

QUALITATIVE AND QUANTITATIVE PROPERTIES FOR THE SPACE $l_{p,q}$

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ABSTRACT. The space $l_{p,q}$ is simply the space l_p but renormed by

$$|x|_{p,q} = (\|x^+\|_p^q + \|x^-\|_p^q)^{\frac{1}{q}}, \quad x \in l_p,$$

where $\|\cdot\|_p$ is the usual l_p norm and x^+ and x^- are the positive and negative parts of x , respectively. Bynum used $l_{p,1}$ and Smith and Turett used $l_{2,1}$ to show that neither normal structure nor uniform normal structure is a self dual property for Banach spaces. In this paper we present some more qualitative and quantitative properties for $l_{p,q}$; in particular, we provide an affirmative answer to a question of Khamsi.

1. Introduction.

In 1972, in order to show that normal structure is not a self dual property for Banach spaces, W. L. Bynum [4] introduced the space $l_{p,q}$ which is simply the space l_p but renormed by

$$|x|_{p,q} = (\|x^+\|_p^q + \|x^-\|_p^q)^{\frac{1}{q}}, \quad x \in l_p,$$

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where $\|\cdot\|_p$ is the usual l_p norm and x^+ and x^- are the positive and negative parts of x , respectively. He then showed that $l_{p,1}$, $1 < p < \infty$, has normal structure, but its dual $l_{p',\infty}$ does not. (Here $p' = \frac{p}{p-1}$ is the conjugate number of p .) (It even lacks asymptotic normal structure [2] [5].) In 1990, using the space $l_{2,1}$, M.A. Smith and B. Turett [21] showed that uniform normal structure is not a self dual property for Banach spaces. In this paper we shall show some more qualitative and quantitative properties for $l_{p,q}$. More precisely, we show in Section 2 that for every $1 < p < \infty$, $l_{p,1}$ is 2-uniformly rotund and hence has uniform normal structure. This presents an affirmative answer to a question of Khamsi [12, p. 349]. We then in Section 3 calculate Bynum's weakly convergent sequence coefficient WCS and another coefficient introduced by Khamsi [12] for $l_{p,q}$. In Section 4 we prove that $l_{p,1}$ ($1 < p < \infty$) satisfies the uniform Opial condition, a notion introduced very recently by Prus [20]. Moreover, we evaluate Opial's modulus for $l_{p,1}$. Finally in Section 5, we make some concluding remarks concerning certain kinds of geometrical properties of Banach spaces and raise an open question.

2. Uniform Normal Structure.

Let $(X, \|\cdot\|)$ be a Banach space and A be a bounded closed convex subset of X with more than one point. Recall that the (self-Chebyshev) radius and the diameter of A are the numbers

$$\text{rad}(A) := \inf_{x \in A} \sup_{y \in A} \|x - y\| \quad \text{and} \quad \text{diam}(A) := \sup_{x \in A} \sup_{y \in A} \|x - y\|,$$

respectively. We say that X has normal structure if for every such A , $\text{rad}(A) < \text{diam}(A)$. The normal structure coefficient $N(X)$ of X is defined as the number

$$\inf \left\{ \frac{\text{diam}(A)}{\text{rad}(A)} : A \text{ as above} \right\}.$$

If $N(X) > 1$, then X is said to have uniform normal structure. Both notions, normal and uniform normal structure play important roles in fixed point theory of nonlinear operators (cf. for example, Kirk [13] and Casini and Maluta [6]).

Recall also that a Banach space X is said to be 2-uniformly rotund (2-UR) (Sullivan [22]) if given any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such

that for all $x, y, z \in B_X$, the closed unit ball of X , with $A(x, y, z) \geq \epsilon$, then

$$\|x + y + z\| \leq 3(1 - \delta).$$

Here $A(x, y, z)$ is the volume enclosed by $\{x, y, z\}$, i.e., the number

$$\sup \left\{ \left| \begin{array}{ccc} 1 & 1 & 1 \\ f(x) & f(y) & f(z) \\ g(x) & g(y) & g(z) \end{array} \right| : f, g \in B_{X^*} \right\}.$$

It is clear that uniform convexity implies 2-uniform rotundity. It is also known (Amir [1]) that 2-uniform rotundity implies uniform normal structure.

