{1,k}}{R_k} z + a_{1,k} \right) - h_{l_0}\left(\frac{r_{1,k}}{R_k} z + a_{1,k} \right) \right|$$

$$+ \left| h_{l_0}\left(\frac{r_{1,k}}{R_k} z + a_{1,k} \right) - P_k(z) \right|. \tag{6}$$

But $z \in B_k$ if and only if $w := \frac{r_{1,k}}{R_k} z + a_{1,k} \in K_{1,k}$. So,

$$\left| f\left(\frac{r_{1,k}}{R_k} z + a_{1,k} \right) - h_{l_0}\left(\frac{r_{1,k}}{R_k} z + a_{1,k} \right) \right| \leq \frac{1}{k}. \tag{7}$$

Moreover, if we change variables, take (5) into account and remark that $|w| > R_k$ when $w \in K_{1,k}$, we have that

$$\left| h_{l_0}\left(\frac{r_{1,k}}{R_k} z + a_{1,k} \right) - P_k(z) \right| \leq \left| F_1(w) - P_k\left(\frac{w - a_{1,k}}{r_{1,k}/R_k} \right) \right| + \sum_{j=2}^{N(l_0)} |\alpha_j^{(l_0)}||F_j(w)|$$

$$\leq e^{-|w|^{1/4}} + \left(\sum_{j=2}^{N(l_0)} |\alpha_j^{(l_0)}|^2 \right)^{1/2} \left(\sum_{j=2}^{N(l_0)} |F_j(w)|^2 \right)^{1/2}$$

$$\leq e^{-|w|^{1/4}} + \sqrt{H} e^{-|w|^{1/4}} \left(\sum_{j=2}^{\infty} (1/3^j)^2 \right)^{1/2}$$

$$\leq (1 + \sqrt{H}) e^{-R_k^{1/4}}. \tag{8}$$

Now, by (7) and (8), we conclude that for any $z \in B_k$,

$$\left| f\left(\frac{r_{1,k}}{R_k} z + a_{1,k}\right) - P_k(z) \right| \leq \frac{1}{k} + (1+\sqrt{H})e^{-R_k^{1/4}} \to 0 \quad (k \to \infty)$$

whence, as in the dense-lineability part, the proof of (a) implies (d) is finished.

It only remains to prove that (a) implies (e). To do this, we are going to slightly modify the previous constructions.

First of all, consider the functions $\{f_n\}_n$ constructed in the proof of (a) implies (c), but only for $n \geq 2$. Observe that, in this construction, each function $g_n(z)$ $(n \geq 2)$ is defined as 0 on all compact sets of the form $K_{1,k}$, hence functions $\{f_n\}_{n\geq 2}$ are "small" in these sets. Now, if we define $M_d = \mathrm{span}\{f_n,\, n \geq 2\}$ and proceed as in the proof of (a) implies (c) (note that $\{P_n(z)\}_{n\geq 2}$ is still a dense sequence in $H(\mathbb{C})$), we conclude that M_d is a dense linear subspace of functions bounded on A and, except of the null function, universal with respect to similarities.

At this point, we divide $\mathbb{N}$ into infinitely many strictly increasing sequences $\{p(n,j)\}_j$ $(n \in \mathbb{N})$. Now we proceed as in the spaceability part, but defining G_n in the next way:

$$G_n(z) := \begin{cases} z^n & \text{if} \quad z \in \overline{\mathbb{D}} \\ 0 & \text{if} \quad z \in F \\ 0 & \text{if} \quad z \in K_{j,k},\ k \geq 2,\ j \leq k \\ 0 & \text{if} \quad z \in K_{1,p(j,k)},\ j \neq n,\ k \in \mathbb{N} \\ P_k\left(\frac{z - a_{1,p(n,k)}}{r_{1,p(n,k)}/R_{p(n,k)}}\right) & \text{if} \quad z \in K_{1,p(n,k)},\ k \in \mathbb{N}, \end{cases}$$

that is, we focus our attention on the sequence of compacta $K_{1,k}$ and divide it into infinitely many subsequences. In one of them we look for the approximation property and in the others, we look for controlling the function. Now, we get the functions F_n as in the proof of (a) implies (d). Let $M_s := \overline{\mathrm{span}}\{F_n\}$ which has infinite dimension and every non-zero function is bounded on A and universal with respect to similarities.

