SOOCHOW JOURNAL OF MATHEMATICS
Volume 21, No. 1, pp. 107-115, January 1995

REMARKS ON MULTIVALUED NONEXPANSIVE MAPPINGS

BY

GENARO LÓPEZ ACEDO* AND HONG-KUN XU

Abstract. Convergence of fixed point sets of multivalued nonexpansive mappings is studied under both the Mosco and Hausdorff senses. A characterization for *-nonexpansive multivalued mappings is given. Also a counterexample is constructed to show a negative answer to a question raised by A. Canbtti, G. Marino and P. Pibtramala.

Let H be a Hilbert space, C a bounded closed convex subset of H and T: $C \to C$ is a (single-valued) nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$). Then for each fixed $x_0 \in C$ and $\lambda \in [0, 1)$, the mapping $T_\lambda : C \to C$ defined by

$$T_\lambda x = (1 - \lambda)x_0 + \lambda Tx, \quad x \in C \tag{1}$$

is a contraction on C. Hence, Banach's Contraction Principle yields a unique $x_\lambda \in C$ such that $T_\lambda x_\lambda = x_\lambda$; namely,

$$x_\lambda = (1 - \lambda)x_0 + \lambda Tx_\lambda. \tag{2}$$

An elegant result in the fixed point theory of (single-valued) nonexpansive mappings is Browder's theorem [1] which states that the approximating curve x_λ defined by (2) converges strongly as $\lambda \to 1$ to a fixed point of T. This result

Received February 22, 1994; revised April 27, 1994.

AMS Subject Classification. Primary 47H09, 47H10.

KeyWords▷ Multivalued nonexpansive mapping, fixed point, Hausdorff metric, Mosco convergence, multivalued *-nonexpansive mapping.

*Research partially supported by DGICYT under project PB 90-903 and the Junta de Andalucia under project 1241.

was extended by Reich [11] to a framework of a uniformly smooth Banach space. For recent progress along the line, the reader is referred to [12], [7], [14].

Now we turn to the multivalued case. For a metric space (X, d), we use $CB(X)$ to denote the family of all nonempty closed bounded subsets of X, $K(X)$ the family of all nonempty compact subsets of X, and H the Hausdorff metric on $CB(X)$ induced by the metric d of X; that is, for A, $B \in CB(X)$,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,E) = \inf\{d(x,y) : y \in E\}$ is the distance from a point $x \in X$ to a subset $E \subset X$. Now recall that a multivalued mapping $T : X \to CB(X)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq d(x,y), \quad x, y \in X.$$

Recall also that a sequence $\{A_n\}$ in $CB(X)$ is said to converge to an element $A \in CB(X)$ under the Mosco sense if

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A,$$

where $\limsup_{n\to\infty} A_n = \{x \in X$: there are subsequences $\{n_k\}$ and $\{x_{n_k}\}$ with $x_{n_k} \in A_{n_k}$ such that $x_{n_k} \to x\}$ and $\liminf_{n\to\infty} A_n = \{x \in X$: there exists $x_n \in A_n$ for each n such that $x_n \to x\}$. It is not hard to see that if $H(A_n, A) \to 0$ $(A_n, A \in CB(X))$, then $A_n \to A$ under the sense of Mosco. Assume now H and C are as above and $T : C \to K(C)$ is nonexpansive. For each fixed $x_0 \in C$ and $\lambda \in [0,1)$, we define the mapping $T_\lambda : C \to K(C)$ by the same formula (1) above. Then T_λ is a multivalued contraction and hence has a (nonunique, in general) fixed point $x_\lambda \in C$ (see[8]); i.e.,

$$x_\lambda \in (1-\lambda)x_0 + \lambda T x_\lambda. \tag{3}$$

Let $y_\lambda \in Tx_\lambda$ be such that

$$x_\lambda = (1-\lambda)x_0 + \lambda y_\lambda. \tag{4}$$

A natrual question now gives rise to whether Browder's theorem can be extended to the multivalued case. The following simple example presents a negative answer.

