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A HISTORICAL REVIEW OF THE CLASSIFICATIONS OF LIE ALGEBRAS

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Abstract. The problem of Lie algebras' classification, in their different varieties, has been dealt with by theory researchers since the early $20^{\rm th}$ century. This problem has an intrinsically infinite nature since it can be inferred from the results obtained that there are features specific to each field and dimension. Despite the hundreds of attempts published, there are currently fields and dimensions in which only partial classifications of some families of algebras of low dimensions have been obtained. This article intends to bring some order to the achievements of this prolific line of research so far, in order to facilitate future research.

1. Introduction

About 1870, the Norwegian mathematician Sophus Marius Lie (1842–1899) began to study some types of geometric transformations that promised to have considerable relevance in the subsequent study of symmetries. No one could have imagined at that moment the impact of his discoveries: the gestation of key tools for the development of Modern Physics and, in particular, the Theory of Relativity. We can currently state that Lie Theory, regarding Lie groups as well as Lie algebras, has proved to be the key to solving many problems related to Geometry and to Differential Equations, which links theoretical Mathematics to the real world.

Owing precisely to their tangible applications, scientists of different disciplines have used specific examples of Lie algebras over different fields and in different dimensions, everyone according to their needs. However, mathematicians are usually more interested in generality than in obtaining a few examples. That is why it was reasonable to expect the immediate interest aroused in the mathematical community by the apparent complexity and the amazing elegance of the new algebraic structures. As shown throughout this document, hundreds of publications attest the attempts made so far to categorize these objects. However, not only has a general classification of Lie algebras not been reached, but the objective has been achieved in very few families of algebras for any dimension and field. The different strategies followed by each author have fallen by the wayside, as well as

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partial classifications given by each research group, some of them wrong and corrected in subsequent articles, or discoveries used to make classification easier. We especially point out here the search for invariants, in most cases designed to enable a specific classification but proved useful later for other reasons, which provided a real advance of scientific knowledge. Many studies have been written, in the field of Lie algebras, on analysis or representation of properties. However, specific attention will be given in this article to the issue of classifications because it would be impossible to collect in a reasonable space all the publications related to Lie Theory.

Even if we limit ourselves to classifications, in order to enable the subsequent presentation of the different research lines and the achievements so far, we must first clarify some key concepts. The first one is actually that of classification. In Mathematics, to classify is to find a certain property allowing the definition of an equivalence relation among the elements of a set so that this set will be divided into disjoint classes. However, it is interesting that on finishing a classification we are able to choose an element as representative of each equivalence class. Furthermore, in the ideal situation it would also be useful to be able to assign in a simple way each element to its corresponding class.

Regarding the elements we want to classify, we must not forget that every Lie algebra is a vector space upon which a new binary operator is defined. This means that in each case a field must be set over which a vector space is defined and, also, that a particular dimension can be chosen for the vector space. In the case of Lie algebras, for the reasons already suggested, additional properties distinguishing between those algebras that verify them and those which do not are often considered. Every time such a property is described there is a chance to classify the Lie algebras that verify it, also in every field and dimension, as happened in the general case. Sometimes, these properties are also useful to classify wide families of algebras; this fact supports their importance. However, the classifications generated from the consideration of different properties are not compatible in general. This fact makes the following description difficult and intricate sometimes.

Finally, we must make another short comment about the techniques used to classify. Although it is not the purpose of this work to describe all the possible strategies for classifying Lie algebras, we consider it essential to recognize the role played by computing in this field. Of course, the first classifications were not made with the help of computers, what considerably reduced the mathematicians' chances and significantly increased the likelihood of making mistakes. Taking into account that the increase in dimension involves a considerably larger increment in the complexity of the operations performed, it is logical to admit that computing power, as well as the development of increasingly versatile and effective symbolic computation packages, make possible studies that were unfeasible years ago. Therefore, on these and other grounds, progress in the classifications of Lie algebras is not only determined by the consideration of new families of algebras but also because over the course of time the same problems can be tackled but in upper dimensions.

This article consists of four parts, apart from this introduction. The primary classification development is described in the next section. Afterwards, we describe the partial classifications comprising subsets of the whole collection of algebras to be studied. Such works are completed with other classifications partially related to the traditional Lie algebras. Finally, we present some brief concluding remarks.

2. Heart of the classifications

Classifying all Lie algebras of dimension less than 4 is an elementary exercise. However, when considering dimension 4, complete classifications are much harder, and subsequent classifications usually refer to subclasses. As it is already well known, there exist three different types of Lie algebras: the *semi-simple*, the *solvable*, and those which are neither semi-simple nor solvable. So, determining the classification of Lie algebras, in general, is equivalent to revealing the classification of each of these three types. However, by the Levi–Maltsev Theorem, which is the combination of the results formulated firstly by Levi [98] in 1905, and later by Maltsev [107] (note that it can also be written Malcev) in 1945: any finite-dimensional Lie algebra over a field of characteristic zero can be expressed as a semidirect sum (the *Levi-Maltsev decomposition*) of a semi-simple subalgebra (called the *Levi factor*) and its radical (its maximal solvable ideal). It reduces the task of classifying all Lie algebras to obtaining the classification of semi-simple and of solvable Lie algebras. Nevertheless, as we will see later, this procedure is not entirely valid when dealing with fields of a given positive characteristic.

Let us consider first the standard case, over the complex and the real fields. With respect to the first problem, the classification of semi-simple Lie algebras was completely solved by the well-known Cartan Theorem: any semi-simple complex or real Lie algebra can be decomposed into a direct sum of ideals which are simple subalgebras being mutually orthogonal with respect to the Cartan-Killing form. So, the problem of classifying semi-simple Lie algebras is then equivalent to that of classifying all non-isomorphic simple Lie algebras; and the classification of simple Lie algebras was already obtained by Killing, Cartan, and others in the last decade of the 19^{th} century [44]. Hence, it can be admitted that the problem of the classification of semi-simple Lie algebras is at present totally solved. Indeed, mainly Killing and Cartan, although other authors also worked in this subject, classified simple Lie algebras in five different classes (the so-called *simple classical Lie algebras*): the algebras belonging to the *linear special group*, those *odd orthogonal* algebras, the *even orthogonal* algebras, the *symplectic* algebras, plus five Lie algebras having no relation among them and not belonging to any of the previous classes, which were called by the authors *exceptional* or *exotic* Lie algebras.