Observe that where the functions $\{f_n\}_n$ approximate the sequence $\{P_k(z)\}_k$, the functions $\{F_n\}_n$ are "small" and vice-versa.

Finally, define $M_{max} := \mathrm{span}\{M_d \cup M_s\}$, which is dense (because it contains M_d) and with maximal dimension (because it contains M_s and $\dim(M_s) = \mathfrak{c}$), and continue as in the proof of [10, Theorem 3.4]. $\square$

3. Boundedness universality of prefixed similarities

In 2012 Bernal [7, Theorem 3.5 and Remark 3.6] proved that, for a prescribed unbounded subset $A \subset \mathbb{C}$ and a prescribed sequence of similarities $\{z \mapsto a_n + b_n z\}$ such that $a_n \to \infty$ and $a_n/b_n \to \infty$ $(n \to \infty)$, the following conditions are equivalent:

(a) There exists an entire function bounded on A and universal under the above similarities, that is, the set $\{f(a_n + b_n z) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$.

(b) The set of entire functions bounded on A and universal under the similarities $\{a_n + b_n z\}$ is dense-lineable.

(c) There is an Arakelian set $F \subset \mathbb{C}$ with $A \subset F$ and such that for every $R > 0$ there exists $n \in \mathbb{N}$ with $F \cap B(a_n, R|b_n|) = \varnothing$.

Observe that conditions $a_n \to \infty$ and $a_n/b_n \to \infty$ $(n \to \infty)$ are natural. In 1995, Bernal and Montes [11] proved that there exists a universal entire function under a prescribed sequence of similarities $\{\varphi_n(z) = a_n + b_n z\}_n$ if and only if $\{\varphi_n\}_n$ is *run-away* (that is, for any compact set $K \subset \mathbb{C}$ there is $n \in \mathbb{N}$ such that $K \cap \varphi_n(K) = \varnothing$) if and only if $\min\{|a_n|, |\frac{a_n}{b_n}|\}$ is unbounded.

In order to preserve the run-away notion, we introduce the next concept.

Definition 3.1. *Let G be an open subset of $\mathbb{C}$, $H(G,G) := \{\varphi : G \to G :\ \varphi$ holomorphic in $G\}$ and $A \subset G$. We say that a sequence $\{\varphi_n\}_n \subset H(G,G)$ is* run-away outside A *if for any compact subset $K \subset G$ there is $n \in \mathbb{N}$ such that $(K \cup A) \cap \varphi_n(K) = \varnothing$.*

It is clear that every sequence $\{\varphi_n\}_n$ run-away outside a set A is run-away. For the converse, we have that if $\{\varphi_n\}_n$ is run-away then $\{\varphi_n\}_n$ is also run-away outside A for any set A bounded in G. If A is not bounded, this is not true; for instance, $\{\varphi_n(z) = nz + n^2\}_n \subset \mathrm{Aut}(\mathbb{C})$ is run-away but it is not run-away outside any $A = \{z \in \mathbb{C} :\ \Re z \geq 0,\ |\Im z| \leq \varepsilon\}$ $(\varepsilon > 0)$.

Now, we are going to give a complete characterization of the existence of algebraically many functions universal under a prescribed sequence $\{\varphi_n\}_n$ of similarities and bounded on a set A. The following theorem extends the above result of Bernal.

