Holomorphic operators generating dense images*

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Abstract

The existence of infinite dimensional closed linear spaces of holomorphic functions f on a domain G in the complex plane such that Tf has dense images on certain subsets of G, where T is a continuous linear operator, is analyzed. Necessary and sufficient conditions for T to have the latter property are provided and applied to obtain a number of concrete examples: infinite order differential operators, composition operators and multiplication operators, among others.

Key words and phrases: dense images, infinite dimensional closed linear spaces, non-relatively compact subsets, residual sets, differential operators, composition operators, multiplication operators.

2000 Mathematics Subject Classification: Primary 47B38. Secondary 30D40, 47B33, 47E05.

1 Introduction

In this paper we are concerned with the existence of holomorphic functions that, under the action of certain operators, have dense images on prefixed subsets of the domain of definition. A classical interpolation theorem due to Weierstrass (see [17, Chapter 15]) asserts that, if (a_n) is a sequence of distinct points in a domain G of the complex plane $\mathbb{C}$ without accumulation points in G and $(b_n) \subset \mathbb{C}$, then there is a holomorphic function f in G such that $f(a_n) = b_n$ for all n. In particular, if we choose as (b_n) an enumeration of all complex rational numbers, one obtain a function f such that the sequence $(f(a_n))$ is dense in $\mathbb{C}$. Equivalently, if A is a subset of G that is not relatively

*This work is supported in part by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 and by MEC DGES Grants BFM2003-03893-C02-01 and MTM2004-21420-E.

compact, then there is a holomorphic function f in G with dense image $f(A)$. Kierst and Szpilrajn [15] started the study of this kind of phenomenons under the point of view of the topological size, showing the residuality of the subset of such functions f for certain sets A. In [4] and [5], holomorphic operators are introduced in this topic, see below. In this paper, we try to find an additional linear structure –not only a topological structure– in the set of functions with dense images when certain operators act on them.

Let us fix the notation that will be used along this paper. Firstly, $\mathbb{N}$ will be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbol (n_k) will stand for a strictly increasing sequence in $\mathbb{N}_0$. If A is a subset of $\mathbb{C}$ then $\overline{A}$, A^0, ∂A denotes, respectively, its closure in $\mathbb{C}$, its interior in $\mathbb{C}$, and its boundary in the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. As usual, $\mathbb{D}$ is the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. If f is a complex valued function defined on a set $A \subset \mathbb{C}$, then $\|f\|_A := \sup_{z\in A} |f(z)|$. If G is a domain (:= nonempty, connected open subset) of $\mathbb{C}$, then $H(G)$ denotes the space of holomorphic functions on G. It becomes a completely metrizable space (hence a Baire space) when it is endowed with the compact open topology [14, pp. 238–239]. We denote by $\mathcal{K}(G)$ the family of compact subsets K of G such that $G \setminus K$ has no non-relatively compact connected components, and by $\mathcal{K}_1(G)$ the subfamily of compacta $K \subset G$ such that $\mathbb{C} \setminus K$ is connected. It is always possible to construct an exhaustive sequence $(K_n)_n$ of compact subsets of G –that is, $\bigcup_{n\in\mathbb{N}_0} K_n = G$ and $K_n \subset K_{n+1}^0$ for all $n \in \mathbb{N}_0$– contained in $\mathcal{K}(G)$ [9]. If $(A_n)_{n\geq 0}$ is a sequence of subsets of G, then it is said that $(A_n)_{n\geq 0}$ tends to ∂G provided that, given a compact subset $K \subset \mathbb{C}$, there exists $n_0 \in \mathbb{N}_0$ such that $K \cap A_n = \emptyset$ for all $n \geq n_0$. The symbol $NRC(G)$ will stand for the family of all subsets of G which are not relatively compact in G.

In 1995, it was proved [4] that, if $A \in NRC(G)$, then there are many functions $f \in H(G)$ such that $\overline{f^{(j)}(A)} = \mathbb{C}$ for every $j \in \mathbb{N}_0$, and in 2002, this result was extended [8] by considering sums of infinite order differential operators and integral operators instead of the differential operators $D^j f := f^{(j)}$. In fact, it was proved that for these operators T on $H(G)$ the set

$$M(T, A) := \{f \in H(G) : \overline{(Tf)(A)} = \mathbb{C}\}$$

is residual (in fact, G_δ-dense). In [5], the study of this boundary behavior on plane sets was translated to large classes of operators. Following [5], a –not necessarily linear– continuous operator $T : H(G) \to H(G)$ is a *dense-image* operator (in short, a DI operator) if the set $M(T, A)$ is residual in $H(G)$ for any $A \in NRC(G)$. Hence, we can say that the *topological size* of $M(T, A)$ is large for these kinds of operators.