M.A. Smith and B. Turett [21] showed that $l_{2,1}$ is 2-UR and hence has uniform normal structure, which indicates that uniform normal structure is not a self dual property for Banach spaces. In this section we show that for each $1 < p < \infty$, $l_{p,1}$ is 2-UR and has uniform normal structure, which leads to an affirmative answer to Khamsi's question [12, p. 349].

Theorem 1. *For each $1 < p < \infty$, $l_{p,1}$ is 2-uniformly rotund.*

Proof. We only sketch the proof since it is a slight refinement of the proof to Theorem 2 of Smith and Turett [21]. For simplicity we write $|\cdot|$ for $|\cdot|_{p,1}$ and $\|\cdot\|$ for $\|\cdot\|_p$. What we need to prove is

$$(1) \quad \lim_{n \rightarrow \infty} A(x_n, y_n, z_n) = 0$$

for any sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ of norm-one elements in $l_{p,1}$ with $\lim_{n \rightarrow \infty} |\frac{1}{3}(x_n + y_n + z_n)| = 1$.

Noticing that the inequality $(\alpha^p + \beta^p)^{\frac{1}{p}} \geq \alpha + p^{-1}\beta^p$ holds for all $0 \leq \alpha, \beta \leq 1$ such that $\alpha^p + \beta^p \leq 1$, we get sequences $\{X_n\}$, $\{Y_n\}$ and $\{Z_n\}$ of norm-one elements in $l_{p,1}$ such that (see Step 1 of [21, p. 227])

$$\lim_{n \rightarrow \infty} |x_n - X_n| = 0, \quad \lim_{n \rightarrow \infty} |y_n - Y_n| = 0, \quad \lim_{n \rightarrow \infty} |z_n - Z_n| = 0,$$

(and hence $\lim_{n \rightarrow \infty} |\frac{1}{3}(X_n + Y_n + Z_n)| = 1$) and for all $i, n \in N$, $sgn X_n(i) \cdot sgn Y_n(i) \neq -1$, $sgn X_n(i) \cdot sgn Z_n(i) \neq -1$, $sgn Y_n(i) \cdot sgn Z_n(i)$

$\neq -1$, and for all $n \in N$, $X_n^+, X_n^-, Y_n^+, Y_n^-, Z_n^+, Z_n^-$ are all nonzero elements in $l_{p,1}$. Set $M_n = (X_n + Y_n + Z_n)/3$. From the above properties of $\{X_n\}$, $\{Y_n\}$ and $\{Z_n\}$, we have

$$M_n^+ = (X_n^+ + Y_n^+ + Z_n^+)/3 \quad \text{and} \quad M_n^- = (X_n^- + Y_n^- + Z_n^-)/3.$$

Denote by δ_p the modulus of convexity of l_p . Then (cf. [7] for the case $p \geq 2$ and [16] for $1 < p \leq 2$)

$$(2) \quad \delta_p(\epsilon) \geq d\epsilon^q,$$

where $q = \max(p, 2)$ and $d = p^{-1}2^{-p}$ if $p \geq 2$ and $d = \frac{p-1}{8}$ if $1 < p \leq 2$.

From (2) and Step 2 of [21, p. 229], it follows that

$\lim_{n \rightarrow \infty} \text{dist}(V_n^+, \text{span}\{M_n^+\}) = 0$, where $\{V_n\}$ is any one of the sequences $\{X_n\}$, $\{Y_n\}$ and $\{Z_n\}$. Now by the same argument as in Step 3 of [21, p. 230], we arrive at the desired conclusion (1). \square

Corollary 1. *For each $1 < p < \infty$, the space $l_{p,1}$ has uniform normal structure.*

