SEMIMARTINGALE ATTRACTORS FOR ALLEN-CAHN SPDES DRIVEN BY SPACE-TIME WHITE NOISE I: EXISTENCE AND FINITE DIMENSIONAL ASYMPTOTIC BEHAVIOUR

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Abstract. We delve deeper into the study of semimartingale attractors that we recently introduced in Allouba and Langa [4]. In this article we focus on second order SPDEs of the Allen-Cahn type. After proving existence, uniqueness, and detailed regularity results for our SPDEs and a corresponding random PDE of Allen-Cahn type, we prove the existence of semimartingale global attractors for these equations. We also give some results on the finite dimensional asymptotic behavior of the solutions. In particular, we show the finite fractal dimension of this random attractor and give a result on determining modes, both in the forward and the pullback sense.

1. Introduction and organization of the article

The analysis of qualitative properties of ordinary and partial differential equations is the key point in dynamical system theory. When a phenomenon from Physics, Chemistry, Biology, Economics can be described by a system of differential equations (in which the existence of global solutions can be assured), one of the most interesting problems is to describe the asymptotic behavior of the system when time grows to infinity. The study of the asymptotic dynamics of the system gives us relevant information about "the future" of the phenomenon described in the model. In this context, the concept of *global attractor* has become a very useful tool to describe the long-time behavior of many important differential equations (see, among others, Ladyzhenskaya [25], Babin and Vishik [9], Hale [24], Temam [32], Robinson [30]). A new difficulty appears when a random term is added to the deterministic equation, a white noise for instance, and the resulting stochastic partial differential equation must be treated in a different way. Firstly, the equation becomes non-autonomous, which makes necessary the introduction of a two-sided time dependent process instead of a semigroup. Moreover, the strong dependence on the random term adds another difficulty. The rapidly growing theory of *random dynamical systems* (Arnold [8]) has become the appropriate tool for the study of many important random and stochastic equations. In this framework, Crauel and Flandoli [13] (see also Schmalfuss [31]) introduced the concept of a random attractor as a proper generalization of the corresponding deterministic global attractor. The theory of random attractors is turning out to be very helpful in the understanding of the long-time dynamics of some stochastic ordinary and partial differential equations. On the other hand, one of the most important results in the

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theory of global attractors for deterministic PDEs claims that the fractal, and so the Hausdorff, dimension of this set is finite (Constantin and Foias [11], Contantin et al. [12], Ladyzhenskaya [26]; see also the books of Temam [32] and Robinson [30]). That is, although the trajectories depend on an infinite number of degrees of freedom, the finite dimensionality of the attractors leads to the idea that the asymptotic behavior can be described by a finite number of time-dependent coordinates. This makes, for example, really interesting the study of the dynamics on the global attractor. There are also some results which generalize the finite-dimensionality of attractors to the stochastic case (Debussche [18], [19]).

In this paper we show how all the theory of finite dimensional random attractors can be generalized to the situation in which the partial differential equation is affected by a space-time white noise, and we characterize this randomness in the attractor as one coming from semimartingale-type solutions (see Definition 2.1). Some of these results were recently sketched in Allouba and Langa [4]. Here, we prove in details the existence of a finite dimensional random attractor associated to the random dynamical system corresponding to a space-time white noise driven stochastic PDE of Allen-Cahn type; and we give a determining modes result for such a SPDE, both in the forward and pullback sense. In the course of our proof, we also give detailed proofs and discussions of existence, uniqueness, and regularity (both weak and strong) results for our SPDE as well as for an associated Allen-Cahn type random PDE. The lack of regularity caused by our driving space-time white noise causes several difficulties in the SPDEs we study. These difficulties are not present in the traditional case of noises that are only white in time (see Remark 3.2 and Remark 3.3 below).

Before spelling out the organization of this paper, we wanted to highlight two key features of this work:

i) Our solutions are weak semimartingales (see Definition 2.1 and Section 3.4 below), and this characterizes the randomness in our attractors as one coming from some type of semimartingale solutions (not simply random processes); thus we call our random attractors semimartingale attractors. This characterization is crucial and will lead to several new stochastic analytic aspects of these random attractors, like the notion of semimartingale decomposition of semimartingale attractors (e.g., [5]).

ii) As in Walsh [33], we regard space-time white noise as a continuous orthogonal martingale measure, which we think will lead to a richer structure of the noise, and so to new aspects of the SPDE under consideration, even compared to cylindrical noise. One such aspect is the notion of semimartingale measure attractors (to which we devote a separate paper), which is built upon the notion of semimartingale measure introduced in Allouba [3].

The paper is organized as follows: in the next Section we write the general theory of random attractors and give the definition of weak semimartingales; Section 3 develops the existence, uniqueness, and regularity (both weak and strong) of solutions for a stochastic PDE of Allen-Cahn type with space-time white noise and for a corresponding random PDE; we follow by proving the existence of a semimartingale attractor associated to these equations. Finally, we show the dependence of the asymptotic behavior of the model on a finite number of degrees of freedom, by proving, with probability one, the finite fractal dimensionality of the semimartingale attractor and some results on determining modes, both in the forward and the pullback sense. Some conclusions are then given, placing the results here in the context of our ongoing research program. We also include some technical results in a final Appendix. Throughout this article we will denote by K a constant that may change its value from line to line.

2. Semimartigale global attractors

2.1. **Definitions.** Proceeding toward a precise statement of our results, let us recall some definitions associated with random attractors. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\theta_t : \Omega \to \Omega,\ t \in \mathbb{R}\}$ a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = \text{id}$, $\theta_{t+s} = \theta_t \theta_s$, for all $s, t \in \mathbb{R}$. The flow θ_t together with the probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a *measurable dynamical system*. Furthermore, we suppose that the shift θ_t is ergodic.

A *random dynamical system* (RDS) (Arnold [8]) on a complete metric (or Banach) space $(\mathbb{B}, d)$ with Borel σ-algebra $\mathcal{B}$, over θ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable map $\mathbb{R}_+ \times \Omega \times \mathbb{B} \ni (t, \omega, \xi) \mapsto \Phi(t, \omega)\xi \in \mathbb{B}$ such that $\mathbb{P}$–a.s.

i) $\Phi(0, \omega) = \text{id}$ (on $\mathbb{B}$)
ii) $\Phi(t + s, \omega) = \Phi(t, \theta_s\omega) \circ \Phi(s, \omega)$, $\forall\, t, s \in \mathbb{R}_+$ (cocycle property).

A RDS is continuous (differentiable) if $\Phi(t, \omega) : \mathbb{B} \to \mathbb{B}$ is continuous (differentiable). A random set $K(\omega) \subset \mathbb{B}$ is said to *absorb* the set $D \subset \mathbb{B}$ if there exists a random time $t_D(\omega)$ such that

$$t \geq t_D(\omega) \to \Phi(t, \theta_{-t}\omega)D \subset K(\omega), \mathbb{P}\text{–a.s.}$$

$K(\omega)$ is forward invariant if $\Phi(t, \omega)K(\omega) \subseteq K(\theta_t\omega)$, for all $t \in \mathbb{R}_+$, $\mathbb{P}$–a.s. Now, let $\text{dist}(\cdot, \cdot)$ denote the Hausdorff semidistance

$$\text{dist}(B_1, B_2) = \sup_{\xi_1 \in B_1} \inf_{\xi_2 \in B_2} d(\xi_1, \xi_2), \qquad B_1, B_2 \subset \mathbb{B}.$$

A random set $\mathcal{A}(\omega) \subset \mathbb{B}$ is said to be a *random attractor* associated with the RDS Φ if $\mathbb{P}$–a.s.

i) $\mathcal{A}(\omega)$ is compact and, for all $\xi \in \mathbb{B}$, the map $\xi \mapsto \text{dist}(\xi, \mathcal{A}(\omega))$ is measurable,
ii) $\Phi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega)$, $\forall t \geq 0$ (invariance), and
iii) for all $D \subset \mathbb{B}$ bounded (and nonrandom) $\lim_{t\to\infty} \text{dist}(\Phi(t, \theta_{-t}\omega)D, \mathcal{A}(\omega)) = 0$.