In this paper we are interested in the *algebraic size* of $M(T, A)$, where now T is continuous and *linear*. Our aim is to determine when $M(T, A)$ is large also in this sense, see below.

Assume that $G \subset \mathbb{C}$ is a simply connected domain, and denote by I the identity operator on $H(G)$. Let $(\varphi_n) \subset \mathrm{Aut}\,(G) := \{\text{automorphisms of } G\}$ be a *run-away* sequence, that is, for any compact subset $K \subset G$ there is $m \in \mathbb{N}$ with $K \cap \varphi_m(K) = \varnothing$. In 1995, Montes and the first author [7] showed the existence of an *infinite-dimensional closed linear subspace* F of $H(G)$ such that for any $f \in F \setminus \{0\}$ the set $\{f \circ \varphi_n : \ n \in \mathbb{N}\}$ is dense in $H(G)$. In particular, one obtains that for every prescribed set $A \in NRC(G)$ there exists a subspace F as above such that $F \setminus \{0\} \subset M(I, A)$. To see this, observe that, for any sequence $(a_n) \subset A$ with $a_n \to t \in \partial G$ and any $z_0 \in G$, there exists a run-away sequence $(\varphi_n) \subset \mathrm{Aut}\,(G)$ with $\varphi_n(z_0) = a_n$ $(n \in \mathbb{N})$. Indeed, the case $G = \mathbb{D}$, $z_0 = 0$ is clear, just by considering $\psi_n(z) = (z + a_n)/(1 + \overline{a_n} z)$; then $(\psi_n) \subset \mathrm{Aut}\,(\mathbb{D})$ is run-away because $\lim_{n\to\infty} |a_n| = |t| = 1$ (see [6]). For the general case it suffices to take $\varphi_n = h^{-1} \circ \psi_n \circ h$, where $h : G \to \mathbb{D}$ is an isomorphism with $h(z_0) = 0$ and $(\psi_n) \subset \mathrm{Aut}\,(\mathbb{D})$ is run-away with $\psi_n(0) = h(a_n)$ $(n \in \mathbb{N})$.

Unfortunately, if $G \subset \mathbb{C}$ is a domain with finite connectedness such that its complement has more than two components then it supports only finitely many automorphisms [13], so there are no run-away sequences. Hence the above reasoning does not work in general.

These facts motivate the following natural question: *If $G \subset \mathbb{C}$ is any domain, T is a continuous linear operator on $H(G)$ and $A \in NRC(G)$, does an infinite-dimensional closed subspace F of $H(G)$ exist satisfying $F \setminus \{0\} \subset M(T, A)$?*

In the Section 2 of this paper we will provide general and, in some sense, minimal conditions on T for the existence of such a subspace F, even without loss of residuality for each $M(T, A)$, see Theorems 2.1–2.2. Several classical examples –including differential, composition and multiplication operators– will be analyzed in Section 3.

2 Existence of large subspaces

Firstly, we need to introduce a sort of "continuity near the boundary" for operators, compare with [5, Condition (P) before Theorem 3.4].

Definition 2.1. We say that a continuous linear operator $T : H(G) \to H(G)$ is *boundary pointwise stable* if and only if the following property holds: For each compact set $K \subset G$ there exists a compact subset $L \subset G$ such that for each point $a \in G \setminus L$ and each positive number $\varepsilon > 0$ there are a set $B \in \mathcal{K}_1(G)$ with $B \subset G \setminus K$ and a number $\delta > 0$ such that, if $f \in H(G)$ and $\|f\|_B < \delta$, then $|Tf(a)| < \varepsilon$.

For instance, using Cauchy's integral formula for derivatives it is easy to verify that the derivative operator D ($Df := f'$) is boundary pointwise stable; see Section 3 for more examples. Note that the notion of stability in [5, Condition (P)] is slightly more restrictive (the set B is a closed ball there) than the one defined here, but it is easy to check that all results in [5] hold with this new definition.