Remark 1. Let (P) be a property for a Banach space X . Recall that X is said to have the property super- (P) if any Banach space Y that is finitely representable in X has the property (P) . (Y is finitely representable in X if for every finite dimensional subspace Y_0 of Y and $\epsilon > 0$ there exist a subspace X_0 of X and an isomorphism $T : Y_0 \rightarrow X_0$ such that $(1 + \epsilon)^{-1}\|y\| \leq \|Ty\| \leq (1 + \epsilon)\|y\|$ for all $y \in Y_0$.) Let $1 < p < \infty$. Khamsi [12, p. 349] asked if $l_{p,1}$ has super-normal structure. Amir [1] proved that if X has uniform normal structure and is super-reflexive, then X does have super-uniform normal structure. Hence Corollary 1 asserts that $l_{p,1}$ (as it is super-reflexive) has super-uniform normal structure. This presents an affirmative answer to Khamsi’s question.

3. Geometrical Coefficients.

Suppose $(X, \|\cdot\|)$ is a Banach space which is not Schur (i.e., weak convergence and strong convergence for sequences do not coincide). Then Bynum [5] defined the weakly convergent sequence coefficient of X as the number

$$WCS(X) := \inf \left\{ \frac{\text{diam}_a \{x_n\}}{r_a \{x_n\}} : \{x_n\} \text{ a weakly (not strongly) convergent sequence in } X \right\},$$

where $\text{diam}_a\{x_n\}$ and $r_a\{x_n\}$ are the asymptotic diameter and Chebyshev radius of $\{x_n\}$, respectively; namely,

$$\text{diam}_a\{x_n\} = \lim_{n \rightarrow \infty} (\sup\{\|x_i - x_j\| : i, j \geq n\})$$

and $r_a\{x_n\} = \inf\{\limsup_{n \rightarrow \infty} \|x_n - x\| : x \in \overline{\text{co}}\{x_n\}\}$. It is immediately clear that $1 \leq WCS(X) \leq 2$ and $WCS(X) > 1$ implies the weak normal structure of X .

Recall that a Banach space X is said to have a (Schauder) finite dimensional decomposition (FDD in short) if there exists a sequence $\{X_n\}$ of finite dimensional subspaces of X such that every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for all n . To a Banach space X with an FDD, Khamsi [12] associated another coefficient $\beta_p(X)$ of X for $p \in [1, \infty)$ by

$$\beta_p(X) := \inf\{\lambda > 0 : (\|x\|^p + \|y\|^p)^{\frac{1}{p}} \leq \lambda\|x + y\| \text{ for all } x, y \in X \\ \text{with } \max \text{supp}(x) + 1 < \min \text{supp}(y)\},$$

where $\text{supp}(x)$ is the support of x , i.e., the set $\{i \in N : \text{the } i\text{-th component } x_i \text{ of } x \text{ is different from zero}\}$. The relation between $WCS(X)$ and $\beta_p(X)$ is the following.

Theorem 2. *If X is a Banach space with an F.D.D., and $p \in [1, \infty)$, then*

$$(3) \quad WCS(X)\beta_p(X) \geq 2^{\frac{1}{p}}.$$

Proof. Let $\{x_n\}$ be any weakly convergent sequence in X . By a translation if necessary, we may assume that the weak limit of $\{x_n\}$ is zero. We may also assume that $\lim \|x_n\|$ exists (otherwise we pass to a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{n_i \rightarrow \infty} \|x_{n_i}\| = \limsup_{n \rightarrow \infty} \|x_n\|$). Then by a standard method (cf. [12]), there are a sequence $\{u_i\}$ of successive blocks and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim \|x_{n_i} - u_i\| = 0$. From the definition of $\beta_p(X)$, it follows that for all $i \neq j$,

$$(\|u_i\|^p + \|u_j\|^p)^{\frac{1}{p}} \leq \beta_p(X)\|u_i - u_j\|,$$

which immediately implies that $2^{\frac{1}{p}} \leq \beta_p(X)\text{diam}_a\{x_n\}/r_a\{x_n\}$. Since $\{x_n\}$ is arbitrary, the proof is complete. \square

The following corollary improves upon the main result, Theorem 3 of Khamsi [12].

