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CRITICAL ELEMENTS OF PROPER DISCRETE MORSE FUNCTIONS

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Abstract: The aim of this paper is to study the notion of critical element of a proper discrete Morse function defined on non-compact graphs and surfaces. It is an extension to the non-compact case of the concept of critical simplex which takes into account the monotonous behaviour of a function at the ends of a complex. We show how the number of critical elements are related to the topology of the complex.

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Introduction

Morse theory looks for links between global properties of a smooth manifold and critical points of a function defined on it (see [8]). For instance, the so-called Morse inequalities relate the Betti numbers of the manifold and the numbers of such critical points. R. Forman [5] introduced the notion of discrete Morse function defined on a finite CW-complex and, in this combinatorial context, he developed a discrete Morse theory as a tool for studying the homotopy type and homology groups of these complexes. He also proved the corresponding Morse inequalities, analogous to the classical ones obtained in the smooth case. This theory has shown to have many applications (for instance, see [6]).

In this paper we study the topological properties of an infinite graph or a non-compact surface which can be deduced by imposing restrictions to the number of critical simplices or decreasing rays of a proper discrete Morse function defined on them. This is the first step in the study of the more general problem of finding optimal discrete Morse function under the above conditions defined on those spaces. One basic tool on this study will be the generalized Morse inequalities obtained by the authors in [1]. These inequalities are a non-trivial generalization of the corresponding ones in the finite case, because the behaviour of the function at the infinite is taken into account. We think that the study of the generalization of Morse theory is interesting because it reflects homological aspects of such complexes related to its ends, expressed in terms of its combinatorial structure, but not explicitly by its compactification. Indeed, in the non-compact context new phenomena can appear like the existence of proper discrete Morse functions with no critical simplices.

We begin presenting in Sec. 1 the basic notions concerning finite discrete Morse theory for later use, namely the definitions of discrete Morse function, critical simplex, discrete gradient field and Morse inequalities. We also include in this section some notions corresponding to discrete Morse theory on infinite complexes. In particular, we introduce the definitions of proper discrete Morse function, decreasing ray, critical element as well as two results obtained by the authors: the Morse inequalities in the infinite 2-dimensional case and the obstruction for the existence of proper discrete Morse functions in infinite 2-complexes. In Sec. 2, we introduce the notion of critical array as a main tool for the results of the paper. Finally, in Sec. 3 we characterize those graphs and surfaces which admit discrete Morse functions with restrictions to the components of the critical array.

1. Preliminaries

Through all this paper, we consider infinite simplicial complexes which are locally finite. For terminology and background concerning these objects, we refer to [4].

An *end* of an infinite complex M is an equivalence class $[K, C]$ of pairs (K, C) where $K \subset M$ is compact, C is a component of $M - K$ whose closure is not compact in M and such that $[K_1, C_1] = [K_2, C_2]$ if there exists (K, C) with $K_1 \cup K_2 \subset K$ and $C \subset C_1 \cap C_2$. For instance, $\mathbf{R}$ has two ends and $\mathbf{R}^n$ has one end. If M is a compact connected surface, then $M \times [0, +\infty)$ has one end and $M \times \mathbf{R}$ has two ends.

A *discrete Morse function defined on* M is a function $f : M \longrightarrow \mathbf{R}$ such that, for any p-simplex $\sigma^{(p)} \in M$:

(M1) $\text{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} \leq 1$.

(M2) $\text{card}\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geq f(\sigma)\} \leq 1$.

where $\tau^{(p+1)} > \sigma$ is indicating that the p-simplex σ is face of the $p+1$-simplex τ.

A p-simplex $\sigma \in M$ is said to be *critical* with respect to f if:

(C1) $\text{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} = 0$.

(C2) $\text{card}\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geq f(\sigma)\} = 0$.

From the above definitions, it can be deduced that $\sigma^{(p)}$ is a non-critical simplex if and only if it verifies one of the following conditions:

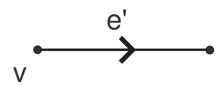
(NC1) There exists a simplex $\tau^{(p+1)} > \sigma^{(p)}$ such that $f(\tau^{(p+1)}) \leq$ $\leq f(\sigma^{(p)})$.

(NC2) There exists a simplex $\upsilon^{(p-1)} < \sigma^{(p)}$ such that $f(\upsilon^{(p-1)}) \geq$ $\geq f(\sigma^{(p)})$.

