ON CHARACTERIZATIONS OF CLASSICAL POLYNOMIALS

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Abstract. It is well known that the classical families of Jacobi, Laguerre, Hermite, and Bessel polynomials are characterized as eigenvectors of a second order linear differential operator with polynomial coefficients, Rodrigues formula, etc. In this paper we present an unified study of the classical discrete polynomials and q-polynomials of the q-Hahn tableau by using the difference calculus on linear-type lattices. We obtain in a straightforward way several characterization theorems for the classical discrete and q-polynomials of the q-Hahn tableau. Finally, a detailed discussion of the Marcelln et. al. characterization is presented.

1. Introduction

The classical polynomials (those of Hermite, Laguerre, Jacobi, and Bessel) are the most important instances of orthogonal polynomials. One of the reasons is because they satisfy not only a three-term recurrence relation (TTRR)

$$\begin{gathered} xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad \gamma_n \neq 0, \\ P_{-1}(x) = 0, \quad P_0(x) = 1, \end{gathered} \tag{1.1}$$

but also other useful properties: they are the eigenvectors of a second order linear differential equation with polynomial coefficients, their derivatives also constitute an orthogonal family, their generation functions can be given explicitly, among others (see for instances [1, 8, 24, 25] or the more recent work [3]). Among all these properties there are very important ones that characterize the classical families.

In fact not every property characterizes the classical polynomials. The simplest example is the TTRR (1.1). It is well known (see e.g. [8]) that the TTRR characterizes the orthogonal polynomials if $\gamma_n \neq 0$ for all $n \in \mathbb{N}$. This is the so-called Favard Theorem (for a review see [18]). Nevertheless there exist several families that satisfy the TTRR but not a linear differential equation with polynomial coefficients, or a Rodrigues-type formula. In fact only few families of orthogonal polynomials satisfy these properties as we will show. For reviews on the characterization theorems see [1, 3, 8].

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The oldest characterization is the so called Hahn characterization —unless this was firstly observed and proved for the Jacobi, Laguerre, and Hermite polynomials by N. Sonin in 1887—. In [11], Hahn proved the following

Theorem 1.1 (Sonin-Hahn [11, 19])**.** *Given a sequence of orthogonal polynomials $(P_n)_n$, it is a classical sequence if an only if the sequence of their derivatives $(P_n')_n$ is an orthogonal sequence.*

In fact the following theorem holds (see the nice survey paper [1] and also [19, 20])

Theorem 1.2. *The following properties are equivalent:*

(1) $(P_n)_n$ *is a classical orthogonal polynomial sequence (COPS),*
(2) *The sequence of their derivatives* $(P_n')_n$ *is an COPS*[1]*,*
(3) $(P_n)_n$ *satisfies the second order linear differential equation with polynomial coefficients (Bochner* [7]*)*

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda P_n(x) = 0,$$

where $\deg(\sigma) \le 2$, $\deg(\tau) = 1$*, and are independent of* n*, and* λ *is a constant independent of* x*.*
(4) $(P_n)_n$ *can be expressed by the Rodrigues formula (Tricomi* [27] *and Cryer* [9]*)* $P_n(x) = \dfrac{B_n}{\rho(x)}\dfrac{d^n}{dx^n}[\sigma^n(x)\rho(x)]$.
(5) *The polynomials are orthogonal with respect to a weight function* ρ *that satisfies the Pearson differential equation* $[\sigma(x)\rho(x)]' = \tau(x)\rho(x)$*, where the polynomials* σ *and* τ *are such that* $\deg(\sigma) \le 2$, $\deg(\tau) = 1$ *(Hildebrandt* [14]*).*
(6) *There exist three sequences* $(a_n)_n$*,* $(b_n)_n$*,* $(c_n)_n$*, and a polynomial* σ*,* $\deg(\sigma) \le 2$*, such that (Al-Salam & Chihara* [2]*)*

$$\sigma(x)P_n'(x) = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), \quad n \ge 1. \tag{1.2}$$

(7) *There exist two sequences* $(f_n)_n$ *and* $(g_n)_n$ *such that the following relation for the monic polynomials holds (Marcellán et al* [19]*)*

$$P_n(x) = \frac{P_{n+1}'(x)}{n+1} + f_nP_n'(x) + g_nP_{n-1}'(x), \quad n \ge 1. \tag{1.3}$$

The proof of this theorem can be found in the appendix A.

A natural extension of the classical polynomials are the so-called discrete polynomials (those of Charlier, Meixner, Kravchuk, and Hahn, see e.g. [8, 24, 25]) and the q-polynomials (see e.g. [6, 24, 25]). In fact, Hahn in 1949 [13] posed the problem of finding all the orthogonal polynomial sequences that satisfy the conditions 2–5 from theorem 1.2 but instead of using the

[1]Notice that this is not the Hahn theorem. In the Hahn theorem the orthogonality of both sequences it is impossed whereas here a more restrictive conditions is supposed: $(P_n)_n$ or $(P_n')_n$ is a classical family.

derivatives, he use the linear operator $L_{q,w}$

$$L_{q,w}f(x) = \frac{f(qx+w)-f(x)}{(q-1)x+w}, \qquad q,w \in \mathbb{R}^+.$$

Hahn solved the problem for the case when $q \in (0,1)$ and $w = 0$, that leads to the q-Hahn tableau (see e.g. [16] and [5]). The case $w = q = 1$, leads to the classical discrete polynomials of Charlier, Meixner, Kravchuk, and Hahn (see [8, 17, 24]). A complete study of the characterization theorems for these two cases has been performed using a functional approach in the papers [10] (discrete case) and [21] ("q" case). The main aim of the present paper is twice: on one hand to present a very simple and unified approach to the afore said two cases using the theory of difference equations on lattices presented in [24, 25], and on the other hand to complete the study started in [10, 20, 21].

The structure of the paper is as follows: In section 2 we introduce the "linear" lattices $x(s)$ and characterize them. In section 3 the characterization theorem is presented and proved for any linear-type lattice and, as corollaries, the corresponding theorems for the uniform lattice $x(s) = s$ and the q-linear lattice $x(s) = c_1 q^s + c_2$ are obtained. Finally, in Section 4, we discuss each case in details as well as the classical case (that can be obtained taking an appropriate limit $q \to 1^-$). In particular, some problems related with the Marcellán et al. characterization [19] are discussed.

2. The linear-type lattices $x(s)$

Definition 2.1. *We say that $x(s)$ is a linear-type lattice if*

$$x(s+\zeta) = F(\zeta)x(s) + G(\zeta), \quad \forall s,\zeta \in \mathbb{C}, \quad F(\zeta) \neq 0. \tag{2.1}$$

Obviously for the linear lattice $x(s) = s$ we have $F(\zeta) = 1$ and $G(\zeta) = \zeta$. Another important instance of the linear-type lattice is the q-linear lattice, ($q \neq \{0, \pm 1\}$), i.e., the functions of the form $x(s) = Aq^s + B$. In this case $x(s+\zeta) = F(\zeta)x(s) + G(\zeta)$, where $F(\zeta) = q^\zeta$ and $G(\zeta) = B(1 - q^\zeta)$.

