Turning waves and breakdown for incompressible flows

Angel Castro[a], Diego Córdoba[a], Charles L. Fefferman[b,1], Francisco Gancedo[c], and María López-Fernández[d]

[a]Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain; [b]Department of Mathematics, Princeton University, 1102 Fine Hall, Washington Road, Princeton, NJ 08544; [c]Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637; and [d]Institut für Mathematik, Universität Zürich Winterthurerstrasse 190 CH-8057 Zurich, Switzerland

Contributed by Charles L. Fefferman, January 29, 2011 (sent for review November 30, 2010)

We consider the evolution of an interface generated between two immiscible, incompressible, and irrotational fluids. Specifically we study the Muskat and water wave problems. We show that starting with a family of initial data given by $(\alpha,f_0(\alpha))$, the interface reaches a regime in finite time in which is no longer a graph. Therefore there exists a time t^* where the solution of the free boundary problem parameterized as $(\alpha,f(\alpha,t))$ blows up: $\|\partial_\alpha f\|_{L^\infty}(t^*)=\infty$. In particular, for the Muskat problem, this result allows us to reach an unstable regime, for which the Rayleigh–Taylor condition changes sign and the solution breaks down.

1. Introduction

Here we study two problems of fluids mechanics concerning the evolution of two incompressible fluids of different characteristics in 2D. We consider that both fluids are immiscible and of different constant densities ρ^1 and ρ^2, modeling the dynamics of an interface that separates the domains $\Omega^1(t)$ and $\Omega^2(t)$. That is, the liquid density $\rho=\rho(x,t),(x,t)\in\mathbb{R}^2\times\mathbb{R}^+$, is defined by

$$\rho(x,t)=\begin{cases}\rho^1, & x\in\Omega^1(t)\\ \rho^2, & x\in\Omega^2(t)=\mathbb{R}^2-\Omega^1(t),\end{cases}\qquad [1]$$

and satisfies the conservation of mass equation

$$\rho_t+v\cdot\nabla\rho=0,\quad \nabla\cdot v=0,\qquad [2]$$

where $v=(v_1(x,t),v_2(x,t))$ is the velocity field. With a free boundary parameterized by

$$\partial\Omega^j(t)=\{z(\alpha,t)=(z_1(\alpha,t),z_2(\alpha,t))\colon \alpha\in\mathbb{P}\},$$

we consider open curves vanishing at infinity

$$\lim_{\alpha\to\infty}(z(\alpha,t)-(\alpha,0))=0,$$

or periodic in the space variable

$$z(\alpha+2k\pi,t)=z(\alpha,t)+2k\pi(1,0).$$

The scalar vorticity, $\nabla^\perp\cdot v$, has the form

$$\nabla^\perp\cdot v(x,t)=\omega(\alpha,t)\delta(x-z(\alpha,t)),\qquad [3]$$

i.e., the vorticity is a Dirac measure on z defined by

$$<\nabla^\perp\cdot v,\eta>=\int_{\mathbb{R}}\omega(\alpha,t)\eta(z(\alpha,t))d\alpha,$$

with $\eta(x)$ a test function. The system is closed by using one of the following fundamental fluid motion equations:

Darcy's law

$$\frac{\mu}{\kappa}v=-\nabla p-g\rho(0,1),\qquad [4]$$

or

Euler equations

$$\rho(v_t+v\cdot\nabla v)=-\nabla p-g\rho(0,1).\qquad [5]$$

Here $p=p(x,t)$ is pressure, g gravity, μ viscosity, and κ permeability of the isotropic medium.

The Muskat problem (1) is given by Eqs. **1**, **2**, and **4**, which considers the dynamics of two incompressible fluids of different densities throughout porous media and Hele–Shaw cells (2, 3). In this last setting, the fluid is trapped between two fixed parallel plates that are close enough together so that the fluid essentially only moves in two directions (4).

Taking $\rho^1=0$, Eqs. **1–3** and **5** are known as the water waves problem (see ref. 5 and references therein), modeling the dynamics of the contour between an inviscid fluid with density ρ^2 and vacuum (or air) under the influence of gravity.

Condition **3** (deduced by [**4**], assumed for [**5**]) allows us to write the evolution equation in terms of the free boundary as follows. One could recover the velocity field from [**3**] by means of Biot–Savart law

$$v(x,t)=\nabla^\perp\Delta^{-1}(\nabla^\perp\cdot v)(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{(x-z(\alpha,t))^\perp}{|x-z(\alpha,t)|^2}\omega(\alpha,t)d\alpha,$$

applying the Dirac measure with amplitude ω. Taking limits on the above equation approaching the boundary in the normal direction inside Ω^j, the velocity is shown to be discontinuous in the tangential direction, but continuous in the normal, and given by the Birkhoff–Rott integral of the amplitude ω along the interface curve:

$$\mathrm{BR}(z,\omega)(\alpha,t)=\frac{1}{2\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\omega(\beta,t)d\beta,$$

where PV denotes principal value. This fact yields the curve velocity from which one can subtract any term c in the tangential without modifying the geometry of the interface

$$z_t(\alpha,t)=\mathrm{BR}(z,\omega)(\alpha,t)+c(\alpha,t)\partial_\alpha z(\alpha,t).\qquad [6]$$

Understanding the problem as weak solutions of [**1**, **2**, and **4**] or [**1–3** and **5**], the continuity of the pressure on the free boundary follows. Therefore, taking limits in Darcy's law from both sides and subtracting the results in the tangential direction, it is easy to close the system for Muskat (in this paper we consider two fluids with the same viscosity):

$$\omega(\alpha,t)=-(\rho^2-\rho^1)\frac{\kappa g}{\mu}\partial_\alpha z_2(\alpha,t).\qquad [7]$$

Author contributions: A.C., D.C., C.L.F., F.G., and M.L.-F. designed research, performed research, and wrote the paper.

The authors declare no conflict of interest.