With respect to the classification of solvable Lie algebras, in spite of the first attempts by Lie [99, 100] and Bianchi [27], it can be said that Dozias was one of the first authors who faced that problem seriously, in 1963: he classified in his Ph.D. thesis the solvable Lie algebras of dimensions less than 6 over the field of the real numbers [57]. In this same year Mubarakzjanov (see [111, 112, 113] and [119], too) also classified these algebras up to dimension 6 over the field of real numbers.

Some decades later, in 1990, Patera and Zassenhaus obtained the classification of solvable Lie algebras up to dimension 4 over any perfect field [120], and in that same year Turkowski [161] dealt with solvable Lie algebras of dimension 6, including the classification of the Levi decomposable algebras, which are semidirect sum of semisimple and solvable Lie algebras (see [160, 162]). De Graaf [52], for his part, illustrated in 2005 some ideas which can be used for obtaining a classification of small-dimensional solvable Lie algebras and got the classification of such algebras of 3 and 4 dimensions over fields of any characteristic, by using the computer algebra system Magma and the Gröbner bases as a tool. This author showed that in dimension 4 his classification differs slightly from the one found in [120], since he found a few more Lie algebras. However, de Graaf's techniques are not applicable for a complete classification for dimension 5, and the problem of classifying all solvable Lie algebras is at present unsolved, and many authors think that completely new ideas are necessary to move forward, since the number of infinite parametrized families of non-isomorphic algebras increases drastically with the dimension (see [145]).

Of course, there are other ways to face the problem. For instance, even in the general case of all Lie algebras, for small dimensions, it is possible to consider specific kinds of isomorphisms to classify. So, Morozov [110] and Vranceneau [167] classified Lie algebras over $\mathbb{C}$ and of dimension less than or equal to 6 and Dixmier [56] gave the classification over $\mathbb{R}$ for such dimensions. Nevertheless, it is usually more effective any attempt through the consideration of the subsets described in the following section. Anyway, before that, it is worthwhile emphasizing that different approaches can be used to reach the same goal or even to present the results; this makes the comparison of classifications a hard task.

As a way of example, let us mention that higher order generalizations of Lie algebras have been conceived as Lie n-algebras, as L_∞-algebras, or as quasi-free differential graded commutative algebras. In this sense, the same well-known classification of complex 3-dimensional Lie algebras was given in [117], but the isomorphism classes were this time combined into families (*strata*) and ordinary Lie algebras were interpreted as L_∞-algebras; since then, a stratification of the moduli space of 3-dimensional Lie algebras is connected with families of Lie algebras split in two classes: *jump deformations* and *smooth deformations*. This property may be used to ease the classifications in future, but its first effect is the introduction of a non-comparable terminology and the generation of new challenge. We are about to see that this situation is common to many other properties and concepts which will be discussed from now on.

3. Classic subtypes

In the following subsections, we present an overview on some relevant subclasses of the total set of Lie algebras. Due to their historical importance, the nilpotent Lie algebras (and some subtypes) deserve special attention, but there are some other classes which should be named within this section. Here we only skretch out some concepts and the actual status of their classifications. Later we will be more explicit regarding historical matters and methods. Note that it is not always possible the description of a clear timeline, above all, because the number of Lie theorists has increased exponentially and, nowadays, their research take place simultaneously all around the globe.

3.1. **Classical and non-solvable Lie algebras.** The first step in the classification of Lie algebras was the consideration of classical Lie algebras. They were totally outlined over $\mathbb{C}$ and $\mathbb{R}$, but the problem is not so easy over an algebraically closed field of characteristic different from zero. This "new" Lie theory emerged around 1935 from the studies by Witt, who defined a simple Lie algebra (now called the *Witt algebra* W_1) whose behavior was totally different from the Lie algebras studied till then, over $\mathbb{C}$ or $\mathbb{R}$. Less than ten years later, Jacobson [87] and Zassenhaus [171] put some order in these new algebras, but it was not until the 21st century when a clear classification came. In fact, [123] is a survey on these specific classifications of simple finite-dimensional Lie algebras over algebraically closed fields of characteristic $p > 0$. Roughly speaking, and being p a prime greater than 3, the simple Lie algebras are either *classical* or *finite dimensional Cartan Lie type* (and their deformations) or *Melikyian algebras.* If the characteristic is big enough, some other interesting properties hold; for instance, for finite-dimensional Lie algebras over an algebraically closed field of characteristic $p > 7$, the existence of non-singular Casimir operators (i.e., dealing with a restricted Lie algebra) is equivalent to the decomposition of the algebra as a direct sum of classical simple Lie algebras (see [60]). Some other interesting papers describe the Kostrikin-Shafarevich conjecture and simple Lie algebras over an algebraically closed field, also those of type A_l, B_l, C_l, and D_l ([28, 88, 97, 108, 154]); we will see later the generalization of these algebras to the Kac-Moody type.

Regarding these Lie algebras over finite fields, the state of the art was updated by Premet and Strade [124] in 2007, establishing the complete classification when the characteristic is (a prime number) greater than 3: for $p \geq 5$, any finite-dimensional simple Lie algebra is of classical, Cartan, or Melikyan type (see [151, 153]). However, the classifications in characteristics 2 and 3 are still incomplete and with remarkable computational problems to test isomorphisms (see [53]). In some sense, they encourage the mathematicians' efforts. Recently, Vaughan-Lee [164] and Eick [59] have classified all simple Lie algebras up to dimension 9 over the field with two elements ($\mathbb{Z}/2\mathbb{Z}$).