Proposition 2.2. *Let $q \neq \{0, \pm 1\}$. The function $x(z)$ is a q-linear lattice of z if and only if it satisfies $x(z+1) = qx(z) + C$.*

Proof. A straightforward computation shows that if $x(z)$ is a q-linear function of n, i.e., $x(z) = cq^z + d$ then it satisfies the recurrence formula $x(z+1) = qx(z) + C$, where $C = d(1-q)$ is a constant. But the general solution of the difference equation $x(z+1) = qx(z) + C$ is $x(z) = Aq^z + D$, where A and D are, in general, non-zero constants. $\square$

Notice that for the linear-type lattices, if $Q_m(x(s))$ is a polynomial of degree m in $x(s)$, $Q_m(x(s+\alpha))$ is also a $m-$th degree polynomial in $x(s)$, i.e., $Q_m(x(s+\alpha)) = \widetilde{Q}_m(x(s))$. Moreover, for the linear-type lattices we have the following

Lemma 2.3. *Let $x(s)$ be a linear-type lattice and $Q_m(x(s))$ a polynomial of degree m in $x(s)$. Then*

$$\frac{\Delta Q_m(x(s+\alpha))}{\Delta x(s+\beta)} = R_{m-1}(x(s)), \qquad \forall \alpha, \beta \in \mathbb{C},$$

where $R_{m-1}(x(s))$ is again a polynomial in $x(s)$ but of degree $m-1$ and $\Delta f(s) = f(s+1) - f(s)$.

Proof. It is sufficient to prove the lema for the powers $x^n(s)$. Since $x(s)$ is a linear-type lattice

$$\frac{\Delta x^n(s+\alpha)}{\Delta x(s+\beta)} = \frac{\Delta(F(\alpha)x(s)+G(\alpha))^n}{F(\beta)\Delta x(s)} = \sum_{k=0}^{n} \binom{n}{k} \frac{F(\alpha)^k G(\alpha)^{n-k}}{F(\beta)} \frac{\Delta x^k(s)}{\Delta x(s)}.$$

But $\Delta x^k(s)/\Delta x(s)$ is a polynomial of degree $k-1$ in $x(s)$ and therefore $\Delta x^n(s+\alpha)/\Delta x(s+\beta)$ also is. □

To conclude this section let point out the following

Remark 2.4. *From Proposition 2.2 and Definition 2.1 it follows that the only linear-type lattices are those corresponding to $F(1) = 1$ (the linear lattice $x(s) = C_1 s + C_2$) and the ones when $F(1) = q \neq \{0, \pm 1\}$ (the q-linear lattices $x(s) = c_1 q^s + c_2$).*

3. The characterization theorem for classical polynomials

In the sequel we will assume that $(P_n[x(s)])_n$ is a sequence of orthogonal polynomials on a linear-type lattice $x(s)$. For sake of simplicity we will denote $P_n(s) := P_n[x(s)]$. Since $P_n(s)$ are orthogonal they satisfy the TRRR

$$\begin{gathered} x(s)P_n(s) = \alpha_n P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s), \\ P_{-1}(s) = 0, \quad P_0(s) = 1. \end{gathered} \tag{3.1}$$

Let us point out that if $\gamma_n \neq 0$, for all $n \in \mathbb{N}$, then the above TTRR defines an orthogonal polynomial sequence. Nevertheless there are several examples for which $\gamma_n = 0$ for some $n_0 \in \mathbb{N}$ (e.g. the Hahn and q-Hahn polynomials). In this case we have a finite family of orthogonal polynomials (see e.g. [8, 25]). In the first case, i.e., when $\gamma_n \neq 0$, for all $n \in \mathbb{N}$ we say that it is a quasi-definite case [8] (also called the regular case) whereas in the second one, we get a weak-quasi-definite case or weak-regular case. Here we will deal with the "classical" polynomials and we will assume that $\gamma_n \neq 0$ for all $n \in \mathcal{N}$ where by $\mathcal{N}$ we denote the set $\mathcal{N} = 1, 2, \dots, n_0$ for some $n_0 \in \mathbb{N}$ or $\mathcal{N} := \mathbb{N}$.

Here we will use the notation of the theory of difference calculus on non-uniform lattices (for more details see [25, §13] or [24, chapter 3]).

Let $s = a, a+1, a+2, \dots$. We will define the forward and backward differences in $x(s)$ by

$$\frac{\Delta y[x(s)]}{\Delta x(s)}, \qquad \frac{\nabla y[x(s)]}{\nabla x(s)},$$

respectively, where $\nabla f(s) = f(s) - f(s-1)$, $\Delta f(s) = f(s+1) - f(s)$.

For the operator Δ we have

$$\Delta\{f(s)g(s)\} = g(s)\{\Delta f(s)\} + f(s+1)\{\Delta g(s)\}. \tag{3.2}$$

Thus the following formula of summation by parts holds

$$\sum_{s=a}^{b} f(s)\Delta g(s) = f(s)g(s)\Big|_{a}^{b+1} - \sum_{s=a}^{b}\Big(\Delta f(s)\Big)g(s+1). \tag{3.3}$$

Also we define the k-th forward difference of a function $f(s)$ by

$$\Delta^{(k)} f(s) := \frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)} f(s), \quad x_m(s) = x\left(s + \frac{m}{2}\right).$$

Remark 3.1. *Notice that the differences $\Delta^{(k)}P_n(s)$ can be written in the linear-type lattice, up to a constant factor, as $(\Delta/\Delta x(s))^k P_n(s)$. Moreover, the operator $\Delta/\Delta x(s)$ for the q-linear lattice $x(s) = c_1 q^s$ becomes into the classical Jackson operator $\mathcal{D}_q$ defined by*

$$\mathcal{D}_\varsigma P(x) = \frac{P(\varsigma x) - P(x)}{x(\varsigma - 1)}, \qquad \varsigma \neq 0, \pm 1. \tag{3.4}$$

Next we state the Hahn-Lesky theorem:

Theorem 3.2. *Given a sequence of orthogonal polynomials $(P_n)_n$, it is a classical sequence if an only if*

- *The sequence of their finite differences $(\Delta P_n)_n$ is an orthogonal sequence* [17, 10].
- *The sequence of their q-differences $(\mathcal{D}_q P_n)_n$ is an orthogonal sequence* [13, 21].

Notice that since we are deal with linear lattices the statement of the theorem can be replaced by the following equivalent one:

Theorem 3.2. *A sequence of orthogonal polynomials $(P_n)_n$ is classical if and only if the sequence of their finite differences $(\Delta/\Delta x(s) P_n)_n$ is an orthogonal sequence.*

The standard proof of this theorem can be found in [17] for the linear lattice $x(s) = s$, and in [10] using the functional technique developed by Maroni. For the q-linear lattice $x(s) = q^s$ it has been done by Hahn in [13] and using a functional approach in [21].

We start with the following

Definition 3.3. *We say that the sequence $(P_n)_n$ is a classical family on the linear-type lattice if they are orthogonal with respect to the discrete measure $\rho(s)\nabla x_1(s)$, i.e.,*

$$\sum_{s=a}^{b-1} P_n(s)P_m(s)\rho(s)\nabla x_1(s) = \delta_{nm}d_n^2, \quad \Delta s = 1, \tag{3.5}$$

where ρ is the solution of the Pearson-type equation

$$\frac{\Delta}{\Delta x(s-1/2)}[\sigma(s)\rho(s)] = \tau(s)\rho(s), \tag{3.6}$$

and σ and τ are fixed polynomials on $x(s)$ of degree at most 2 and exactly 1. The function ρ is usually called the orthogonalizing weight function of the polynomial family $(P_n)_n$.