We are now ready to state the first of our main results.

Theorem 2.1. *Let $G \subset \mathbb{C}$ be a domain, $T : H(G) \to H(G)$ a continuous linear operator and $A \in NRC(G)$. Suppose that T satisfies the following conditions:*

(A) *T is boundary pointwise stable.*

(B) *For every compact subset $K \subset G$ there exist a point $a \in A \setminus K$ and a function $h \in H(G)$ such that $Th(a) \neq 0$.*

Then there exists an infinite-dimensional closed linear subspace F of $H(G)$ with $F \setminus \{0\} \subset M(T, A)$.

Proof. Firstly, we fix a dense subset $(q_n)_{n\geq 0}$ of $\mathbb{C}$, a sequence $(\varrho_m)_{m\geq 0}$ of positive numbers such that $\sum_{m\geq 0} \varrho_m < 1$, and an exhaustive sequence of compact subsets $(K_j)_{j\geq 0} \subset \mathcal{K}(G)$. Without loss of generality we can assume that $\overline{\mathbb{D}} \subset K_0 \subset G$. Fix also a bijective mapping $i : (m, n) \in \mathbb{N}_0^2 \mapsto i(m, n) \in \mathbb{N}_0$ such that i is nondecreasing in m and n. Finally we define the sequence $(p_i)_{i\geq 0}$ as $p_{i(m,n)} = q_n$ for all $m \geq 0$.

1. Given $M_0 := K_0$, let L_0 be the compact subset given by the stability of T. By (B) and the exhaustivity of (K_j), there exist $k_0 \in \mathbb{N}_0$, a point a_0 and a function $h_0 \in H(G)$ such that $a_0 \in A \setminus K_{k_0} \subset G \setminus L_0$ and $Th_0(a_0) \neq 0$. Let $\varepsilon_0 := 1$. By (A), there exist a compact set $B_0 \subset G \setminus M_0$ with $B_0 \in \mathcal{K}_1(G)$ and a $\delta_0 > 0$ such that for each function $f \in H(G)$ we have that $\|f\|_{B_0} < \delta_0$ implies $|Tf(a_0)| < \varepsilon_0 = 1$.

Now, we proceed by induction to construct sequences $(\delta_n)_{n\geq 0} \subset (0, +\infty)$, $(k_n)_{n\geq 0} \subset \mathbb{N}_0$, $(B_n)_{n\geq 0} \subset \mathcal{K}_1(G)$, $(a_n)_{n\geq 0} \subset A$ and $(h_n)_{n\geq 0} \subset H(G)$. Assume that $\delta_0, \delta_1, \ldots, \delta_{n-1}$, $k_0, k_1, \ldots, k_{n-1}$, $B_0, B_1, \ldots, B_{n-1}$, $a_0, a_1, \ldots, a_{n-1}$,

$h_0, h_1, \ldots, h_{n-1}$ have been already determined, and let M_n be the compact set $M_n := K_n \cup \left(\bigcup_{j=0}^{n-1} B_j\right)$. Let L_n be the compact subset given by the stability condition as applied on M_n. Again by (B) and the exhaustivity of (K_j), there exist $k_n \geq k_{n-1}$, a point a_n and a function $h_n \in H(G)$ satisfying $a_n \in A \setminus K_{k_n} \subset G \setminus L_n$ and $Th_n(a_n) \neq 0$. Let $\varepsilon_n := 1/2^n$. By (A), there exist a set $B_n \subset G \setminus M_n$ with $B_n \in \mathcal{K}_1(G)$ and a $\delta_n \in (0, \delta_{n-1})$ such that, for each function $f \in H(G)$,

$$\|f\|_{B_n} < \delta_n \quad \Longrightarrow \quad |Tf(a_n)| < \frac{1}{2^n}. \tag{1}$$

Now, we define $g_n(z) := \dfrac{p_n}{Th_n(a_n)} \cdot h_n(z)$ $(z \in G)$. Then $g_n \in H(G)$; and $Tg_n(a_n) = p_n$ for all $n \geq 0$.