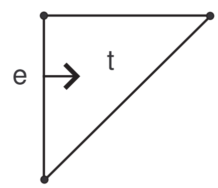
It is important to point out that both conditions can not be verified simultaneously by a non-critical simplex.

In the n-dimensional cases with $n \leq 2$, we can draw these two conditions as follows:

First,



means $f(v) \geq f(e')$ and we say that v and e' are *matched*. On the other hand,



means $f(e) \geq f(t)$ and we say that e and t are *matched*.

A *proper* discrete Morse function is a function f verifying $f^{-1}([a,b])$ is a finite set of simplices for any $a, b \in \mathbf{R}$, $a < b$.

In the finite context, Forman proved the Morse inequalities for discrete Morse functions [5]:

Theorem 1.1. *Let f be a discrete Morse function defined on a finite cw-complex M and let b_p be the p-th Betti number of M with $p = 0, \ldots, \dim(M)$. Then*

(I1) $m_p(f) - m_{p-1}(f) + \cdots \pm m_0(f) \geq b_p - b_{p-1} + \cdots \pm b_0$,

(I2) $m_p(f) \geq b_p$,

(I3) $m_0(f) - m_1(f) + m_2(f) - \cdots \pm m_{\dim(M)}(f) = b_0 - b_1 + b_2 - \cdots \pm b_{\dim(M)}$,

where $m_p(f)$ denotes the number of critical p-simplices of f on M.

Given an infinite simplicial complex M, it is said that a (either finite or infinite) sequence of simplices of M,

$$\alpha_0^{(i-1)}, \beta_0^{(i)}, \alpha_1^{(i-1)}, \beta_1^{(i)}, \ldots, \beta_r^{(i)}, \alpha_{r+1}^{(i-1)}, \ldots$$

is a *i-path* if it verifies that the $(i-1)$-simplices

$$\alpha_{r-1}^{(i-1)} \text{ and } \alpha_r^{(i-1)}$$

are faces of the i-simplex

$$\beta_{r-1}^{(i)},$$

for any $r \in \mathbf{N}$.

An infinite i-path will be called *i-ray*. Given two i-rays contained in the same complex, we say they are *equivalent* or *cofinal* if they coincide from a common $(i-1)$-simplex.

Given a discrete Morse function f defined on M, we say that an i-path (resp., i-ray),

$$\alpha_0^{(i-1)}, \beta_0^{(i)}, \alpha_1^{(i-1)}, \beta_1^{(i)}, \ldots, \beta_r^{(i)}, \alpha_{r+1}^{(i-1)}(, \ldots)$$

is a *decreasing i-path (resp., i-ray)* if it verifies that:

$$f(\alpha_0^{(i-1)}) \geq f(\beta_0^{(i)}) > f(\alpha_1^{(i-1)}) \geq f(\beta_1^{(i)}) > \cdots \geq$$
$$\geq f(\beta_r^{(i)}) > f(\alpha_{r+1}^{(i-1)})(\geq \cdots)$$

From now on, d_i will denote the number of non equivalent decreasing $i+1$-rays of f in M.

A *i-critical element* of f on M is either a i-critical simplex of f on M or a decreasing $(i+1)$-ray with $1 \leq i \leq n$. For instance, a 0-critical element is either a critical vertex or a decreasing 1-ray.

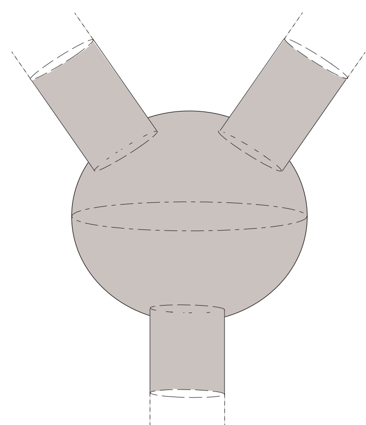
Another key concept that we shall need later is the notion of discrete vector field. Given a simplicial complex M, a *discrete vector field V* defined on M is a collection of pairs $(\alpha^{(p)} < \beta^{(p+1)})$ of simplices of M such that every simplex is in, at most, one pair of V. We can visualize discrete vector fields in low dimensional complexes by considering arrows as we did in the definition of matched simplices, where the first figure is indicating that the vertex v and the edge e' verify that $(v, e') \in V$ and, analogously, the second one is indicating that the edge e and the triangle t verify that $(e, t) \in V$.