Now we are ready to enunciate our main result:

Theorem 3.4. *Let $x(s)$ be a linear-type lattice and let $\sigma(s)$ and $\rho(s)$ be two functions such that $a^k\sigma(a)\rho(a) = b^k\sigma(b)\rho(b) = 0$, forall $k \le 0$. Then, the following properties are equivalent*

(1) *$(P_n)_n$ is a classical orthogonal polynomial sequence (COPS).*
(2) *The sequence of their differences $\left(\Delta^{(1)}P_n\right)_n$ also is an COPS.*
(3) *$(P_n)_n$ satisfies the second order linear difference equation with polynomial coefficients*

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n(s)}{\nabla x(s)} + \tau(s)\frac{\Delta P_n(s)}{\Delta x(s)} + \lambda P_n(s) = 0, \tag{3.7}$$

where $\deg(\sigma) \le 2$, $\deg(\tau) = 1$, are independent of n and λ is a constant independent of x.

(4) *$(P_n)_n$ can be expressed by the Rodrigues-type formula*[2]

$$P_n(s) = \frac{B_n}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}[\rho_n(s)]. \tag{3.8}$$

(5) *The polynomials are orthogonal with respect to a weight function ρ that satisfies the Pearson-type difference equation (3.6), where $\deg(\sigma) \le 2$, $\deg(\tau) = 1$.*
(6) *There exist three sequences $(a_n)_n$, $(b_n)_n$, $(c_n)_n$, and a polynomial ϕ, $\deg(\phi) \le 2$, such that*

$$\phi(x)\frac{\Delta P_n(s)}{\Delta x(s)} = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), \quad n \ge 1.$$

(7) *There exist three sequences $(e_n)_n$, $(f_n)_n$, $(g_n)_n$ such that the following relation holds for all $n \ge 1$*

$$P_n(x) = e_n\frac{\Delta P_{n+1}(s)}{\Delta x(s)} + f_n\frac{\Delta P_n(s)}{\Delta x(s)} + g_n\frac{\Delta P_{n-1}(s)}{\Delta x(s)}, \quad e_n \ne 0,\ g_n \ne \gamma_n,$$

where γ_n is the corresponding coefficient of the TTRR (1.1).

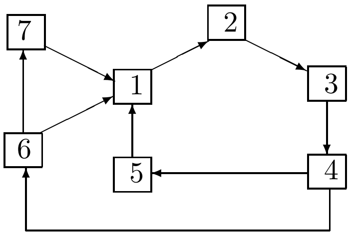
As a simple consequence of the above theorem we have the following

Corollary 3.5 ([10, 21])**.** *The discrete polynomials on the linear lattice $x(s) = s$ are classical. The q-polynomials in the q-linear lattice (or exponential lattice) $x(s) = c_1q^s + c_2$ are classical.*

[2] The operator $\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}$ in the linear type lattices can be rewritten in the form ∇^n for the linear lattice and $q^{-n(n+1)/2}\left(\frac{\nabla}{\nabla x(s)}\right)^n$ for the q-linear ones.

Proof. It follows from the fact that $x(s) = s$ and $x(s) = c_1 q^s + c_2$ are linear-type lattices. $\square$

Let us prove the Theorem 3.4. The idea of the proof is summarized in the next figure:



We start proving that (1)→(2):

Proposition 3.6. *Let $x(s)$ be a linear-type lattice and let $(P_n)_n$ be a classical family orthogonal with respect to a weight function ρ, solution of the Pearson-type equation (3.6) and such that*[3]

$$\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0. \tag{3.9}$$

Then the sequence $\left(\Delta^{(1)} P_n(s)\right)_n$, where $\Delta^{(1)} P_n(s) = \dfrac{\Delta P_n(s)}{\Delta x(s)}$, is also a classical orthogonal family with respect to the function $\rho_1(s)\Delta x(s)$, where the weight function is $\rho_1(s) = \sigma(s+1)\rho(s+1)$.

Proof. Let $Q_k(s)$ be an arbitrary k-th degree polynomial on $x(s)$, $k < n$. The orthogonality conditions for $(P_n)_n$ yield, for all $k < n$,

$$\begin{aligned}
0 &= \sum_{s=a}^{b-1} P_n(s) Q_{k-1}(s) \tau(s) \rho(s) \nabla x_1(s) \qquad \text{(from (3.6))} \\
&= \sum_{s=a}^{b-1} P_n(s) Q_{k-1}(s) \Delta(\sigma(s)\rho(s)) \quad \text{(from (3.3), (3.9))} \\
&= -\sum_{s=a}^{b-1} \Delta(P_n(s) Q_{k-1}(s)) \sigma(s+1)\rho(s+1)
\end{aligned}$$

Applying the Leibniz rule (3.2)

$$\begin{aligned}
0 = &-\sum_{s=a}^{b-1} (\Delta P_n(s)) Q_{k-1}(s) \sigma(s+1)\rho(s+1) + \\
&\sum_{s=a}^{b-1} P_n(s+1)(\Delta Q_{k-1}(s)) \sigma(s+1)\rho(s+1) \quad (s \to s-1, \text{ and } (3.9))
\end{aligned}$$

[3]This condition leads to the so-called discrete orthogonal polynomials, i.e., polynomials with a discrete orthogonality of the form (3.5). For the q-linear lattices (3.5) becomes into the q-Jackson integral (see e.g. [5, 15, 16]). For the continuous orthogonality see [24, §3.10].

$$= -\sum_{s=a}^{b-2}\Big(\frac{\Delta P_n(s)}{\Delta x(s)}\Big)Q_{k-1}(s)\sigma(s+1)\rho(s+1)\nabla x_1(s+1/2)+$$
$$\sum_{s=a+1}^{b} P_n(s)\Big(\frac{\Delta Q_{k-1}(s-1)}{\Delta x(s-1/2)}\Big)\sigma(s)\rho(s)\nabla x_1(s)$$

Next, we use Lemma 2.3 as well as the conditions (3.9), then

$$0 = -\sum_{s=a}^{b-2}\Big(\frac{\Delta P_n(s)}{\Delta x(s)}\Big)Q_{k-1}(s)\sigma(s+1)\rho(s+1)\nabla x_1(s+1/2)+$$
$$\sum_{s=a}^{b-1} P_n(s)\underbrace{\Big(R_{k-2}(s)\Big)\sigma(s)}_{\text{degree} \leq n}\rho(s)\nabla x_1(s) \quad \text{(from (3.9), (3.5))}$$
$$= -\sum_{s=a}^{b-2}\Big(\frac{\Delta P_n(s)}{\Delta x(s)}\Big)Q_{k-1}(s)\sigma(s+1)\rho(s+1)\nabla x_1(s).$$

Thus, $\Delta P_n(s)/\Delta x(s)$ is orthogonal with respect to $\rho_1(s)\nabla x_1(s+1/2) = \sigma(s+1)\rho(s+1)\Delta x(s)$. We only need now to prove that $\Delta^{(1)}P_n(s)$ is a classical family. For doing this notice that the weight function $\rho_1(s)$ satisfy the Pearson type equation (see e.g. [24, §3.2.2])

$$\frac{\Delta}{\Delta x_1(s-1/2)}\left[\sigma(s)\rho_1(s)\right] = \tau_1(s)\rho_1(s)$$

where τ_1 is a first degree polynomial on $x(s)$ given by

$$\tau_1(s) = \frac{\sigma(s+1) - \sigma(s) + \tau(s+1)\Delta x_1(s)}{\Delta x(s)}.$$