2. Consider the set $M := \overline{\mathbb{D}} \cup \left(\bigcup_{j=0}^{\infty} B_j\right)$. Observe that, by construction, the sets $B_1, B_2, \ldots, B_n, \ldots$ are closed, pairwise disjoint, tend to ∂G and belong to $\mathcal{K}_1(G)$. From this, we derive that M is a relatively closed subset of G and that $G_\infty \setminus M$ is connected and locally connected in G_∞, the one-point compactification of G. Furthermore, since (M_n) is exhaustive –because (K_n) is– and $B_n \subset G \setminus M_n$ $(n \geq 0)$, we conclude that for every compact subset $K \subset G$ there exists a neighborhood V of the infinity point of G_∞ such that no component of M intersects both K and V. On the other hand, given $m \in \mathbb{N}$, the functions $\epsilon_m : M \to (0, +\infty)$ and $F_m : M \to \mathbb{C}$ defined as

$$\epsilon_m(z) = \begin{cases} \varrho_m & \text{if} \quad z \in \overline{\mathbb{D}} \\ \delta_{i(m,n)} & \text{if} \quad z \in B_{i(m,n)} \text{ and } n \in \mathbb{N}_0 \\ \varrho_m \delta_{i(k,n)} & \text{if} \quad z \in B_{i(k,n)} \text{ and } k, n \in \mathbb{N}_0 \text{ with } k \neq m, \end{cases}$$

$$F_m(z) = \begin{cases} z^m & \text{if} \quad z \in \overline{\mathbb{D}} \\ g_{i(m,n)}(z) & \text{if} \quad z \in B_{i(m,n)} \text{ and } n \in \mathbb{N}_0 \\ 0 & \text{if} \quad z \in B_{i(k,n)} \text{ and } k, n \in \mathbb{N}_0 \text{ with } k \neq m, \end{cases}$$

are continuous on M. In addition, F_m is holomorphic in M^0. Hence, the Nersesjan tangential approximation theorem (see [11, p. 157] or [16]) guarantees the existence of a function $f_m \in H(G)$ such that

$$|f_m(z) - F_m(z)| < \epsilon_m(z) \qquad \text{for all } z \in M.$$

Thus, we have to our disposal a sequence of functions $(f_m)_{m \geq 0} \subset H(G)$ satisfying the next properties:

$$\|f_m(z) - z^m\|_{\overline{\mathbb{D}}} < \varrho_m \quad \text{for all } m \geq 0 \tag{2}$$

$$\|f_m(z) - g_{i(m,n)}(z)\|_{B_{i(m,n)}} < \delta_{i(m,n)} \quad \text{for all } m, n \geq 0 \text{ and} \tag{3}$$

$$\|f_m(z)\|_{B_{i(k,n)}} < \varrho_m \delta_{i(k,n)} \quad \text{for all } m, n \geq 0 \text{ and all } k \neq m. \tag{4}$$

¿From (1), (3), (4), and taking in mind that $T(f_m - g_{i(m,n)})(a_{i(m,n)}) = Tf_m(a_{i(m,n)}) - Tg_{i(m,n)}(a_{i(m,n)}) = Tf_m(a_{i(m,n)}) - p_{i(m,n)} = Tf_m(a_{i(m,n)}) - q_n$, we obtain

$$|Tf_m(a_{i(m,n)}) - q_n| < \frac{1}{2^{i(m,n)}} \leq \frac{1}{2^n} \quad (m, n \geq 0) \tag{5}$$

and

$$|Tf_m(a_{i(k,n)})| < \frac{\varrho_m}{2^{i(k,n)}} \leq \frac{\varrho_m}{2^n} \quad (m, n \geq 0,\ k \neq m). \tag{6}$$

Note that in the last inequality the homogeneity of T has also been used.

3. Let E be the linear span of $(f_m)_{m\geq 0}$ and denote by F its closure in $H(G)$. Obviously F is a closed linear subspace of $H(G)$.

¿From the property (2) and by using a well known basis perturbation theorem (see [10, p. 46]) as in [7, Second step of the proof of Theorem 1.2] it can be shown that $(f_m)_m$ is a basic sequence in $L^2(\mathbb{T})$, the Hilbert space of all square-integrable complex functions on $\mathbb{T}$ endowed with the norm $\|f\|_2 := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt\right)^{1/2}$. Therefore the functions f_m are linearly independent, so F is an infinite-dimensional vector space. In addition, $(f_m)_{m\geq 0}$ is equivalent to the basic sequence $(z^m)_{m\geq 0}$. In particular, the linear mapping $S : \sum_{m=0}^\infty c_m f_m \in X \mapsto \sum_{m=0}^\infty c_m z^m \in Y$ is a topological isomorphism. Here X and Y are, respectively, the closure in $L^2(\mathbb{T})$ of the linear span of (f_m) and of (z^m).