Corollary 2. *If $\beta_p(X) < 2^{\frac{1}{p}}$ for some $p \in [1, \infty)$, then $WCS(X) > 1$ and hence X has weak normal structure.*

Remark 2. Strict inequality in (3) may occur even for a reflexive Banach space X . For example, let $X = l_\infty^2 \oplus l_2$ with norm

$$\|(x, y)\| = |(\|x\|, \|y\|)|,$$

$|\cdot|$ being any p -norm in \mathbb{R}^2 with $1 < p < \infty$. Choosing $u = (x_1, 0)$, $v = (x_2, 0)$, where $x_1 = (1, 0)$ and $x_2 = (0, 1)$, we see that $\beta_2(X) \geq \sqrt{2}$. However $WCS(X) = \sqrt{2}$ and $\beta_2(X)WCS(X) = 2 > \sqrt{2}$.

Remark 3. (i) The condition $\beta_p(X) < 2^{\frac{1}{p}}$ for some $p \in [1, \infty)$ does not imply reflexivity of X .

(ii) Even for a reflexive Banach space X , $\beta_p(X) < 2^{\frac{1}{p}}$ does not imply uniform normal structure. For example, let $X = (\sum_{n=1}^\infty \oplus l_n^\infty)_2$. Then $\beta_2(X) = 1 < \sqrt{2}$ and X lacks uniform normal structure (cf. Maluta [18]).

Theorem 3. *Assume $1 < p < \infty$ and $q \geq 1$. Then*

$$\beta_q(l_{p,q}) = \begin{cases} 2^{\frac{p-q}{pq}}, & \text{if } q \leq p; \\ 1, & \text{if } q > p. \end{cases}$$

Proof. First consider the case $q \leq p$. Given any $u, v \in l_{p,q}$ with $\max \text{supp}(u) + 1 < \min \text{supp}(v)$, we have $\max \text{supp}(u^+) + 1 < \min \text{supp}(v^+)$ and $\max \text{supp}(u^-) + 1 < \min \text{supp}(v^-)$. For simplicity, we write $|\cdot|$ for $|\cdot|_{p,q}$ and $\|\cdot\|$ for $\|\cdot\|_p$. Then we have

$$\begin{aligned} |u|^q + |v|^q &= (\|u^+\|^q + \|u^-\|^q) + (\|v^+\|^q + \|v^-\|^q) \\ &= (\|u^+\|^q + \|v^+\|^q) + (\|u^-\|^q + \|v^-\|^q) \\ &\leq 2^{\frac{p-q}{p}} [(\|u^+\|^p + \|v^+\|^p)^{\frac{q}{p}} + (\|u^-\|^p + \|v^-\|^p)^{\frac{q}{p}}] \\ (4) \quad &= 2^{\frac{p-q}{p}} (\|u^+ + v^+\|^q + (\|u^- + v^-\|^q)) \\ &= 2^{\frac{p-q}{p}} |u + v|^q, \text{ that is,} \\ &(|u|^q + |v|^q)^{\frac{1}{q}} \leq 2^{\frac{p-q}{pq}}, \end{aligned}$$

which shows that $\beta_q(l_{p,q}) \leq 2^{\frac{p-q}{pq}}$. Since for $u = e_1$ and $v = e_3$, equality in (4) does hold, we therefore must have $\beta_q(l_{p,q}) = 2^{\frac{p-q}{pq}}$.