Thus ρ_1 satisfies a difference equation of the form (3.6). This complete the proof. □

In the same way, using induction we have

Corollary 3.7. *Let $x(s)$ be a linear-type lattice and let $(P_n)_n$ be a classical family. Then, the sequence of their k-th finite differences $\Delta^{(k)}P_n(s)$, where $\Delta^{(k)} := \frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)}$, also is a classical family.*

Now we prove that (1)+(2)→(3):

Proposition 3.8. *Let $x(s)$ be a linear-type lattice. If the sequences $(P_n)_n$ and $\big(\Delta^{(1)}P_n\big)_n$ are classical, then $(P_n)_n$ satisfies the second order linear difference equation of hypergeometric type (3.7).*

Proof. Let $k < n$. Then, using the orthogonality of $\Delta^{(1)}P_n$,

$$
\begin{aligned}
0 &= \sum_{s=a}^{b-2} \frac{\Delta P_n(s)}{\Delta x(s)}\frac{\Delta Q_k(s)}{\Delta x(s)}\sigma(s+1)\rho(s+1)\nabla x_1(s+1/2) \qquad \text{(from (3.9))}\\
&= \sum_{s=a}^{b-1} \frac{\Delta P_n(s)}{\Delta x(s)}\Delta Q_k(s)\sigma(s+1)\rho(s+1) \quad \text{(from (3.3), (3.9))}\\
&= -\sum_{s=a}^{b-1} Q_k(s)\Delta\Big(\frac{\Delta P_n(s-1)}{\Delta x(s-1)}\sigma(s)\rho(s)\Big) \quad (\Delta f(s) = \nabla f(s+1))\\
&= -\sum_{s=a}^{b-1} Q_k(s)\Delta\Big(\frac{\nabla P_n(s)}{\nabla x(s)}\sigma(s)\rho(s)\Big) \qquad \text{(from (3.2))}\\
&= -\sum_{s=a}^{b-1} Q_k(s)\Big(\sigma(s)\rho(s)\Delta\frac{\nabla P_n(s)}{\nabla x(s)} + \frac{\nabla P_n(s+1)}{\nabla x(s+1)}\Delta[\sigma(s)\rho(s)]\Big) \text{ (from (3.6))}\\
&= -\sum_{s=a}^{b-1} Q_k(s)\left(\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n(s)}{\nabla x(s)} + \tau(s)\frac{\Delta P_n(s)}{\Delta x(s)}\right)\rho(s)\nabla x_1(s).
\end{aligned}
$$

But, since the lattice $x(s)$ is of the linear type,

$$
Q(s) := \sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla P_n(s)}{\nabla x(s)} + \tau(s)\frac{\Delta P_n(s)}{\Delta x(s)}
$$

is a polynomial of degree n in $x(s)$. Therefore, it should be, up to a constant factor (in general depending on n) the polynomial $P_n(s)$. Thus $Q(s) = -\lambda P_n(s)$. □

Remark 3.9. *The proof of the last proposition in the linear lattice $x(s) = s$ can be found in the first Russian edition of the book* [25].

The last proposition is very important because it gives a very simple method for finding the classical polynomials on the linear-type lattice. In fact, it was the key in the proofs of Hahn and Lesky for proving the Theorem 3.2.

The solutions of the difference equation (3.7) have been extensively studied (see e.g. [6, 24, 25]). In particular they can be written by the Rodrigues-type formula (3.8) [24, 25], so (3)→(4). Let us mention that from the Rodrigues-type formula (3.8) one can obtain an explicit expression for the classical polynomials in terms of the hypergeometric or basic hypergeometric series as it is shown in several previous works (see e.g. [6, 24]).

Another consequence of the Rodrigues formula is the following: Putting $n = 1$ in (3.8) we obtain

$$
P_1(s) = \frac{B_1}{\rho(s)}\frac{\Delta}{\Delta x(s-1/2)}[\sigma(s)\rho(s)] \quad \Rightarrow \frac{\Delta}{\Delta x(s-1/2)}[\sigma(s)\rho(s)] = \rho(s)\tau(s),
$$

i.e. the Pearson-type equation (3.6) thus (4)→(5).

Remark 3.10. *Notice that from the above results the equivalence of (1)–(5) in Theorem 3.4 follows.*

Now we prove that (6)→(2).

Proposition 3.11. *Let $x(s)$ be a linear-type lattice and $\phi(s)$ a polynomial such that $\deg(\phi) \le 2$. If $(P_n)_n$ is an OPS and there exist three sequences $(a_n)_n$, $(b_n)_n$, and $(c_n)_n$, $\forall n \in \mathcal{N}$, such that*

$$\phi(x)\frac{\Delta P_n(s)}{\Delta x(s)} = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \tag{3.10}$$

then $(P_n)_n$ is a classical family.

Proof. We start computing the following sum for all $k < n-1$

$$\begin{aligned}
&\sum_{s=a}^{b-1} Q_k(s)\frac{\Delta P_n(s)}{\Delta x(s)}\phi(s)\rho(s)\Delta x(s)\\
=&\sum_{s=a}^{b-1} Q_k(s)\left[a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)\right]\rho(s)\Delta x(s)\\
=&F(-1/2)\sum_{s=a}^{b-1} Q_k(s)\left[a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)\right]\rho(s)\nabla x_1(s) = 0.
\end{aligned}$$

Therefore the sequence $(\frac{\Delta P_n(s)}{\Delta x(s)})_n$ is an OPS, and then by Theorem 3.2 P_n is a classical family. ☐

Remark 3.12. *From the above proposition it follows that $\phi(s)\rho(s) = \rho_1(s) = \sigma(s+1)\rho(s+1)$. Therefore, comparison with the Pearson-type equation leads to the expression $\phi(s) = \sigma(s) + \tau(s)\Delta(s-1/2)$. Notice also that since $(P_n)_n$ is an orthogonal family then the relation (3.10), usually called the structure relation of Al-Salam & Chihara type, is equivalent to the following relations (I is the identity operator)*

$$\begin{aligned}
L_n P_n(x) &:= \left(\phi(x)\frac{\Delta}{\Delta x(s)} + \psi_1(x,n)I\right)P_n(x) = \widetilde{c}_n P_{n-1}(x), \quad \deg(\psi_1) = 1,\\
R_n P_n(x) &:= \left(\phi(x)\frac{\Delta}{\Delta x(s)} + \psi_2(x,n)I\right)P_n(x) = \widetilde{a}_n P_{n+1}(x), \quad \deg(\psi_2) = 1.
\end{aligned}$$

The operators L_n and R_n are usually called the lowering and raising operators for the polynomial family $(P_n)_n$.

Proposition 3.13 ((7)→(2))**.** *Let $x(s)$ be a linear-type lattice. If $(P_n)_n$ is an monic OPS and there exist three sequences $(e_n)_n$, $(f_n)_n$, and $(g_n)_n$, $e_n \ne 0$, $g_n \ne \gamma_n$, $\forall n \in \mathcal{N}$, such that*

$$P_n(x) = e_n\frac{\Delta P_{n+1}(s)}{\Delta x(s)} + f_n\frac{\Delta P_n(s)}{\Delta x(s)} + g_n\frac{\Delta P_{n-1}(s)}{\Delta x(s)}, \tag{3.11}$$

then $(P_n)_n$ is a classical family.