In the knowledge edge, lots of attempts are directed to get original ideas to understand the old structures whereas others are looking for new formal entities to which one can reproduce the known properties. So, some authors have tried the inclusion of a new operation in the Lie algebra, sometimes expecting more chances to obtain a proper classification. Although we will deal with this topic later, a specific example is constituted by the restricted Lie algebras. The classification of simple restricted Lie algebras was obtained in [28] and the one of simple (not necessarily restricted) Lie algebras of characteristic $p > 7$ in [150].

Another strategy could be to take into account different divisions in the total set of Lie algebras, and to classify one of these subsets. This is the case of the *irreducible quasi-simple Lie algebras* (or *extended affine Lie algebras*). Basically, the extended affine Lie algebras are simple Lie algebras or affine Lie algebras or toroidal Lie algebras. The classification of these algebras was given in Allison et al. [3] in 1997.

The researchers can also consider known types of algebras and analyzing them in depth or combining them with other types to classify the elements belonging to all of them simultaneously. In the case of *symplectic Lie algebras* (one of the five simple types), which admit affine structures, Ovando [118] gave the classification of real 4-dimensional symplectic Lie algebras, completing the result from [109]. On the other hand, when considering this type together with others, some more recent classifications emerged. The classification of (complex) symplectic characteristically nilpotent Lie algebras was given by Burde in [36] up to dimension 10. And Campoamor determined in [40] all the six-dimensional real solvable and non-nilpotent Lie algebras endowed with a symplectic form that decompose as the direct sum of two ideals or are indecomposable solvable algebras with a four-dimensional *nilradical* (i.e., its maximal nilpotent ideal). And as a new result, the set of all *semisimple involutory Lie algebras* was classified in [82].

Given a Lie algebra, the centralizer of a subalgebra is constituted by the elements in the algebra which commute with all the elements in the subalgebra. [94] classified all finite dimensional Lie algebras over an algebraically closed field of characteristic 0, whose nonzero elements have Abelian centralizers. This type of algebras is called *commutative transitive*, and they are neither simple nor solvable.

Despite their theoretical (and sometimes practical, too) interest, these above classifications did not produce significant progress in the traditional ones. Moreover, the appearance of these non-classical types of algebras generates new challenges when studying algebras closely related to them. For instance, *quasiclassical Lie algebras* are finite-dimensional non-semisimple Lie algebras which can be obtained as Lie algebras of skew-symmetric elements of associative algebras with involution. Indecomposable quasi-classical Lie algebras of dimension $n \leq 9$ which have a non-trivial Levi subalgebra were obtained in [39]. Additionally, solvable indecomposable quasi-classical Lie algebras of dimension $n \leq 6$ were also classified in [39]. Obviously, this point is not a finishing line, but a crossroad where new possibilities emerge.

Partially due to the difficulties of obtaining a complete classification of solvable Lie algebras, some authors considered the idea of a classification of solvable extensions of certain classes of Lie algebras (we will come back to the extensions later). In particular, the most relevant were the analysis of all solvable, non-nilpotent algebras with a given nilradical. So, Rubin and Winternitz started a research line (see [130, 114, 157, 147, 148, 146]) which has been dealing with the classification of solvable Lie algebras with a given nilradical, such as the *Abelian algebras* (also called *1-step* as we will deduce later), the *Heisenberg algebras*, the *algebras of strictly upper triangular matrices*, and so on (for arbitrary finite dimensions).

The traditional way of facing the problem is not finished, either. In these last years, many specialists have been working on the classifications and the ideas of those classical types. So, Turkowski [162] gave in 1992 the classification of 9-dimensional real Lie algebras neither being semi-simple nor solvable and not decomposable as a direct sum of algebras of lower dimension (this classification was completed by Campoamor in [38]). Nowadays, there exists the opportunity of enriching the traditional ideas with the newest. So, in 2007, Strade classified [152] all non-solvable Lie algebras up to dimension 6 over any finite field by using methods from the classification of finite-dimensional simple Lie algebras over algebraically closed fields of prime characteristic.

Finally, there is still the main open problem: the classification of the most significant part of the whole set of Lie algebras. The following statement, quite recent, may provide us with a clue: The classification of non-degenerate solvable Lie algebras of maximal rank (over a field of characteristic zero) can be reduced to classifying a nilpotent Lie algebra of maximal rank (see [172]), and this latter classification was given by Santharoubane [133]. Next we face the core of this issue.

3.2. **General nilpotent Lie algebras.** The previously cited Malcev [107] had already reduced in 1945 the classification of complex solvable Lie algebras to the classification of one subset, the *nilpotent Lie algebras*. To do it, Malcev defined a particular type of algebra, that he called *splittable algebra*, whose structure is completely determined from its maximal nilpotent ideal and proved that an arbitrary solvable Lie algebra is contained in a unique minimal splittable algebra. The relation between an algebra and its splittings led him to the construction of all solvable Lie algebras with a given splitting. So, in this way, the classification of all solvable Lie algebras had been reduced to the classification of the nilpotent Lie algebras.

With respect to the classification of nilpotent Lie algebras, many attempts have been made on this topic, and lots of lists of algebras have been published with bigger or less fortune. The earliest one was given by an Engel's pupil, Umlauf, who classified in 1891 all the nilpotent Lie algebras up to dimension 6 over the field of complex numbers [163]. He observed that there only exists a finite number of them, although when studying dimensions 7, 8, and 9 he found that they appeared some infinite families of such algebras. Long after Umlauf, several authors dealt with this type of algebras, trying to get its classification. So, in the decade of the fifties in the 20$^{\text{th}}$ century, Vranceanu [167], in 1950, and Chevalley, Morosov, and Dixmier, in 1957, can be cited. In particular, Dixmier [56] classified in 1957 the nilpotent Lie algebras over a commutative field up to dimension 5, and in this same year Morosov [110], by using the method of nilpotent elements of semi-simple Lie algebras, previously introduced by himself around 1940, classified the nilpotent Lie algebras over the field of complex numbers up to dimension 6, and made a first attempt to classify, also, the nilpotent Lie algebras of such a dimension over the field of real numbers (and any other field of characteristic zero). Their classifications depend heavily on the property that a nilpotent Lie algebra of dimension n contains a maximal Abelian ideal of dimension greater than $\frac{1}{2}\left((8n+1)^{\frac{1}{2}}-1\right)$.

