Contour dynamics for 2D active scalars

by D. Cordoba and F. Gancedo

Introduction

One line of research in the mathematical analysis of fluid mechanics is focused on solving problems which involve the possible formation and propagation of singularities. In these scenarios it becomes crucial to understand the role played by the singularities in the formation of patterns. For this purpose we present two physical models that are of interest from this mathematical point of view as well as for their applications in physics. It is an enormous challenge to approach these problems which require the combination of analytic techniques, asymptotics, numerics and modeling. With the more sophisticated numerical tools now available, the subject has gained considerable momentum. Recently numerical simulations indicate a possible singularity formation in the free boundary of a fluid domain which is a weak solution to a family of incompressible equations.

Successful analysis of singularities in incompressible flows would solve a major problem of mathematics and would establish a new method for addressing blow-up formation in non-linear PDE. A fluid dynamic understanding of these singularities could lead to important insights on the structure of turbulence, one of the major open scientific problems of classical physics.

On the search for singularities we study the simplest possible models that capture the non-local structure of an incompressible flow: active scalars. These remarkable examples (see [5]) are solutions $\rho = \rho(x,t)$ to the following non-linear equation:

$$(\partial_t + u \cdot \nabla)\rho = 0 \tag{1}$$

with $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$ and $u = (u_1, u_2)$ the fluid velocity. This transport equation reveals that the quantity ρ moves with the fluid flow and is conserved along trajectories. The velocity field is divergence free providing the incompressibility condition and is determined by the active scalar by singular integral operators as follows:

$$u_i(x,t) = P.V. \int_{\mathbb{R}^2} K_i(x-y)\rho(y,t)dy \tag{2}$$

where $P.V.$ denotes principal value and K_i is a classical Calderon-Zymund kernel (see [16]). The above identity indicates the non-local structure of the equation and that the velocity is at the same level as the scalar ρ: $\|u\|_{L^p}(t) \leq C\|\rho\|_{L^p}(t)$ for $1 < p < \infty$.

The incompressibility of the flow yields the system to be conservative in such a way that the L^p norms of ρ are constants for all time: $\|\rho\|_{L^p}(t) = \|\rho\|_{L^p}(0)$ for $1 \leq p \leq \infty$.

A fundamental property of the active scalars is that the level sets move with the flow, i.e. there is no transfer of flow along a level set. Then a natural solution, with finite energy, can be given by

$$\rho(x,t) = \begin{cases} \rho^1 & \text{if } x \in \Omega^1(t) \\ \rho^2 & \text{if } x \in \Omega^2(t) = \mathbb{R}^2 \backslash \Omega^1(t) \end{cases} \tag{3}$$

for $\Omega^i(t)$ connected regions. This represents an evolution problem for a fluid with different characteristics ρ^i which remain constant inside each domain $\Omega^i(t)$. These solutions start initially with a jump on the boundary of $\Omega^i(t)$ and the equation of the contour dynamics are highly singular.

Below we consider solutions of type (3) for the following different physical scenarios: one given by the quasi-geostrophic equation and other modeled by Darcy's law. These systems satisfy equations (1) and (2) in a weak sense with completely different outcomes regarding well-possed and regularity issues.

The 2D surface quasi-geostrophic equation

The 2D surface quasi-geostrophic equation that we will address below has the property that the velocity u in the form (2) is given by

$$u = (-R_2\rho, R_1\rho)$$

where the scalar ρ represents the temperature of the fluid and R_j are the Riesz transforms:

$$R_j\rho(x,t) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{(x_j - y_j)\cdot\rho(y,t)}{|x-y|^3} dy.$$

The symbols of these operators in the Fourier side are $\widehat{R_j} = i\frac{\xi_j}{|\xi|}$, therefore the energy of the system is conserved due to the fact that

$$\|u\|_{L^2} = \|(-R_2(\rho-\rho_\infty), R_1(\rho-\rho_\infty))\|_{L^2} = \|\rho-\rho_\infty\|_{L^2},$$

for ρ_∞ the constant value of ρ at infinity.

This equation, which we will denote by QG, has applications in meteorology and oceanography, and is a special case of the more general 3D quasi-geostrophic equation. There has been high scientific interest in understanding the behavior of the QG equation because it is a plausible model to explain the formation of fronts of hot and cold air. In a different direction, Constantin, Majda and Tabak [4] proposed this system as a 2D model for the 3D vorticity intensification and they showed that there is a geometric and analytic analogy with 3D incompressible Euler equations.