Our goal is to prove that $F \setminus \{0\} \subset M(T, A)$. Let $f \in F \setminus \{0\}$ and let $f = \displaystyle\sum_{m=0}^{\infty} \alpha_m f_m$ its representation on $L^2(\mathbb{T})$. As $f \neq 0$, there exists some $k \geq 0$ such that $\alpha_k \neq 0$, in fact we can suppose that $\alpha_k = 1$ because if $f \in M(T, A)$ then $\lambda f \in M(T, A)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Since $F = \overline{E}$, there is a sequence $\left(h_l := \displaystyle\sum_{m=0}^{N_l} \alpha_m^{(l)} f_m\right)_{l\geq 0} \subset E$ converging to f in $H(G)$ and we always can assume that $\alpha_k^{(l)} = 1$ $(l \geq 0)$. Indeed, if this where not the case, we would decompose $h_l = h_l^* + (\alpha_k^{(l)} - 1)f_k$; then each h_l^* has the desired property for its kth-coefficient and $h_l^* \to f$ in $H(G)$ because $(\alpha_k^{(l)} - 1)f_k \to 0$ $(l \to \infty)$ compactly, which in turn is true since $(f_m)_m$ is a basis of $L^2(\mathbb{T})$, from where one derives in particular that $\alpha_k^{(l)} \to \alpha_k = 1$ $(l \to \infty)$. As

$B_{i(k,n)}$ is compact, there is an integer $l \geq 0$ such that $\|h_l - f\|_{B_{i(k,n)}} < \delta_{i(k,n)}$, hence, by (1) and the linearity of T, we have

$$|Th_l(a_{i(k,n)}) - Tf(a_{i(k,n)})| < \frac{1}{2^{i(k,n)}} \leq \frac{1}{2^n}.$$

Then, by (5), (6) and the triangle inequality,

$$\begin{aligned}
|Tf(a_{i(k,n)}) - q_n| &\leq |Tf(a_{i(k,n)}) - Th_l(a_{i(k,n)})| + |Th_l(a_{i(k,n)}) - q_n| \\
&\leq \frac{1}{2^n} + |Tf_k(a_{i(k,n)}) - q_n| + \sum_{\substack{m=0 \\ m\neq k}}^{N_l} |\alpha_m^{(l)} Tf_m(a_{i(k,n)})| \\
&\leq \frac{1}{2^n} + \frac{1}{2^n} + \sum_{\substack{m=0 \\ m\neq k}}^{N_l} |\alpha_m^{(l)}| \varrho_m \frac{1}{2^n} = \frac{C}{2^n},
\end{aligned}$$

where C is a finite constant (to be determined later) not depending on n. Hence

$$\lim_{n\to\infty} \big(Tf(a_{i(k,n)}) - q_n\big) = 0.$$

Finally, (q_n) is a dense subset of $\mathbb{C}$, so $\{Tf(a_{i(k,n)}) : \ n \in \mathbb{N}\}$ $(\subset (Tf)(A))$ is also dense and we have that $f \in M(T, A)$, as desired.

It remains only to determine the constant C above. Since (h_l) tends to f uniformly on compacta in G as $l \to \infty$, we have that, in particular, $h_l \to f$ uniformly on $\overline{\mathbb{D}}$, hence $h_l \to f$ in $L^2(\mathbb{T})$, so $S(h_l) \to S(f)$ in $L^2(\mathbb{T})$. Therefore the sequence $(S(h_l))_{l\geq 0}$ is bounded in $L^2(\mathbb{T})$, or equivalently, there exists a constant $M \in (0, +\infty)$ such that $\left(\sum_{m=0}^{\infty} |\alpha_m^{(l)}|^2\right)^{1/2} \leq M$ for all $l \in \mathbb{N}_0$, where we have set $\alpha_m^{(l)} := 0$ for $m > N_l$. Thus, we get

$$\sum_{\substack{m=0 \\ m\neq k}}^{N_l} |\alpha_m^{(l)}| \varrho_m \leq \left(\sum_{m=0}^{\infty} |\alpha_m^{(l)}|^2\right)^{1/2} \left(\sum_{m=0}^{\infty} \varrho_m^2\right)^{1/2} \leq M \sum_{m=0}^{\infty} \varrho_m \leq M \qquad (l \in \mathbb{N}_0).$$