Vergne and Safiullina are mainly who made in the decade of the sixties of the 20$^{\text{th}}$ century the biggest efforts in the search of the classification of these nilpotent algebras, getting notable results in many cases. So, Safiullina, in 1964, used the previously cited Morozov's approach to supply new results related to the classification of complex nilpotent Lie algebras of dimension 7 (see [131]), although it was really Vergne, in 1966, who gave a great impulse to the study of these algebras in her Ph.D. thesis [165], later published in [166] (we will come back to this work later). Indeed, Vergne not only obtained the complete classifications of nilpotent Lie algebras of dimension less than 7 over the real and complex fields, but introduced the filiform Lie algebras, which constitute the most structured subclass into the nilpotent Lie algebras. In fact, Vergne showed that the variety of nilpotent Lie algebras of a given dimension n has an irreducible component of dimension exceeding n^2 consisting entirely of Lie algebras of maximal class. Thus Lie algebras of maximal class are rather abundant and may not be easily classified. Apart from the group Bourbaki [30], other authors who also dealt in this decade with nilpotent Lie algebras were Chong Yun Chao and Shedler. The first one proved that there exist uncountably many non-isomorphic nilpotent real Lie algebras for any given dimension $n \geq 10$ and that there exist uncountably many non-isomorphic solvable and non-nilpotent real Lie algebras for any given dimension $n \geq 11$ (see [47]). On the other hand, Shedler, in his Ph.D. thesis [140], reached a classification of nilpotent Lie algebras of dimension 6 over any field.

Continuing with the 20$^{\text{th}}$ century, there are also several authors in the decade of seventies that dealt with the classification of the nilpotent Lie algebras. Among them, we highlight Favre [61] and Gauger [72] in 1973, de la Harpe [55] in 1974, and Skjelbred and Sund in 1977. Specifically, Favre introduced the concept of *system of weights* of a Lie algebra and used it to study the class of nilpotent Lie algebras for which such a concept is particularly well adapted. For his part, Gauger gave a method to study nilpotent Lie algebras as quotiens of free nilpotent Lie algebras. Skjelbred and Sund, by their own, classified real nilpotent six-dimensional Lie algebras in [144], and later, in [143], improved their results by constructing all nilpotent Lie algebras of dimension n given those algebras of dimension $n - 1$ and their automorphism groups. However, it is not until the next decade when it occurred another great advance in the study of the classification of nilpotent Lie algebras. So, in 1981, Beck and Kolman [22] gave the classification of nilpotent Lie algebras of dimension 6 over the field of the real numbers; later, in 1983, Cerezo [45] classified both real and complex nilpotent Lie algebras of dimension 6, and in this same year Nielsen [115] compared the classifications of Morozov, Vergne, Skjelbred and Sund, and Umlauf, and gave for the first time a complete and non-redundant list for nilpotent Lie algebras of dimension 6 over the field of the real numbers.

In 1986, Magnin [105] studied the classification of real nilpotent Lie algebras of dimension 7 and two years after Romdhani [129] also got the classification of complex and real nilpotent Lie algebras of the same dimension. To do it, he only used the elementary techniques of Linear Algebra, and excluded all algebras which decompose into a direct sum of lesser-dimensional factors.

In 1989 Ancochea and Goze [10], by using a new invariant of filiform Lie algebras introduced by themselves, namely the *characteristic sequence*, got an incomplete classification of the nilpotent Lie algebras of dimension 7. It is convenient to note that Seeley [137] also classified the complex nilpotent Lie algebras of dimension 6 and 7, following a previous similar project in dimension 5 by Grunewald and O'Halloran [81], and by using the Iwasawa decomposition for $GL_6(\mathcal{C})$. Some years later, Goze and Remm [80] would rectify the classification given in [10] (another verification of these lists has been published in [106]).

In 2007, de Graaf [53] retook the comparison of all the classifications of the nilpotent 6-dimensional Lie algebras. Note that Morozov [110] and other authors gave their classifications over a field of characteristic zero, whilst de Graaf analyzed them over any field of characteristic different from 2. In fact, when considering finite fields, the evolution of these classifications appreciably differs from the described above. They are universally accepted up to dimension 5, and Schneider [136] classified all nilpotent Lie algebras of dimension 6 and 7 over the finite fields with characteristic 2, 3, and 5, by using the software GAP. Additionally, he gave all nilpotent Lie algebras of dimension 8 and 9 over the field with two elements ($\mathbb{Z}/2\mathbb{Z}$). Finally, as we commented before, and by using Gröbner bases, de Graaf [53] obtained his constructive classification for dimension 6.

It can be of interest noting that all the previous lists of nilpotent Lie algebras were obtained by using different invariants. By introducing a new invariant, Carles [43] compared in 1989 the list of Safiullina, Romdhani, and Seely, and observed some mistakes in all of them. Later on, in 1993, based on his own Ph.D. thesis and on these results by Carles, Seeley [138] published his list over complex field. Other contributions to this subject were made by Tsagas [158], who classified in 1999 the real nilpotent Lie algebras of dimension 8, and this same author, together with Kobotis and Koukouvinos [159], who dealt with those real nilpotent Lie algebras of dimension 9 having a maximum Abelian ideal of dimension 7. Since then, the classifications of both real or complex nilpotent Lie algebras of dimension greater and equal to 8 have not been obtained, in spite of the frequent attempts made by several authors to get them. So the problem is unsolved at present and it has made that other subjects related with these algebras are being now considered. To that respect, it is quite significative the opinion of some experts, like Shalev and Zelmanov [139], who think that it is already not possible to obtain in an explicit way the classification of nilpotent Lie algebras of bigger dimensions. Anyway, it is really true that in order to continue studying the problem of classifying Lie algebras, in general, it is totally necessary to use the computer and a systematic approach; logically, the more the computer is involved, the more systematic the methods have to be.