An interesting approach is to study the dynamics of the $\alpha - patches$ (see [7]) which are weak solutions of a family of equations that interpolate 2D incompressible Euler and QG. An $\alpha - patch$ $(0 < \alpha < 1)$ consists in a 2D region $\Omega(t)$ (bounded and connected) that moves with the velocity given by

$$u(z(\gamma,t),t) = \frac{C_\alpha}{2\pi}\int_{-\pi}^{\pi} \frac{\partial_\gamma z(\eta,t)}{|z(\gamma,t)-z(\eta,t)|^\alpha} d\eta \tag{4}$$

where $z(\gamma,t)$ is the position of the boundary of the domain $\Omega(t)$ parameterized for $\gamma \in [-\pi,\pi]$, and C_α depends on α, ρ^1 and ρ^2. The evolution of its boundary satisfies

$$z_t(\gamma,t) = u(z(\gamma,t),t), \tag{5}$$

and they are weak solutions of (1) and the following equation

$$u = -\nabla^{\perp}(-\Delta)^{\frac{\alpha}{2}-1}\rho, \tag{6}$$

where the symbol of $(-\Delta)^{\beta}$ is $|2\pi\xi|^{2\beta}$ and $x^{\perp} = (-x_2, x_1)$.

In the limiting case $\alpha = 0$, the identity (6) becomes the Biot-Savart law and therefore we can recover the 2D incompressible Euler equation. With this notation the scalar ρ represents the vorticity of the two dimensional flow. This case has been studied analytically with success by Chemin and Bertozzi-Constantin in [3] and [2] where they show global existence and therefore no singularity formation.

In the case $\alpha = 1$, for a temperature ρ satisfying (4), the velocity blows up logarithmically at the free boundary $z(\gamma, t)$. Thus, there is difficulty even in deriving the evolution problem. This issue can be solved since the normal component of the velocity is well-defined for a regular contour. For this type of free boundary problem the tangential component of the velocity does not modify the shape of the interface (in this case it blows-up), and the dynamics is given by the normal component.

Rodrigo in [13] gave a closed system for the *patch* problem for QG where the front is represented by a function. He proved local-existence for a periodic and infinitely differentiable contour by using the Nash-Moser iteration. This tool was applied for infinitely differentiable initial data because the operator involved loses two derivatives.

In the case for which the free boundary is not parameterized as a function, the fact that the interface collapses leads to a singularity in the fluid. Then it becomes crucial to get control of the evolution of the following quantity:

$$\mathcal{F}(z)(\gamma, \eta, t) = \frac{|\eta|}{|z(\gamma, t) - z(\gamma - \eta, t)|} \quad \forall\, \gamma, \eta \in [-\pi, \pi],$$

with

$$\mathcal{F}(z)(\gamma, 0, t) = \frac{1}{|\partial_\gamma z(\gamma, t)|},$$

which measures the arc-chord condition of the curve. Let us point out that the operators involved in the equations are ill-defined otherwise. Also, one could modify the contour equation (4) as follows:

$$z_t(\gamma, t) = \frac{C_1}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma z(\gamma, t) - \partial_\gamma z(\eta, t)}{|z(\gamma, t) - z(\eta, t)|} d\eta + c(\gamma, t)\partial_\gamma z(\gamma, t). \tag{7}$$

A curve that satisfies this new equation also yields a solution of the *patch* problem for QG: the terms introduced in the evolution system are tangential and therefore they change the parametrization of the interface but not the shape. Choosing $c(\gamma, t)$ in a wise fashion, enables the length of the tangent vector to $z(\gamma, t)$ be a function in the variable t only:

$$A(t) = |\partial_\gamma z(\gamma, t)|^2.$$

With this property it is easy to obtain the following two identities:

$$\partial_\gamma^2 z(\gamma, t) \cdot \partial_\gamma z(\gamma, t) = 0, \qquad \partial_\gamma^3 z(\gamma, t) \cdot \partial_\gamma z(\gamma, t) = -|\partial_\gamma^2 z(\gamma, t)|^2. \tag{8}$$

The first equality gives extra cancellation in the system (7) and the second one is a kind of ad hoc integration by parts. Both are used in [10] to obtain a proof of local-existence for a patch convected by the QG equation within the chain of Sobolev spaces. Similar results follow for the $\alpha - patch$.