[1]To whom correspondence should be addressed. E-mail: cf@math.princeton.edu.

In a similar way for water waves, Euler equations yield

$$\omega_t(\alpha,t) = -2\partial_t \mathrm{BR}(z,\omega)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - \partial_\alpha\left(\frac{|\omega|^2}{4|\partial_\alpha z|^2}\right)(\alpha,t) + \partial_\alpha(c\omega)(\alpha,t) + 2c(\alpha,t)\partial_\alpha \mathrm{BR}(z,\omega)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2g\partial_\alpha z_2(\alpha,t). \quad [8]$$

Then, the two contour equations are set by [**6** and **7**] and [**6** and **8**].

For these models, the well-posedness turns out to be false for some settings. Rayleigh (6) and Saffman and Taylor (2) gave a condition that must be satisfied for the linearized model in order to exist a solution locally in time: The normal component of the pressure gradient jump at the interface has to have a distinguished sign. This quantity is known as the Rayleigh–Taylor condition. It reads as

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t))\cdot\partial_\alpha^\perp z(\alpha,t) > 0,$$

where $\nabla p^j(z(\alpha,t),t)$ denotes the limit gradient of the pressure obtained approaching the boundary in the normal direction inside $\Omega^j(t)$.

An easy linearization around a flat contour $(\alpha,f(\alpha,t))$, allows us to find

$$f_t = \frac{1}{2}H(\omega)$$

where H is the Hilbert transform which symbol on the Fourier side is given by $\widehat{H} = -i\ \mathrm{sign}\ (\xi)$. The equations

$$\omega = -(\rho^2-\rho^1)\frac{\kappa g}{\mu}\partial_\alpha f, \qquad \text{(linear Muskat)}$$

$$\omega_t = 2g\partial_\alpha f, \qquad \text{(linear water waves)}$$

show the parabolicity of the Muskat problem when the denser fluid is below ($\rho^2 > \rho^1$) and the dispersive character of water waves.

1. There is a wide literature on the Muskat problem and the dynamics of two fluids in a Hele–Shaw cell. There are works considering the case of a viscosity jump neglecting the effect of gravity (7, 8). Local existence in a more general situation (with discontinuous viscosity and density) is shown in ref. 9 and also treated in ref. 10. A different approach to prove local existence can be found in ref. 11 for the setting we are considering in this paper. The Rayleigh–Taylor stability depends upon the sign of $(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t)$ (11) indicating that the heavier fluid has to be below in the stable case. If the lighter fluid is below, the problem has been shown to be ill-posed (11). Global-existence results for small initial data can be found in refs. 7 and 11–14. For large initial curves and parameterized by $(\alpha,f(\alpha,t))$, there are maximum principles for the L^∞ and L^2 norms of f, and decay rates, together with global existence for Lipschitz curves if $\|\partial_\alpha f\|_{L^\infty}(0) < 1$ (15, 16, 17).
2. The water waves problem has been extensively considered (see refs. 5 and 18 and references therein). For sufficiently smooth free boundary, the Rayleigh–Taylor condition remains positive with no bottom considerations (19), a fact that was used to prove local existence (19). The Rayleigh–Taylor stability can play a different role for the case of non-"almost"-flat bottom (20). Recently, for small initial data, exponential time of existence has been proven in two dimensions (21) and global existence in the three-dimensional case (two-dimensional interface) (22, 23).

2. Rayleigh–Taylor Breakdown for Muskat

This section is devoted to show the main ingredients to prove the Theorem 2.1. We consider the function

$$F(z)(\alpha,\beta) = \frac{|\beta|^2}{|z(\alpha)-z(\alpha-\beta)|^2}, \qquad \alpha,\beta\in\mathbb{R},$$

and in the periodic setting

$$F(z)(\alpha,\beta) = \frac{\|\beta\|^2}{2(\cosh(z_2(\alpha)-z_2(\alpha-\beta)) - \cos(z_1(\alpha)-z_1(\alpha-\beta)))}, \quad \alpha,\beta\in\mathbb{T}, \quad [9]$$

where $\|x\| = \mathrm{dist}(x,2\pi Z)$. If $F(z)\in L^\infty(\mathbb{R}^2)$, then the curve z satisfies the arc-chord condition. We say that the Rayleigh–Taylor (R-T) of the solution of the Muskat problem breaks down in finite time if for initial data z_0 satisfying $\sigma(\alpha,0) = (\rho^2-\rho^1)\partial_\alpha z_1(\alpha,0) > 0$ there exists a time $t^* > 0$ for which $\sigma(\alpha,t^*)$ is strictly negative in a nonempty open interval.

Theorem 2.1. *There exists a nonempty open set of initial data in H^4, satisfying Rayleigh–Taylor and arc-chord conditions, for which the Rayleigh–Taylor condition of the solution of the Muskat problem [1, 2, and 4] breaks down in finite time.*

After choosing the appropriate tangential term and a integration by parts, the contour equation reads

$$z_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{(z_1(\alpha,t)-z_1(\beta,t))}{|z(\alpha,t)-z(\beta,t)|^2}(\partial_\alpha z(\alpha,t)-\partial_\alpha z(\beta,t))d\beta.$$

For a 2π periodic interface, removing the principal value at infinity, the equation becomes

$$z_t(\alpha) = \frac{(\rho^2-\rho^1)}{4\pi}\times\int_{\mathbb{T}}\frac{\sin(z_1(\alpha)-z_1(\alpha-\beta))(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta))}{\cosh(z_2(\alpha)-z_2(\alpha-\beta))-\cos(z_1(\alpha)-z_1(\alpha-\beta))}d\beta. \quad [10]$$

From now on, we shall use the periodic configuration.