Since (NC1) and (NC2) can not be verified simultaneously by a non critical simplex, it can be deduced that every discrete Morse function $f : M \longrightarrow \mathbf{R}$ induces a discrete vector field on M. Namely, given a simplex $\sigma^{(p)}$ of M, it can be critical or not. If it is not critical, there is a unique simplex τ of consecutive dimension such that either $\tau^{(p-1)} < \sigma^{(p)}$ and $f(\tau) \geq f(\sigma)$ or $\tau^{(p+1)} > \sigma^{(p)}$ and $f(\tau) \leq f(\sigma)$. So we can consider the pair $(\tau < \sigma)$ or $(\sigma < \tau)$, depending on the case. If σ is critical there is no simplex in M matched with it. Thus, each simplex of M is either the first simplex of a pair or the second simplex of a pair or it is not in any pair. Hence, this set of pairs verifies the definition of discrete vector field on M. Essentially, a pair $\tau^{(p-1)} < \sigma^{(p)}$ is in this vector field if and only if $f(\sigma^{(p)}) \leq f(\tau^{(p-1)})$. This vector field is called the *gradient vector field induced by f*. Note that not all vector fields correspond to a discrete Morse function.

Now, let M be a non-compact connected triangulated surface without boundary. So, denoting the Betti numbers of M by b_i, we have that $b_0 = 1$ and $b_2 = 0$. Moreover, we suppose that $b_1 < +\infty$. In these conditions, we deduce that the number of ends of M is finite too. Thus, we can express M as the union of $\widehat{M}$ and a finite number of infinite cylinders, that is,

$$M = \widehat{M} \cup \bigcup_{j=1}^{n} C_j, \quad (1)$$

where C_j denotes an infinite cylinder and $\widehat{M}$ is a compact surface with boundary. For instance, the following figure represents a surface with three ends obtained by gluing three cylinders to a sphere minus three open disks.



The authors proved in [1] generalized versions of the Morse inequalities for infinite graphs and non-compact surfaces without boundary:

Theorem 1.2. *Let M be a non-compact connected triangulated 2-manifold ($b_0 = 1$, $b_2 = 0$) without boundary such that $b_1 < +\infty$. Let f be a discrete Morse function defined on M with a finite number of critical simplices and a finite number of decreasing i-rays, $i = 1, 2$. It holds that:*

(a) $m_0 + d_0 \geq 1$; $m_1 + d_1 \geq b_1$;

(b) $m_1 + d_1 - m_0 - d_0 \geq b_1 - 1$; $m_2 - m_1 - d_1 + m_0 + d_0 \geq 1 - b_1$;

(c) $m_0 + d_0 - m_1 - d_1 + m_2 = 1 - b_1$,

where d_i denotes the number of non equivalent decreasing $i+1$-rays of f in M, $i = 0, 1$

Notice that the corresponding result for infinite graphs can be obtained from the above theorem with the obvious changes.

As it was done in the compact case in [6], taking into account the inequalities of the Th. 1.2, it is interesting to investigate the existence of discrete Morse function with as less critical elements as possible. These functions are called *optimal* discrete Morse functions.

Finally, given a 1-ray r and a 2-ray r' in a non compact 2-dimensional simplicial complex, we say that r is *adjacent* to r' or conversely, r' is *bounded* by r, if every edge of r is contained in some triangle of r' but it is not an edge of r'.

The following result proved by the authors in [3], states an obstruction for the existence of proper discrete Morse functions in infinite 2-complexes.

Theorem 1.3. *Let M be a non-compact 2-simplicial complex and let f be a proper discrete Morse function defined on M. It holds that there is not any increasing 1-ray adjacent to a decreasing 2-ray.*

Intuitively, the proof of the above theorem is based on the fact that the discrete Morse function considered is proper and hence its values on every monotonous ray are non-bounded. Therefore, if we consider the restriction of f to the decreasing 2-ray and its adjacent increasing 1-ray, we obtain non-bounded decreasing and increasing sequences respectively. So there exist triangles in the decreasing 2-ray such that f on any of them is less or equal than F on two of its edges and it implies that f is not a discrete Morse function.

2. The critical array of a proper discrete Morse function

In order to not consider trivial cases, we restrict our study to discrete Morse functions with a finite number of critical simplices.