In [7] numerically possible candidates are shown that lead to a singularity for the family of equations (6). For the particular case $\alpha = 1$ there is a formation of a corner that develops a high increase on the curvature at the same point where it reaches the minimum distance between two *patches* (see figure 1).

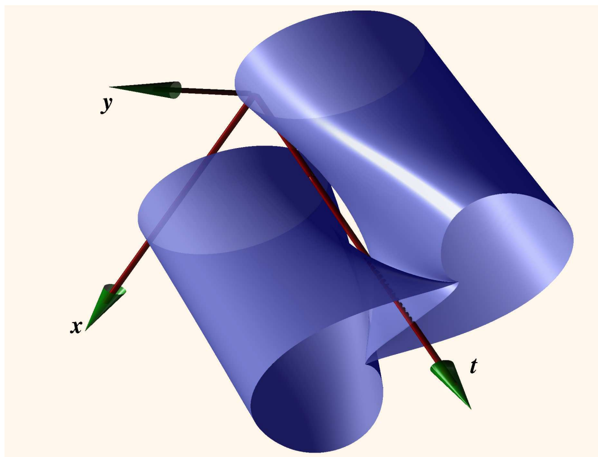

Figure 1: The evolution of two *patches* with $\alpha = 1$.

Furthermore, by re-scaling the spatial variable in the following form

$$z = (t_0 - t)^{\delta} y$$

where $\delta = \frac{1}{\alpha}$ and introducing a new variable $\tau = -log(t_0 - t)$, the contour dynamic equations (4) and (5) become

$$\frac{\partial y}{\partial \tau} - \delta y = \frac{C_\alpha}{2\pi} \int_{-\pi}^{\pi} \frac{\partial_\gamma y(\gamma, t) - \partial_\gamma y(\eta, t)}{|y(\gamma, t) - y(\eta, t)|^\alpha} d\eta. \tag{9}$$

Solutions of (9) independent of τ represent solutions of (6) with the property that the maximum curvature grows as

$$\kappa = \frac{1}{R} \sim \frac{C}{(t_0 - t)^{\frac{1}{\alpha}}} \quad \text{when} \quad t \to t_0,$$

and the minimum distance of the two *patches* satisfies

$$d \sim C(t_0 - t)^{\frac{1}{\alpha}} \quad \text{when} \quad t \to t_0.$$

These singularities are self similar and stable, occurring in one single point where the curvature blows-up at the same time the two level sets collapse (see fig. 2).

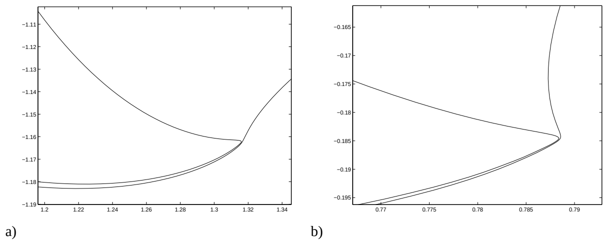

Figure 2: Detail of the corner region at t=16.515 for $\alpha = 0.5$ (a) and t=4.464 for $\alpha = 1$ (b). Observe that the singularity is point-like in both cases.

Darcy's law

The evolution of fluids in porous media is an important topic in fluid mechanics encountered in engineering, physics and mathematics. This phenomena has been described using the experimental Darcy's law that, in two dimensions, is given by the following momentum equation:

$$\frac{\mu}{\kappa}u = -\nabla p - (0, \mathrm{g}\rho). \tag{10}$$

Here u is the incompressible velocity, p is the pressure, μ is the dynamic viscosity, κ is the permeability of the isotropic medium, ρ is the liquid density, and g is the acceleration due to gravity.

The Muskat problem [11] models the evolution of an interface between two fluids with different viscosities and densities in porous media by using Darcy's law. This problem has been considered extensively without surface tension, in which case the pressures of the fluids are equal on the interface. Saffman and Taylor [14] made the observation that the one phase version (one of the fluids has zero viscosity) was also known as the Hele-Shaw cell equation, which, in turn, is the zero-specific heat case of the classical one-phase Stefan problem.