The steps of the proof are as follows:

1. First, for any initial curve $z_0(\alpha) = z(\alpha,0)$ in H^4 that satisfy R-T

$$(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,0) > 0$$

and the arc-chord condition then the solution to the Muskat problem $z(\alpha,t)$ becomes analytic for $0 < t < T$. Moreover, $z(\alpha,t)$ is real analytic in a strip

$$S(t) = \{\alpha + i\zeta\colon |\zeta| < ct\}$$

for $t\in(0,T)$ where c depends only on

$$\inf(0) = \inf_\alpha \frac{\partial_\alpha z_1(\alpha,0)}{|\partial_\alpha z(\alpha,0)|^2}.$$

The proof follows by controlling the quantities extended on $S(t)$:

$$F(z)(\alpha+i\zeta,\beta,t)$$

and $g(\alpha+i\zeta,t)$ by using [**9**] and formula

$$g(\alpha,t) = \int_{\mathbb{T}}[\sin(z_1(\alpha,t)-z_1(\alpha-\beta,t))]/[\cosh(z_2(\alpha,t)-z_2(\alpha-\beta,t))-\cos(z_1(\alpha,t)-z_1(\alpha-\beta,t))]d\beta,$$

respectively. The norms

MATHEMATICS

$$\|F(z)\|_{L^\infty(S)}(t) = \sup_{\alpha+i\zeta\in S(t),\beta\in\mathbb{T}} |F(z)(\alpha+i\zeta,\beta)|,$$

$$\|z\|^2_{L^2(S)}(t) = \sum_{\pm}\int_{\mathbb{T}} |z(\alpha\pm ict,t)|^2 d\alpha,$$

$$\|z\|^2_{H^j(S)}(t) = \|z\|^2_{L^2(S)}(t) + \sum_{\pm}\int_{\mathbb{T}} |\partial^j_\alpha z(\alpha\pm ict,t)|^2 d\alpha,$$

$$\text{for } j\in\mathbb{N},$$

$$\inf(t) = \inf_{\alpha+i\zeta\in S(t)} \Re\left(\frac{\partial_\alpha z_1(\alpha+i\zeta,t)}{|\partial_\alpha z(\alpha+i\zeta,t)|^2}\right).$$

Then the quantity

$$\|z\|^2_{\mathrm{RT}}(t) = \|z\|^2_{H^4(S)}(t) + \|F(z)\|_{L\infty(S)}(t) + 1/(\inf(t) - c - K\|\Im(g)\|_{H^2(S)}(t))$$

satisfies

$$\frac{d}{dt}\|z\|_{\mathrm{RT}}(t) \le C\|z\|^k_{\mathrm{RT}}(t),$$

for C, K, and k universal constants. It yields

$$\|z\|_{\mathrm{RT}}(t) \le \frac{\|z\|_{\mathrm{RT}}(0)}{(1 - C\|z\|^k_{\mathrm{RT}}(0)t)^{1/k}},$$

providing control of the analyticity and $T = 1/(C\|z\|^k_{\mathrm{RT}}(0))$.

2. Second, there is a lower bound on the strip of analyticity, which does not collapse to the real axis as long as the Rayleigh–Taylor is greater than or equal to 0. Then there is a time T and a solution of the Muskat problem $z(\alpha,t)$ defined for $0 < t \le T$ that continues analytically into a complex strip if $(\rho^2 - \rho^1)\partial_\alpha z_1 \ge 0$, where T is either a small constant or it is the first time a vertical tangent appears, whichever occurs first. We redefine the strip

$$S(t) = \{\alpha + i\zeta\colon |\zeta| < h(t), \quad 0 < h(0)\},$$

and the quantity $\|z\|^2_S = \|z\|^2_{H^4(S)} + \|F(z)\|_{L^\infty(S)}$ with this new $S(t)$. For an $h(t)$ decreasing [the expression of $h(t)$ is chosen later], we consider the evolution of the most singular quantity

$$\sum_{\pm}\int |\partial^4_\alpha z(\alpha\pm ih(t), \quad t)|^2 d\alpha.$$

Taking a derivative in t, one finds

$$\frac{d}{dt}\sum_{\pm}\int |\partial^4_\alpha z(\alpha\pm ih(t))|^2 d\alpha \le \frac{h'(t)}{10}$$
$$\sum_{\pm}\int \Lambda(\partial^4_\alpha z)(\alpha\pm ih(t))\cdot\overline{\partial^4_\alpha z}(\alpha\pm ih(t))d\alpha$$
$$-10h'(t)\int\Lambda(\partial^4_\alpha z)(\alpha)\cdot\overline{\partial^4_\alpha z}(\alpha)d\alpha$$
$$+2\sum_{\pm}\Re\int \partial^4_\alpha z_t(\alpha\pm ih(t))\cdot\overline{\partial^4_\alpha z}(\alpha\pm ih(t))d\alpha.$$

Estimating in a wise way, one obtains

$$\frac{d}{dt}\sum_{\pm}\int |\partial^4_\alpha z(\alpha\pm ih(t))|^2 d\alpha \le C\|z\|^k_S(t)$$
$$-10h'(t)\int\Lambda(\partial^4_\alpha z)(\alpha)\cdot\overline{\partial^4_\alpha z}(\alpha)d\alpha + (C\|z\|^k_S(t)h(t)$$
$$+\frac{1}{10}h'(t))\int\Lambda(\partial^4_\alpha z)(\alpha\pm ih(t))\cdot\overline{\partial^4_\alpha z}(\alpha\pm ih(t))d\alpha.$$

Therefore, choosing

$$h(t) = h(0)\exp(-10C\int_0^t \|z\|^k_S(r)dr)$$

eliminates the most dangerous term. The other terms are easily controlled, giving finally

$$\frac{d}{dt}\sum_{\pm}\int |\partial^4_\alpha z(\alpha\pm ih(t))|^2 d\alpha \le C\|z\|^{k+2}_S(t),$$

which allows us to reach a regime for which the boundary z develops a vertical tangent at time T.

3. Third, it is shown the existence of a large class of analytic curves for which there exist a point where the tangent vector is vertical and the velocity indicates that the curve is going to turn up and reach the unstable regime.