Consequently, it is enough to choose $C := M + 2$, and we are done. □

As a consequence of Theorem 2.1 we obtain the next general statement, in which it is asserted that, under pointwise stability, the properties "$M(T, A)$ is not empty", "$M(T, A)$ is topologically large", "$M(T, A)$ is algebraically large" (for any $A \in NRC(G)$) are equivalent. In addition, it is provided a condition –see (a) below– that is easy to check and is equivalent to the mentioned ones.

Theorem 2.2. *Let $G \subset \mathbb{C}$ be a domain and let $T : H(G) \to H(G)$ be a continuous linear operator that is boundary pointwise stable. Then the following conditions are equivalent:*

(a) *T collapses at no point outside some compact set, that is, there is a compact subset $K \subset G$ with the property that for every $a \in G \setminus K$ there exists $h \in H(G)$ such that $Th(a) \neq 0$.*

(b) *For every $A \in NRC(G)$, there exists an infinite-dimensional closed linear subspace F of $H(G)$ with $F \setminus \{0\} \subset M(T, A)$.*

(c) *For every $A \in NRC(G)$, the set $M(T, A)$ is not empty.*

(d) *The operator T is dense-image.*

Proof. It is trivial that (b) implies (c) and that (d) implies (c). That (c) implies (d) is due to [5, Theorem 3.4].

Assume now that (c) holds and, by way of contradiction, that (a) is not true. Then we can select an increasing exhausting sequence $(K_n)_{n\geq 0}$ of compact sets in G as well as a sequence of points $(a_n)_{n\geq 0}$ such that $a_n \in G \setminus K_n$ $(n \geq 0)$ and $Th(a_n) = 0$ for all $h \in H(G)$. In particular, for $A := \{a_n : n \in \mathbb{N}_0\}$ and $h \in M(T, A)$, we would have $Th(A) = \{0\}$, which is absurd. Thus, (c) implies (a).

Finally, suppose that (a) is true, that is, there is a compact set $K_0 \subset G$ such that, for every $a \in G \setminus K_0$, there exists a function $h \in H(G)$ with $Th(a) \neq 0$. Fix a set $A \in NRC(G)$ and a compact set $K \subset G$. Since $K_0 \cup K$ is compact, we have that there exists at least one point $a \in A \setminus (K_0 \cup K)$. Then $a \in G \setminus K_0$, so we can find $h \in H(G)$ with $Th(a) \neq 0$. Moreover, $a \in A \setminus K$, so condition (B) in Theorem 2.1 is fulfilled. Since T is boundary pointwise stable, from the mentioned theorem we obtain (b). □

Theorem 2.2 motivates the following definition.

Definition 2.2. We say that a linear continuous operator T on $H(G)$ has *large dense images*, or that T is an *LDI operator*, if for each $A \in NRC(G)$ the set $M(T, A)$ is residual and contains, except for the zero function, an infinite-dimensional closed linear subspace F of $H(G)$.

Remarks 2.3. 1. The condition (a) in Theorem 2.2 is easily satisfied. For instance, it suffices that the range of T contains the constants.

2. The notion of LDI operator can be stated, equivalently, in terms of sequences, as follows: T is an LDI operator if and only if for each sequence

$(a_n)_n \subset G$ tending to ∂G the set $M(T,(a_n)_n)$ is residual and there exists an infinite-dimensional closed subspace F such that $F \setminus \{0\} \subset M(T,(a_n)_n)$. Just take into account that $M(T,B) \subset M(T,A)$ if $B \subset A$ and that every set $A \in NRC(G)$ contains a sequence tending to the boundary.

3. In the terminology of the recent paper [12] (see also [1] and [2]), an operator T on $H(G)$ is LDI if and only if, for every $A \in NRC(G)$, the set $M(T,A)$ is residual and *spaceable*.