Proof. For a sake of simplicity we will suppose that $(P_n)_n$ is a monic sequence. Since $(P_n)_n$ is an OPS they satisfy a TTRR (3.1). Taking the difference to both sides of (3.1), using (3.2) as well as the linearity property (2.1) we get

$$P_n(s)+[F(1)x(s)+G(1)]\frac{\Delta P_n(s)}{\Delta x(s)}=\frac{\Delta P_{n+1}(s)}{\Delta x(s)}+\beta_n\frac{\Delta P_n(s)}{\Delta x(s)}+\gamma_n\frac{\Delta P_{n-1}(s)}{\Delta x(s)}.$$

Then substituting the value of $P_n(s)$ from (3.11) we find

$$\begin{aligned}F(1)x(s)\frac{\Delta P_n(s)}{\Delta x(s)}= &\ (1-e_n)\frac{\Delta P_{n+1}(s)}{\Delta x(s)}+(\beta_n-G(1)-f_n)\frac{\Delta P_n(s)}{\Delta x(s)}\\ &+(\gamma_n-g_n)\frac{\Delta P_{n-1}(s)}{\Delta x(s)}.\end{aligned}$$

If $g_n\neq\gamma_n$, $\forall n\in\mathcal{N}$, then from the Favard theorem (see e.g. [8]) the sequence $(\frac{\Delta P_n(s)}{\Delta x(s)})_n$ is an OPS, and therefore by Theorem 3.2) P_n is a classical family. $\square$

To conclude the proof we should show that if $(P_n)_n$ is a classical family, then (3.10) and (3.11) take place. The first one follows directly from the Rodrigues-type formula as it is shown in [3, 4] so (4)→(6), and the second one follows from the first one, i.e., (6)→(7) (see [3, 4]). For the sake of completeness we will present it here and alternative proof for the second case taken from [3] (the first relation can be proven using the same ideas and we leave it as an exercise to the reader). In fact we will show that (1)+(2)→(7).

Let be $Q_n(s)=\Delta P_{n+1}(s)/\Delta x(s)$. Using the linearity of $x(s)$ we have $P_n(s)=\sum_{k=0}^{n}c_{n,k}Q_k(s)$. Since $(P_n)_n$ is a classical family, then $(Q_n)_n$ also is, and therefore

$$c_{n,k}=\frac{\left(\displaystyle\sum_{s=a}^{b-2}P_n(s)Q_k(s)\rho_1(s)\Delta x(s)\right)}{d_{1\,k}^2},\quad \rho_1(s)=\rho(s+1)\sigma(s+1),$$

where $d_{1\,k}^2$ is the square of the norm of Q_k. Using the condition (3.9) the numerator becomes

$$\begin{aligned}&\sum_{s=a-1}^{b-2}P_n(s)Q_k(s)\rho_1(s)\Delta x(s)=\sum_{s=a-1}^{b-2}P_n(s)\Delta[P_{k+1}(s)]\rho_1(s)\\ &=P_n(s)P_{k+1}(s)\rho_1(s)\Big|_{a-1}^{b-1}-\sum_{s=a-1}^{b-2}P_{k+1}(s+1)\Delta[P_n(s)\rho_1(s)]\end{aligned}$$

$$= -\sum_{s=a-1}^{b-2} P_{k+1}(s+1)P_n(s+1)\Delta[\rho_1(s)] - \sum_{s=a-1}^{b-2} P_{k+1}(s+1)\Delta[P_n(s)]\rho_1(s)$$

$$= -\sum_{s=a}^{b-1} P_{k+1}(s)P_n(s)\tau(s)\rho(s)\nabla x_1(s) - \sum_{s=a}^{b-2} P_{k+1}(s+1)\frac{\Delta P_n(s)}{\Delta x(s)}\rho_1(s)\Delta x(s).$$

where we use the condition (3.9), the formula (3.2) as well as the Pearson-type equation (3.6). Now, from the orthogonality of the classical polynomials we conclude that the first sum vanishes for all $k < n-2$. But the second one also vanishes for all $k < n-2$ since $\Delta P_n(s)/\Delta x(s)$ is an orthogonal sequence with respect to $\rho_1(s)\Delta x(s)$ and $P_{k+1}(s+1)$ is a polynomial of degree $k+1$ in $x(s)$.

This completes the proof of Theorem 3.4. □

Remark 3.14. *Notice that if we consider monic polynomials, then for the linear lattice $x(s)$, $e_n = 1/(n+1) \neq 0$ and $F(1) = 1$ and for the q-linear one $e_n = (1-q)/(1-q^{n+1}) \neq 0$ and $F(1) = q$.*

It is important to notice that in the proof of Proposition 3.11 there is not any restriction on c_n but for the classical "continuous", discrete and q cases the condition $c_n \neq 0$ was imposed (see e.g. [10, 19, 21]). A similar situation happens in the proof of the Proposition 3.13, in the same aforesaid papers the condition $g_n \neq 0$ is imposed. Nevertheless, we see from the proof presented here that a more restricted condition should be imposed: $g_n \neq \gamma_n$. Notice that since $\gamma_n \neq 0$ (by Favard theorem) the last conditions implies the first one $\gamma_n \neq 0$. In the next section we will discuss what happens if these conditions are not fulfilled.

4. The classical polynomials: further discussion

4.1. **The q-linear lattices: The q-Hahn Tableau.** Here we will discuss the q-case. The classical case follows from the limit $q \to 1^-$. For the sake of simplicity and without lost of generality we will consider the most simple q-lattice $x(s) = q^s$. In the following we will use the classical notation

$$\mathcal{D}_q P(x) = \frac{\Delta P(s)}{\Delta x(s)}, \qquad \mathcal{D}_{1/q} P(x) = \frac{\nabla P(s)}{\nabla x(s)}, \quad x(s) := x = q^s,$$

where $\mathcal{D}_\varsigma$ denotes, as before, the classical q-Jackson derivative (3.4). With this notation we have that (3.7), (3.10), and (3.11) become

$$\phi(x)\Theta P_n(x) - \sigma(x)\Theta^\star P_n(x) - x(1-q)q^{-1/2}\lambda_n P_n(x) = 0, \quad x := q^s, \quad (4.1)$$

$$\phi(x)\mathcal{D}_q P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad x := q^s, \quad (4.2)$$

$$P_n(x) = \frac{1}{[n+1]_q}\mathcal{D}_q P_{n+1}(x) + f_n \mathcal{D}_q P_n(x) + g_n \mathcal{D}_q P_{n-1}(x), \quad x := q^s, \quad (4.3)$$

respectively.