Remark 2.1. Note that $\Phi(t, \theta_{-t}\omega)\xi$ can be interpreted as the position at $t = 0$ of the trajectory which was in ξ at time $-t$. Thus, the attraction property holds *from* $t = -\infty$.

We have the following theorem about existence of random attractors due to Crauel ([15], Theorem 3.3):

Theorem 2.1. *There exists a global random attractor $\mathcal{A}(\omega)$ iff there exists a random compact set $K(\omega)$ attracting every bounded nonrandom set $D \subset \mathbb{B}$.*

Moreover, Crauel [15] proved that random attractors are unique and, under the ergodicity assumption on θ_t, there exists a deterministic compact set $K \subset \mathbb{B}$ such that $\mathbb{P} - a.s.$ the random attractor is the omega-limit set of K, that is,

$$\mathcal{A}(\omega) = \bigcap_{n\geq 0} \overline{\bigcup_{t\geq n} \Phi(t, \theta_{-t}\omega)K}.$$

Our SPDEs solutions are weak semimartingale, which we now define.

Definition 2.1. We call a random field $U(t,x,\omega)$, $x \in G \subset \mathbb{R}^d$, a weak semimartingale sheet (or simply a weak semimartingale) if there exists a $p \geq 0$ such that the L^2 scalar product $(U(t),\varphi)$ is a semimartingale in time for each fixed $\varphi \in C_c^p(G)$. If $\mathcal{A}$ is a random attractor corresponding to a SPDE whose solutions are weak semimartingales, then $\mathcal{A}$ is called a semimartingale attractor.

2.2. **Finite dimensional asymptotic behavior.** Here we obtain some results on the finite dimensional asymptotic behavior of trajectories associated to a random dynamical system, which we will apply below to the solutions for Allen-Cahn type SPDEs in (3).

2.2.1. *The random squeezing property.* Suppose the existence of a random compact absorbing set $K(\omega)$ such that, for some random variable $r(\omega)$, we have that $\mathbb{P}$-a.s. $K(\omega) \subset B(0,r(\omega))$. Moreover, suppose that the $r(\omega)$ is a *tempered* random variable, that is, $\mathbb{P}$-a.s.

$$\lim_{t\to+\infty} \frac{r(\theta_t\omega)}{e^{\epsilon t}} = 0,$$

for all $\epsilon > 0$.

Let $\mathbf{P} : \mathbb{B} \to \mathbf{P}\mathbb{B}$ be a finite-dimensional orthogonal projector and let $\mathbf{Q} = \mathbf{I} - \mathbf{P}$ be its counterpart. In what follows, the main hypothesis (H) is the following: Suppose there exist $0 < \delta < 1$ and a random variable $c(\omega)$ with finite expectation,

$$\mathbb{E}_{\mathbb{P}}(c(\omega)) < \ln(1/\delta), \tag{1}$$

such that, for $\tau \in \mathbb{R}$

$$|\mathbf{Q}(\Phi(1,\theta_\tau\omega)u - \Phi(1,\theta_\tau\omega)v)| \leq \delta \exp\left(\int_\tau^{\tau+1} c(\theta_s\omega)ds\right)|u-v|, \tag{2}$$

for all $u, v \in K(\theta_\tau\omega)$, where $|\cdot|$ denotes the norm in $\mathbb{B}$.

This property is called *the random squeezing property* (RSP) in Flandoli and Langa [21], and it was first used in Debussche [18] to prove that the random attractor associated to a RDS has finite Hausdorff dimension $\mathbb{P}$-a.s.

Proposition 2.1. *(*[18]*,* [21]*)*
Suppose that (1)*,* (2) *hold. Then,* $\mathbb{P}$*-a.s.*

$$d_f(\mathcal{A}(\omega)) < +\infty,$$

where

$$d_f(K) \doteq \limsup_{\epsilon\to 0} \frac{\log N_\epsilon(K)}{\log(1/\epsilon)}$$

denotes the fractal dimension of a compact set $K \subset \mathbb{B}$*, where* $N_\epsilon(K)$ *is the minimum number of balls of radius* ϵ *necessary to cover* K*.*

2.2.2. *Forward and Pullback determining modes.* The following theorem shows the dependence of the asymptotic behavior, starting with two initial data, on a finite number of degrees of freedom (Langa [28], Theorem 2, and Flandoli and Langa, Theorem 2):

Theorem 2.2. *Suppose* (1) *and* (2) *hold. Then,*

a) *(Forward determining modes)*
given $u_0, v_0 \in \mathbb{B}$, suppose that for some $\alpha \geq 0$, we have, $\mathbb{P}$-a.s., that

$$\lim_{t\to+\infty} |\boldsymbol{P}(\Phi(t,\omega)u_0 - \Phi(t,\omega)v_0)| \leq \alpha.$$

Then,

$$\lim_{t\to+\infty} |\Phi(t,\omega)u_0 - \Phi(t,\omega)v_0| \leq \alpha.$$

b) *(Pullback determining modes)*
On the other hand, if $t \in \mathbb{R}$ and for all $r \leq t$, $\mathbb{P}$-a.s., and $u_0, v_0 \in \mathbb{B}$

$$\lim_{s\to+\infty} |\boldsymbol{P}(\Phi(r+s,\theta_{-s}\omega)u_0 - \Phi(r+s,\theta_{-s}\omega)v_0)| \leq \alpha,$$

then, for all $r \leq t$,

$$\lim_{s\to+\infty} |\Phi(r+s,\theta_{-s}\omega)u_0 - \Phi(r+s,\theta_{-s}\omega)v_0| \leq \alpha.$$

Note that in b) we need a convergence in all final times $r \leq t$ to get the result. In the next result we will write a weaker hypothesis for this result.

Remark 2.2. a) If $\alpha = 0$ we would have a classical determining modes result (cf. Foias and Prodi [22]).

b) Due to the fact that the pullback convergence to the random attractor implies the forward convergence to this set in probability (Crauel and Flandoli [13]), i.e., for all $\epsilon > 0$

$$\lim_{t\to+\infty} \mathbb{P}(\omega \in \Omega : dist(\Phi(t,\omega)D, \mathcal{A}(\theta_t\omega)) > \epsilon) = 0,$$

we get that our hypotheses in the previous theorem implies those in Chueshov et al. [10], Theorem 2.3, so that the assertion there also holds in our case.

Using Proposition 2 in Langa [28], we also get the pullback convergence in the previous theorem under a weaker condition.

Theorem 2.3. *([28]) Suppose that $u(\omega), v(\omega)$ are two random variables on the attractor $\mathcal{A}(\omega)$ such that $\mathbb{P}$–a.s.*

$$\Phi(t,\omega)u(\omega) \neq \Phi(t,\omega)v(\omega), \quad \text{for all } t \in \mathbb{R}_+ \text{ and}$$
$$\lim_{s\to+\infty} |\boldsymbol{P}_0(\Phi(t+s,\theta_{-s}\omega)u(\theta_{-s}\omega) - \Phi(t+s,\theta_{-s}\omega)v(\theta_{-s}\omega))| = 0$$

whenever $u(\omega) \neq v(\omega)$, $\mathbb{P}$–a.s. (where $\boldsymbol{P}_0$ is a projection which is injective between $\bigcup_{t\in\mathbb{R}} \mathcal{A}(\theta_t\omega)$ and its image, see Langa and Robinson [27] for the existence of such (dense) set of projections). Then, for all $r \leq t$ we have that

$$\lim_{s\to+\infty} |\boldsymbol{P}_0(\Phi(r+s,\theta_{-s}\omega)u(\theta_{-s}\omega) - \Phi(r+s,\theta_{-s}\omega)v(\theta_{-s}\omega))| = 0, \qquad \mathbb{P}\text{–a.s.}$$