For the equation

$$z_t(\alpha,t) = u(\alpha,t) = (u_1(\alpha,t), u_2(\alpha,t)),$$

that is,

$$a.\,\partial_\alpha z_1(\alpha) > 0 \quad \text{if } \alpha\ne 0, \qquad b.\,\partial_\alpha z_1(0) = 0,$$
$$c.\,\partial_\alpha z_2(0) > 0, \qquad d.\,\partial_\alpha u_1(0) < 0,$$

for analytic functions $z_1(\alpha)$ and $z_2(\alpha)$ such that $z(\alpha)$ satisfies the arc-chord condition. Here we consider the periodic case (being analogous for an open curve vanishing at infinity). We assume that $z(\alpha)$ is a smooth odd curve satisfying the properties a, b, and c. Differentiating the expression **10** for the horizontal component of the velocity, at $\alpha = 0$, it yields

$$(\partial_\alpha u_1)(0) = \int_{-\pi}^{\pi}[\cos(z_1(\beta))(\partial_\alpha z_1(\beta))^2$$
$$+\sin(z_1(\beta))\partial^2_\alpha z_1(\beta)]/[\cosh(z_2(\beta)) - \cos(z_1(\beta))d\beta$$
$$-\int_{-\pi}^{\pi}\sin(z_1(\beta))\partial_\alpha z_1(\beta)[\sin(z_1(\beta))\partial_\alpha z_1(\beta)$$
$$-\sinh(z_2(\beta))(\partial_\alpha z_2(0) - \partial_\alpha z_2(\beta))]/[(\cosh(z_2(\beta))$$
$$-\cos(z_1(\beta)))^2]d\beta.$$

Integration by parts provides

$$\int_{-\pi}^{\pi}[\sin(z_1(\beta))\partial^2_\alpha z_1(\beta)]/[\cosh(z_2(\beta)) - \cos(z_1(\beta))]d\beta$$
$$= -\int_{-\pi}^{\pi}\cos(z_1(\beta))[(\partial_\alpha z_1(\beta))^2]/[\cosh(z_2(\beta)) - \cos(z_1(\beta))]d\beta$$
$$+\int_{-\pi}^{\pi}\sin(z_1(\beta))\partial_\alpha z_1(\beta)[\sin(z_1(\beta))\partial_\alpha z_1(\beta)$$
$$+\sinh(z_2(\beta))\partial_\alpha z_2(\beta)]/[(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2]d\beta.$$

Therefore, it is easy to obtain that

$$(\partial_\alpha u_1)(0) = \partial_\alpha z_2(0)\int_{-\pi}^{\pi}[\sin(z_1(\beta))\sinh(z_2(\beta))]/[(\cosh(z_2(\beta))$$
$$-\cos(z_1(\beta)))^2]\partial_\alpha z_1(\beta)d\beta$$
$$= 2\partial_\alpha z_2(0)\int_0^{\pi}[\sin(z_1(\beta))\sinh(z_2(\beta))]/[(\cosh(z_2(\beta))$$
$$-\cos(z_1(\beta)))^2]\partial_\alpha z_1(\beta)d\beta \qquad \textbf{[11]}$$

Expression **11** allows us to determine the sign of $(\partial_\alpha u_1)(0)$. One could take

$$z_1(\beta) = -\sin(\beta) + \beta$$

and construct the function $z_2(\beta)$ in the following way: Let β_1, β_2, β_3, and β_4 be real increasing numbers less than π. We pick $z_2(\beta) \leq 0$ for $\beta_2 < \beta < \pi$, $z_2(\beta) < c < 0$ for $\beta_2 < \beta < \beta_4$, and $z_2^*(\beta)$ a smooth function with the following properties

$a.\ z_2^*(\beta)$ is odd, $\quad b.\ (\partial_\beta z_2^*)(0) > 0$,

$c.\ z_2^*(\beta) > 0$ if $\beta \in (0,\beta_1)$, $\quad d.\ z_2^*(\beta) < 0$ if $\beta \in (\beta_1,\beta_2]$.

Also, $z_2^*(\beta)$ is 2π-periodic. For $z_2(\beta) = b z_2^*(\beta)$, $0 \leq \beta \leq \beta_2$, and $b > 0$, the velocity satisfies

$$\begin{aligned}(\partial_\alpha u_1)(0) &< 2(\partial_\alpha z_2)(0) \\ &\times \bigg(\int_0^{\beta_1} [\sin(z_1(\beta))\sinh(z_2(\beta))]/[(\cosh(z_2(\beta)) \\ &- \cos(z_1(\beta)))^2]\partial_\alpha z_1(\beta)d\beta \\ &+ \int_{\beta_3}^{\pi} [\sin(z_1(\beta))\sinh(z_2(\beta))]/[(\cosh(z_2(\beta)) \\ &- \cos(z_1(\beta)))^2]\partial_\alpha z_1(\beta)d\beta \bigg) \\ &= 2(\partial_\alpha z_2)(0)\bigg(\int_0^{\beta_1} [\sin(z_1(\beta))\sinh(b z_2^*(\beta))]/[(\cosh(b z_2^*(\beta)) \\ &- \cos(z_1(\beta)))^2]\partial_\alpha z_1(\beta)d\beta + A \bigg),\end{aligned}$$

where $A < 0$. The constant b large enough yields $(\partial_\alpha u_1)(0) < 0$.
Rectifying the curve on the interval $[\beta_2,\beta_3]$, it is easy to obtain a smooth curve. Finally, convolving with the heat kernel the vertical component, the curve $z(\alpha)$ is approximated by an analytic one.