Next consider the case $q > p$. We only need to show $\beta_q(l_{p,q}) \leq 1$. This is equivalent to

$$(5) \quad |u|^q + |v|^q \leq |u + v|^q,$$

for all $u, v \in l_{p,q}$ such that $\max \text{supp}(u) + 1 < \min \text{supp}(v)$. However given any such a pair $\{u, v\}$, noting that the inequality

$$\left(\sum_n a_n^p\right)^{\frac{q}{p}} + \left(\sum_n b_n^p\right)^{\frac{q}{p}} \leq \left(\sum_n (a_n + b_n)^p\right)^{\frac{q}{p}}$$

holds true for all finite sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative numbers, we deduce that

$$\begin{aligned} \|u^+\|^q + \|v^+\|^q &\leq \|u^+ + v^+\|^q, \text{ and} \\ \|u^-\|^q + \|v^-\|^q &\leq \|u^- + v^-\|^q. \end{aligned}$$

Summing these last two inequalities yields (5). \square

Corollary 3. (i) (Khamisi [12]) $\beta_p(l_p) = 1$ and $\beta_1(l_{p,1}) = 2^{\frac{p-1}{p}}$.

(ii) $WCS(l_{p,q}) = \min \left\{ 2^{\frac{1}{p}}, 2^{\frac{1}{q}} \right\}$.

Proof. (i) is a direct consequence of Theorem 3. Combining Theorems 2 and 3, we get $WCS(l_{p,q}) \geq \min \left\{ 2^{\frac{1}{p}}, 2^{\frac{1}{q}} \right\}$. Moreover, considering the sequence $\{e_{2n} - e_{2n+1}\}$ in the case $q \leq p$ and the sequence $\{e_n\}$ in the case $q > p$, we see that $WCS(l_{p,q}) \leq \min \left\{ 2^{\frac{1}{p}}, 2^{\frac{1}{q}} \right\}$. Hence (ii) is proven. \square

4. The Uniform Opial Condition.

Recall that a Banach space $(X, \|\cdot\|)$ is said to satisfy Opial's condition ([19]) if for any sequence $\{x_n\}$ converging weakly to x , we have for all $y \in X, y \neq x, \limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$.

It is known that a Hilbert space and all the l_p spaces for $1 \leq p < \infty$ enjoy this property. Opial's condition plays an important role in fixed point theory of nonexpansive mappings (cf. Opial [19] and Dulst [8]). By a gauge we mean a continuous strictly increasing function $\varphi : [0, \infty) \rightarrow \mathbb{R}^+$

such that $\varphi(0) = 0$ and $\lim_{r \rightarrow \infty} \varphi(r) = \infty$. With a gauge φ , we associate a (possibly multivalued) duality map $J_\varphi : X \rightarrow X^*$, the dual space of X , defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\}, \quad x \in X.$$

A space X is said to have a weakly continuous duality map if there exists a gauge φ such that the duality map J_φ is single-valued and (sequentially) continuous from X , with the weak topology, to X^* , with the weak* topology. Every l_p space ($1 < p < \infty$) has a weakly continuous duality map with the gauge $\varphi(t) = t^{p-1}$. Browder [3] initiated the study of nonlinear operators via duality maps and proved that a space with a weakly continuous duality map satisfies Opial's condition.

In 1992, Prus [20] introduced the notion of the uniform Opial condition. A Banach space $(X, \|\cdot\|)$ is said to satisfy the uniform Opial condition if for every $c > 0$, there exists an $r = r(c) > 0$ such that

$$1 + r \leq \liminf_{n \rightarrow \infty} \|x + x_n\|$$

for all $x \in X$ with $\|x\| \geq c$ and sequences $\{x_n\}$ in X such that $\{x_n\}$ weakly converges to 0 and $\liminf_{n \rightarrow \infty} \|x_n\| \geq 1$. In [17], Lin, Tan and Xu defined Opial's modulus of X , denoted r_X , by

$$r_X(c) := \inf \left\{ \liminf_{n \rightarrow \infty} \|x + x_n\| - 1 \right\}, \quad c \geq 0,$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and all weakly null sequences $\{x_n\}$ in X with $\liminf_{n \rightarrow \infty} \|x_n\| \geq 1$. It is easily seen that the uniform Opial condition implies Opial's condition and that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$. Lin, Tan and Xu [17] proved that a space X with a weakly continuous duality map must satisfy the uniform Opial condition and calculated that Opial's modulus of l_p is $r_{l_p}(c) = (1 + c^p)^{\frac{1}{p}} - 1$. In this section we show that $l_{p,1}$ does satisfy the uniform Opial condition. This presents an example of a Banach space that satisfies the uniform Opial condition but fails to have a weakly continuous duality map.