We shall introduce in this section the key notion of critical array which will be a central tool in Sec. 3. Essentially, the critical array of a discrete Morse function defined on a complex is an ordered n-uple which contains the information concerning the numbers of critical elements. More precisely,

Definition 2.1. Let M be a connected and non-compact surface without boundary and let f be a proper discrete Morse function defined on M

such that $m_i < +\infty$ and $d_i < +\infty$. The **critical array** of the pair (M, f) is the ordered 5-uple,

$$C(M, f) = (m_0, m_1, m_2; d_0, d_1).$$

Remark 2.2. (a) Notice that the notion of critical array for infinite graphs can be stated as follows,

$$C(G, f) = (m_0, m_1; d_0)$$

where f is a discrete Morse function defined on an infinite graph G.

(b) Given a compact surface M, it is convenient to point out that the critical array of every pair (M, f) is given by $(m_0, m_1, m_2; 0, 0)$. However, since there are non-compact surfaces which admit this kind of critical arrays, in order to avoid any confusion, we shall define the critical array associated to a compact surface as (m_0, m_1, m_2), that is, not considering the last two components. The same considerations can be stated for finite graphs.

Proposition 2.3. *Let M be a connected and non-compact surface without boundary and let f be a proper discrete Morse function defined on M. It holds that $(0, 0, 0; 0, 0)$ can not be the critical array of any pair (M, f).*

Proof. Due to the fact that, from Th. 1.2 (a), we have that $m_0 + d_0 \geq \geq 1$ in the connected case and hence, there is at least a critical element, namely, a critical vertex or a decreasing 1-ray. $\diamondsuit$

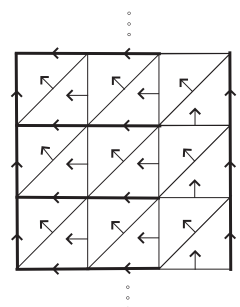
Proposition 2.4. *If the 5-uple $(m_0, m_1, m_2; d_0, d_1)$ is the critical array of a pair (M, f) then $d_0 \geq d_1$.*

Proof. Given any decreasing 2-ray r', there exists a compact subcomplex $\widehat{M}$ as in (1) which contains all the critical simplices of f on M and such that there not bifurcations of decreasing 2-rays in $M - \widehat{M}$ (see [1] Lemma 3.3). Hence, r' is contained in a infinite cylinder C_j. Thus, either f is monotonous when restricted to the two 1-rays r_1, r_2 adjacent to r' or all of vertices and edges of r_1 or r_2 are paired with simplices which not belong to r'.

In the first case, by means of Th. 1.3 we get that r_1 and r_2 must be decreasing and hence, r' give rise to two decreasing 1-rays. In the second case, let $\hat{r}$ be the union of r' and all the finite decreasing 2-paths starting from an edge belonging to r_1 or r_2. Notice that $\hat{r}$ is a thickening of a tree contained in the 1-skeleton of the first baricentric subdivision of C_j and thus $\hat{r}$ is homologically trivial. Let us denote by $\partial\hat{r}$ the set of edges and vertices which belong to one triangle of $\hat{r}$ but are not paired with any edge r triangle of $\hat{r}$. Since there are no critical simplices of f in

C_j, given an edge $e \in \partial\hat{r}$, it must be paired with one of its vertices. So, we have obtained that $\partial\hat{r}$ is a forest with at most two components. It is convenient to point out that when f is restricted to a component of $\partial\hat{r}$, there is a unique (up to equivalence) 1-ray where f is monotonous. By reasoning in a similar way as we did in the intuitive proof of Th. 1.3, we conclude that f must be decreasing on every 1-rays contained in $\partial\hat{r}$.

Notice that if $\hat{r}$ contains all triangles of C_j, then $\partial\hat{r}$ consists of a unique tree which contains all vertices of C_j. In this case we have $d_0 = d_1 = 1$ in C_j. See the following figure



If there exist more than one decreasing 2-ray in C_j and r' is one of them, then $\partial r'$ has two components and hence it contains two decreasing 1-rays. Moreover, since two different decreasing 2-rays could be separated by a decreasing 1-ray, then we conclude that $d_0 \geq d_1$ in C_j. Notice that we obtain $d_0 = d_1$ in C_j when every decreasing 1-ray separates two decreasing 2-rays. $\Diamond$

3. Critical elements of proper discrete Morse functions

In this section, we study how the topology of M and the numbers of critical simplices and decreasing rays of a discrete Morse function defined on M are related. In order to get it we use the notion of critical array as a basic tool containing all these numbers. Being more precise, we focus our attention on values of the components of the critical array and we show how the lowest number of critical elements of any discrete Morse function on M is small when the topology of M is simple. Moreover, in the non-compact case it is possible to push critical simplices to the infinity along decreasing rays. Note that this process does not change the number of critical elements, but only its nature: critical vertices shall correspond to decreasing 1-ray and critical edges shall correspond to decreasing 2-rays.