Example 1.[10] Let $C = [0,1] \times [0,1]$ be the square in the real plane and $T : C \to K(C)$ be defined by

$$T(a,b) = \text{the triangle with vertices } (0,0),(a,0),(0,b), \quad (a,b) \in C.$$

Then it is easy to see that for any $(a_i, b_i) \in C$, $i = 1, 2$,

$$H(T(a_1,b_1), T(a_2,b_2)) = \max\{|a_1 - a_2|, |b_1 - b_2|\} \leq \|(a_1,b_1) - (a_2,b_2)\|,$$

showing that T is nonexpansive. It is also easy to see that the fixed point set of T is $F(T) = \{(a,0) : 0 \leq a \leq 1\} \cup \{(0,b) : 0 \leq b \leq 1\}$. Let $x_0 = (1,0)$. Then the map T_λ defined by (1) has the fixed point set

$$F(T_\lambda) = \{(a,0) : 1 - \lambda \leq a \leq 1\}.$$

Let

$$x_\lambda = \begin{cases} (\frac{1}{n}, 0), & \text{if } \lambda = 1 - \frac{1}{n}; \\ (1,0), & \text{otherwise.} \end{cases}$$

Then $\{x_\lambda\}$ satisfies (3) but is not convergent.

The same example also shows that the net $\{F(T_\lambda)\}$ of fixed point sets of the T_λ's does not converge as $\lambda \to 1$ to the fixed point set $F(T)$ of T under either the Hausdorff metric or the Mosco sense. However, this will be so if we put some restrictions on the fixed point set $F(T)$ of T. First recall that a Banach space X is said to satisfy *Opial's property* [9] if for any sequence $\{x_n\}$ in X, the condition that $\{x_n\}$ converges weakly to x implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \quad \forall y \in X, \quad y \neq x.$$

Spaces saisfying this property include all Hilbert spaces and ℓ^p for $1 < p < \infty$. Also it is known [3] that any separable Banach space can be equivalently renormed so that it possesses Opial's property.

Theorem 1. *Let C be a nonempty closed bounded convex subset of a Banach space X satisfying Opial's property and $T : C \to K(C)$ be a nonexpansive mapping such that $F(T) = \{z\}$. Then for any $x_0 \in C$, the net $\{F(T_\lambda)\}$ of fixed point sets of the T_λ's weakly converges as $\lambda \to 1$ to the fixed point set $F(T)$ of T under the Mosco sense, i.e.,*

$$w - \limsup_{\lambda \to 1} F(T_\lambda) = w - \liminf_{\lambda \to 1} F(T_\lambda) = F(T).$$

Proof. It is sufficient to show that

(i) $F(T) \supseteq w - \limsup_{\lambda \to 1} F(T_\lambda)$ and

(ii) $w - \liminf_{\lambda \to 1} F(T_\lambda) \supseteq F(T)$.

To show (i), we assume that $x \in w - \limsup_{\lambda \to 1} F(T_\lambda)$, which means that there exist a sequence $\lambda_n \in [0,1)$ converging to 1 and a sequence $\{x_n\}$ such that $x_n \in F(T_{\lambda_n})$ and $x_n \to x$ weakly. Let $y_n \in Tx_n$ be such that $x_n = (1 - \lambda_n)x_0 + \lambda_n y_n$. Choose $z_n \in Tx$ satisfying

$$\|y_n - z_n\| \leq H(Tx_n, Tx) \leq \|x_n - x\|. \tag{5}$$

Since Tx is compact, we may assume that $z_n \to z_\infty \in Tx$ strongly. Noting that $\|x_n - y_n\| \to 0$, we obtain by (5) that

$$\limsup \|x_n - z_\infty\| \leq \limsup \|x_n - x\|. \tag{6}$$

Since $x_n \to x$ weakly, it follows from (6) and Opial's property that $x = z_\infty$ and $x \in Tx$. This concludes the proof of (i). Next we show (ii). For each $\lambda \in [0,1)$, choose any $x_\lambda \in F(T_\lambda)$ and $y_\lambda \in Tx_\lambda$ satisfying (4). Then by the same proof as above, we see that every weak cluster point of $\{x_\lambda\}$ is a fixed point of T. But, by assumption, $F(T) = \{z\}$. Hence $\{x_\lambda\}$ converges weakly as $\lambda \to 1$ to z.

If the unique fixed point z of T is such that $Tz = \{z\}$, then we have the following strong convergence result.