4. Fourth, with the initial data found in 3 and no assumption on the R-T condition, we use a modification of Cauchy–Kowalewski theorems (24, 25) to show that there exists an analytic solution for the Muskat problem in some interval $[-T,T]$ for a small enough $T > 0$. Here we are forced to change substantially the method in ref. 26 because, in this case, the curve cannot be parameterized as a graph, so we have to deal with the arc-chord condition. Then, with $\{X_r\}_{r>0}$, a scale of Banach spaces given by real functions that can be extended analytically on the complex strip $S_r = \{\alpha + i\zeta \in \mathbb{C}: |\zeta| < r\}$ with norm

$$\|f\|_r = \sum_{\pm} \int |f(\alpha \pm ir)|^2 d\alpha + \int |\partial_\alpha^4 f(\alpha \pm ir)|^2 d\alpha,$$

and $z^0(\alpha)$ a curve satisfying the arc-chord condition and $z^0(\alpha) \in X_{r_0}$ for some $r_0 > 0$, we prove the existence of a time $T > 0$ and $0 < r < r_0$ so that there is a unique solution to the Muskat problem in $C([-T,T];X_r)$. This result allows us to find solutions that do not satisfy the R-T but shrink the strip of analyticity. We extend Eq. **10** as follows:

$$z_t(\alpha + i\zeta,t) = G(z(\alpha + i\zeta,t)),$$

with

$$G(z)(\alpha,t) = \frac{(\rho^2 - \rho^1)}{4\pi} \times \int_{\mathbb{T}} \frac{\sin(z_1(\alpha,t) - z_1(\alpha-\beta,t))(\partial_\alpha z(\alpha,t) - \partial_\alpha z(\alpha-\beta,t))}{\cosh(z_2(\alpha,t) - z_2(\alpha-\beta,t)) - \cos(z_1(\alpha,t) - z_1(\alpha-\beta,t))} d\beta.$$

For $0 < r' < r$ and the open set O in S_r given by

$$O = \{z \in X_r: \|z\|_r < R, \quad \|F(z)\|_{L^\infty(S_r)} < R^2\}, \qquad [12]$$

the function G for $G: O \to X_{r'}$ is a continuous mapping and there is a constant C_R (depending on R only) such that

$$\|G(z)\|_r \leq \frac{C_R}{r - r'}\|z\|_r, \qquad [13]$$

$$\|G(z^2) - G(z^1)\|_{r'} \leq \frac{C_R}{r - r'}\|z^2 - z^1\|_r, \qquad [14]$$

and

$$\sup_{\alpha + i\zeta \in S_r, \beta \in \mathbb{T}} |G(z)(\alpha + i\zeta) - G(z)(\alpha + i\zeta - \beta)| \leq C_R|\beta|, \qquad [15]$$

for $z, z^j \in O$. For initial data $z^0 \in X_{r_0}$ satisfying arc-chord, we can find a $0 < r_0' < r_0$ and a constant R_0 such that $\|z^0\|_{r_0'} < R_0$ and

$$[\cosh(z_2^0(\alpha + i\zeta) - z_2^0(\alpha + i\zeta - \beta)) - \cos(z_1^0(\alpha + i\zeta) - z_1^0(\alpha + i\zeta - \beta))]/(\|\beta\|^2) > \frac{1}{R_0^2}, \qquad [16]$$

for $\alpha + i\zeta \in S_{r_0'}$. We take $0 < r < r_0'$ and $R_0 < R$ to define the open set O as in [**12**]. Therefore we can use the classical method of successive approximations:

$$z^{n+1}(t) = z^0 + \int_0^t G(z^n(s))ds,$$

for $G: O \to X_{r'}$ and $0 < r' < r$. We assume by induction that

$$\|z^k\|_r(t) < R, \quad \text{and} \quad \|F(z^k)\|_{L^\infty(S_r)}(t) < R$$

for $k \leq n$ and $0 < t < T$ with $T = \min(T_A,T_{CK})$ and T_{CK} the time obtaining in the proofs in refs 24 and 25. We get $\|z^{n+1}\|_r(t) < R$ that follows using [**13** and **14**]. The time T_A is to yield $\|F(z^{n+1})\|_{L^\infty(S_r)}(t) < R$. Then, using the induction hypothesis and [**15**], we can control the quantity taking $0 < T_A < (R_0^{-2} - R^{-2})(C_R^2 + 2R_0C_R)^{-1}$.

5. Fifth, all the results above allow us to prove that there is a nonempty set of initial data in H^4 satisfying the arc-chord and R-T conditions, such that the solution of the Muskat problem reaches the unstable regime: The R-T becomes strictly negative on a nonempty interval. We pick initial data as in 3. We apply the local-existence result in 4 to get an analytic solution $z(\alpha,t)$ on $[-T,T]$. Then we consider a time $0 < \delta < T$ and a curve $\omega_\delta^\epsilon(\alpha,t)$, solving the Muskat problem with initial datum $z(\alpha,-\delta) + \eta_\delta^\epsilon(\alpha)$. The function η_δ^ϵ has a small H^4 norm, i.e.,

$$\|\omega_\delta^\epsilon(\cdot,-\delta) - z(\cdot,-\delta)\|_{H^4} = \|\eta_\delta^\epsilon\|_{H^4} \leq \epsilon.$$

The time δ is small enough so that $\omega_\delta^\epsilon(\alpha,-\delta)$ satisfies R-T: $(\rho^2 - \rho^1)\partial_\alpha(\omega_\delta^\epsilon)_1(\alpha,-\delta) > 0$. Then we apply the local-existence result in 1 that $\omega_\delta^\epsilon(\alpha,t)$ becomes analytic for some time $-\delta < t$. With 2, we assure the existence and analyticity of the solution even if $\partial_\alpha(\omega_\delta^\epsilon)_1(\alpha,t) \leq 0$ for some time t. Then, we show that both solutions are close in the H^4 topology as time evolves. We can apply to ω_δ^ϵ the local-existence result in 4 if it is needed. Then, with δ and ε small enough, we find the desired result.