Let us start by considering the one-dimensional case, studying how critical elements are related to the topological nature of a graph. In particular, we aim to understand in the following result how the non-existence of critical simplices of a discrete Morse function defined on a graph and its homology are strongly linked.

Theorem 3.1. *Let M be a connected graph. Then, M admits a discrete Morse function without critical simplices if and only if M is an infinite tree.*

Proof. First, by applying finite discrete Morse inequalities (Th. 1.1) for a discrete Morse function f defined on M such that $m_i(f) = 0$ with $i = 0, 1$, we would get that $m_0(f) \geq b_0 = 1$, so we conclude that M has to be an infinite graph. Now, by applying generalized discrete Morse inequalities for f (1-dimensional version of Th. 1.2), we get that $0 = = m_1(f) \geq b_1$ and hence $b_1 = 0$. So we conclude that M has to be an infinite tree.

Conversely, we are going to define a discrete Morse function f with no critical simplices on an infinite tree M. Let v_0 be a vertex of M. Then, we select any ray $R = v_0, e_0, v_1, e_1, \ldots$ in M starting from v_0. First, we define the desired function f in a decreasing way on R, that is, such that:

$$f(v_0) \geq f(e_0) > f(v_1) \geq f(e_1) > \ldots$$

In the rest of M we define f in an increasing way when we are moving away from any vertex of R, including v_0, by paths (finite or not) different of ray R.

Finally, it is necessary to check that f is a discrete Morse function with no critical simplices. Given a vertex v in the ray R, there is only one edge e (which is in R) incident with v such that $f(v) \geq f(e)$ and the other edges e' incident with v verify that $f(e') > f(v)$.

If we consider now a vertex w which is not in R, there is a unique path in $M - R$ connecting a vertex of R and w. Then, by using the above construction, f has to be increasing on this path and so, there is an edge σ incident with w such that $f(\sigma) \leq f(w)$. Notice that σ is the only edge verifying this condition because f has been defined in a increasing way as we are moving away from R.

Next, let e be an edge in the ray R. Since f is decreasing on R, its two vertices v and v' verify either $f(v) \geq f(\sigma) > f(v')$ or $f(v) < f(\sigma) \leq$ $\leq f(v')$.

Now, if e is an edge which is not in R, since f is increasing as we are moving away from R, it holds either $f(v) < f(e) \leq f(v')$ or $f(v') < f(e) \leq f(v)$, where v and v' denote the two vertices of e.

Consequently, f is a discrete Morse function on M and since there is not any non-paired simplex, we conclude that $m_i(f) = 0$ with $i = 0, 1$. ♢

The following result states the relationship between the non-existence of critical simplices of a discrete Morse function defined on a graph and the number of critical elements at the infinity, that is, the number of decreasing rays d_0.

Corollary 3.2. *Let M be a connected graph. If M admits a discrete Morse function without critical simplices then $d_0 = 1$.*

Proof. By applying discrete Morse inequalities we get that $d_0 = 1 - b_1$ and by means of theorem 3.1 it holds that $b_1 = 0$, so we conclude that $d_0 = 1$. ♢

It is convenient to point out that the above corollary implies that, under the assumption of non-existence of critical simplices, the corresponding critical array $C(M, f) = (0, 0; d_0)$ is much simpler, namely, $C(M, f) = (0, 0; 1)$. Moreover, this result states that the non-existence of critical simplices of a discrete Morse function on a graph implies the optimality of this function, that is, it has as less critical elements as possible on a tree.

In the next result we alternatively reduce to zero the number of each kind of critical elements and study the way it is reflected in the topology of the considered complex.