Theorem 2. *Let C be a nonempty closed convex subset of a Hilbert space H and T: $C \to K(C)$ be a nonexpansive mapping with a unique fixed point z. Suppose in addition that $Tz = \{z\}$. Then $H(F(T_\lambda), F(T)) \to 0$ as $\lambda \to 1$.*

Proof. First we observe that $\{F(T_\lambda)\}$ is uniformly bounded. In fact, given any $x_\lambda \in F(T_\lambda)$, we have some $y_\lambda \in Tx_\lambda$ such that $x_\lambda = (1-\lambda)x_0 + \lambda y_\lambda$. However,

$$\|y_\lambda - z\| = d(y_\lambda, Tz) \le H(Tx_\lambda, Tz) \le \|x_\lambda - z\|.$$

Hence

$$\|x_\lambda - z\| \le \lambda\|y_\lambda - z\| + (1-\lambda)\|x_0 - z\| \le \lambda\|x_\lambda - z\| + (1-\lambda)\|x_0 - z\|.$$

This implies that $\|x_\lambda - z\| \le \|x_0 - z\|$ and $\{x_\lambda\}$ is uniformly bounded. Now choose $x_\lambda \in F(T_\lambda)$ such that

$$H(F(T_\lambda), F(T)) = \sup_{x \in F(T_\lambda)} \|x - z\| < \|x_\lambda - z\| + 1 - \lambda.$$

We shall show that $\|x_\lambda - z\| \to 0$ as $\lambda \to 1$. Indeed, we have $y_\lambda \in Tx_\lambda$ satisfying (4). Since $\|y_\lambda - z\| = d(y_\lambda, Tz) \le H(Tx_\lambda, Tz) \le \|x_\lambda - z\|$, we obtain

$$\left\|\frac{x_\lambda - (1-\lambda)x_0}{\lambda} - z\right\| \le \|x_\lambda - z\|; \quad \text{that is,}$$

$$\left\|\frac{x_\lambda - x_0}{\lambda} + (x_0 - z)\right\|^2 \le \|(x_\lambda - x_0) + (x_0 - z)\|^2,$$

which leads to

$$\|\lambda - x_0\|^2 \le \frac{2\lambda}{1+\lambda}\langle x_\lambda - x_0, z - x_0\rangle \le \|x_\lambda - x_0\|\|z - x_0\|.$$

Therefore,

$$\|x_\lambda - x_0\| \le \|z - x_0\|. \tag{7}$$

From the proof of theorem 1, we know that $x_\lambda \to z$ weakly as $\lambda \to 1$. It then easily follows from (7) that $\limsup_{\lambda\to 1}\|x_\lambda\| \le \|z\|$. On the other hand, due to the lower weak continutiy of the norm of H, we have $\liminf_{\lambda\to 1}\|x_\lambda\| \ge \|z\|$. Therefore, we have $\lim_{\lambda\to 1}\|x_\lambda\| = \|z\|$ and

$$\|x_\lambda - z\|^2 = \|x_\lambda\|^2 - 2\langle x_\lambda, z\rangle + \|z\|^2 \to 0 \quad \text{as} \quad \lambda \to 1.$$

This completes the proof of the theorem.

Corollary 1. [10] *Let the assumptions of theorem* 2 *be satisfied. Then*

$$w - \limsup_{\lambda - 1} F(T_\lambda) = \| \cdot \| - \liminf_{\lambda \to 1} F(T_\lambda) = F(T).$$

Remark 1. The example above shows that the conclusions of theorems 1 and 2 are not valid if the fixed point set $F(T)$ of T is not a singleton. However, it is an open question whether the restriction $Tz = \{z\}$ in Theorem 2 can be removed. We also do not know if Theorem 2 is valid outside a Hilbert space.

Next we let (X, d) be a metric space. A multivalued map $f : X \to K(X)$ is said to be *-nonexpansive [4] if for all $x, y \in X$ and $u_x \in f(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \in f(x)\}$, there exists $u_y \in f(y)$ with $d(y, u_y) = \inf\{d(y, w) : w \in f(y)\}$ such that

$$d(u_x, u_y) \leq d(x, y).$$

It is obvious that this notion is identical with the notion of nonexpansiveness for singlevalued mappings. But they are different for multivalued mappings (see [13]). We now give a characterization of multivalued *-nonexpansive mappings. Denote by P_f the map $x \mapsto \{u_x \in f(x) : d(x, u_x) = \inf\{d(u, x) : u \in f(x)\}\}$. Note that $P_f(x)$ is nonempty for $f(x)$ is compact.