3. Turning Water Waves

In this section, we prove for the water wave problem ($\rho^1 = 0$ and [**1–3** and **5**]) that with initial data given by a graph $(\alpha,f_0(\alpha))$, the interface reaches a regime in finite time where it only can be parameterized as $z(\alpha,t) = (z_1(\alpha,t),z_2(\alpha,t))$, for $\alpha \in \mathbb{R}$, with $\partial_\alpha z_1(\alpha,t) < 0$ for $\alpha \in I$, a nonempty interval. Therefore there exists a time t^* where the solution of the free boundary problem reparameterized by $(\alpha,f(\alpha,t))$ satisfies $\|f_\alpha\|_{L^\infty}(t^*) = \infty$.

Theorem 3.1. *There exists a nonempty open set of initial data $(\alpha,f_0(\alpha))$, with $f_0 \in H^5$, such that in finite time t^* the solution of*

MATHEMATICS

the water waves problem ($\rho^1 = 0$ and **[1–3** *and* **5]**) *given by $(\alpha f(\alpha,t))$ satisfies $\|f_\alpha\|_{L^\infty}(t^*) = \infty$. The solution can be continued for $t > t^*$ as $z(\alpha,t)$ with $\partial_\alpha z_1(\alpha,t) < 0$ for $\alpha \in I$, a nonempty interval.*

In order to prove this theorem, we consider a curve $z^*(\alpha) \in H^5$ with the same properties as in point 3 of the previous section. Then, we pick $z(\alpha,t^*) = z^*(\alpha)$ and $\omega(\alpha,t^*) = -\partial_\alpha z_2^*(\alpha)$ as a datum for the initial value problem. It is easy to find the same properties for the velocity, because the tangential direction does not affect the evolution. Picking the appropriate $c(\alpha,t)$ and applying the local-existence result in ref. 18 (note that in this case it is not necessary analyticity, just H^5 regularity), there exists a solution of the water waves problem with $z(\alpha,t) \in C([t^* - \delta, t^* + \delta]; H^5)$, $\omega(\alpha,t) \in C([t^* - \delta, t^* + \delta]; H^4)$, and $\delta > 0$ small enough. Then, the initial datum $(z_0(\alpha),\omega_0(\alpha)) = (\alpha f_0(\alpha),\omega_0(\alpha))$ is given by $(z(\alpha,t^*-\delta),\omega(\alpha,t^*-\delta))$.

4. Muskat Breakdown

In this section, we show that there exists a smooth initial data in the stable regime for the Muskat problem such that the solution turns to the unstable regime and later it breaks down. The outline of the proof is to construct a curve in the unstable regime which is analytic except in a single point. We show that, as we evolve backward in time, the curve becomes analytic and is as close as we desired (in the H^k topology with k large enough) to the curve from part 3 of Section 2.

Here we will work in the periodic setting and will consider the equation

$$\partial_t z(\zeta,t) = \int_{w\in\Gamma_+(t)} \frac{\sin(z_1(\zeta,t) - z_1(w,t))}{\cosh(z_2(\zeta,t) - z_2(w,t)) - \cos(z_1(\zeta,t) - z_1(w,t))} \times (\partial_\zeta z(\zeta,t) - \partial_\zeta z(w,t))dw, \qquad \textbf{[17]}$$

where $\zeta \in \Omega(t)$,

$$\Omega(t) = \{\zeta \in \mathbb{C}/2k\pi: |\Im\zeta| < h(\Re z,t)\},$$

$h(x,t)$ is a positive periodic function with period 2π and smooth for fixed time t, and

$$\Gamma_\pm(t) = \{\zeta \in \mathbb{C}/2k\pi: \zeta = x + ih(x,t)\}.$$

This equation is equivalent to [**1**, **2**, and **4**] for holomorphic functions.

In order to prove the result, we will need the following theorem:

Theorem 4.1. *Let $h(x,t)$ be a positive, smooth, and periodic function with period 2π for fixed time $t \in [t_0 - \delta, t_0]$. Let $z(x,t_0)$ be a curve satisfying the following properties:*

- $z_1(x,t_0) - x$ and $z_2(x,t_0)$ are periodic with period 2π;
- $z(\zeta,t_0)$ is real for ζ real;
- $z(\zeta,t_0)$ is analytic in $\zeta \in \Omega(t_0)$;
- $z(\zeta,t) \in H^k(\Gamma_\pm(t_0))$ with k a large enough integer.
- Complex arc-chord condition:

$$|\cosh(z_2(\zeta,t_0) - z_2(w,t_0)) - \cos(z_1(\zeta,t_0) - z_1(w,t_0))| \geq [||\Re(\zeta - w)|| + |\Im(\zeta - w)|]^2,$$

for ζ, $w \in \overline{\Omega}(t_0)$, where $\|x\| = \text{distance}(x, 2k\pi)$.
- Generalized Rayleigh–Taylor condition: $\mathrm{RT}(\zeta,t_0) > 0$, where

$$\mathrm{RT}(\zeta,t) = \Re\left(\frac{-2\pi\partial_\zeta z_1(\zeta,t)}{(\partial_\zeta z_1(\zeta,t))^2 + (\partial_\zeta z_2(\zeta,t))^2}(1 + i\partial_x h(\Re\zeta,t))^{-1}\right. + \Im\left(\left\{\mathrm{PV}\int_{w\Gamma_+(t)}[\sin(z_1(\zeta,t) - z_1(w,t))]/[\cosh(z_2(\zeta,t) - z_2(w,t)) - \cos(z_1(\zeta,t) - z_1(w,t))]dw + i\partial_t h(\zeta,t)\right\}\right. \left. \times (1 + i\partial_x h(\Re\zeta,t))^{-1}\right).$$

Then, for small enough δ, there exists a solution for Eq. **17** *in the time interval $t \in [t_0 - \delta, t_0]$, satisfying*

- $z_1(x,t) - x$ and $z_2(x,t)$ are periodic with period 2π;
- $z(\zeta,t)$ is real for ζ real;
- $z(\zeta,t)$ is analytic in $\zeta \in \Omega(t_0)$;
- $z(\zeta,t) \in H^k(\Gamma_\pm(t))$ with k a large enough integer.