3.3. **Some specific nilpotent Lie algebras.** A lot of progress has been made in the study of other classifications concerning some particular properties of nilpotent Lie algebras. For instance, if we refer to *filiform Lie algebras* over a field of complex numbers, we have to cite several works following the seminal paper by Vergne [165]. Firstly, Ancochea and Goze [9] already reached in 1988 the classification of complex filiform Lie algebras of dimension 8. However, this list of algebras was incomplete and the authors rectified it themselves in 1992 [11], almost at the same time as

Seeley [137], although in an independent way. In 1991, Gómez classified those of dimension 9 in his Ph.D. thesis, later published by Echarte (his adviser) and himself in [58], and Boza, Echarte, and Núñez [31] those of dimension 10 in 1994. Later, Gómez, Jiménez, and Khakimdjanov [75], in 1996, gave the classification of these algebras on dimension 11 as well as some corrections of previous classifications. Benjumea et al. [25] also tried to tackle bigger dimensions by obtaining in 1996 the laws of the filiform Lie algebras of dimension 13 and 14. Regarding the explicit lists, Boza, Fedriani, and Núñez [32] got in 1998 the classification of complex filiform Lie algebras of dimension 12, and also showed in 2003 ([34]) an explicit classification of such algebras of dimension 11, in a different way as the one used in [75]. Although [33] supplied a method valid for every dimension over the complex field, the computations that involves are too hard to be nowadays tacked successfully.

As it was explained before for other classes, some subsets of the filiform Lie algebras have also been studied. In this sense, as a recent illustration, Bosko computed in [29] all the filiform Lie algebras (called in the paper as of maximal class) for the Schur multiplier $t(\mathfrak{g}) \leq 16$, in both the complex and real fields. This type of algebras does not exist for dimension 14, 15, and 16. Remember that in mathematical group theory, the Schur multiplier (or Schur multiplicator) is the second homology group $H_2(G, \mathbb{Z})$ of a group G. And notice that the Schur multiplier of the Lie algebra has also been used in [116] to classify nilpotent Lie algebras in general.

We would like to cite here some authors who have got partial classifications concerning other particular properties of nilpotent Lie algebras. Let us start with a special subset within the nilpotent class, the constituted by the *metabelians*, Lie algebras whose nilpotency index is $p = 2$. These algebras are also called *2-step*, because its nilpotency sequence is zero for $\mathfrak{g}^2$ (as $\mathfrak{g}^1 = \mathfrak{g}$, the proper algebra, this means that its derived algebra is null). According to this agreement, the filiform Lie algebras of dimension 2 are also 2-step algebras (and on dimension n, the filiform Lie algebras are n-step). Summarizing, Scheuneman [135], Gauger [72], Revoy [127], and the previously cited Fernández and Núñez classified the 2-step nilpotent Lie algebras in 1967, 1973, 1980, and 2001, respectively. The more recent classification of 2-step nilpotent Lie algebras (over $\mathbb{C}$) is obtained in [126]: there, the five non-isomorphic algebras are indecomposable. Moreover, the classification of low-dimensional (metabelian) nilpotent Lie algebras, over arbitrary fields (including the characteristic 2 case) can be found in [155]. Finally, the classification of metabelian Lie algebras $\mathfrak{g}$ of type FP_m, under the additional assumption that $\mathfrak{g}$ is a split extension of Abelian Lie algebras, was completed in [95]. With respect to the complex $(n-3)$-filiform Lie algebras, they were classified in [37].

Secondly, let us pay attention to those researches dealing with *characteristically nilpotent Lie algebras*, which were introduced by Dixmier and Lister in 1957 [56], as the nilpotent Lie algebras whose derivations are all nilpotent. Some characteristically nilpotent algebras are also metabelian. Besides, all the filiform algebras non-derived from a solvable are also characteristically nilpotent. But there are characteristically nilpotent algebras which are neither metabelian nor filiform. A very interesting monograph on these algebras, where its historical evolution was commented, can be consulted in [4]. A particularly relevant algebra within this case is the so-called algebra of Bratzlavsky, which appeared in dimension 6.

Let us consider now the central commutator ideal as a way to classify nilpotent algebras. Gauger [72] classified nilpotent Lie algebras of 2-dimensional central commutator ideals over an algebraically closed field of characteristic different from 2. In [21], this classification was extended to the case of any arbitrary field K. Furthermore, when K was real closed, a list with all these algebras up to dimension 11 was given. On the other hand, the classification of nilpotent Lie algebras with 2-dimensional non-central commutator ideals over any field has been published in [21]. However, the problem of classifying Lie algebras over a field of characteristic different from 2 with central commutator subalgebra of dimension 3 is "hopeless", also called "wild" (see [23]).

Delimited by the multipliers, all nilpotent Lie algebras $\mathfrak{g}$ such that $t(\mathfrak{g}) \leq 6$ was classified in [84]. Then, a classification for $t(\mathfrak{g}) = 7, 8$ was given in [83].

Kath determined in [92] the *admissible nilpotent Lie algebras*, i.e., all the nilpotent Lie algebras $\mathfrak{g}$ with $\dim[\mathfrak{g}, \mathfrak{g}] = 2$. In [170], the studied algebras were those $\mathfrak{g}$ nilpotent such that $\dim(\mathfrak{g}'/\mathfrak{g}'') = 3$ and $\mathfrak{g}'' \neq 0$ (where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}'' = [\mathfrak{g}', \mathfrak{g}']$). There, $\mathfrak{g}$ was decomposed as the central direct sum of ideals H and U. The classification of U is easily seen to be given by the direct sum of a generalized Heisenberg Lie algebra and an Abelian Lie algebra, and the classification of H over the complex field was given in the paper.

Other research groups have studied similar topics, but considering *solvable extensions of nilpotent Lie algebras*. They are solvable Lie algebras with a specific n-dimensional nilradical $\mathbb{N}_{n-2}$ (of degree of nilpotency $n-1$ and with an $(n-2)$-dimensional maximal Abelian ideal). Given n, such a solvable algebra is unique up to isomorphisms. Another way of facing these extensions is provided by the graded Lie algebras. There are other possibilities of incorporating a new operation to the algebra, but this issue is pertinent here because it can be used to classify the algebras through their nilradicals, as we are about to see.