Theorem 3.1. *Let $A \subset \mathbb{C}$ and $\{\varphi_n(z) = a_n + b_n z\}_n \subset \mathrm{Aut}(\mathbb{C})$. Then the following conditions are equivalent:*

(a) *There exists a proper Arakelian set $F \subset \mathbb{C}$ such that $A \subset F$ and $\{\varphi_n\}_n$ is run-away outside F.*

(b) *There exists an entire function universal under $\{\varphi_n\}_n$ and bounded on A.*

(c) *The set of entire functions universal under $\{\varphi_n\}_n$ and bounded on A is dense-lineable.*

(d) *The set of entire functions universal under $\{\varphi_n\}_n$ and bounded on A is spaceable.*

(e) *The set of entire functions universal under $\{\varphi_n\}_n$ and bounded on A is maximal dense-lineable.*

Proof. It is trivial that (e) $\Rightarrow$ (c) $\Rightarrow$ (b) and that (d) $\Rightarrow$ (b). Assume (b) and let f be an entire function universal under $\{\varphi_n\}_n$ bounded on A. As in the proof of Theorem 2.2 the set $F := \{z \in \mathbb{C} : |f(z)| \leq \sup_A |f|\}$ is an Arakelian set such that $A \subset F$, $F \neq \mathbb{C}$. Let us see that $\{\varphi_n\}_n$ is run-away outside F. By way of contradiction, suppose that there is a compact set $K \subset \mathbb{C}$ such that $(K \cup F) \cap \varphi_n(K) \neq \varnothing$ for all $n \in \mathbb{N}$. Then there exists a sequence $\{z_n\}_n \subset K$ such that $\{\varphi_n(z_n)\}_n \subset K \cup F$. In the following define $g(z) = 2 + \sup_{K\cup F} |f| \in H(\mathbb{C})$. By the universality of f, there is $n_0 \in \mathbb{N}$ such that

$$1 \geq \sup_K |f(\varphi_{n_0}(z)) - g(z)| \geq |f(\varphi_{n_0}(z_{n_0})) - (2 + \sup_F |f|)| \geq 2,$$

which is a contradiction. Therefore we have (a).

Assume now that (a) holds. As F is a proper closed subset of $\mathbb{C}$, there is $R_0 \in (0,1)$, $z_0 \in \mathbb{C} \setminus F$ such that $\overline{B}(z_0, R_0) \cap F = \varnothing$.

There is $n_0 \in \mathbb{N}$ such that $B(0,n) \cap F \neq \varnothing$ for all $n \geq n_0$. Let $R_1 := n_0 + 1$. By (a), there is $N_1^{(1)} \in \mathbb{N}$ such that $(F \cup \overline{B}(0,R_1)) \cap \varphi_{N_1^{(1)}}(\overline{B}(0,R_1)) = \varnothing$. We define $B_1 = \overline{B}(0,R_1)$ and $K_{1,1} = \varphi_{N_1^{(1)}}(\overline{B}(0,R_1)) = \overline{B}(a_{N_1^{(1)}}, |b_{N_1^{(1)}}|R_1)$.

We choose R_2 $(> R_1)$ with $K_{1,1} \subset \overline{B}(0,R_2)$. By (a), there is $N_1^{(2)} \in \mathbb{N}$ such that $(F \cup \overline{B}(0,R_2)) \cap \varphi_{N_1^{(2)}}(\overline{B}(0,R_2)) = \varnothing$. Let $R_2' (> R_2)$ such that

$\varphi_{N_1^{(2)}}(\overline{B}(0,R_2)) \subset \overline{B}(0,R_2')$. Again, by (a), there is $N_2^{(2)} \in \mathbb{N}$ such that $(F \cup \overline{B}(0,R_2')) \cap \varphi_{N_2^{(2)}}(\overline{B}(0,R_2')) = \varnothing$. We define $B_2 = \overline{B}(0,R_2)$ and $K_{n,2} = \varphi_{N_n^{(2)}}(\overline{B}(0,R_2)) = \overline{B}(a_{N_n^{(2)}}, |b_{N_n^{(2)}}|R_2)$ $(n = 1,2)$. Observe that $K_{1,2} \cap K_{2,2} = \varnothing$ because $R_2' > R_2$.

We choose now $R_3 > R_2' (> R_2)$ and continue as above to obtain, with two intermediate steps, the sets $B_3 = \overline{B}(0,R_3)$, $K_{n,3} = \varphi_{N_n^{(3)}}(\overline{B}(0,R_3)) = \overline{B}(a_{N_n^{(3)}}, |b_{N_n^{(3)}}|R_3)$ $(n = 1,2,3)$.