The general polynomial solution of (4.1) is [6, 24].

$$P_n(s) = {}_3\varphi_2\left(\begin{array}{c} q^{-n}, q^{s_1+s_2-\bar{s}_1-\bar{s}_2+n-1}, x\, q^{-\bar{s}_2} \\ q^{s_1-\bar{s}_2}, q^{s_2-\bar{s}_2} \end{array} \middle| \, q \, , \, q \right), \tag{4.4}$$

where the basic hypergeometric series ${}_3\varphi_2$ is defined by

$${}_r\varphi_p\left(\begin{array}{c} a_1, \dots, a_r \\ b_1, \dots, b_p \end{array} ; q \, , \, z \right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_p;q)_k} \frac{z^k}{(q;q)_k} \left[(-1)^k q^{\frac{k(k-1)}{2}} \right]^{p-r+1},$$

being $(a;q)_k = \prod_{m=0}^{k-1}(1-aq^m)$, $(a;q)_0 := 1$, the q-shifted factorial. It corresponds to the functions

$$\sigma(x) = C(x-q^{s_1})(x-q^{s_2}), \quad \phi(x) = C'(x-q^{\bar{s}_1})(x-q^{\bar{s}_2}), \quad Cq^{s_1}q^{s_2} = C'q^{\bar{s}_1}q^{\bar{s}_2},$$

and the eigenvalues are given by

$$\lambda_n = -\frac{C\, q^{-n+\frac{3}{2}}}{c_1^2(1-q)^2}(1-q^n)(1-q^{s_1+s_2-\bar{s}_1-\bar{s}_2+n-1}).$$

In particular, choosing $\phi = aq(x-1)(bx-c)$ and $\sigma = q^{-1}(x-aq)(x-cq)$, we obtain the big q-Jacobi polynomials introduced by Hahn in [13], i.e.,

$$p_n(x;a,b,c;q) = {}_3\varphi_2\left(\begin{array}{c} q^{-n}, abq^{n+1}, x \\ aq, cq \end{array} \middle| q; q \right),$$

and, in the particular case $c = q^{-N-1}$, the aforesaid $q-$Hahn polynomials $Q_n(x;a,b,N|q)$ are deduced.

Remark 4.1. *The general solution of the equation (4.1) defines the so-called q-Hahn tableau* [16]*. A detailed study of this class has been done in* [5]*. In particular, in* [5] *comparison with the q-analog of the Askey tableau* [15] *and the Nikiforov & Uvarov tableau* [26] *has been performed and all possible limit cases obtained from (4.4) have been analyzed, identifying them with several known classical families of q-polynomials as well as two new ones.*

In the following we will use the notation introduced in [21]

$$\phi(x) = \widehat{a}x^2 + \overline{a}x + \widetilde{a}, \quad \psi(x) := q^{-1/2}\tau(s) = \widehat{b}x + \bar{b}, \quad \widehat{b} \neq 0. \tag{4.5}$$

In the paper [21] the values of the coefficients of the TTRR (3.1), and the structure relations (4.2) and (4.3) have been obtained in terms of the coefficients of ϕ and ψ defined in (4.5). In particular,

$$\gamma_n = -\frac{q^{n-1}[n]_q\Big([n-2]_q\widehat{a}+\widehat{b}\Big)}{\Big([2n-1]_q\widehat{a}+\widehat{b}\Big)\Big([2n-2]_q\widehat{a}+\widehat{b}\Big)^2\Big([2n-3]_q\widehat{a}+\widehat{b}\Big)} \times \tag{4.6}$$

$$\times\left[q^{n-1}\Big([n-1]_q\overline{a}+\overline{b}\Big)\Big(q^{n-1}\widehat{a}\overline{b}-\overline{a}([n-1]_q\widehat{a}+\widehat{b})\Big)+\widetilde{a}\Big([2n-2]_q\widehat{a}+\widehat{b}\Big)^2\right], \; n \geq 1,$$

$$c_n = -\frac{[n]_{q^{-1}}\Big([n-1]_q\widehat{a}+\widehat{b}\Big)}{[n]_q}\gamma_n, \quad n \geq 1, \tag{4.7}$$

and

$$g_n = -\frac{q^{n-2}[n-1]_q\widehat{a}}{[n-2]_q\widehat{a}+\widehat{b}}\,\gamma_n\,, \quad n \geq 2, \tag{4.8}$$

where we use the standard notation for the q-numbers

$$[x]_\varsigma = \frac{\varsigma^x - 1}{\varsigma - 1}.$$

From the above relations it follows that if we want to have an infinite orthogonal polynomial sequence $(P_n)_{n\geq 0}$ (the so called *quasi-definite* or regular case) γ_n should be different from zero for all $n \geq 0$. But, as we already pointed out, there exist some examples when $\gamma_n = 0$ for some n_0 (e.g. the q-Hahn and q-Kravchuk polynomials for $n = N+1$). In these cases we have a *finite* family of polynomials (strictly speaking this case does not constitute a regular case) that corresponds to a *weak-regular* case. Notice that from formula (4.6) it follows that the corresponding family exists, at least in the weak-regular sense, if the square bracket in (4.6) is different from zero and a sufficient condition is

$$[n]_q\widehat{a} + \widehat{b} \neq 0, \qquad \text{for } n \in 1, 2, \ldots, n_0. \tag{4.9}$$

The last condition is usually called the *admissibility condition* (for a detailed study of this condition see [22, 23] and references therein). That this condition was necessary was established in [21].

Now, from the expression (4.7) and taking into account that $\gamma_n \neq 0$ for all $n \in \mathcal{N}$, the condition $c_n \neq 0$, for all $n \in \mathcal{N}$, follows. This condition is equivalent to the admissibility condition.

Let now analyze the expression (4.8). In this case we see that for the quasi-definite case $g_n \neq 0$. But in our proof we see that $g_n \neq \gamma_n$ for all $n \in \mathcal{N}$. Thus, the following question arises: what happens if $g_n = \gamma_n$ for $n = 1, 2, \ldots, n_0$?

To answer this question we use (4.8). Then

$$g_n = \gamma_n \qquad \Longleftrightarrow \qquad [2n-3]_q\widehat{a} + \widehat{b} = 0, \qquad \forall n = 2, 3, \ldots,$$

which is in contradiction with the admissibility condition (4.9).

Remark 4.2. *In* [21] *the condition $g_n \neq 0$ for all $n \in \mathcal{N}$ was imposed but not the more restrictive one $g_n \neq \gamma_n$, from where the first one immediately follows. Of course in* [21] *the admissibility condition $[n]_q\widehat{a} + \widehat{b} \neq 0$ it is assumed and it implies that $g_n \neq \gamma_n$ for all $n \in \mathcal{N}$.*

From the above discussion follows that the q-classical polynomials are completely characterized by the relation (4.3) with the restriction $g_n \neq \gamma_n$ for all $n \in \mathcal{N}$. Moreover, if $g_n = \gamma_n$ for all $n = 1, 2, \ldots, n_0$, then the corresponding orthogonal polynomial sequence, if such a sequence exists, is not a classical one.

4.2. **The linear lattice $x(s) = s$.** For the linear lattice $x := x(s) = s$ the second order linear difference equation is

$$[\sigma(x) + \tau(x)]\Delta P_n(x) - \sigma(x)\nabla P_n(x) + \lambda_n P_n(x) = 0, \tag{4.10}$$

where

$$\sigma(x) = A(x - x_1)(x - x_2), \quad \phi(x) := \sigma(x) + \tau(x) = A(x - \bar{x}_1)(x - \bar{x}_2),$$

and its general solution is of the form

$$P_n(x) = {}_3\mathrm{F}_2\left(\begin{array}{c} -n, x_1 + x_2 - \bar{x}_1 - \bar{x}_2 + n - 1, x_1 - x \\ x_1 - \bar{x}_1, x_1 - \bar{x}_2 \end{array} \middle| 1 \right), \tag{4.11}$$

and

$$\lambda_n = -A\, n\, (x_1 + x_2 - \bar{x}_1 - \bar{x}_2 + n - 1).$$

Here ${}_3\mathrm{F}_2$ is the generalized hypergeometric series

$${}_p\mathrm{F}_q\left(\begin{array}{c} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $(a)_k = \prod_{m=0}^{k-1}(a + m)$, $(a)_0 := 1$, is the Pochhammer symbol.