A *graded Lie algebra* (or *color Lie algebra*) is a Lie algebra endowed with a gradation which is compatible with the Lie bracket, this is, a Lie algebra which is also a non-associative graded algebra under the bracket operation. For example, a choice of Cartan decomposition endows any semisimple Lie algebra with the structure of a graded Lie algebra, and any parabolic Lie algebra is also a graded Lie algebra. Graded Lie algebras became a topic of interest in Physics in the context of "supersymmetries", while in Mathematics they have their main place in the context of deformation theory. In the field of Lie Theory, graded algebras came hand in hand with the analysis of irreducible component (a very interesting issue, but out of the scope of this paper).

The first systematic results on naturally graded nilpotent Lie algebras were indeed obtained when studying the variety of nilpotent Lie algebra laws. Then they eased the first estimation on the number of irreducible components of the variety with the introduction of filiform Lie algebras in [166]. There exist only two naturally graded models with the maximal nilpotence index, namely L_n and Q_n (and the latter only in even dimensions). This is, every n-dimensional naturally graded filiform Lie algebra is isomorphic to L_n or Q_n (see [166]). By the way, the non-existence of Q_n in odd dimensions gave even more theoretical attractiveness (see [5]) to the topic, although the publications here have only developed already in the 21$^{\text{st}}$ century. The only selected attempt on this issue during the last century was due to Goze and Khakimdjanov, who gave in [79] the complete classification of graded (not necessarily naturally graded) filiform Lie algebras. On the other hand, the naturally graded complex nilpotent Lie algebras of characteristic sequence $(n-2,1,1)$ (also called naturally graded quasi-filiform, where n is the dimension of the algebra) were classified in [70] (correcting the classification previously published in [74]). A complete classification of the quasifiliform Lie algebras admitting a nonzero diagonalizable derivation was presented; however, some pathological algebras were missing in those classifications and being found in [7].

The whole class of complex Lie algebras having a naturally graded nilradical with characteristic sequence $(n-2,1,1)$ has been recently classified in [7]. Note that not all these algebras are solvable. The solvable Lie algebras having either L_n or Q_n as nilradical were classified in [147, 5]. And the solvable Lie algebras having a nilradical of the type $L_n \oplus C$ (which corresponds to one of the two possible decomposable quasi-filiform naturally graded algebras) were classified in [169], following a similar procedure to that of [147]. Wang et al. [169] compute all finite-dimensional indecomposable complex solvable Lie algebras having $N_{n,1} \oplus \mathbb{C}$ as nilradical. Note that this nilpotent Lie algebra is naturally graded and also quasifiliform. For his part, Campoamor, a very active author in this field during these last years, classified in [41] the solvable Lie algebras (both in the complex and real fields) with an $\mathbb{N}$-graded nilradical of maximal nilpotency index. And in [7], the authors have classified the solvable Lie algebras with an indecomposable radical of arbitrary dimension, starting from their derivation algebra. Then, they classify the solvable Lie algebras having a decomposable quasi-filiform naturally graded nilradical using the classification of indecomposable nilradical.

According to the accepted notation, a quasi-filiform Lie algebra is a Lie algebra of dimension n and of nilpotency class $n-2$. Completing previous results, [71] classified all naturally graded and even-dimensional quasi-filiform Lie algebras. On the other hand, the classification of solvable Lie algebras having filiform (i.e., of maximal nilindex) naturally graded Lie algebras was started in [147] and completed in [5]. A complete classification of all solvable Lie algebras with n-dimensional nilradical which is nilpotent of nilindex $n-3$ was given in [146]. For real solvable Lie algebras with nilradical Q_{2k}, a complete classification was reached in [5]. Other authors dealt with classes like naturally graded nilradicals with maximal nilindex and a Heisenberg subalgebra of codimension 1 [5], a specific sequence of quasi-filiform nilradicals [169], or all solvable extensions of $\mathbb{N}$-graded filiform nilradicals [41].

In [86], one can find a classification of seven-dimensional solvable Lie algebras with nilradical of minimal dimension four. Note that the nilradical of a solvable Lie algebra of dimension n has dimension at least $\frac{n}{2}$ (see [86]).

More or less at the same time, the explicit classification of complex graded quasi-filiform Lie algebras of length $n-1$ was given in [76] (for dimension $n \leq 15$) and [78] (for $n > 15$). These papers went beyond the classification of complex graded quasi-filiform Lie algebras when the length is maximal (i.e., the length is $\dim(g) = n$), which was obtained in 2001 by Gómez, Jiménez, and Reyes [77].

Three-dimensional color Lie algebras are essentially graded as Lie color algebras by groups larger than $\mathbb{Z}_2$. Their classification was given in [141, 142] and presented in terms of commutation relations between generators.

A particular class of five-dimensional real solvable Lie algebras is introduced and classified in [168]: those whose coadjoint orbits are only orbits of zero and maximal dimension, and whose derived ideal is three-dimensional and commutative.

Lu classified in [104] the finite-dimensional complex solvable Lie algebras with non-degenerate symmetric invariant bilinear form.

4. Generalizations

As we have commented before, the historical development of classifications of Lie algebras has led to multiple changes of direction. The structures analyzed may be similar to the original, but sometimes the differences focus on the additional structures imposed, or on the generalization (mainly towards the infinite case), or on other concepts related to the main topic. Each of the following subsections is devoted to one of these three aspects.

4.1. **Additional structures over Lie algebras.** We are now considering operations which are "compatible" with the bracket product. The works developed till now seem to be quite indicative of the kind of research that classifications involve. We are trying to take a picture of the current state of the art.

Metric Lie algebras are those having non-degenerate invariant inner product (invariant symmetric bilinear form). In 2006, Kath and Olbrich [93] gave a complete classification of metric (real) Lie algebras of index 3, whereas the classification of all nilpotent indecomposable metric Lie algebras up to dimension 10 was given in [92].