3 Examples of LDI operators

1. Let $G \subset \mathbb{C}$ be a domain. If $\Phi(z) = \sum_{n\geq 0} a_n z^n$ is an entire function of subexponential (exponential) type, that is, $\lim_{n\to\infty}(n!|a_n|)^{1/n} = 0$ ($\limsup_{n\to\infty}(n!|a_n|)^{1/n} < +\infty$, resp.), then $\Phi(D) = \sum_{n\geq 0} a_n D^n$ defines a continuous linear operator on $H(G)$ ($H(\mathbb{C})$, resp.) [3, §6.4], where $D^0 = I$. By using Cauchy's estimates we can show that $\Phi(D)$ is boundary pointwise stable (take $L = K$ in Definition 2.1, and choose as B a small closed disk around a); and it is clear that if $\Phi \not\equiv 0$ then the range of $\Phi(D)$ contains all constants. Hence, by Theorem 2.2 and Remark 2.3.1 we obtain the following.

Theorem 3.1. *If $\Phi \not\equiv 0$ is an entire function of subexponential type and $G \subset \mathbb{C}$ is a domain, then the differential operator $\Phi(D)$ is LDI. If $G = \mathbb{C}$, then the same holds even if Φ is of exponential type.*

In particular, if $\Phi(z) \equiv 1$ ($\Phi(z) \equiv z$, $\Phi(z) \equiv e^{bz}$ with $b \in \mathbb{C}\setminus\{0\}$, resp.), we obtain that the identity operator I (the differential operator D, the translation operator $\tau_b f(z) := f(z+b)$, resp.) has large dense images (in the translation case, we are assuming that $G = \mathbb{C}$). In the case that $G \subset \mathbb{C}$ is a domain such that $b + G = G$ (for instance, $G = \{z : |\Im z| < r\}$, $b \in \mathbb{R}$) we can show that τ_b is boundary pointwise stable, just by taking $L = -b + K$, $\delta = \varepsilon$ and $B \in \mathcal{K}_1(G)$ with $a + b \in B \subset G \setminus K$ in Definition 2.1. And it is evident that all the constants are in the range of τ_b. Hence, by Theorem 2.2, τ_b is an LDI operator. This can also be derived from Theorem 3.2 or Theorem 3.5, see below.

It is not possible to apply Theorem 2.2 to the antiderivative operator D_a^{-N} on $H(G)$ ($N \in \mathbb{N}$, $a \in G$, G simply connected) defined as $D_a^{-N} f =$ [the unique function $g \in H(G)$ such that $D^N g = f$ and $g(a) = (Dg)(a) = \cdots = (D^{N-1}g)(a) = 0$]. This is so because D_a^{-N} is not boundary pointwise stable. We do not know whether D_a^{-N} is an LDI operator. Nevertheless, these operators are DI (see [5] or [8]). The same problem arises if we consider the general case of the Volterra operator V_φ generated by an analytic function $\varphi : G \times G \to \mathbb{C}$, namely, $V_\varphi f(z) := \int_a^z \varphi(z,t) f(t) dt$.

2. If $\varphi \in H(G,G) := \{f \in H(G) : \ f(G) \subset G\}$, then the composition operator defined as $C_\varphi : f \in H(G) \mapsto f \circ \varphi \in H(G)$ is a continuous linear operator. We recall that a self-map $\psi : X \to X$ on a topological space X is said to be *proper* whenever the preimage under ψ of any compact subset is again a compact subset. This topological property characterizes the LDI composition operators, as the following theorem shows.

Theorem 3.2. *The composition operator C_φ on $H(G)$ is LDI if and only if φ is proper. In particular, if $G = \mathbb{C}$, we have that C_φ is LDI if and only if φ is a non-constant polynomial.*

Proof. By the last example, the identity operator is LDI. On the other hand, $M(C_\varphi, A) = M(I, \varphi(A))$ for every $A \subset G$. But it is easy to see that for a continuous self-map $\varphi : G \to G$, it is proper if and only if $\varphi(A) \in NRC(G)$ for every $A \in NRC(G)$. The part 'only if' is trivial because $M(T,B)$ is empty if $B \subset G$ is relatively compact in G. As for the case $G = \mathbb{C}$, just take into account that the Casorati-Weierstrass theorem prevents φ to be proper if it is transcendental. □

The conclusion of Theorem 3.2 holds specially when $\varphi \in \mathrm{Aut}\,(G)$. For instance, if $r > 0$, $\alpha \in [0, 2\pi)$ and we denote $r\mathbb{D} := \{z \in \mathbb{C} : \ |z| < r\}$, then the rotation operator R_α defined on $H(r\mathbb{D})$ as $(R_\alpha f)(z) = f(ze^{i\alpha})$ is an LDI operator.