We continue inductively to construct sequences $B_k = \overline{B}(0,R_k)$, $K_{n,k} = \overline{B}(a_{N_n^{(k)}}, |b_{N_n^{(k)}}|R_k)$ $(k \in \mathbb{N},\ n = 1,\dots,k)$ pairwise disjoints satisfying the same conditions as Lemma 2.1. Now we continue as in the proof of Theorem 2.2 to get (c), (d) and (e). □

Observe that the geometric construction in the last proof together with the construction of [7, Proof of Theorem 3.5], shows the equivalence between condition (c) of Bernal's result and condition (a) in our Theorem 3.1.

Finally, we characterize the translation case. Recall that $\varphi(z) = a + bz \in \mathrm{Aut}(\mathbb{C})$ is run-away (that is, the sequence of iterates $\{\varphi_n = \varphi \circ \overset{(n)}{\cdots} \circ \varphi\}_n$ is run-away) if and only if $b = 1$, $a \neq 0$ (see [11]). If $\theta \in (-\pi,\pi]$, the ray from the origin with slope θ will be denoted by L_θ.

Given a subset $A \subset \mathbb{C}$ and $\theta \in (-\pi,\pi]$, we define the *radial inscribed radius* of A as

$$\rho_\theta(A) = \sup\{\, r > 0 : \text{ there exist a closed ball } B \text{ with radius } r \text{ and center in } L_\theta \text{ such that } B \subset A\}.$$

For instance, $\rho_0(\{z \in \mathbb{C} : |\Im z| \le \varepsilon\}) = \varepsilon$ $(\varepsilon > 0)$ and $\rho_\theta(\{z :\ |\arg z| \le \alpha\}) = +\infty$ $(\alpha \in (0,\pi],\ \theta \in (-\alpha,\alpha))$.

Corollary 3.2. *Let $A \subset \mathbb{C}$ and $a \in \mathbb{C}$, $a \neq 0$. The following statements are equivalent:*

(a) *There exists an Arakelian set $F \subset \mathbb{C}$ such that $A \subset F$ and $\rho_\theta(\mathbb{C} \setminus F) = +\infty$, where $\theta = \arg a$.*

(b) *There exists an Arakelian set $F \subset \mathbb{C}$ such that $A \subset F$ and the translation $\varphi(z) = z + a$ is run-away outside F.*

(c) *There exists an entire function f bounded on A such that the set $\{f(z+na) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$.*

(d) *The set of entire functions f bounded on A such that $\{f(z+na) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$, is maximal dense-lineable and spaceable.*

Proof. By Theorem 3.1, we have the equivalence between (b), (c) and (d). It only remains to show that the translation $\varphi(z) = z + a$ is run-away outside F if and only if $\rho_\theta(\mathbb{C} \setminus F) = +\infty$ ($\theta = \arg a$).

If φ is run-away outside F then for any $r > 0$ there is $n \in \mathbb{N}$ such that $(\overline{B}(0,r) \cup F) \cap \overline{B}(na,r) = \varnothing$. So $\overline{B}(na,r) \subset \mathbb{C} \setminus F$ and, as r is arbitrary, $\rho_\theta(\mathbb{C} \setminus F) = +\infty$.

Assume $\rho_\theta(\mathbb{C} \setminus F) = +\infty$ ($\theta = \arg a$). Fix $r > 0$. We have $\rho_\theta(\mathbb{C} \setminus (F \cup \overline{B}(0,r))) = +\infty$. Then there is $c \in \mathbb{C}$, $\arg c = \theta$, such that $\overline{B}(c, r + |a|) \subset \mathbb{C} \setminus (F \cup \overline{B}(0,r))$. As $\arg c = \arg a$, there exists $n_0 \in \mathbb{N}$ such that $n_0|a| \leq |c| < (n_0+1)|a|$. Hence,

$$|n_0 a - c| = \left| n_0|a|e^{i\theta} - |c|e^{i\theta} \right| = |c| - n_0|a| \leq (n_0+1)|a| - n_0|a| = |a|.$$