A *Frobenius Lie algebra* is a Lie algebra equipped with a non-degenerate skew-symmetric bilinear form that is a coboundary. Csikós and Verhóczki [50] gave in 2007 a classification of Frobenius algebras of dimension 4 over a field of characteristic different from 2 and of dimension 6 over an algebraically closed field of characteristic zero.

In 2008, Diatta [51] gave a list of 24 families of indecomposable 5-dimensional solvable Lie algebras that admit a left-invariant contact 1-form. After determining the indecomposable solvable Lie algebras that have both a contact structure and a hypo structure, the complete classification of solvable Lie algebras of dimension 5 having a hypo structure was given. In fact, the solvable Lie algebras that admit a hypo structure were classified in [48]. The hypo structures are the $SU(2)$-structures induced naturally on orientable hypersurfaces of Calabi-Yau manifolds of (real) dimension 6.

Regarding the *Lie algebras admitting a complex product structure*, we would like to recall that product structures on manifolds have been considered by many authors from different points of view. In 2008, [12] gave a complete classification of 6-dimensional nilpotent Lie algebras admitting a complex product structure.

In 2010, a classification of all *pseudo-Riemannian Lie algebras* of dimension 4 (for any field of characteristic different from 2) was given in [46].

In 2011, the authors of [96] have given a classification of non-Abelian Lie algebras satisfying that $\mathfrak{PM}$ is a singleton. Note that $\mathfrak{PM}$ is the quotient of the set of all inner products on the Lie algebra up to isometry and scaling.

The following type has some more history. According to [122], the *Lie algebras with Abelian complex structures* correspond to 2-step solvable Lie algebras (i.e., the metabelian ones). This fact already justifies the interest of their classification, also in the real field, since a classification for the algebras with Abelian structures imply a partial classification of the metabelian class. In fact, real nilpotent Lie algebras admitting complex structures and Abelian complex structures were studied in [49] and [132], respectively, giving the corresponding classification for dimension 6. The 4-dimensional solvable real Lie algebras admitting an Abelian complex structure were already listed in [149], and later 6-dimensional real Lie algebras (in general) admitting an Abelian structure have been classified in [13].

In 1985, Filippov [69] introduced the theory of n-Lie algebras; it was based on the $(2n-1)$-fold Jacobi type identity. He also gave a classification for n-Lie algebras of lower dimensions (i.e., the dimension is here at most $n+1$) over a field of characteristic zero. Some years later, Ling [101] proved that there is a unique $(n+1)$-dimensional simple n-Lie algebra, for any n greater than 2, over any algebraically closed field of characteristic zero. Already in the 21$^{\text{st}}$ century, the papers dealing with this topic have increased several times over. Rui-Pu Bai is one of the most active researchers in this research line. His more recent papers are the following: In [16], it was given the classification of all solvable 3-Lie algebras with the m-dimensional filiform 3-Lie algebra N $(m \geq 5)$ as a maximal hypo-nilpotent ideal. By using the "realization" theory, all the complex 3-Lie algebras in dimension less than or equal to 5 are obtained in [15]. The corresponding one for the field of characteristic 2 was given in [19]. A complete classification of six dimensional 4-Lie algebras over a complete field of characteristic zero was given in [17]. After some attempts by several authors, the complete classification of $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 2 was given in [20]; they also described the solvability and nilpotency of these algebras. Other authors have also contributed to the recent development of the topic. So, the classification of $(n+3)$-dimensional metric n-Lie algebras was given in [73], while the one of the $(n+2)$-dimensional metric n-Lie algebras in [125]. Finally, a complete classification of $(n+2)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero has been given in [18].

4.2. **Infinite Lie algebras.** A Lie algebra is said to be infinite whenever its underlying vector space is infinite-dimensional. This concept has helped the scientific community broaden both the viewpoint and the target. For example, [102] studied the structure of infinite dimensional solvable filiform Lie algebras over $\mathbb{C}$. According to this paper, all the solvable infinite-dimensional filiform Lie algebras belong to a generalization of the Bratlazki's algebras.

Before facing a well-structured kind of infinite algebra, let us choose another example. A non-associative algebra is called automorphic if it admits a group of automorphisms which transitively permutes its one-dimensional subspaces. According to Kac's handbook [90], automorphic Lie algebras are infinite dimensional Lie algebras generalising graded Lie algebras. The study of automorphic algebras (associated to the dihedral group D_n) started from the finite dimensional algebra $sl_2(\mathbb{C})$, whose classification was completed in [103].

4.2.1. *Cartan type and Kac-Moody algebras.* The introduction of the *Kac-Moody algebras* by Kac and Moody, in 1968, undoubtedly produced a different (but complementary) research line in the study of the classifications of nilpotent Lie algebras. Note that these Kac-Moody algebras are of infinite dimension, in general, and constitute a generalization of the semi-simple Lie algebras. By using the system of weights previously introduced by Bratzlavski [35] in 1972 and Favre [61] in 1973, Santharoubane canonically associated in 1982 a Kac-Moody algebra $g(A)$ with each finite nilpotent Lie algebra g. Then it is said that g is of the type A, and this fact was used to reduce the problem of classifying all nilpotent Lie algebras to obtain the classification of the Kac-Moody algebras of each type (see [134]).

The first family of Kac-Moody algebras are the simple Lie algebras of types A_l, B_l, C_l, and D_l, with $l \geq 1$, $l \geq 2$, $l \geq 3$, and $l \geq 4$, respectively. These algebras were first dealt with by Favre and Santharoubane [62]. Later, other authors made their research on other types of these algebras. So, the types E_6, E_7, and E_8 were studied by Agrafiotou and Tsagas in [2], and the type F_4 by Favre and Tsagas [63]). The second family of Kac-Moody algebras is formed by affine Lie algebras. Santharoubane studied in 1982 [133] those of rank 2 and of the types $A_1^{(1)}$ and $A_2^{(2)}$, and Kanagavel dealt in [91] with the non-twisted affine algebras of rank 3 and of the types $A_2^{(1)}$, $B_2^{(1)}$, and $G_2^{(1)}$. Later, Agrafiotou [1] studied the nilpotent Lie algebras of maximal rank and the Kac-Moody algebras of the type $D_5^{(1)}$. Afterwards, Fernández and Núñez, between the years 2000 and 2002, got the classification of the metabelian nilpotent Lie algebras of maximal rank [65] and of the Kac-Moody algebras of types $F_4^{(1)}$ [66] and $E_6^{(1)}$ [67]. Finally, Fernández [64] got the classification of the nilpotent Lie algebras of maximal rank and the Kac-Moody algebras of the type $D_4^{(3)}$.