A particular choice $x_1 = 0$, $x_2 = N + \alpha$, $\bar{x}_1 = -\beta - 1$, and $\bar{x}_2 = N - 1$ leads to the Hahn polynomials. Taking several limits from (4.11) we can obtain the other classical families: Kravchuk, Meixner, and Charlier (see e.g. [3, 15, 24, 26]). In this case the structure relations are

$$\phi(x)\Delta P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x),$$
$$P_n(x) = \frac{1}{n+1}\Delta P_{n+1}(x) + f_n \Delta P_n(x) + g_n \Delta P_{n-1}(x). \tag{4.12}$$

Next we compute γ_n. For doing that we use the expression (we are using monic polynomials) $\gamma_n = l_n - l_{n+1} - l_n\beta_n$ obtained when we identify the coefficients of x^{n-1} in the TTRR (1.1), where k_n and l_n are the coefficients of the monomials x^{n-1} and x^{n-2} in $P_n(x) = x^n + k_n x^{n-1} + l_n x^{n-2} + \cdots$, $n \geq 3$. To compute the values of k_n and l_n we substitute P_n in the second order linear difference equation (4.10) and identify the coefficients of the monomials x^{n-1} and x^{n-2} (for more details see [3]). All these yield

$$\gamma_n = -\frac{(p + a(n-2))\, n}{(p + 2a(n-1))^2\, (p + a\,(2n-3))\, (p + a\,(2n-1))} \times$$
$$\Big[c(p + 2a(n-1))^2 - bp \left(q + p(n-1) + a(n-1)^2 \right)$$
$$+ a\Big(q + p(n-1)a(n-1)^2 \Big)^2 - b^2\, (p + a(n-1))\, (n-1) \Big], \quad n \geq 1,$$

where the notation $\sigma(x) = ax^2 + bx + c$ and $\tau(x) = px + q$ has been used.

From the above expression we see that the corresponding orthogonal polynomial sequence exists (at least in the weak-regular sense) when the expression in the square bracket is different from zero and a sufficient condition for this is $p + na \neq 0$, for all $n = 1, 2, \dots, n_0$.

But now, using the expression (see e.g. [3, page 108]) $c_n = \lambda_n \gamma_n / n$, we see that for all $n \geq 1$, $c_n \neq 0$. The condition $p + na \neq 0$ for all $n \in \mathcal{N}$ is the admissibility condition in this case.

Let us now analyze the structure relation (4.12). In this case [3, page 109] $g_n = -\frac{(n-1)a\gamma_n}{p+(n-2)a}$, therefore in the quasi-definite case $g_n \neq 0$. If $\gamma_n = g_n$ for all n, then we obtain that $p + (2n-3)a = 0$, for all n which is in contradiction with the admissibility condition.

Remark 4.3. *In* [10] *the condition* $g_n \neq 0$ *for all* $n \in \mathcal{N}$ *was imposed but not the more restrictive one* $g_n \neq \gamma_n$*, from where the first one immediately follows. For the discrete case in* [10] *the admissibility condition* $p + na \neq 0$ *it is assumed and therefore* $g_n \neq \gamma_n$ *for all* $n \in \mathcal{N}$*.*

From the above discussion also follows that the classical discrete polynomials are completely characterized by the relation (4.12) with the restriction $g_n \neq \gamma_n$ for all $n \in \mathcal{N}$. Moreover, if $g_n = \gamma_n$ for all $n \in \mathcal{N}$, then the corresponding orthogonal polynomial sequence, if such a sequence exists, is not a classical one.

4.3. **The classical case.** The classical case can be obtained from the q-case taking the limit $q \to 1^-$. Nevertheless the Theorem 1.2 can be proven using the same scheme section 3. The only difference is that here one uses he standard integral calculus and integration by parts instead of the calculus with the difference operator. Of particular interest is the proof of property 7 so we will provide it here: Taking derivatives of the TTRR (1.1) and using (1.3), we have the expression

$$xP_n'(x) = \frac{n}{n+1}P_{n+1}'(x) + (\beta_n - f_n)P_n'(x) + (\gamma_n - g_n)P_{n-1}'(x), \qquad (4.13)$$

from where, if $g_n \neq \gamma_n$, $\forall n \in \mathcal{N}$, and using the Favard theorem the sequence $(P_n')_n$ is an OPS, and therefore by the Sonin-Hahn Theorem 1.1 P_n is a classical family. Notice again that the condition $g_n \neq \gamma_n$ should be imposed. Using the formulas in [20] it is easy to see that this condition is equivalent to the condition $n\sigma''/2 + \tau' = 0$ which is nothing else that the admissibility condition for the classical polynomials [20]. Let us point out that the more restrictive condition $\gamma_n \neq g_n$ for all $n \in \mathbb{N}$ was not considered in [19] (they considered only the regular case, i.e., $\gamma_n \neq 0$). As in the cases already discussed we conclude that the classical *continuous* polynomials are completely characterized by the relation (1.3) with the restriction $g_n \neq \gamma_n$ for all $n \in \mathbb{N}$. Moreover, if $g_n = \gamma_n$ for $n = 1, 2, \ldots, n_0$, then the corresponding orthogonal polynomial sequence, if such a sequence exists, is not a classical one.

4.4. **The Marcellán et al. characterization.** At this point the following question arises: what happens if we do not impose the condition $g_n \neq \gamma_n$, $\forall n = 1, 2, \ldots, n_0$? There is any family of orthogonal polynomials, necessarily non classical, that satisfies the TTRR (1.1) where $\gamma_n \neq 0$ for $n \in \mathcal{N}$, and the relation (1.3) with $g_n = \gamma_n$ for all $n \in \mathcal{N}$? i.e.,

$$P_n(x) = \frac{P'_{n+1}(x)}{n+1} + f_n P'_n(x) + \gamma_n P'_{n-1}(x)\,. \tag{4.14}$$

To answer this question we can use (4.13) but rewritten in the form[4]

$$P'_{n+1}(x) = \frac{n+1}{n}(x - \beta_n + f_n) P'_n(x),$$

that leads to

$$P'_n(x) = n \prod_{j=1}^{n-1} (x - \beta_j + f_j), \qquad n \geq 2.$$

Therefore, substituting the last expression in (4.14) we get, denoting $\xi_j = \beta_j - f_j$,

$$P_n(x) = [(x - \xi_n)(x - \xi_{n-1}) + n f_n (x - \xi_{n-1}) + (n-1)\gamma_n] \prod_{j=1}^{n-2} (x - \xi_j).$$

But this implies that for $n \geq 3$, two consecutive polynomials have common zeros that is a contradiction. Therefore there is not any family of orthogonal polynomials that satisfy (4.14).