So, for any $z \in \overline{B}(n_0 a, r)$,

$$|z - c| \leq |z - n_0 a| + |n_0 a - c| \leq r + |a|.$$

Therefore, $\varphi_{n_0}(\overline{B}(0,r)) = \overline{B}(n_0 a, r) \subset \overline{B}(c, r+|a|) \subset \mathbb{C} \setminus (F \cup \overline{B}(0,r))$. Hence $\varphi_{n_0}(\overline{B}(0,r)) \cap (F \cup \overline{B}(0,r)) = \varnothing$ and φ is run-away outside F. □

4. Further results and remarks

(1) Observe that by the proof of Theorem 2.2 we can obtain universal functions that are bounded at "any" prefixed proper Arakelian set that covers A. On the other hand, if A is bounded then A is sub-Arakelian and, trivially, any entire function is bounded on A. So, from Theorem 2.2, we deduce

Theorem 4.1. *The set of entire functions universal under similarities is maximal dense lineable and spaceable.*

See also [12] for spaceability (even for a prescribed sequence of similarities) and [5] (or [16], [9]) for dense-lineability. As a consequence, if A is not sub-Arakelian, there exist many universal functions under similarities which can not be bounded in A.

(2) Theorem 2.2 tells us that the algebraic size of the set of universal entire functions under similarities which are bounded on a sub-Arakelian set A is as large as possible. But, simultaneously, whenever A is unbounded, this set is not very large in the topological sense, because it is of the first category in the Baire space $H(\mathbb{C})$, i.e., it is a countable union of sets whose closures have empty interiors. In fact, the bigger set

$$\mathcal{B}(A) := \{f \in H(\mathbb{C}) : \ f \text{ is bounded on } A\}$$

is of the first category, because $\mathcal{B}(A) = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ with $\mathcal{B}_n = \{f \in H(\mathbb{C}) : |f(z)| \le n \text{ on } A\}$ and each $\mathcal{B}_n$ is closed and has empty interior. Indeed, $\mathcal{B}_n \cap \{\text{nonconstant polynomials}\} = \varnothing$ and the second set in the intersection is dense in $H(\mathbb{C})$. Otherwise, if A is bounded, it is clear (see [11]) that the set of universal entire functions bounded on A is residual (its complement is of the first category).

(3) In 2006, Niess [25] gave a necessary and sufficient condition for the existence of an entire function f whose translates $z \mapsto f(z + a_n)$ (where $\{a_n\}_n \subset \mathbb{C}$) are dense in $H(\mathbb{C})$ and such that f is bounded on every (straight) line: there exists a subsequence $\{a_{n_k}\}_k$ such that, for every $R > 0$ and every line L, there is $k_0 \in \mathbb{N}$ with $L \cap B(a_{n_k}, R) = \varnothing$ for all $k \ge k_0$.

If we consider in Theorem 2.2 the Arakelian set $A := \mathbb{C} \setminus \{z = x+iy : \ 0 < y < xe^{-x}\}$, then for any straight line L the set $L \setminus A$ is bounded. Therefore, we obtain the following.

Theorem 4.2. *The set of universal entire functions under similarities that are bounded on any straight line is maximal dense-lineable and spaceable.*

(4) Observe that the universal functions given by Theorem 2.2 are not only bounded in A but $\lim_{z\to\infty, z\in F} f(z) = 0$, where F is any prefixed Arakelian set that covers A. Just consider that in the proof the sets B°_{n+1} are bounded and that $|f_n(z)| < e^{-|z|^{1/4}}$ $(z \in F \setminus B^\circ_{n+1})$.

(5) Finally, we note that it is possible to establish the spaceability and maximal dense-lineability of the set of harmonic functions in $\mathbb{R}^N$ universal under similarities and bounded on any Arakelian set F; just use the harmonic analogue of Arakelian's approximation theorem given in [3, Theorem 1.1].

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