3. Generalized Allen-Cahn SPDEs and Random PDEs

3.1. **Definitions.** In this part we consider the SPDE

$$(3) \qquad \begin{cases} \dfrac{\partial U}{\partial t} = \Delta_x U + f(U) + \dfrac{\partial^2 W}{\partial t \partial x}, & (t,x) \in \mathcal{O}_L \doteq (0,+\infty) \times (0,L); \\ U(t,0) = U(t,L) = 0, & 0 < t < \infty; \\ U(0,x) = u_0(x), & 0 < x < L; \end{cases}$$

where $W(t,x)$ is the Brownian sheet corresponding to the driving space-time white noise, written formally as $\partial^2 W/\partial t\partial x$. As noted earlier, we treat white noise as a continuous orthogonal martingale measure, which we denote by $\mathcal{W}$. The drift $f:\mathbb{R}\to\mathbb{R}$ is of the form:

$$f(u)=\sum_{k=0}^{2p-1} a_k u^k,\ \text{with } p\in\mathbb{N},\ \text{and } a_{2p-1}<0, \tag{4}$$

It is not difficult to prove the following elementary inequalities for f (see Temam [32]), which we use in the proof of Theorem 3.2:

i) There exists $K>0$ such that $f'(v)\le K,\ \forall\, v\in\mathbb{R}$.
ii) There exist $c_1, c_0>0$ such that $f(v)v\le -c_1v^{2p}+c_0,\ \forall\, v\in\mathbb{R}$.
iii) There exist $k_1,k_0>0$ such that $|f(v)|\le k_1|v|^{2p-1}+k_0,\ \forall\, v\in\mathbb{R}$.

We denote the SPDE (3) by $e_{AC}(f,u_0)$. We collect here definitions and conventions that are used throughout this article (see Walsh [33] for a whole setting of this type of SPDEs; see also Allouba [2, 3]). Filtrations are assumed to satisfy the usual conditions (completeness and right continuity), and any probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ with such a filtration is termed a usual probability space.

Definition 3.1 (Strong and Weak Solutions to $e_{AC}(f,u_0)$)**.** We say that the pair $(U,\mathcal{W})$ defined on the usual probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ is a continuous or L^2-valued solution to the stochastic PDE $e_{AC}(f,u_0)$ if $\mathcal{W}$ is a space-time white noise on $\mathcal{C}_L\doteq\mathbb{R}_+\times[0,L]$; the random field $U(t,x)$ is $\mathcal{F}_t$-adapted ($U(t,\cdot)\in\mathcal{F}_t\ \forall t$), with either $U\in C(\mathcal{C}_L;\mathbb{R})$ (a continuous solution) or $U\in C(\mathbb{R}_+;L^2(0,L))$ (an L^2-valued solution); and the pair $(U,\mathcal{W})$ satisfies either one of the following two formulations:

(TFF) the test function formulation

$$\begin{aligned}(U(t)-u_0,\varphi)-\int_0^t(U(s),\varphi'')ds&=\int_0^t(f(U(s),\varphi)ds\\&+\int_0^L\int_0^t\varphi(x)\mathcal{W}(ds,dx);\ 0\le t<\infty,\ \text{a.s. }\mathbb{P},\end{aligned}$$

for every $\varphi\in\Theta_0^L\doteq\{\varphi\in C^\infty(\mathbb{R};\mathbb{R}):\varphi(0)=\varphi(L)=0\}$, where $(\cdot,\cdot)$ is the L^2 inner product on $[0,L]$, or

(GFF) the Green function formulation

$$\begin{aligned}U(t,x)=\int_0^L\int_0^t f(U(s,y))G_{t-s}(x,y)dsdy+\int_0^L\int_0^t G_{t-s}(x,y)\mathcal{W}(ds,dy)\\+\int_0^L G_t(x,y)u_0(y)dy;\ 0\le t<\infty\ \text{a.s. }\mathbb{P},\end{aligned}$$

where $G_t(x,y)$ is the fundamental solution to the deterministic heat equation ($u_t=u_{xx}$) with vanishing boundary conditions.

A solution is said to be strong if the white noise $\mathcal{W}$ and the usual probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ are fixed a priori and $\mathcal{F}_t$ is the augmentation of the natural filtration for $\mathcal{W}$ under $\mathbb{P}$. It is termed a weak solution if we are allowed to choose the usual probability space and the white noise $\mathcal{W}$ on it, without requiring that the filtration be the augmented natural filtration of $\mathcal{W}$. We say pathwise uniqueness holds for $e_{AC}(f,u_0)$ if whenever $(U^{(1)},\mathcal{W})$ and $(U^{(2)},\mathcal{W})$ are two solutions to $e_{AC}(f,u_0)$ on the same probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$, and with respect to the same white noise $\mathcal{W}$, then $\mathbb{P}\left[U^{(1)}(t,x)=U^{(2)}(t,x);0\le t<\infty,x\in[0,L]\right]=1$.

We often simply say that U solves $e_{AC}(f,u_0)$ (weakly or strongly) to mean the same thing as above.

Remark 3.1. As it is well known (Walsh [33]), if the drift and diffusion coefficients are locally bounded random fields (in our case they trivially are for continuous solutions since the diffusion coefficient $a \equiv 1$ and the drift f is clearly locally Lipschitz under our conditions in (4), then the two formulations (GFF) and (TFF) are equivalent.

3.2. **Existence and uniqueness of solutions.** Let $\beta \geq 0$; let $Z_\beta(t,x)$ be the pathwise-unique strong solution to (3) with $f(Z_\beta) = -\beta Z_\beta$ and $Z_\beta(0,x) \equiv 0$, which is Hölder continuous with $\alpha_{\text{time}} = 1/4 - \epsilon$ in time and $\alpha_{\text{space}} = 1/2 - \epsilon$ in space, $\forall \epsilon > 0$ (a standard result as in [33] pp. 321-322). Let $V_\beta = U - Z_\beta$, for any solution U to (3). We see then that V_β satisfies

$$
\begin{aligned}
V_\beta(t,x) &= U(t,x) - \int_0^L \int_0^t G_{t-s}(x,y)\left[\mathcal{W}(ds,dy) - \beta Z_\beta(s,y)dsdy\right] \\
&= \int_0^L G_t(x,y)u_0(y)dy + \int_0^L \int_0^t \left[f\left(V_\beta + Z_\beta(s,y)\right) + \beta Z_\beta(s,y)\right] G_{t-s}(x,y)dsdy \\
&\doteq \int_0^L G_t(x,y)u_0(y)dy + I_\beta(t,x) = M(t,x) + I_\beta(t,x).
\end{aligned}
$$

That is, V_β solves the random PDE:

$$
\begin{cases}
\dfrac{\partial V_\beta}{\partial t} = \Delta_x V_\beta + f\left(V_\beta + Z_\beta\right) + \beta Z_\beta, (t,x) \in \mathcal{O}_L; \\
V_\beta(t,0) = V_\beta(t,L) = 0, 0 < t < \infty; \\
V_\beta(0,x) = u_0(x), x \in [0,L].
\end{cases}
\tag{5}
$$

Our first result gives detailed existence, uniqueness, and comparative regularity results of our SPDE $e_{AC}(f,u_0)$ in (3) and the associated random PDE (5).

Theorem 3.1. *Suppose f satisfies* (4).