Theorem 4. *For each $1 < p < \infty$, $l_{p,1}$ satisfies the uniform Opial condition on, with Opial's modulus $r_{l_{p,1}}(c) = (1 + c^p)^{\frac{1}{p}} - 1$, $c \geq 0$.*

Proof. Write $|\cdot|$ for $|\cdot|_{p,1}$, $\|\cdot\|$ for $\|\cdot\|_p$ and $f(t_1, t_2, t_3, t_4)$ for the function $(t_1^p + t_3^p)^{\frac{1}{p}} + (t_2^p + t_4^p)^{\frac{1}{p}}$. Then it is easily seen that for any $c > 0$, the infimum of f over the region $D_c := \{(t_1, t_2, t_3, t_4) : 0 \leq t_1, t_2, t_3, t_4, t_1 + t_2 \geq 1 \text{ and } t_3 + t_4 \geq c\}$ is achieved at the point $(\frac{1}{2}, \frac{1}{2}, \frac{c}{2}, \frac{c}{2})$ with the value $(1 + c^p)^{\frac{1}{p}}$. Now suppose $\{x_n\}$ is a sequence in $l_{p,1}$ such that $\{x_n\}$ weakly converges to 0 and $\liminf_{n \rightarrow \infty} |x_n| \geq 1$ and x is an element of X with norm at least c . Choose a subsequence $\{z_n\}$ of $\{x_n\}$ such that $\lim_k |z_k + x| = \liminf_{n \rightarrow \infty} |x_n + x|$ and that $\lim_k |z_k|$, $B := \lim_k |z_k^+|$ and $C := \lim_k |z_k^-|$ exist. Since $\lim_k |z_k| = \lim_k (|z_k^+| + |z_k^-|) \geq \liminf_{n \rightarrow \infty} |x_n| \geq 1$, we have $B + C \geq 1$. By Bynum's proof of his Theorem 4 [5], we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} |x_n + x| &= \lim_k |z_k + x| \\ &= \lim_k (\|(z_k + x)^+\| + \|(z_k + x)^-\|) \\ &= (B^p + \|x^+\|^p)^{\frac{1}{p}} + (C^p + \|x^-\|^p)^{\frac{1}{p}} \\ &= f(B, C, \|x^+\|, \|x^-\|) \\ &\geq \inf_{D_c} f(t_1, t_2, t_3, t_4) = (1 + c^p)^{\frac{1}{p}}. \end{aligned}$$

It follows that $r_{l_{p,1}}(c) \geq (1 + c^p)^{\frac{1}{p}} - 1$. Finally, by considering the sequence $\{e_n\}$, we conclude that $r_{l_{p,1}}(c) = (1 + c^p)^{\frac{1}{p}} - 1$. \square

Remark 4. Considering the elements $x = e_1 - e_2$ and $y = e_1 - e_3$, we see that the dual space $l_{p',\infty}$ of $l_{p,1}$ is not strictly convex and hence $l_{p,1}$ is not smooth. Therefore, $l_{p,1}$ demonstrates a class of Banach spaces which satisfy the uniform Opial condition but fail to have a weakly continuous duality map.

5. Concluding Remarks and an Open Question.

A Banach space $(X, \|\cdot\|)$ is said to satisfy the weak (or non-strict) Opial condition ([19] and [8]) if for any sequence $\{x_n\}$ converging weakly to x , we have for all $y \in X$,

$$\liminf_{n \rightarrow \infty} \|x_n - x\| \leq \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

Lemma 2.2 of Prus [20] shows that for each $1 < p < \infty$, $l_{p,\infty}$ satisfies the weak Opial condition; however, it fails to satisfy Opial's condition for it

lacks normal structure [4]. This presents a class of Banach spaces that satisfy the weak Opial condition but fail to satisfy Opial’s condition. We are indebted to the anonymous referee for letting us know that the family of spaces $X_a := (l_2, \max\{a\|\cdot\|_2, \|\cdot\|_\infty\})$ [$0 < a < 1$] has the non-strict Opial condition, but fails Opial’s condition for all a .