Now, let $\underline{z}(x,t)$ be the solution of the Muskat problem with $\underline{z}(x,0) = \underline{z}^0(x)$, where $\underline{z}^0(x)$ is the particular initial data from part 3 of the Section 2. We shall define this solution as the unperturbed solution. Let us denote the Rayleigh–Taylor function

$$\sigma_1^0(x,t) \equiv \frac{-2\pi\partial_x \underline{z}_1(x,t)}{(\partial_x\underline{z}_1(x,t))^2 + (\partial_x\underline{z}_2(x,t))^2}.$$

Notice the minus sign in the right-hand side of the previous expression. One can check the following properties of this Rayleigh–Taylor function:

1. $\sigma_1^0(\cdot,t)$ is analytic on $\{x + iy: x \in \mathbb{T}, |y| \leq c_b\}$ with $|\sigma_1^0(x+iy,t)| \leq C$, for all $x + iy$ as above and for all $t \leq [0,\tau]$;
2. $\sigma_1^0(0,0)$ is real for $x \in \mathbb{T}$, $t \in [0,\tau]$;
3. σ_1^0 has a priori bounded C^{k_0} norm as a function of $(x,t) \in \mathbb{T} \times [0,\tau]$ (k_0 large enough);
4. $\sigma_1^0(0,0) = 0$;
5. $\partial_x\sigma_1^0(0,0) = 0$;
6. $\partial_x^2\sigma_1^0(0,0) = -c_2 < 0$;
7. $\partial_t\sigma_1^0(0,0) = c_1 > 0$.

In this setting, we define the following weight functions

$$h(x,t) = A^{-1}(\tau^2 - t^2) + (A^{-1} - (\tau - t))\sin^2\left(\frac{x}{2}\right) \quad \text{for } t \in [\tau^2,\tau]. \qquad \textbf{[18]}$$

$$\hbar(x,t) = \frac{1}{4}\left(A^{-1}\tau^2 + A^{-1}\sin\left(\frac{x}{2}\right)\right) + A^{-2}\tau t + At\sin\left(\frac{x}{2}\right) \quad t \in [0,\tau^2], \qquad \textbf{[19]}$$

with $x \in \mathbb{T}$. First we choose the parameters A large enough and then τ small enough, then one can show that

$$\sigma_1^0(x,t) + \partial_t h(x,t) - A^{\frac{1}{2}}h(x,t) \geq c\tau^2 \quad \text{for } x \in \mathbb{T}, t \in [\tau^2,\tau] \qquad \textbf{[20]}$$

and

$$\sigma_1^0(x,t) + \partial_t \hbar(x,t) - A^{\frac{1}{2}}\hbar(x,t) \geq \frac{1}{2}A^{-2}\tau \quad \text{for } x \in \mathbb{T}, t \in [0,\tau^2]. \qquad \textbf{[21]}$$

The inequalities **20** and **21** are one of the main ingredients of the proof of the following results.

Theorem 4.2. *Let $z(x,t)$ be a solution of the Muskat equation in the interval $t \in [0,\tau]$. Let $h(x,t)$ and $\hbar(x,t)$ as in the expressions* **18** *and* **19**, *and k a large enough integer. Assume that $z(x,t)$ satisfies*

- $z_1(x,t) - x$ and $z_2(x,t)$ are periodic with period 2π;
- $z(\zeta,t)$ is real for ζ real;
- $z(\zeta,t)$ is analytic in $\zeta \in \Omega(t)$;
- $z(\zeta,t) \in H^k(\Gamma_\pm(t))$ with k a large enough integer.
- Complex arc-chord condition:

$$|\cosh(z_2(\zeta,t) - z_2(w,t)) - \cos(z_1(\zeta,t) - z_1(w,t))| \geq [\|\Re(\zeta - w)\| + |\Im(\zeta - w)|]^2,$$

for $\zeta, w \in \overline{\Omega}(t)$.

Here, in the definition of $\Omega(t)$ and $\Gamma_\pm(t)$, we use $h(x,t)$ if $t \in [\tau^2,\tau]$ and $\hbar(x,t)$ if $t \in [0,\tau^2]$. Then

$$\frac{1}{2}\frac{d}{dt}\left(\int_{w\in\Gamma_+(t)} |\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)|^2 d\Re\zeta\right) \geq -C(A)\lambda^2,$$

if $t \in [\tau^2,\tau]$

$$\int_{w\in\Gamma_+(t)} |\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)|^2 d\Re\zeta \leq \lambda^2$$

and $\lambda \leq \tau^{50}$.

In addition,

$$\frac{1}{2}\frac{d}{dt}\left(\int_{w\in\Gamma_+(t)} |\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)|^2 d\Re\zeta\right) \geq -C(A)\tau^{-1}\lambda^2,$$

if $t \in [0,\tau^2]$

$$\int_{w\in\Gamma_+(t)} |\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)|^2 d\Re\zeta \leq \lambda^2$$

and $\lambda \leq \tau^{50}$.

This theorem implies that for all $\gamma > 0$ there is $\varepsilon > 0$ such that

$$\int_{w\in\Gamma_+(t)} |\partial_\zeta^k z(\zeta,t) - \partial_\zeta^k \underline{z}(\zeta,t)|^2 d\Re\zeta \leq \gamma$$

for $t \in [0,\tau]$ if

$$\int_{w\in\Gamma_+(t)} |\partial_\zeta^k z(\zeta,\tau) - \partial_\zeta^k \underline{z}(\zeta,\tau)|^2 d\Re\zeta \leq \varepsilon$$

and $z(x,t)$ satisfies the requirements of the theorem.