(i) (*Strong Regularity*) *If $u_0 : [0,L] \to \mathbb{R}$ is Lipschitz continuous and deterministic. Then, the SPDE $e_{AC}(f,u_0)$ has a strong, pathwise-unique, a.s. α-Hölder continuous solution with $\alpha_t = 1/4 - \epsilon$ in time and $\alpha_x = 1/2 - \epsilon$ in space, for all $\epsilon > 0$. On the other hand, under the same conditions on u_0, the random PDE* (5) *has an a.s. $C^{1,2}((0,\infty) \times (0,L); \mathbb{R})$ unique solution.*

(ii) (*Weak Regularity*) *For all $0 \leq s < T$, we have:*

a) *if $u_0 \in L^2(0,L)$, there exist a.s.ũnique solutions U and V to $e_{AC}(f,u_0)$ and* (5)*, respectively, such that*

$$V \in C([s,\infty); L^2(0,L)) \cap L^2(s,T; H_0^1(0,L)) \cap L^{2p}(s,T; L^{2p}(0,L)),$$

and

$$U \in C([s,\infty); L^2(0,L));$$

b) *if $u_0 \in H_0^1(0,L)$, then the a.s. unique solutions U and V are such that*

$$V \in C([s,\infty); H_0^1(0,L)) \cap L^2(s,T; H^2(0,L)) \cap L^{2p}(s,T; L^{2p}(0,L)),$$

and

$$U \in C([s,\infty); L^2(0,L)).$$

and hence, $V \in C([s+\varepsilon,\infty); H_0^1(0,L)) \cap L^2(s+\varepsilon,\infty; H^2(0,L))$, for every $u_0 \in L^2(0,L)$ and $\varepsilon > 0$.

Remark 3.2. In addition to the existence, uniqueness, and regularity for the SPDE $e_{AC}(f,u_0)$, our proof of Theorem 3.1 gives detailed strong, as well as weak, regularity results for the random PDE (5) associated with our SPDE (3). The strong regularity results are for completeness, and they are not needed for the rest of the paper. Two points are worth emphasizing: 1. solutions to the random PDE (5) are typically much smoother than solutions to the Allen-Cahn SPDE $e_{AC}(f,u_0)$ and 2. while increasing the regularity of the initial function u_0 has a considerable effect on smoothing out the random PDE solution (if u_0 is Lipschitz then the solution V is in $C^{1,2}((0,\infty)\times(0,L))$; the most regularity we can guarantee for the Allen-Cahn SPDE solution is Hölder continuity (with Hölder exponents 1/4 in time and 1/2 in space) regardless of how smooth the initial data is. This of course is a direct result of the fact that the driving noise is white in both space and time. In the case of a time only white noise, the solution U to the Allen-Cahn SPDE driven by such noises is spacially much smoother than our solutions (typically at least in $H^1(0,L)$, e.g., see [13]). For more on the effects of our rougher noise on the proof of the existence of the attractor see Remark 3.3 below.

Proof. (of Theorem 3.1) We note that when $p=1$ in (4) f is Lipschitz and the strong existence, pathwise uniqueness, and Hölder regularity for $e_{AC}(f,u_0)$ follow from standard results (see [33]).

We now turn to the case $p>1$. For simplicity and without loss of generality, we assume $\beta=0$. Let $Z \doteq Z_0$ and $V \doteq V_0$. Clearly, the existence and uniqueness for $e_{AC}(f,u_0)$ is equivalent to the existence and uniqueness for the corresponding random PDE (5). This is because Z is the pathwise-unique strong solution (see [33]) to the standard heat SPDE and $V+Z$ is a solution to $e_{AC}(f,u_0)$ if and only if V solves (5). Furthermore $Z(t,x)$ is a.s. α-Hölder-continuous with $\alpha_t=1/4-\epsilon$ in time and $\alpha_x=1/2-\epsilon$ in space, for all $\epsilon>0$ (see [33]), and it vanishes at 0 and L. For the rest of the proof, we fix $\omega\in\Omega$, and treat the path-by-path deterministic version of our random PDE (5). Following the proof of Theorem 1.1 in Temam [32], Chapter III—and for the usual Sobolev spaces $H_0^1(0,L):=\{v\in H^1(0,L), v(0)=v(L)=0\}$ and $H^2(0,L)$—we have $\mathbb{P}$-a.s. that there is a unique continuous (in (t,x)) solution V to (5) satisfying (5) if $u_0:[0,L]\to\mathbb{R}$ is deterministic and continuous. This implies that $|f(V+Z)|\le C_1<\infty$ on $[0,t]\times[0,L]$; thus $I_0(t,\cdot)\in C^1(0,L)$ with $|DI_0(t,x)|\le CC_1t^{\frac{1}{2}}$ (the smoothness for I_0 is obtained throughout as in Theorems 2 to 5 in Chapter 1 of [23]) and hence $V(t,\cdot)\in C^1(0,L)$ for every t (the first term in (5), M, is in $C^2(0,L)$ whenever u_0 is continuous on $[0,L]$). If additionally u_0 is Lipschitz on $[0,L]$; then $f(V+Z)$ is Hölder continuous on $[0,L]$, uniformly locally in t. To see this, remember that when u_0 is Lipschitz on $[0,L]$ then, with M as defined as in (5), we have $M\in C^2(0,L)$ and

$$DM(t,x)=\int_0^L u_0(y)\frac{\partial}{\partial x}G_t(x,y)dy\le K. \tag{6}$$

The bound in (6) again follows from standard analysis methods as in Chapter 1 in [23] (see also Lemma A.3 below for a probabilistic proof of this fact on $\mathbb{R}^d$, $d\ge1$).

The bound in (6) and the bound that we have for $DI_0(t,x)$ imply that V, and hence $f(V+Z)$, is Hölder continuous on $[0,L]$, uniformly locally in t. This, in turns implies that $I_0(t,\cdot)\in C^2(0,L)$ and hence $V(t,\cdot)\in C^2(0,L)$ for every t. The temporal regularity for V is proved similarly and we omit it, and we obtain that $V\in C^{1,2}((0,\infty)\times(0,L);\mathbb{R})$. It is then clear that $U(t,x)=V(t,x)+Z(t,x)$ is the pathwise-unique (because uniqueness holds a.s. for both V and Z) strong solution (because the white noise $\mathcal{W}$ is fixed throughout) of (3), and that U is $\mathbb{P}$ a.s. Hölder continuous under our conditions on u_0 with $\alpha_t = 1/4 - \epsilon$ in time and $\alpha_x = 1/2 - \epsilon$ in space, for all $\epsilon > 0$ (since both V and Z are)

In addition, we also get $\mathbb{P}$-a.s. that for all $0 \le s < T$:

a) if $u_0 \in L^2(0, L)$, there exists a unique solution

$$V \in C([s, \infty); L^2(0, L)) \cap L^2(s, T; H_0^1(0, L)) \cap L^{2p}(s, T; L^{2p}(0, L)),$$

b) if $u_0 \in H_0^1(0, L)$, then there exists a unique solution

$$V \in C([s, \infty); H_0^1(0, L)) \cap L^2(s, T; H^2(0, L)) \cap L^{2p}(s, T; L^{2p}(0, L)),$$

and hence, $V \in C([s + \varepsilon, \infty); H_0^1(0, L)) \cap L^2(s + \varepsilon, \infty; H^2(0, L))$, for every $u_0 \in L^2(0, L)$ and $\varepsilon > 0$. Again, the assertions about the existence, uniqueness and weak regularity of U (part ii) a) and b) in Theorem 3.1) easily follow from the corresponding results for V (parts a) and b) above), the regularity of Z, and the fact that $U(t, x) = V(t, x) + Z(t, x)$. □

3.3. **Growth rates for Z_β.** In this subsection, we obtain asymptotic growth rates of interest related to Z_β.