Proposition 3.3. *Let M be a connected graph and let f be a discrete Morse function defined on M such that $m_0(f)$ and $m_1(f)$ are finite. Then:*

(i) *If M admits the critical array $(0, m_1; d_0)$ then M is infinite and $d_0 = 1 + m_1 - b_1 \geq 1$.*

(ii) *If M admits the critical array $(m_0, 0; d_0)$ then M is a tree and $d_0 = 1 - m_0$.*

(iii) *If M admits the critical array $(m_0, m_1; 0)$ then M either a finite graph or an infinite graph such that every 1-ray is increasing.*

Proof. (i) Let us suppose that M admits the critical array $(0, m_1; d_0)$. By using finite discrete Morse inequalities, Th. 1.1, it holds that $m_0 \geq b_0 = 1$

for every finite complex, so M has to be an infinite graph. Moreover, by applying generalized Morse inequalities (1-dimensional version of Th. 1.2) to M we get that $d_0 \geq 1$ and $d_0 - m_1 = 1 - b_1 \geq 1$.

(ii) Let us suppose that M admits the critical array $(m_0, 0; d_0)$. By means of discrete Morse inequalities, finite or generalized, we get that $0 = m_1 \geq b_1$ and hence M is a tree, finite or not, and finally we conclude that $m_0 + d_0 = 1$.

(iii) Let us suppose that M admits the critical array $(m_0, m_1; 0)$. This is the general critical array for a finite graph. If it is the critical array corresponding to a pair (M, f) with M an infinite graph then there is not any decreasing 1-ray for f on M and since $m_i(f)$ are finite, we conclude that the behaviour of f towards the ends is increasing, that is, f is increasing on every 1-ray of M. $\Diamond$

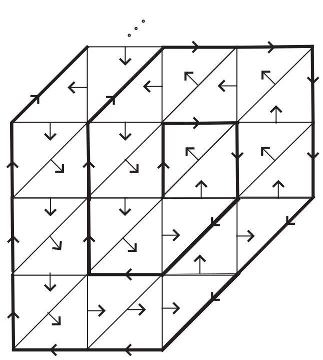
At this point we are going to extend our study to the two-dimensional case, by considering the critical array of a discrete Morse function defined on a orientable connected non-compact surface.

First we characterize those orientable surfaces which admit a discrete Morse function with no critical simplices.

Theorem 3.4. *Let M a orientable connected non-compact surface without boundary. Then, M admits a proper discrete Morse function without critical simplices if and only if M is the plane $\mathbf{R}^2$ or the cylinder $S^1 \times \mathbf{R}$.*

Proof. By applying generalized discrete Morse inequalities we get that $d_1 \geq b_1$ and by means of Prop. 2.4 it holds that $d_0 \geq d_1$. Thus, it follows that $1 - b_1 \geq 0$ and hence $b_1 \leq 1$ so we conclude that M is either $\mathbf{R}^2$ or $S^1 \times \mathbf{R}$. Conversely we are going to define discrete Morse functions with no critical simplices on both surfaces.

On $\mathbf{R}^2$ we consider the following discrete gradient vector field:



For $S^1 \times \mathbf{R}$ we can consider the gradient vector field indicated in the figure of the proof of the Prop. 2.4. $\Diamond$

The next result states that, in absence of critical simplices, the numbers of decreasing rays are linked.

Corollary 3.5. *Let M be an orientable connected non-compact surface without boundary which admits a proper discrete Morse function with no critical simplices. Then, either $d_0 = d_1 + 1$ or $d_0 = d_1$.*

Proof. By means of the proof of Th. 3.4 we get that $b_1 \leq 1$. Taking into account that $m_i = 0$ and by using discrete Morse inequalities, we get that $d_0 - d_1 = 1 - b_1$. Hence, we conclude that either $d_0 = d_1 + 1$ or $d_0 = d_1$. $\Diamond$

Remark 3.6. Notice that the two options of the above corollary characterize $\mathbf{R}^2$ and $S^1 \times \mathbf{R}$ respectively.

Remark 3.7. Notice that by using the above proof in the non-orientable case, we obtain that $b_1(M) \leq 1$ and hence the only possible non-orientable surface which could admit a proper discrete Morse function with no critical simplices is the open Moebius band, that is, M is homeomorphic to $P_2(\mathbf{R}) - \{p_0\}$. However we do not know any example of such functions.

In the following result we shall see how using only the condition of non-existence of critical edges, that is, $m_1 = 0$, we get strong restrictions for the topology of the surfaces which admit such kind of discrete Morse functions.