The problem considers fluids with different constant viscosities μ^1, μ^2, and densities ρ^2, ρ^1. Therefore using Darcy's law, we find that the vorticity is concentrated on the free boundary $z(\gamma, t)$, and is given by a Dirac distribution as follows:

$$w(x,t) = \varpi(\gamma, t)\delta(x - z(\gamma, t)),$$

with $\varpi(\gamma, t)$ the vorticity strength. Then $z(\gamma, t)$ evolves with an incompressible velocity field coming from the Biot-Savart law:

$$u(x,t) = \nabla^{\perp}\Delta^{-1}\omega(x,t). \tag{11}$$

It can be explicitly computed on the contour $z(\gamma, t)$ and is given by the Birkhoff-Rott integral of the amplitude ϖ along the interface curve:

$$BR(z, \varpi)(\gamma, t) = \frac{1}{2\pi}P.V.\int \frac{(z(\gamma,t) - z(\eta,t))^{\perp}}{|z(\gamma,t) - z(\eta,t)|^2}\varpi(\eta,t)d\eta. \tag{12}$$

Using Darcy's law, we close the system with the following formula:

$$\varpi(\gamma,t) = (I + A_\mu T)^{-1}\big(- 2\mathrm{g}\kappa \frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\gamma z_2\big)(\gamma,t), \tag{13}$$

where

$$T(\varpi) = 2BR(z,\varpi)\cdot\partial_\gamma z, \quad A_\mu = \frac{\mu^2-\mu^1}{\mu^2+\mu^1}. \tag{14}$$

Baker, Meiron and Orszag [1] shown that the adjoint operator T^*, acting on ϖ, is described in terms of the Cauchy integral of ϖ along the curve $z(\gamma,t)$, and whose real eigenvalues have absolute values strictly less than one. This yields that the operator $(I + A_\mu T)$ is invertible so that equation (13) gives an appropriate contour dynamics problem.

The first important question to be asked is whether local-existence is guaranteed. However such a result turns out to be false for general initial data. Rayleigh [12] and Saffman-Taylor [14] gave a condition that must be satisfied in order to have a solution locally in time, namely that the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This is known as the Rayleigh-Taylor condition. Siegel, Caflish and Howison [15] proved ill-posedness in a 2-D case when this condition is not satisfied (unstable case and same densities). On the other hand, they showed global-in-time solutions when the initial data are nearly planar and the Rayleigh-Taylor condition holds initially.

Recently in [6] we have obtained local existence in the 2D case when the fluid has different densities and viscosities. In our proof it is crucial to get control of the norm of the inverse operators $(I + A_\mu T)^{-1}$. The arguments rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\delta}$ curves, for which we need precise estimates obtained with arguments involving conformal mappings, the Hopf maximum principle and Harnack inequalities. We then provide bounds in the Sobolev spaces H^k for ϖ obtaining

$$\begin{aligned}\frac{d}{dt}(\|z\|^2_{H^k} + \|\mathcal{F}(z)\|^2_{L^\infty})(t) \le -K \int_{\mathbb{T}} \frac{\sigma(\gamma)}{|\partial_\gamma z(\gamma)|^2}\partial^k_\gamma z(\gamma)\cdot\Lambda(\partial^k_\gamma z)(\gamma) d\gamma \\ + \exp C(\|z\|^2_{H^k} + \|\mathcal{F}(z)\|^2_{L^\infty})(t),\end{aligned} \tag{15}$$

where $K = -\kappa/(2\pi(\mu_1+\mu_2))$, $\sigma(\gamma,t)$ is the difference of the gradients of the pressure in the normal direction (Rayleigh-Taylor condition), and the operator Λ is the square root of the Laplacian. When $\sigma(\gamma,t)$ is positive, there is a kind of heat equation in the above inequality but with the operator Λ in place of the Laplacian. Then, the most singular terms in the evolution equation depend on the Rayleigh-Taylor condition. In order to integrate the system we study the evolution of

$$m(t) = \min_{\gamma\in\mathbb{T}} \sigma(\gamma,t), \tag{16}$$

which satisfies the following bound

$$|m'(t)| \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^k})(t).$$

Using the pointwise estimate $f\Lambda(f) \geq \frac{1}{2}\Lambda(f^2)$ in estimate (15), we obtain

$$\frac{d}{dt}E_{RT}(t) \leq C \exp C E_{RT}(t),$$

where E_{RT} is the energy of the system given by

$$E_{RT}(t) = \|z\|^2_{H^k}(t) + \|\mathcal{F}(z)\|^2_{L^\infty}(t) + (m(t))^{-1}.$$

Here we point out that it is completely necessary to consider the evolution of the Rayleigh-Taylor condition to obtain bona fide energy estimates.