Lemma 3.1. *Let Z_β be as in the proof of Theorem 3.1. Then Z_β may be rewritten as*

$$Z_\beta(t, x) = \int_0^L \int_0^t G_{\beta, t-s}(x, y) \mathcal{W}(ds, dy),$$

where G_β is the fundamental solution to the noiseless version of (3) *with $f(Z_\beta) = -\beta Z_\beta$, with Dirichlet boundary conditions (*[33]*). Let $\hat{Z}_\beta(t) \doteq \sup_{0 \le x \le L} Z_\beta(t, x)$. Then,*

i) *For each $0 < p < 3$ and $0 \le \gamma < 1 \wedge (3 - p)$, there exists a constant $K > 0$ such that*

$$\int_0^L \int_0^t G_{\beta, t-s}^p(x, y) ds dy \le K(x \wedge (L - x))^\gamma t^{(3-p-\gamma)/2}; \; x \in (0, L),\, t > 0, \beta \ge 0. \tag{7}$$

ii) $\mathbb{P}[\hat{Z}_\beta(t) > t^{\frac{1}{4}+\epsilon}] \le K t^{-\epsilon} \to 0$ *as $t \to \infty$, for every $\epsilon > 0$ and every $\beta \ge 0$ for some universal constant $K > 0$.*

iii) *If*

$$Z_\beta^\varphi(t) \doteq (Z_\beta(t), \varphi) - \int_0^t (Z_\beta(s), \varphi'') ds + \int_0^t (\beta Z_\beta(s), \varphi) ds; \quad 0 \le t < \infty,$$

then, for every $\beta \ge 0$ and $\varphi \in \Theta_0^L$, $Z_\beta^\varphi(t)/t \to 0$ as $t \to \infty$ $\mathbb{P}$-a.s.

Proof. i) The Green function, G_β is easily found to be

$$G_{\beta,t}(x, y) = \frac{e^{-\beta t}}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left(-\frac{(2nL + y - x)^2}{4t} \right) - \exp\left(-\frac{(2nL + y + x)^2}{4t} \right) \right\}.$$

It is clearly enough to prove the estimate on G_β for $\beta = 0$, and we will denote G_0 by simply G. Now, let B^x be the scaled Brownian motion

$B^x = \sqrt{2}\tilde{B}^{x/\sqrt{2}}$, starting at some $x \in \mathbb{L} \doteq (0, L)$, where $\tilde{B}^x$ is a standard Brownian motion starting at x. Let $\tau^x_{\mathbb{L}} \doteq \inf\{t > 0; B^x_t \notin \mathbb{L}\}$; then $G_t(x, y)$ is the density of B^x killed at $\tau^x_{\mathbb{L}}$. We easily have

a) $G_r(x, y) = \mathbb{E}\left[G_{r/4}(B^x_{r/4}, y)1_{[\tau^x_{\mathbb{L}} \geq r/4]}\right]$

b) $\int_0^L G^p_{r/4}(\xi, y)dy \leq K \int_0^L \left[\frac{1}{\sqrt{\pi r}} e^{-(\xi-y)^2/r}\right]^p dy \leq \frac{K}{|r|^{(p-1)/2}}$

where we simply used the Markov property to obtain (3.3) a). Now, applying Hölder inequality to (3.3) a) and then using (3.3) b) we get

$$\begin{aligned}\int_0^L G^p_{t-s}(x, y)dy \quad &\leq \mathbb{E}\left[\int_0^L \left|G_{(t-s)/4}(B^x_{(t-s)/4}, y)\right|^p dy\, 1_{[\tau^x_{\mathbb{L}} \geq (t-s)/4]}\right] \\ &\leq \frac{K\mathbb{P}\left[\tau^x_{\mathbb{L}} \geq (t-s)/4\right]}{|t-s|^{(p-1)/2}}\end{aligned}$$

But we also have

$$\mathbb{P}[\tau^x_{\mathbb{L}} \geq r] \leq (\mathbb{P}[\tau^x_{\mathbb{L}} \geq r])^\gamma \leq K\left[\frac{x \wedge (L-x)}{\sqrt{r}}\right]^\gamma$$

where the inequalities in (3.3) follow from the standard facts: $\mathbb{P}[\tau^x_{\mathbb{L}} \geq r] \leq 1$ and $\mathbb{P}[\tau^x_{\mathbb{L}} \geq r] \leq Kx \wedge (L-x)/\sqrt{r}$. Finally, (8) and (3.3) give us

$$\begin{aligned}&\int_0^t \int_0^L G^p_{t-s}(x, y)dyds \leq \int_0^t \tfrac{K\mathbb{P}[\tau^x_{\mathbb{L}} \geq (t-s)/4]}{(t-s)^{(p-1)/2}}ds \\ &\leq K[x \wedge (L-x)]^\gamma \int_0^t \tfrac{ds}{(t-s)^{(p+\gamma-1)/2}} = K[x \wedge (L-x)]^\gamma t^{(3-p-\gamma)/2}\end{aligned}$$

ii) Using (7) with $p = 2$ and $\gamma = 0$ along with Chebyshev and Burkholder inequalities; we have, for any $\beta \geq 0$, that

$$\mathbb{P}[\left|\hat{Z}_\beta(t)\right| > t^{\frac{1}{4}+\epsilon}] \leq \frac{\mathbb{E}_{\mathbb{P}}|\hat{Z}_\beta(t)|}{t^{\frac{1}{4}+\epsilon}} \leq K\frac{\mathbb{E}_{\mathbb{P}}\left(\int_0^L \int_0^t \sup_{0\leq x\leq L} G^2_{\beta,t-s}(x, y)dsdy\right)^{1/2}}{t^{\frac{1}{4}+\epsilon}} \leq Kt^{-\epsilon},$$

iii) First, note that for every test function $\varphi \in \Theta^L_0$,

$$Z^\varphi_\beta(t) = \int_0^L \int_0^t \varphi(y)\mathcal{W}(ds, dy); \quad 0 \leq t < \infty, \varphi \in \Theta^L_0.$$

Using Doob's maximal inequality, we now have

$$\mathbb{P}\left[\sup_{2^n\leq t\leq 2^{n+1}} \frac{\left|Z^\varphi_\beta(t)\right|}{t} > \epsilon\right] \leq \frac{1}{\epsilon^2}\mathbb{E}_{\mathbb{P}}\left[\sup_{2^n\leq t\leq 2^{n+1}} \left(\frac{Z^\varphi_\beta(t)}{t}\right)^2\right]$$

$$\leq \frac{1}{2^{2n}}\mathbb{E}_{\mathbb{P}}\left[\sup_{2^n\leq t\leq 2^{n+1}} (Z^\varphi_\beta)^2\right] \leq \frac{4}{2^{2n}}\mathbb{E}_{\mathbb{P}}(Z^\varphi_\beta)^2(2^{n+1}) \leq \frac{8K_\varphi L}{2^n};\ \forall\epsilon > 0, n \geq 1,$$

where K_φ is the bound on φ^2. An easy application of Borel-Cantelli lemma gives us that, for every $\varphi \in \Theta^L_0$,

$$Z^\varphi_\beta(t)/t \to 0 \text{ as } t \to \infty,\ \mathbb{P}-\text{a.s.}$$

The proof is complete. □

3.4. **Existence of the semimartingale attractor.** Let U be the solution to $e_{AC}(f,u_0)$ on $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$. For any functions $J(t,x)$ and $j(x)$ let

$$J^{\varphi}(t) \doteq (J(t),\varphi) \text{ and } j^{\varphi} = (j,\varphi);\ \forall\varphi\in\Theta_0^L. \tag{8}$$

Then, by the assumptions on f we easily have that $\{U^{\varphi}(t); t\in\mathbb{R}_+\}$ is a semimartingale on $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ for each φ since (5) gives $\mathbb{P}$–a.s.

$$U^{\varphi}(t) = u_0^{\varphi} + \int_0^t U^{\varphi''}(s)ds + \int_0^t (f(U(s)),\varphi)ds + \int_0^L\int_0^t \varphi(x)\mathcal{W}(ds,dx);\ 0\le t<\infty.$$

So, by Definition 2.1 the random field $U(t,x,\omega) \doteq U(t,x,\omega;u_0)$ is a weak semimartigale sheet, starting at $u_0(x)$. Now, set

$$\Phi_U(t-s,\omega)U(s,\cdot,\omega) = U(t,\cdot,\omega). \tag{9}$$

In particular, we can define in $\mathbb{B}=L^2(0,L)$ the random dynamical system

$$\Phi_U(t,\omega)u_0 = U(t,\cdot,\omega;u_0). \tag{10}$$

As in Definition 2.1, we call a random attractor $\mathcal{A}$ associated with Φ_U a semimartingale attractor (as we mentioned on p. 2, our noise setting allows us to to treat a related type of attractors we call semimartingale measure attractor. More on this in an upcoming article). When we want to emphasize the type of semimartingales captured by the attractor, we say weak-semimartingale functional attractor. Our result for $\mathcal{A}$ can now be stated as