Recall that a Banach space $(X, \|\cdot\|)$ is nearly uniformly convex (NUC) [10] if it is reflexive and its norm is uniformly Kadec-Klee, i.e., for each $\epsilon > 0$ there exists δ such that for any sequence $\{x_n\} \subset B_X$, the closed unit ball of X , the conditions that $\{x_n\}$ weakly converges to x and $\inf\{\|x_n - x_m\| : n \neq m\} \geq \epsilon$ imply $\|x\| \leq 1 - \delta$.

It is known that any k -UR Banach space must be NUC (cf. for example, Kirk [14]). The dual notion of NUC is nearly uniformly smooth (NUS); namely a Banach space is NUS if and only if X^* is NUC. Thus, from Theorem 1, $l_{p,\infty}$ is NUS for $1 < p < \infty$, which together with Bynum [4] [5] shows that NUS Banach spaces need not have normal (even asymptotic normal) structure although NUC spaces do have normal structure [9].

Recently, A. Jimenez-Melado and E. Llorens-Fuster [10] introduced the notion of orthogonal convexity that implies the fixed point property for nonexpansive mappings (FPP). Let $(X, \|\cdot\|)$ be a Banach space, for $x, y \in X$ and $\lambda > 0$, we set

$$M_\lambda(x, y) = \{z \in X : \max\{\|z - x\|, \|z - y\|\} \leq \frac{1}{2}(1 + \lambda)\|x - y\|\}.$$

If A is a nonempty bounded subset and $\{x_n\}$ is a bounded sequence of X then we write $|A| = \sup\{\|z\| : z \in A\}$, $D[\{x_n\}] = \limsup_{n \rightarrow \infty} (\limsup_{m \rightarrow \infty} \|x_n - x_m\|)$, and $A_\lambda[\{x_n\}] = \limsup_{n \rightarrow \infty} (\limsup_{m \rightarrow \infty} |M_\lambda(x_n, x_m)|)$. The Banach space X is called orthogonally convex if for each sequence $\{x_n\}$ in X weakly convergent to zero, with $D[\{x_n\}] > 0$, there exists $\lambda > 0$ such that $A_\lambda[\{x_n\}] < D[\{x_n\}]$.

By Corollary 5 and Theorem 1 of [15], we see that for each $1 < p < \infty$, $l_{p,\infty}$ is orthogonally convex (and hence has the FPP).

Bynum [4] observed that for every $1 < p, q < \infty$, the space $l_{p,q}$ is uniformly convex. This together with Corollary 1 shows that for every $p > 1$ and $q \geq 1$, $l_{p,q}$ has uniform normal structure. However the following question remains open.

Question. *What is the exact value of the normal structure coefficient $N(l_{p,q})$ of $l_{p,q}$ for $p > 1$ and $q \geq 1$?*

We only have a lower bound for $N(l_{p,q})$ in some cases.

Proposition. For $p \geq q \geq 1$, $N(l_{p,q}) \geq 2^{\frac{q-p}{pq}} \min\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}$.

Proof. By the Hölder inequality, it is easily deduced that $\|x\|_p \leq |x|_{p,q} \leq 2^{\frac{p-q}{pq}} \|x\|_p$ for $p \geq q$ and $x \in l_{p,q}$. Hence the Banach-Mazur distance $d(l_p, l_{p,q})$ between l_p and $l_{p,q}$ is at most $2^{\frac{p-q}{pq}}$. From Theorem 5 of [5] and Corollary 2.10 (e) of [1], it follows that

$$N(l_{p,q}) \geq \frac{N(l_p)}{d(l_p, l_{p,q})} \geq 2^{\frac{q-p}{pq}} \min\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}. \quad \square$$

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