In the case where the viscosities are the same, the free boundary is given by a fluid with different densities. In order to simplify the notation, one could take $\mu/\kappa = \mathrm{g} = 1$ in Darcy's law and then apply the rotational operator to obtain the vorticity given by $\omega = -\partial_{x_1}\rho$. The Biot-Savart law (11) yields the velocity field in terms of the density as follows:

$$u(x,t) = P.V. \int_{\mathbb{R}^2} H(x-y)\rho(y,t)dy - \frac{1}{2}\left(0, \rho(x,t)\right),$$

where the Calderon-Zygmund kernel $H(\cdot)$ is defined by

$$H(x) = \frac{1}{2\pi}\left(-2\frac{x_1 x_2}{|x|^4}, \frac{x_1^2 - x_2^2}{|x|^4}\right).$$

By means of Darcy's law, we can find the following formula for the difference of the gradients of the pressure in the normal direction: $\sigma(\gamma,t) = \mathrm{g}(\rho^2 - \rho^1)\partial_\gamma z_1(\gamma,t)$. A wise choice of parameterizing the curve is that for which we have $\partial_\gamma z_1(\gamma,t) = 1$ (for more details see [8]). This yields the denser fluid below the less dense fluid if $\rho^2 > \rho^1$ and therefore the Rayleigh-Taylor condition holds for all time. An additional advantage is that we avoid a kind of singularity in the fluid when the interface collapses due to the fact that we can take $z(\gamma,t) = (\gamma, f(\gamma,t))$ which implies $\mathcal{F}(z)(\gamma,\eta) \leq 1$ obtaining the arc-chord condition for all time. Then the character of the interface as the graph of a function is preserved, and in [8] this fact has been used to show local-existence in the stable case ($\rho^2 > \rho^1$), together with ill-posedness in the unstable situation ($\rho^2 < \rho^1$).

Currently we are studying the long-time behavior of the stable case for which we can show that the L^∞ norm of any interface decays, and numerical simulations of the dynamics of the contour develop a regularity effect (see [9]).

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References

[1] G. Baker, D. Meiron and S. Orszag. Generalized vortex methods for free-surface flow problems. *J. Fluid Mech.* 123 477-501, 1982.

[2] A. L. Bertozzi and P. Constantin. Global regularity for vortex patches. *Comm. Math. Phys.* 152 (1), 19-28, 1993.

[3] J.Y. Chemin. Persistence of geometric structures in two-dimensional incompressible fluids. *Ann. Sci. Ecole. Norm. Sup.* 26 (4), 517-542, 1993.

[4] P. Constantin, A. J. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7, 1495-1533, 1994.

[5] P. Constantin. Scaling exponents for active scalars. *Journal of Statistical Physics*, vol 90, 3-4, 571-595, 1998.

[6] A. Córdoba, D. Córdoba and F. Gancedo. Interface evolution: the Hele-Shaw and Muskat problems. *To appear in Annals of Math.*

[7] D. Córdoba, M. A. Fontelos, A. M. Mancho and J. L. Rodrigo. Evidence of singularities for a family of contour dynamics equations. *Proc. Natl. Acad. Sci. USA* 102, 5949-5952, 2005.

[8] D. Córdoba and F. Gancedo. Contour dynamics of incompressible 3-D fluids in a porous medium with different densities. *Comm. Math. Phys.* 273, 2, 445-471 (2007).

[9] D. Córdoba, F. Gancedo and R. Orive. A note on the interface dynamics for convection in porous media. *Physica D*, 237, 1488-1497, 2008.

[10] F. Gancedo. Existence for the α-patch model and the QG sharp front in Sobolev spaces. *Adv. Math.*, 217, 2569-2598, 2008.

[11] M. Muskat. The flow of homogeneous fluids through porous media. *New York*, Springer 1982.

[12] Lord Rayleigh (J.W. Strutt), On the instability of jets. *Proc. Lond. Math. Soc.* 10, 413, 1879.

[13] J.L. Rodrigo. On the Evolution of Sharp Fronts for the Quasi-Geostrophic Equation. *Comm. Pure and Appl. Math.*, 58, 0821-0866, 2005.

[14] P.G. Saffman and Taylor. The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. London, Ser. A* 245, 312-329, 1958.

[15] M. Siegel, R. Caflisch and S. Howison. Global Existence, Singular Solutions, and Ill-Posedness for the Muskat Problem. *Comm. Pure and Appl. Math.*, 57, 1374-1411, 2004.

[16] E. Stein. Singular Integrals and Differentiability Properties of Function. *Princeton University Press.* Princeton, NJ, 1970.