Lemma 4.3. *Let $z(x,t)$ be a solution of the Muskat problem satisfying the requirements of Theorem 4.2 and close enough to the unperturbed solution in $t \in [0,\tau]$. Let $h(x,t)$ and $\hbar(x,t)$ be as in* ***[18]*** *and* ***[19]*** *with a suitable choice of A and τ. Then $z(x,t)$ satisfies the generalized Rayleigh–Taylor condition in $t \in [0,\tau]$. In particular, the unperturbed solution satisfies the generalized Rayleigh–Taylor condition in $t \in [0,\tau]$*

Theorems 4.1 and 4.2 and Lemma 4.3 allow us to achieve the desired result. Indeed we can choose a curve $z(x,\tau)$ such that

$$\int_{\zeta\in\Gamma_\pm} |\partial_\zeta^k z(\zeta,\tau) - \partial_\zeta^k \underline{z}(\zeta,\tau)|^2 d\Re\zeta \leq \varepsilon,$$

with $0 < \varepsilon < \varepsilon_0$ (ε_0 small enough), satisfying the generalized Rayleigh–Taylor condition by Lemma 4.3 and satisfying the rest of the hypothesis of Theorem 4.1. Because $h(0,\tau) = 0$, $z(x,t)$ is allowed to be nonanalytic at $x = 0$ [maybe $z(x,\tau) \in H^k(\mathbb{T})$ but $z(x,\tau) \notin H^{k+1}(\mathbb{T})$]. By Theorem 4.1, there is a solution $z(x,t)$, analytic in $\Omega(t)$, for some interval $t \in [\tau - \delta,\tau]$ with small enough δ and for all ε. By Theorem 4.2, we can choose ε small enough in such a way that, by Lemma 4.3, $z(x,\tau - \delta)$ satisfies the generalized Rayleigh–Taylor condition. Then we can go further the time $\tau - \delta$. Iterating this argument, we find we can extend $z(x,t)$ to be a solution of the Muskat problem, analytic in $\Omega(t)$ for all $t \in [0,\tau]$ and as close as we want to the unperturbed solution.

ACKNOWLEDGEMENTS A.C., D.C., and F.G. were partially supported by Grant MTM2008-03754 of the Ministerio de Ciencia e Innovación (MCINN) (Spain) and Grant StG-203138CDSIF of the European Research Council. C.F. was partially supported by National Science Foundation (NSF) Grant DMS-0901040 and Office of Naval Research Grant ONR00014-08-1-0678. F.G. was partially supported by NSF Grant DMS-0901810. M.L.-F. was partially supported by Grants MTM2008-03541 and MTM2010-19510 of the MCINN (Spain).

1. Muskat M (1934) Two fluid systems in porous media. The encroachment of water into an oil sand. *Physics* 5:250–264.
2. Saffman P-G, Taylor G (1958) The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc R Soc London Ser A* 245:312–329.
3. Howison S (2000) A note on the two-phase Hele-Shaw problem. *J Fluid Mech* 409:243–249.
4. Hele-Shaw HS (1898) On the motion of a viscous fluid between two parallel plates. *Trans Royal Inst Nav Archit, London* 40:21.
5. Bardos C, Lannes D (2010) Mathematics for 2D interfaces. ArXiv:1005.5329.
6. Rayleigh J-W-S (1878) On the instability of jets. *Proc Lond Math Soc* 10:4–13.
7. Siegel M, Caflisch R, Howison S (2004) Global existence, singular solutions, and ill-posedness for the muskat problem. *Commun Pur Appl Math* 57:1374–1411.
8. Yi F (1996) Local classical solution of Muskat free boundary problem. *J Partial Diff Eqs* 9:84–96.
9. Córdoba A, Córdoba D, Gancedo F (2011) Interface evolution: The Hele-Shaw and Muskat problems. *Ann Math* 173:477–542.
10. Ambrose D (2004) Well-posedness of two-phase Hele-Shaw flow without surface tension. *Eur J Appl Math* 15:597–607.
11. Córdoba D, Gancedo F (2007) Contour dynamics of incompressible 3-D fluids in a porous medium with different densities. *Commun Math Phys* 273:445–471.
12. Constantin P, Pugh M (1993) Global solutions for small data to the Hele-Shaw problem. *Nonlinearity* 6:393–415.
13. Yi F (2003) Global classical solution of Muskat free boundary problem. *J Math Anal Appl* 288:442–461.
14. Escher J, Matioc B-V (2010) On the parabolicity of the Muskat problem: Well-posedness, fingering, and stability results. ArXiv:1005.2512.
15. Córdoba D, Gancedo F (2009) A maximum principle for the Muskat problem for fluids with different densities. *Comm Math Phys* 286:681–696.
16. Constantin P, Córdoba D, Gancedo F, Strain R-M (2010) On the global existence for the for the Muskat problem. ArXiv:1007.3744v1.
17. Córdoba D, Gancedo F, Orive R (2008) A note on the interface dynamics for convection in porous media. *Physica D* 237:1488–1497.
18. Córdoba A, Córdoba D, Gancedo F (2010) Interface evolution: Water wave in 2D. *Adv Math* 223:120–173.
19. Wu S (1997) Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent math* 130:39–72.
20. Lannes D (2005) Well-posedness of the water-waves equations. *J Am Math Soc* 18:605–654.
21. Wu S (2009) Almost global wellposedness of the 2-D full water wave problem. *Invent Math* 177:45135.
22. Germain P, Masmoudi N, Shatah J (2009) Global solutions for the gravity water waves equation in dimension 3. *C R Math Acad Sci* 347:897–902.
23. Wu S (2009) Global well-posedness of the 3-D full water wave problem. ArXiv:0910.2473.
24. Nirenberg L (1972) An abstract form of the nonlinear Cauchy-Kowalewski theorem. *J Differ Geom* 6:561–576.
25. Nishida T (1977) A note on a theorem of Nirenberg. *J Differ Geom* 12:629–633.
26. Sulem C, Sulem P-L, Bardos C, Frisch U (1981) Finite time analyticity for the two- and three-dimensional Kelvin-Helmholtz instability. *Commun Math Phys* 80:485–516.