Theorem 3. *A multivalued map* $f : X \to K(X)$ *is* *-nonexpansive *if and only if the associated map* $P_f : X \to K(X)$ *is nonexpansive.*

Proof. First assume that f is *-nonexpansive. Given any $x, y \in X$ and $u_x \in P_f(x)$. By definiition, there is $u_x \in f(y)$ such that $d(u_x, u_y) \leq d(x, y)$. It follows that

$$\sup_{u_x \to P_f(x)} d(u_x, P_f(y)) \leq \sup_{u_x \to P_f(x)} d(u_x, u_y) \leq d(x, y).$$

The same argument shows that

$$\sup_{u_y \to P_f(y)} d(u_y, P_f(x)) \leq d(x, y).$$

Hence

$$H(P_f(x), P_f(y)) \leq d(x, y)$$

and P_f is nonexpansive. Conversely, we assume that P_f is nonexpansive. Then given any $x, y \in X$ and $u_x \in f(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \to f(x)\}$ (i.e., $u_x \in P_f(x)$). By compactness, we can choose $u_y \in P_f(y)$ such that $d(u_x, u_y) = d(u_x, P_f(y))$. Hence

$$d(u_x, u_y) \leq H(P_f(x), P_f(y)) \leq d(x, y)$$

and f is *-nonexpansive.

Remark 2. Theorem 3 indicates that the fixed point theory of multivalued nonexpansive mappings applies to multivalued *-nonexpansive mappings; in particular, we have the following results whose nonexpansive counterparts were proved in [5] and [6], respectively.

Corollary 2 *Let X be a Banach space satisfying Opial's property, C a nonempty weakly compact convex subset of X , and $T : C \to K(C)$ a *-nonexpansive mapping. Then T has a fixed point.*

Corollary 3. *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X, and $T : C \to K(C)$ a *-nonexpansive mapping. Then T has a fixed point.*

Corollaries 2 and 3 improve upon the corresponding results of [4] and [13].

We conclude the paper with a counterexample that presents a negative answer to a question raised by A. Canbtti, G. Marino and P. Pibtramala [2].

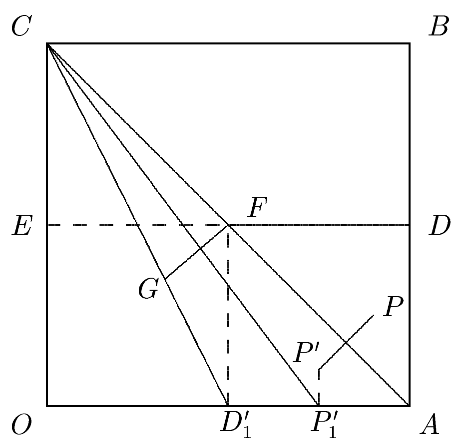
Suppose that H is a Hilbert space and K is a nonempty closed convex subset of H. We denote by $\mathcal{KC}(K)$ the family of all nonempty compact convex subsets of K, and by $d(A, B)$ the distance between two subsets $A, B \subset H$, i.e., $d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$. With each mapping $T : K \to \mathcal{KC}(H)$ one can associate a multivalued mapping $\hat{T} : K \to \mathcal{KC}(H)$ defined as follows:

$$\hat{T}x := \{y \in Tx : d(y, K) = d(Tx, K)\}.$$

The question raised by A. Canbtti, G. Marino and P. Pibtramala (see [2, Remark 1, p. 207]) is whether the nonexpansiveness of T implies that $\hat{T}$ is nonexpansive. The following example shows that the answer is negative.