Theorem 3.2. *Suppose f satisfies* (4)*, $u_0\in L^2(0,L)$, and u_0 is deterministic. Then, the SPDE $e_{AC}(f,u_0)$ possesses a finite dimensional semimartingale attractor $\mathcal{A}\subset L^2(0,L)$.*

Remark 3.3. As we mentioned in Remark 3.2, the fact that our driving noise is white in both space and time leads to a much less spatial regularity for our solutions of $e_{AC}(f,u_0)$ as compared to SPDEs driven by noises that are white only in time: in the case of Allen-Cahn SPDEs with noises that are white in time only, the solution U is typically at least in H^1 (see e.g., [13]); while in our case the solution U of $e_{AC}(f,u_0)$ is not even in H^1 (even if we start with a C^∞ initial function u_0). So, the proofs in [13], for example, use the fact that for their time only white noise one may apply the Laplacian to the noise (not to mention the solution of the SPDE); and one may use directly the H_0^1 norm on the solution U of the SPDE. All of these facts do not apply in our case since neither U nor Z_β are even in H_0^1 let alone H^2, and we must proceed differently. Thus, our proof relies heavily on the regularity of the solution V_β to the associated random PDE (5); and the apriori estimates needed to establish the existence of the attractor are substantially harder in our case, requiring more elaborate fundamental inequalities (see the proof below, along with the modified inequalities in the Appendix, and compare it to the proof in [13]). Also, adding to the difficulty in our case is the combination of this lack of spatial regularity and the order of the nonlinearity in $e_{AC}(f,u_0)$, $2p-1$; which makes proving the existence of the attractor in L^2 more difficult than in the case of Burgers type SPDEs (even those driven by space-time white noise), whose nonlinearity is effectively second order.
Finally, if u_0 is assumed to be Lipschitz; then, by Theorem 3.1, the derivatives in ΔV below exist in the strong classical sense; and solutions to (5) are classical.

Proof. (of Theorem 3.2) In light of Remark 2.1, we look at our white noise $\mathcal{W}$ as a two-sided (in time) space-time white noise on $(\Omega, \mathcal{F}, \mathbb{P})$. I.e., if

$$\Omega = \{\omega \in C(\mathbb{R} \times [0,L]; \mathbb{R}) : \omega(0,x) = \omega(t,0) = 0\}$$

with $\mathbb{P}$ being the product measure of two Brownian-sheet measures on the negative and positive time parts of Ω; then $W(t,x) = \omega(t,x)$ and $\mathcal{W}$ is the white noise corresponding to W. We accordingly extend the time domain of the Z_β and V_β to negative time as well, in the obvious standard way, and we also refer to the extended Z_β and V_β as Z_β and V_β, respectively.

It can easily be checked that Φ_U satisfies properties (i) and (ii). Let $\beta > 0$; then, as in the proof of Theorem 3.1, (5) has a unique solution V_β with the same regularity as V for all $-\infty < s < T$. Multiplying (5) by V_β^{2p-1} and integrating over space $[0,L]$ in (5) (with the standard convention $(\Delta V_\beta, V_\beta^{2p-1}) = -(DV_\beta, DV_\beta^{2p-1})$ for $V_\beta \in H_0^1(0,L)$; e.g., [30]); using Young's and Hölder's inequalities repeatedly and the generalized Poincaré inequality $|v|_{L^p(0,L)} \leq L|Dv|_{L^p(0,L)},\ p \geq 1$ along with its consequence $-(DV, DV^{2p-1}) \leq -((2p-1)/p^2L)|v|^{2p}_{L^{2p}(0,L)}$ (see Lemma A.1 and Lemma A.2); and using elementary inequalities on f and elementary manipulations, we get after collecting terms (with $L^{2p} = L^{2p}(0,L)$) that

$$\frac{1}{2p}\frac{d}{dt}|V_\beta(t)|^{2p}_{L^{2p}} \leq -\frac{2p-1}{p^2L}|V_\beta|^{2p}_{L^{2p}}$$

$$+|V_\beta + Z_\beta|^{4p-2}_{L^{4p-2}}\left\{\left[\sum_{\substack{1\leq i\leq 2p-1\\ i \text{ odd}}}\binom{2p-1}{i}K_{1,i}\frac{4p-i-2}{4p-2}\epsilon_i\right] - c_{1,0}\right\}$$

$$+\sum_{\substack{1\leq i\leq 2p-1\\ i \text{ odd}}}\binom{2p-1}{i}\left[K_{0,i}L^{\frac{4p-i-2}{4p-2}}|Z_\beta(t)|^i_{L^{4p-2}} + \frac{iK_{1,i}}{(4p-2)\epsilon_i^{\frac{4p-i-2}{i}}}|Z_\beta(t)|^{4p-2}_{L^{4p-2}}\right]$$

$$+\sum_{\substack{0\leq i\leq 2p-2\\ i \text{ even}}}\binom{2p-1}{i}c_{0,i}|Z_\beta(t)|^i_{L^i} + \beta\left[\frac{\epsilon(2p-1)}{2p}|V_\beta|^{2p}_{L^{2p}} + \frac{1}{2p\epsilon^{2p-1}}|Z_\beta|^{2p}_{L^{2p}}\right]$$

Choosing the Young's ϵ_i's so that

$$c_{1,0} \geq \sum_{\substack{1\leq i\leq 2p-1\\ i \text{ odd}}}\binom{2p-1}{i}K_{1,i}\frac{4p-i-2}{4p-2}\epsilon_i$$

we get

$$\frac{d}{dt}|V_\beta(t)|^{2p}_{L^{2p}} \leq |V_\beta|^{2p}_{L^{2p}}\left[\beta\epsilon(2p-1) - \frac{4p-2}{pL}\right] + 2p\sum_{\substack{0\leq i\leq 2p-2\\ i \text{ even}}}\binom{2p-1}{i}c_{0,i}|Z_\beta(t)|^i_{L^i}$$

$$+2p\left\{\sum_{\substack{1\leq i\leq 2p-1\\ i \text{ odd}}}\binom{2p-1}{i}\left[K_{0,i}L^{\frac{4p-i-2}{4p-2}}|Z_\beta(t)|^i_{L^{4p-2}} + \frac{iK_{1,i}}{(4p-2)\epsilon_i^{\frac{4p-i-2}{i}}}|Z_\beta(t)|^{4p-2}_{L^{4p-2}}\right]\right\}$$

$$+\frac{\beta}{\epsilon^{2p-1}}|Z_\beta(t)|^{2p}_{L^{2p}}$$

Choosing ϵ such that

$$-\lambda = \left[\beta\epsilon(2p-1) - \frac{4p-2}{pL}\right],$$

where $\lambda = \lambda_1/2$ (λ_1 is the first positive eigenvalue for the Laplace operator), we get

$$\frac{d}{dt}|V_\beta(t)|_{L^{2p}}^{2p} + \lambda|V_\beta(t)|_{L^{2p}}^{2p} \le \frac{\beta}{\epsilon^{2p-1}}|Z_\beta(t)|_{L^{2p}}^{2p} + 2p \sum_{\substack{0\le i\le 2p-2 \\ i \text{ even}}} \binom{2p-1}{i} c_{0,i}|Z_\beta(t)|_{L^i}^i$$

$$+2p\left\{\sum_{\substack{1\le i\le 2p-1 \\ i \text{ odd}}} \binom{2p-1}{i}\left[K_{0,i}L^{\frac{4p-i-2}{4p-2}}|Z_\beta(t)|_{L^{4p-2}}^i + \frac{iK_{1,i}}{(4p-2)\epsilon_i^{\frac{4p-i-2}{i}}}|Z_\beta(t)|_{L^{4p-2}}^{4p-2}\right]\right\},$$

where $|Z_\beta(t)|_{L^0}^0 \doteq 1$. Now, picking β large enough (similarly to [14]), using Gronwall's Lemma, and letting $P(t)$ denote the term on the right hand side of the above inequality; we can deduce that there exists an $s_1(\omega)$ such that if $s < s_1(\omega)$ and $-1 \le t \le 0$,

$$|V_\beta(t)|_{L^{2p}}^{2p} \quad \le \kappa\left[|V_\beta(s)|_{L^{2p}}^{2p}e^{\lambda s} + \int_{-\infty}^0 P(r)e^{\lambda r}dr\right] \le r_0(\omega) = 1 + \kappa\int_{-\infty}^0 P(r)e^{\lambda r}dr.$$