Theorem 3.8. *Let M be a connected non-compact surface without boundary which admits a proper discrete Morse function with no critical edges. Then M is either $\mathbf{R}^2$, $S^1 \times \mathbf{R}$ or $P_2(\mathbf{R}) - \{p_0\}$.*

Proof. Let us suppose that M is orientable and such that it admits a critical array $(m_0, 0, m_2; d_0, d_1)$. It is known that $b_1(M) = 2g + h - 1$ where g is the genus of M, that is ,the number of handles of M and h is the number of ends of M. By using Th. 1.2, we get that

$$d_0 - d_1 + m_0 - m_2 = 2 - 2g - h. \qquad (*)$$

Since $d_0 - d_1 \geq 0$ by means of Prop. 2.4, we obtain that $2 \geq 2g + h$. As M is non-compact, $h \geq 1$ and hence $1 \geq 2g$ and we conclude that $g = 0$. Using the fact that $g = 0$ in the equality $(*)$ we get that $d_0 - d_1 + m_0 + m_2 = 2 - h \geq 0$, so we obtain that $h \leq 2$. Hence, if $h = 1$ M is $\mathbf{R}^2$ and if $h = 2$ M is $S^1 \times \mathbf{R}$.

Reasoning in a similar way in the non-orientable case, taking into account that $b_1(M) = g + h - 1$, we obtain that $2 \geq g + h$. Thus $g = 1$

and $h = 1$, that is, M is homeomorphic to $P_2(\mathbf{R}) - \{p_0\}$. $\Diamond$

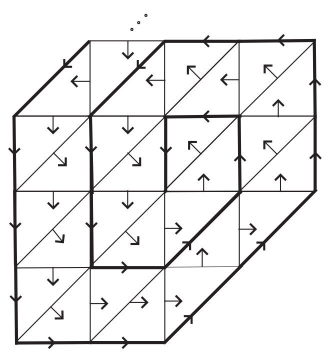
Now we shall study those surfaces which admit a specially simple critical array, which in fact characterize them.

Theorem 3.9. $\mathbf{R}^2$ *is the only connected non-compact surface which admits a proper discrete Morse function whose critical array has only one non-zero component. In fact this critical array must be either* $(1, 0, 0; 0, 0)$ *or* $(0, 0, 0; 1, 0)$.

Proof. Let f be a proper discrete Morse function defined on a connected non-compact surface M such that its corresponding critical array has only a non-zero component. By applying Th. 1.2 we get that $m_0 + d_0 \geq b_0 = 1$ and hence the only non-zero component must be either m_0 or d_0. Thus, by applying again Th. 1.2 we get that $0 = m_1 + d_1 \geq b_1$ and we conclude that $b_1 = 0$, so M must be $\mathbf{R}^2$. Moreover, since $b_1 = 0$, by Th. 1.2 we know that $m_0 + d_0 = 1$ and thus the only possibilities are: $(1, 0, 0; 0, 0)$ and $(0, 0, 0; 1, 0)$.

Finally, we give the discrete gradient vector fields corresponding to proper discrete Morse functions on $\mathbf{R}^2$ whose critical arrays are $(1, 0, 0; 0, 0)$ and $(0, 0, 0; 1, 0)$ respectively.

For the critical array $(1, 0, 0; 0, 0)$ we can consider the following discrete gradient vector field:



For the critical array $(0, 0, 0; 1, 0)$ we can consider the first example of the converse of the proof of Th. 3.4. $\Diamond$

Remark 3.10. Comparing Th. 3.8 with its compact version, namely, the closed disk is the only compact surface which admits a discrete Morse function whose critical array has only a non-zero component (in fact this critical array is $(1, 0, 0)$) we could ask if there is some link between proper discrete Morse function whose corresponding critical arrays are

$(1,0,0;0,0)$ and $(0,0,0;1,0)$. The answer is positive in the sense of we can obtain the discrete gradient vector field associated to the array $(0,0,0;1,0)$ from $(1,0,0;0,0)$ by reversing the flow line along a ray in order to change the nature of the only 0-critical element: from a critical vertex to an ideal point.

Notice that both kind of functions are optimal in the sense that they have as few critical elements as possible. This is a particular case of the more general problem which consists on finding the optimal function on a given non-compact surface, that is, the function where Morse inequalities became equalities. In the compact case these problems have been studied by Lewiner et al. [6, 7].

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