3. Let T be a linear continuous operator on $H(G)$ and $\psi \in H(G)$. Then the generalized multiplication operator defined as

$$M_\psi T : f(z) \in H(G) \mapsto \psi(z) T f(z) \in H(G)$$

is also a linear continuous operator. In particular case $T = I$, we obtain the ordinary multiplication operator M_ψ.

Theorem 3.3. *Let T be a linear continuous operator on $H(G)$ that is boundary pointwise stable and satisfies the condition* (a) *of Theorem 2.2. Let $\psi \in H(G)$ such that the set $Z(\psi)$ of zeros of ψ is finite. Then $M_\psi T$ is LDI.*

Proof. Firstly, we prove that $M_\psi T$ is boundary pointwise stable. Let $K \subset G$ be a compact set and L be the compact subset given by the stability of T. Then $\widetilde{L} := L \cup Z(\psi) \subset G$ is compact. Fix $a \in G \setminus \widetilde{L}$ and $\varepsilon > 0$. Hence $\psi(a) \neq 0$. By stability, there exist a compact set $B \in \mathcal{K}_1(G)$ with $B \subset G \setminus K$ and a $\delta > 0$

such that for each $f \in H(G)$ with $\|f\|_B < \delta$ we have $|Tf(a)| < \varepsilon/|\psi(a)|$. Hence, $|M_\psi Tf(a)| = |\psi(a)| \cdot |Tf(a)| < \varepsilon$, whence $M_\psi T$ is boundary pointwise stable.

On the other hand, since $Z(\psi)$ is finite and T satisfies the condition (a) of Theorem 2.2, the operator $M_\psi T$ satisfies the same condition. Indeed, replace the compact set K assigned to T by the compact set $K \cup Z(\psi)$. Then Theorem 2.2 concludes the proof. □

Corollary 3.4. *Let $\psi \in H(G)$, $\psi \not\equiv 0$. Then M_ψ is LDI if and only if $Z(\psi)$ is finite.*

Proof. That the finiteness of $Z(\psi)$ implies that M_ψ is LDI follows from Theorem 3.3 just by taking $T = I$. As for the converse, assume, by way of contradiction, that $Z(\psi)$ is infinite. Then $Z(\psi) \in NRC(G)$, because $\psi \not\equiv 0$. But $M(M_\psi, Z(\psi)) = \varnothing$ and this contradicts the hypothesis. □

4. We finish this paper with two assertions involving composition, sum or multiplication of operators. This allows to construct new operators with large dense images from known ones.

Theorem 3.5. *Assume that T, $S : H(G) \to H(G)$ are continuous linear operators, in such a way that T is LDI and S is onto. Then TS is LDI. In particular, every onto continuous linear operator is LDI.*

Proof. It is evident that $M(TS, A) = S^{-1}(M(T, A))$ for every set $A \subset G$, so the residuality part is as in [5]. Assume now that $A \in NRC(G)$. Then there exists a infinite-dimensional closed linear space $F \subset M(T, A) \cup \{0\}$. Then, by linearity and continuity, $S^{-1}(F)$ is a closed linear space contained in $M(TS, A) \cup \{0\}$. If $S^{-1}(F)$ were finite-dimensional, then $\dim(S(S^{-1}(F))) = \dim(F)$ would be also finite (note that $F = SS^{-1}(F)$ because S is onto), which is a contradiction. This proves the first part of the statement. The second part follows because the operator $T = I$ is LDI. □

Theorem 3.6. *Let $T, S : H(G) \to H(G)$ be two linear continuous operators with T LDI. Suppose that for each function $f \in H(G)$ and each point $t \in \partial G$, there exists $\lim_{z \to t} (Sf)(z) \in \mathbb{C}$ ($\in \mathbb{C} \setminus \{0\}$). Then $T + S$ ($T \cdot S$, resp.) is LDI.*

The proof is elementary and left to the interested reader.

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