For the linear lattices $x(s) = s$ and $x(s) = q^s$ the situation is the same. We present here the computations only for the q-case, the other case is analogous —in fact the final expression for the polynomials P_n coincide with the one in the classical "continuous" case.

For the q-case we proceed as before, i.e., we take the q-derivatives of the TTRR (3.1) and use the relation (4.3) where $e_n = 1/[n]_q$, $g_n = \gamma_n$, $F(1) = 1$, $G(1) = 0$, we obtain

$$\mathcal{D}_q P_{n+1}(x) = \frac{[n+1]_q}{[n]_q}(x - \xi_n/q), \quad \xi_j = \beta_j - f_j.$$

Substituting it in (4.3) when $g_n = \gamma_n$ we obtain the following expression for the polynomials P_n

$$P_n(x) = [(x - \xi_n/q)(x - \xi_{n-1}/q) + [n]_q f_n (x - \xi_{n-1}/q) + [n-1]_q \gamma_n] \prod_{j=1}^{n-2} (x - \xi_j/q).$$

As before, from this expression follows that for $n \geq 3$, two consecutive polynomials has common zeros, that is in contradiction with the fact that they constitutes an orthogonal sequence.

From the above discussion follows that the structure relation (3.11) when $g_n \neq \gamma_n$ for all $n \in \mathcal{N}$ completely characterizes the classical orthogonal polynomials.

[4] As in section 4.3 we will take the derivative of the TTRR (1.1) but now use (4.14).

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Appendix A. The classical polynomials

In this appendix we will present the proof of the Theorem 1.2. We will follow the same scheme in Section 3 (see figure 1).

As starting point we will use the Pearson equation, i.e., we say that the classical polynomials are the polynomials orthogonal with respect to a continuous weight function ρ supported in the interval (a, b), solution of the Pearson equation

$$[\sigma(x)\rho(x)]' = \tau(x)\rho(x), \tag{A.1}$$

where σ and τ are polynomials of degree at least two and exactly one, respectively, and such that the following boundary conditions hold[5] $\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0$.

(1)→(2): Using the orthogonality of the classical family $(P_n)_n$ with respect to ρ we have that for any polynomial of degree less than or equal to $k-1$, Q_{k-1}, with $k < n$,

$$0 = \int_a^b P_n(x) \underbrace{Q_{k-1}(x)\tau(x)}_{\text{degree}\leq k<n} \rho(x)dx = \int_a^b P_n(x)Q_{k-1}(x)[\sigma(x)\rho(x)]'dx$$

$$= \underbrace{P_n(x)Q_{k-1}(x)\sigma(x)\rho(x)\Big|_a^b}_{=0} - \int_a^b [P_n(x)Q_{k-1}(x)]'\sigma(x)\rho(x)dx$$

$$= \underbrace{-\int_a^b P_n(x) \overbrace{Q'_{k-1}(x)\sigma(x)}^{\text{degree}<n} \rho(x)dx}_{=0} - \int_a^b P'_n(x)Q_{k-1}(x)[\sigma(x)\rho(x)]dx.$$

Thus P'_n is orthogonal to any polynomial of degree $k-1 < n-1$, i.e., $(P'_n)_n$ is also an orthogonal family. Furthermore, since the weight function for the sequence $(P'_n)_n$ is $\rho_1(x) = \sigma(x)\rho(x)$, we have that they satisfy the equation $[\sigma(x)\rho_1(x)]' = [\tau(x) + \sigma'(x)]\rho_1(x)$, i.e., a Pearson equation (A.1).

[5]These conditions follow from the fact that for the classical families the moments $\mu_n = \int_a^b x^n\rho(x)dx$, $n \geq 0$, of the measure associated with $\rho(x)$ are be finite.

(2)→(3): We use now that $(P'_n)_n$ is an orthogonal family with respect to the weight function $\rho_1(x) = \sigma(x)\rho(x)$. Thus

$$\begin{aligned}
0 &= \int_a^b P'_n(x)Q'_k(x)\tau(x)\rho_1(x)dx \\
&= \underbrace{P'_n(x)Q_k(x)\sigma(x)\rho(x)\Big|_a^b}_{=0} - \int_a^b [\sigma(x)\rho(x)P'_n(x)]'Q_k(x)dx \\
&= -\int_a^b Q_k(x)\{\underbrace{[\sigma(x)\rho(x)]'}_{=\tau(x)\rho(x)}P'_n(x) + \sigma(x)\rho(x)P''_n(x)\} \\
&= \int_a^b Q_k(x)[\sigma(x)P''_n(x) + \tau(x)P'_n(x)]\rho(x)dx.
\end{aligned}$$

But since the last integral vanishes for every polynomial Q_k of degree $k < n$ then $\sigma(x)P''_n(x)+\tau(x)P''_n(x)$ should be proportional to P_n, i.e., $\sigma(x)P''_n(x)+\tau(x)P''_n(x) = -\lambda_n P_n$, where λ_n is a constant, in general depending on n.
(3)→(4): The solution of the above differential equation can be written in the following compact form (see e.g. [25, §2] or [24, §1.2]) usually called the Rodrigues formula

$$P_n(x) = \frac{B_n}{\rho(x)}\frac{d^n}{dx^n}[\sigma^n(x)\rho(x)],$$

where B_n is a constant.
(4)→(5): It follows from the Rodrigues formula just putting $n = 1$.
(4)→(6): From the Rodrigues formula the following expression (see e.g. [25, Eq. (7) page 25]) immediately follows

$$\sigma(x)P'_n(x) = \frac{\lambda_n}{n\tau'_n}\left[\tau_n(x)P_n(x) - \frac{B_n}{B_{n+1}}P_{n+1}(x)\right], \quad \tau_n(x) = \tau(x) + n\sigma'(x),$$

from where, using the three-term recurrence relation for the family $(P_n)_n$ the structure relation (1.2) follows.
(6)→(2): Suppose that (1.2) holds where $\deg\sigma \le 2$ and $(P_n)_n$ is an orthogonal family. Notice that the integral

$$\int_a^b Q_k(x)P'_n(x)\sigma(x)\rho(x)dx = \int_a^b Q_k(x)\rho(x)[a_nP_{n+1}(x)+b_nP_n(x)+c_nP_{n-1}(x)]dx$$

vanishes for all $k < n-1$. Then $(P'_n)_n$ is an orthogonal family with respect to the weight function $\rho_1(x) = \sigma(x)\rho(x)$ and therefore by the Sonin-Hahn Theorem 1.1 $(P_n)_n$ is a classical family.
(1)+(2)→(7): For proving this we suppose that $(P_n)_n$ and $(P'_n)_n$ are orthogonal with respect to $\rho(x)$ and $\rho_1(x) = \sigma(x)\rho(x)$, respectively. If $(P_n)_n$ is a monic sequence then

$$P_n(x) = \frac{1}{n+1}P'_{n+1} + f_nP'_n(x) + g_nP'_{n-1}(x) + \sum_{k=1}^{n-2} c_k(n)P'_k(x).$$

But

$$c_k(n) = \frac{\int_a^b P_n(x)P_k'(x)\sigma(x)\rho(x)dx}{\int_a^b [P_k'(x)]^2\sigma(x)\rho(x)dx} = 0,$$

since $\deg P_k'\sigma \leq k+1 < n-2$ and $(P_n)_n$ is and orthogonal family with respect to $\rho(x)$.

Finally the proof (7)→(2) is presented in section 4.3.

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