The at most polynomial growth of the norms of $Z_\beta(t)$ in $P(t)$ in (11) as $t\to-\infty$ (and hence the finiteness of $r_0 = r_0(\omega)$) follows straightforwardly from standard results concerning the elementary SPDE corresponding to Z_β (e.g., see [16] Lemma 4.1 and the ensuing discussion as well as [14, 33]; see also Lemma 3.1 here). On the other hand, if $\mathbb{A} = -\Delta_x$ (the negative Dirichlet Laplacian), then $e^{-t\mathbb{A}}v(x) = \int_0^L v(y)G_t(x,y)dy$, and so using (5) on the interval $[-1,0]$ and applying the operator $\mathbb{A}^{1/8}$ gives us

$$\begin{aligned} &\left|\mathbb{A}^{\frac18}V_\beta(0)\right|_{L^2} \le \left|\mathbb{A}^{\frac18}e^{-\mathbb{A}}V_\beta(-1)\right|_{L^2} \\ &+\int_{-1}^0\left\{\left|\mathbb{A}^{\frac18}e^{\mathbb{A}s}f(V_\beta(s)+Z_\beta(s))\right|_{L^2} + \beta\left|\mathbb{A}^{\frac18}e^{\mathbb{A}s}Z_\beta(s)\right|_{L^2}\right\}ds \end{aligned} \tag{11}$$

To go further, we use the following Sobolev embedding and smoothing properties of the semigroup $(e^{-t\mathbb{A}})_{t\ge0}$ and $e^{-\mathbb{A}}$:

$$|z|_{L^2} \le C_2|z|_{W^{\frac12,1}} \ \forall z\in W^{\frac12,1}(0,L).$$

(12)

$$|e^{-\mathbb{A}t}z|_{W^{s_2,r}} \le C_1\left(t^{\frac{s_1-s_2}{2}}+1\right)|z|_{W^{s_1,r}} \ \forall z\in W^{s_1,r}(0,L),\ -\infty<s_1\le s_2<\infty, r\ge1$$

$$\left|\mathbb{A}^{\frac18}e^{-\mathbb{A}}\right|_{\mathcal{L}(L^2(0,L))} \le C_0.$$

Now, using (12) with $r=1,\ s_1=-1/4,\ s_2=1/2$ we see that

$$\begin{aligned} &\left|\mathbb{A}^{\frac18}e^{\mathbb{A}s}f(V_\beta(s)+Z_\beta(s))\right|_{L^2} \\ &\le C_1C_2\left(t^{-\frac38}+1\right)\left|\mathbb{A}^{\frac18}f(V_\beta(s)+Z_\beta(s))\right|_{W^{-\frac14,1}} \\ &\le C_1C_2\left(t^{-\frac38}+1\right)|f(V_\beta(s)+Z_\beta(s))|_{L^1} \\ &\le C_1C_2\left(t^{-\frac38}+1\right)\left[k_1|V_\beta+Z_\beta|_{L^{2p-1}}^{2p-1}+k_0L\right] \\ &\le C\left(t^{-\frac38}+1\right)\left[|V_\beta|_{L^{2p-1}}^{2p-1}+|Z_\beta|_{L^{2p-1}}^{2p-1}+1\right] \\ &\le \kappa\left(t^{-\frac38}+1\right)\left[|V_\beta|_{L^{2p}}^{2p-1}+|Z_\beta|_{L^{2p-1}}^{2p-1}+1\right] \end{aligned}$$

$$\leq \kappa \left(t^{-\frac{3}{8}}+1\right)\left[r_0^{\frac{2p-1}{2p}}+|Z_\beta|^{2p-1}_{L^{2p-1}}+1\right], \tag{13}$$

where κ depends on L and p. Using (12), (13), and the fact that $|V_\beta|_{L^2} \leq L^{(p-1)/2p}|V_\beta|_{L^{2p}}$ we arrive at

$$\left|\mathbb{A}^{\frac{1}{8}}V_\beta(0)\right|_{L^2} \leq R_0(\omega) = \kappa_0 r_0^{\frac{1}{2p}} + \kappa_1 \int_{-1}^{0}\left(|s|^{-\frac{3}{8}}+1\right)\left[r_0^{\frac{2p-1}{2p}}+|Z_\beta(s)|^{2p-1}_{L^{2p-1}}+1\right] + \beta\left|\mathbb{A}^{\frac{1}{8}}e^{\mathbb{A}s}Z_\beta(s)\right|_{L^2} ds,$$

where the constants κ_0, κ_1 depend on L and p.

Lastly, if we let $K(\omega)$ be the ball in $\mathcal{D}(\mathbb{A}^{\frac{1}{8}})$ of radius $R_0(\omega) + |\mathbb{A}^{\frac{1}{8}}Z_\beta(0,\omega)|_{L^2}$; then $K(\omega)$ is compact because $\mathbb{A}$ has a compact inverse, and it is obviously an attracting set at time 0. The existence of the attractor follows.

To prove the finite dimensionality of the semimartingale attractor, suppose f satisfies (4) and u_0 is Lipschitz continuous and deterministic. First, observe that (2) is a consequence of the driving space-time white noise being additive. Indeed, for any two solutions $U^{(1)}, U^{(2)}$ of

$$\frac{\partial U}{\partial t} = \Delta_x U + f(U) + \frac{\partial^2 W}{\partial t \partial x}$$

with respect to the same white noise $\mathcal{W}$ (this can always be assured since our solutions are strong) and with corresponding initial data $u_0(x), v_0(x)$, we have that the difference $Y(t) = U^{(1)} - U^{(2)}$ satisfies

$$\frac{\partial Y}{\partial t} = \Delta_x Y + f(U^{(1)}) - f(U^{(2)}). \tag{14}$$

I.e., the space-time white noise no more explicitly drives (14).

We then can follow exactly the computations in Debussche [18], Section 3.1 (see also Flandoli and Langa [21]) to verify (2). Indeed, if we call $z_i = \mathbf{Q}\Phi(t,\omega)u_0$, $i = 1, 2$, then we have, for $z = z_1 - z_2$,

$$\frac{dz}{dt} + Az = \mathbf{Q}(f(U^{(1)}) - f(U^{(2)})),$$

and then, as in Debussche [18], we can write

$$\frac{d}{dt}|z|^2 + \lambda_{m+1}|z|^2 \leq |f(U^{(1)}) - f(U^{(2)})|^2_{L^{6/5}},$$

and, for $m_p = 4(p-1)$,

$$|f(U^{(1)}) - f(U^{(2)})|^2_{L^{6/5}} \leq c(|U^1|^2_{L^{6(p-1)}} + |U^2|^2_{L^{6(p-1)}})^{m_p}|U^1 - U^2|^2.$$

But note that we have obtained an absorbing radius $r(\omega)$ for $|U^i|^2_{L^{6(p-1)}}$, $i = 1, 2$, so that

$$\frac{d|z|^2}{dt} + \lambda_{m+1}|z|^2 \leq C\, r(\omega)^{m_p}|U^1 - U^2|^2,$$

which leads straightforwardly to the squeezing property by Gronwall Lemma for m big enough. That $r(\omega)$ is tempered is a consequence of the at most polynomial growth of this random variable.