Among the implications of the Kac-Moody theory, let us highlight just one. Real forms of complex affine Lie algebras can be *almost split* or *almost compact.* The "split" were classified in 1995 for affine Kac-Moody Lie algebras by Back et al. [14], whereas the classification of the second set was obtained in 2003 by Ben Messaoud and Rousseau [24].

4.3. **Other related concepts.** At this point, it would not be needed to remember that a Lie algebra g is a vector space V endowed with an alternating bilinear product which satisfies the Jacobi identity. Sometimes, the algebras come from the study of Lie groups, such as the ones of operators (see, for example, [54]). Anyway, as we have seen, g can be studied from many different points of view, and making use of a number of properties, invariants, superstructures, etc. In order to understand the classifications in the Lie Theory, one has to be aware of the multiple approaches than can be assumed. We have only mentioned some concepts that have shown to be useful with the purpose of classifying Lie algebras, but there are many others, like the representations of Lie algebras (which admit their own classifications). Next we are revising the last concepts that we consider convenient to get an idea of the real situation.

Let us start with the *rigid Lie algebras*. To define them, it is convenient the consideration of the linear group $GL(V)$, the space of all alternating bilinear forms over V, and their orbits; but roughly speaking one can say that a Lie algebra g is rigid if any Lie algebra g' "close" to g is isomorphic to g. A classification of real rigid solvable algebras of dimension 8 was published in [6]. The same authors, jointly with Goze, gave in [8] the real solvable Lie algebras of dimension less than or equal to 8 that are algebraically rigid. Additionally, [121] included the classification of (complex) strongly rigid solvable Lie algebras up to dimension 6.

Let us continue with the *Lie superalgebras*. They are (non-associative) $\mathbb{Z}_2$-graded algebras, defined over a commutative ring (typically $\mathbb{R}$ or $\mathbb{C}$) and whose product (called the *Lie superbracket* or *supercommutator*, satisfies two conditions analogous to the usual for Lie algebras with grading. Just as for Lie algebras, the universal enveloping algebra of the Lie superalgebra can be given by a Hopf algebra structure. Note that the even subalgebra of a Lie superalgebra forms a (normal) Lie algebra as all the signs disappear, and the superbracket becomes a normal Lie bracket. Kac was the person who gave relevance to Lie superalgebras for the first time [89]. He wrote the complete classification of simple (finite-dimensional) Lie superalgebras that are not Lie algebras over an algebraically closed field of characteristic 0. Then, a number of mathematical physicists became interested in these objects because of their many possible physical interpretations. In [85], a complete classification of real and complex Lie superalgebras $g_0 \oplus g_1$ is given when g_0 is a 3-dimensional Lie algebra and g_1 is itself g_0 under the adjoint representation.

Let us consider a graded Lie algebra (graded by $\mathbb{Z}$ or $\mathbb{N}$, say) which is anticommutative and Jacobi in the graded sense. Then this algebra has also a $\mathbb{Z}_2$ grading. This grading is usually said to "roll up" the algebra into odd and even parts, but is not referred to as "super". With respect to the more relevant classifications, there are only three $\mathbb{N}$-*graded Lie algebras* of maximal class (infinite-dimensional filiform Lie algebras), up to isomorphism [139]. And the infinite-dimensional $\mathbb{N}$-graded Lie algebras with 1-dimensional homogeneous spaces and two generators over a field of characteristic zero were classified by Fialowski [68] in 1983. Up to isomorphism, there also exist three such algebras.

A K-algebra is called *kinematic* (or *kinematical* or *quadratic*) if the following condition holds for every element X in the algebra: $X^2 \in K + KX$. Kinematic algebras which appear in the framework of Lie superalgebras or Lie algebras of order 3 were classified in [42]. In this same paper, the supersymmetric and Lie algebras of order 3 extensions of the De Sitter algebra, of the Poincaré algebra, and of the Galilei, the Carroll, and the Newton algebras were derived and classified.

To put an end to this list of related concepts, let us enumerate three more papers that we estimate may be completed soon. In 2007, [156] provided a complete classification of *elementary Lie algebras* over a field which is algebraically closed and of characteristic different from 2 or 3. In 2009, [26] gave a complete list of all simple Lie *algebras of absolute toral rank two* (over an algebraically closed field of prime characteristic $p > 3$); a later application yields a crucial characterization of the Melikyan algebras, completing the classification. In 2010, by using adjoint representation, all complex and *bi-Hermitian structures* were classified on four-dimensional real Lie algebras in [128].

5. Conclusion

After a long journey throughout the history of the classification of Lie algebras it is possible to adopt different attitudes. On the one hand, some will particularly look at the weaknesses: the inherent challenges of the problem, the frequent mistakes made or the impossibility of reaching a complete solution in a reasonable time. However, it is also possible to look at the positive aspects: the wealth of examples provided, the utility of their applications and partial successes due to the efforts of many mathematicians, some of them world-leading scholars.

In this historic moment in which scientific progress is simultaneously achieved in the most distant parts of the world, we think it is convenient to perform an analysis of achievements so far, what is still to be accomplished and even to be raised. In the area under our consideration, Lie algebras' classifications remain a challenge for human intelligence, as no strategy seems to have proved sufficient to solve the problem as a whole. The topicality of their applications and the mathematician's own pride are good reasons to trust that in the coming years discoveries will continue to take place to complete those already published, but they will have to be collected in another survey.

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