Example 2. Let OABC be the unit square $[0,1]\times[0,1]$ in the plane H. Let D and E be the midpoints of the segments $\overline{AB}$ and $\overline{OC}$, respectively. Let K be the triangle $\triangle ADF$. To each point $P \in K$, let P' be the symmetric point of P with respect to the diagonal segment $\overline{AC}$. Let P_1' be the projection of P' onto the segment $\overline{OA}$. Now we define a map $T : K \to \mathcal{KC}(H)$ by setting (see the figure below)

$$T(P) := \text{The segment } \overline{CP_1'}.$$



It is then easy to see that T is a nonexpansive mappipng with the unique fixed point A. We also have the following facts:

(i) $T(A) = \overline{AC}$ and hence $d(T(A), K) = 0$;

(ii) $\hat{T}(A)$ is the segment $\overline{AF}$;

(iii) $T(D) = \overline{CD_1'}$, where D_1' is the midpoint of $\overline{OA}$;

(iv) $\hat{T}(D) = \{G\}$, where G is the nearest point projection of F onto the segment $\overline{CD_1'}$. Hence

$$\begin{aligned} H(\hat{T}(A), \hat{T}(D)) &= \sup\{d(G,M) : M \in \overline{AF}\} \\ &= \text{The length of the segment } \overline{GA} \\ &> \text{The length of the segment } \overline{AD} \\ &= d(A,D), \end{aligned}$$

showing that $\hat{T}$ is not nonexpansive.

Acknowledgement

Part of this work was done while the second author was visiting Universidad de Sevilla. He would like to thank Universidad de Sevilla for financial support and Departamento de Análisis Matemático for hospitality. He aloso wishes to thank Professor K. K. Tan for providing him with a copy of the paper [2]. Both authors are grateful to the anonymous referee for careful reading and good sugestions for the manuscript.

References

[1] F. E. Browder, *Convergenceofapproximantstofixedpointsofnonexpansivenonlinearmappings inBanachspaces*, Arch. Rational Mech. Anal., 24(1967), 82-90.

[2] A. Canbtti, G. Marino and P. Pibtramala, *Fanφsbestapproximationandfixedpointtheorems formultivaluednonexpansivemappingsinHilbertspaces*, Indian J. Math., 33(1991), 203-220.

[3] D. van Dulst, *Equivalentnormsandthefixedpointpropertyfornonexpansivemappings*, J. London Math. Soc., 25(1982), 139-144.

[4] T. Husain and A. Latif, *Fixedpointsofmultivaluednonexpansivemaps*, Math. Japonica, 33(1988), 385-391.

[5] E. Lami Dozo, *MultivaluednonexpansivemappingsandOpialφscondition*, Proc. Amer. Math. Soc., 38(1973),286-292.

[6] T. C. Lim, *Afixedpointtheoremformultivaluednonexpansivemappingsinauniformlyconvex Ba› nachspace*, Bull. Amer. Math. Soc., 80(1974), 1123-1126.

[7] G. Marino and G. Trometta, *Onaℓproximationfixedpointsfornonexpansivemaps*, Indian J. math., 34(1992), 91-98.

[8] S. Nadler, *Multivaluedcontractionmappings*, Pacific J. Math., 30(1969), 475-488.

[9] Z. Opial, *Weakconvegenceofthesequenceofsuccessiveapproximationsfornonexpansivemappings*, Bull. Amer. Math. Soc., 73(1973), 591-597.

[10] P. Pietramala, *Convergenceofapproximatingfixedpointssetsformultivaluednonexpansivemap› pings*, Comment. Math. Univ. Carolinae, 32(1991), 697-701.

[11] S. Reich, *StrongconvergencetheoremsforresolventsofaccretiveoperatorsinBanachspaces*, J. Math. Anal. Appl., 75(1980), 287-292.

[12] S. P. Singh and B. Watson, *OnapproximatingfixedpointsProc▷SymposiaPureMath▷*, 45, pt. 2(1986), 393-395.

[13] H. K. Xu, *Onweaklynonexpansiveand⇀›nonexpansivemultivaluedmappings*, Math. Japonica, 39(1991), 441-445.

[14] H. K. Xu and X. M. Yin, *Strongconvergencetheoremsfornonexpansivenon›selfmappings‹Non› linearAnalysis*, to appear.

Departamento de Análisis Matemático, Facultad de Matemáticas,Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain.

Institute of Applied Mathematics, East China University of Science and Technology, Shanghai 200237, China.