On the other hand, (2) is also true by Da Prato and Zabczyk ([17], p. 336; see also Crauel et al. [14], Section 3.2).

Thus, we can conclude that the semimartingale attractor of the SPDE (3) has $\mathbb{P}$-a.s. finite fractal dimension. This follows as in Langa [28], Proposition 3; which generalizes to the stochastic case Lemma 2.2 in Eden et al. [20] (see also Robinson [30]). □

Also, note that we immediately have, because of the RSP, the determining modes result

Theorem 3.3. *The SPDE* (3) *satisfies a (forward and pullback) determining modes result as in Theorems 2.2, 2.3.*

4. Comments and Conclusions

This article is another step in our work–started in [4] and which is being continued in different directions in [5, 6, 7]—of studying the asymptotic behavior of different types of SPDEs driven by space-time white noise. Here, we build on the theory of random attractors; and we generalize it to our space-time continuous orthogonal local martingale measure noise setting. In so doing, we characterize the randomness of our attractor as one coming from semimartingale-type solutions. We believe this characterization is a key step that allows us to use stochastic analytical tools to gain a deeper understanding of the stochastic aspects of these random attractors; and we are hopeful it will point out more clearly the differences between the attractors associated with SPDEs and those associated with their non-random counterparts. One consequence of this characterization would lead to the notions of semimartingale decomposition of the random attractor, and that of semimartingale measure attractor, based on the notion of semimartingale measures, which generalizes the concept of continuous orthogonal semimartingale measures introduced in Allouba [3] and it is different from the measure which is the law of solutions.

We focus in this article on the stochastic Allen-Cahn equations, driven by space-time white noise; and we give a thorough treatment of the semimartingale functional attractor in this case. In particular, the existence of a finite fractal dimension semimartingale attractor and some results on determining modes have been proved.

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Appendix A: Some inequalities

The first lemma generalizes Poincaré's inequality to all $L^p(0,L)$, $p \geq 1$.

Lemma A.1 (L^p Poincaré's Inequality)**.** *Suppose* $v \in C^1((0,L);\mathbb{R})$*, for some* $L > 0$*, with* $v(0) = 0$*; then*

$$|v|_{L^p} \leq L|Dv|_{L^p}, \text{ for all } p \geq 1.$$

Proof. We have $v(x) = \int_0^x Dv(y)dy$, $0 < x \leq L$; and so using Hölder's inequality we get

$$|v(x)| \leq \begin{cases} L^{\frac{1}{p'}}|Dv|_{L^p}; & p > 1 \text{ and } p' = \frac{p}{p-1}, \\ |Dv|_{L^1}. \end{cases}$$

Consequently,

$$|v|_{L^p} \leq \begin{cases} \left(\int_0^L L^{\frac{p}{p'}} |Dv|_{L^p}^p dy\right)^{\frac{1}{p}} \leq L|Dv|_{L^p}; \quad p > 1, \\ L|Dv|_{L^1}. \end{cases}$$

The proof is complete. □

The second inequality gives us a bound on the Laplacian of a function integrated against an odd power of the same function:

Lemma A.2 (Laplacian and Odd Power Integral Inequality)**.** *Suppose $v \in C^2((0,L);\mathbb{R})$, for some $L > 0$, with $v(0) = v(L) = 0$; then*

$$\int_0^L \frac{\partial^2 v}{\partial x^2} \cdot v^{2p-1} dx \leq -\frac{(2p-1)}{p^2 L} |v|_{L^{2p}}^{2p} \text{ for all } p \geq 1.$$

If $v \in C^1((0,L);\mathbb{R})$, for some $L > 0$, with $v(0) = 0$; then

$$-\int_0^L \frac{\partial v}{\partial x} \cdot \frac{\partial v^{2p-1}}{\partial x} dx \leq -\frac{(2p-1)}{p^2 L} |v|_{L^{2p}}^{2p} \text{ for all } p \geq 1.$$

Proof. Let u be the function given by

$$u(x) \doteq \int_0^x \left(\frac{\partial v^p}{\partial y}\right)^2 dy; \ 0 \leq x \leq L.$$

Then $u'(x) = \left(\frac{\partial v^p}{\partial x}\right)^2$ and we have, using Lemma A.1, that

$$\int_0^L \left(\frac{\partial v^p}{\partial y}\right)^2 dy = |u'|_{L^1} \geq \tfrac{1}{L}|u|_{L^1} = \tfrac{1}{L}\int_0^L \int_0^x \left(\tfrac{\partial v^p}{\partial y}\right)^2 dy dx \tag{15}$$

$$\geq \tfrac{1}{L}\int_0^L \left(\int_0^x \tfrac{\partial v^p}{\partial y} dy\right)^2 dx \tag{16}$$

$$= \tfrac{1}{L}|v|_{L^{2p}}^{2p}. \tag{17}$$

Therefore,

$$\int_0^L \frac{\partial^2 v}{\partial x^2} \cdot v^{2p-1} dx = -\int_0^L \tfrac{\partial v}{\partial x} \cdot \tfrac{\partial v^{2p-1}}{\partial x} dx = -(2p-1)\int_0^L \left(v^{p-1} \cdot \tfrac{\partial v}{\partial x}\right)^2 dx$$

$$= -\tfrac{2p-1}{p^2}\int_0^L \left(\tfrac{\partial v^p}{\partial y}\right)^2 dy \leq -\tfrac{2p-1}{p^2 L}|v|_{L^{2p}}^{2p}.$$

where the last inequality follows from (17). □

We now give a probabilistic proof of (6) in the case of the heat equation on $\mathbb{R}^d$; i.e., when $[0,L]$ is replaced with $\mathbb{R}^d$ and $G_t(x,y)$ is replaced with the fundamental solution to the heat equation on $\mathbb{R}^d$, $p_t(x,y)$.

Lemma A.3. *With the notations above, we have*

$$\int_{\mathbb{R}^d} u_0(y) \frac{\partial}{\partial x} p_t(x,y) dy \leq K. \tag{18}$$

for some universal constant $K > 0$ whenever u_0 is Lipschitz.

Proof. Let $B^x = \left\{B_t^x \doteq \sqrt{2}\tilde{B}_t^{x/\sqrt{2}}; 0 \le t < \infty\right\}$, where $\tilde{B}^x = \left\{\tilde{B}_t^x; 0 \le t < \infty\right\}$ is a standard d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$. Then, $p_t(x,y)$ is the density of the scaled Brownian motion B^x on $\mathbb{R}^d$, $p_t(x,y)$, we have

$$\begin{aligned}|D_j M(t,x)| = \left|\int_{\mathbb{R}^d} u_0(y)\frac{\partial}{\partial x_j}p_t(x,y)dy\right| &= \left|\int_{\mathbb{R}^d} u_0(y)\frac{-(x_j-y_j)}{2t}(4\pi t)^{-d/2}e^{-|x-y|^2/4t}dy\right| \\ &= \left|-\frac{1}{2t}\mathbb{E}\left[(x_j - B_t^{j,x})u_0(B_t^x)\right]\right| \le \frac{1}{t}\mathbb{E}\left|(x_j - B_t^{j,x})\left(u_0(B_t^x) - u_0(x)\right)\right| \\ &\le \frac{1}{t}\left[\mathbb{E}(x_j - B_t^{j,x})^2\mathbb{E}(u_0(B_t^x) - u_0(x))^2\right]^{1/2} \\ &\le \frac{K_1}{t}\left[\mathbb{E}(x_j - B_t^{j,x})^2\mathbb{E}\left|B_t^x - x\right|^2\right]^{1/2} \le K_2,\end{aligned}$$

where $D_j = \partial/\partial x_j$ and $B^{j,x}$ is the j-th component of the d-dimensional B^x, $1 \le j \le d$; and where we have used elementary facts about the Brownian motion B^x, Hölder inequality, and the Lipschitz condition on u_0 to get (19). □

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