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STRUCTURE OF THE PULLBACK ATTRACTOR FOR A NON-AUTONOMOUS SCALAR DIFFERENTIAL INCLUSION

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Abstract. The structure of attractors for differential equations is one of the main topics in the qualitative theory of dynamical systems. However, the theory is still in its infancy in the case of multivalued dynamical systems. In this paper we study in detail the structure and internal dynamics of a scalar differential equation, both in the autonomous and non-autonomous cases. To this aim, we will also show a general result on the characterization of a pullback attractor for a multivalued process by the union of all the complete bounded trajectories of the system.

1. **Introduction.** The study of the structure of the global attractor for nonautonomous equations is a challenging problem which has drawn the attention of several authors over the last years. The main idea consists in applying in the nonautonomous case some methods which are similar to those used in the autonomous case. But this is not an easy task. To begin with, it is necessary to define a suitable concept of nonautonomous equilibria. This is done in the particular case of a nonautonomous Chafee-Infante equation (see [5, Chapter 6]; see also [12]), where a global solution with asymptotic stability is introduced as a suitable concept. In this way, it was proved in [14] (see also [5]) that a nonautonomous Chafee-Infante equation possesses a unique positive, bounded, non-degenerate, complete trajectory, which attracts all positive solutions. Such a solution plays the same role as the unique positive equilibria of the autonomous Chafee-Infante equation. We remark that the structure of the whole attractor for the nonautonomous Chafee-Infante equation is still far from being completely understood.

Our aim in this paper is to extend such kind of results to a nonautonomous differential inclusion in which uniqueness of the Cauchy problem fails. Although we intend to study a

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partial differential inclusion in the near future, as a first step we are considering an ordinary differential inclusion, as this allows us to obtain a better understanding of the problem and of the difficulties that could appear in such type of problems. More precisely, for a differential inclusion governed by a nonautonomous Heaviside function we prove that the pullback attractor consists of three nonautonomous equilibria, two of which are positive and stable, whereas the third one is equal to zero and unstable, and the heteroclinic connections between them. Moreover, there exist only bounded complete trajectories which go from the zero solution to the stable equilibria.

For the development of this canonical model, we give a result on the characterization of backwards bounded pullback attractors for multivalued processes as the union of bounded backwards trajectories. This is done in Section 2. Section 3 develops our nonautonomous multivalued differential equation with Heaviside function, for which a full understanding of the internal dynamics can be obtained. In particular, a strong order relation of solutions is shown. The ideas also serve as the framework to describe in detail the structure of the global attractor in the autonomous case. We think this should be the standard behaviour in both positive and negative cones for a nonautonomous multivalued Chafee-Infante equation, which we plan to study in the near future.

2. **Characterization of pullback attractors for multivalued processes.** First, let us recall briefly the main concepts and results of the theory of pullback attractors for multivalued processes.

Let X be a complete metric space with metric ρ and let $P(X)$ be the set of all non-empty subsets of X. Denote $\mathbb{R}_d = \{(t,s) \in \mathbb{R}^2 : t \geq s\}$.

The map $U : \mathbb{R}_d \times X \to P(X)$ is called a multivalued process if:

1. $U(t,t,,\cdot) = Id$ is the identity map;
2. $U(t,s,x) \subset U(t,\tau,U(\tau,s,x))$ for all $s \leq \tau \leq t$, $x \in X$,
   where $U(\tau,s,B) = \cup_{z\in B} U(\tau,s,z)$ for a subet $B \subset X$.

It is called strict if, moreover, $U(t,s,x) = U(t,\tau,U(\tau,s,x))$ for all $s \leq \tau \leq t$, $x \in X$.

For any $t \in \mathbb{R}$, $B \subset X$ the ω-limit set $\omega(t,B)$ is defined by

$$\omega(t,B) = \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} U(t,\tau,B)}.$$

Denote by $\mathscr{B}(X)$ the set of all non-empty bounded subsets of X and by $O_\varepsilon(B) = \{y \in X : dist(y,B) < \varepsilon\}$ an ε-neighborhood of the set B. Let $dist(A,B) = \sup_{y\in A} \inf_{x\in B} \rho(y,x)$.

Definition 2.1. The familiy of sets $\{K(t)\}_{t\in\mathbb{R}}$ is called pullback attracting if attracts every $B \in \mathscr{B}(X)$ in the pullback sense, that is,

$$dist(U(t,s,B),K(t)) \to 0, \text{ as } s \to -\infty. \tag{1}$$

Definition 2.2. The family of compact sets $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ is called a pullback attractor if:

1. It is pullback attracting.
2. $\mathscr{A}(t) \subseteq U(t,s,\mathscr{A}(s))$, for all $t \geq s$ (negative semi-invariance);
3. $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ is minimal in the sense that if $\{K(t)\}_{t\in\mathbb{R}}$ is a pullback attracting family of closed sets, then $\mathscr{A}(t) \subset K(t)$ for all $t \in \mathbb{R}$.

The pullback attractor is strictly invariant if $\mathscr{A}(t) = U(t,s,\mathscr{A}(s))$, for any $t \geq s$.

Theorem 2.3. [9, p.536] *Let us suppose that there exists a pullback attracting family of compact sets* $\{K(t)\}_{t\in\mathbb{R}}$ *and that the map* $x \mapsto U(t,\tau,x)$ *has closed graph for all* $t \geq \tau$.

Then there exists a pullback attractor $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$, which is defined by

$$\mathscr{A}(t) = \overline{\bigcup_{B\in\mathscr{B}(X)} \omega(t,B)}$$

and satisfies $\mathscr{A}(t) \subset K(t)$ for all $t \in \mathbb{R}$.

We can obtain the strict invariance of the pullback attractor with some additional assumptions.

Definition 2.4. The family of sets $\{K(t)\}_{t\in\mathbb{R}}$ is said to be backwards bounded if there exists τ such that the set $K_\tau = \cup_{t\leq\tau} K(t)$ is bounded.

Lemma 2.5. *If U is a strict process possessing a backwards bounded pullback attractor, then this attractor is strictly invariant.*

Proof. Let t^* be such that $B = \cup_{t\leq t^*} A(t)$ is bounded. Since $\mathscr{A}(t)$ is negatively semi-invariant and U is strict, for $\tau \leq t^*$, $\tau \leq s \leq t$ we have

$$\begin{aligned} U(t,s,A(s)) &\subset U(t,s,U(s,\tau,A(\tau))) \\ &= U(t,\tau,A(\tau)) \subset U(t,\tau,B). \end{aligned}$$

Then, passing to the limit as $\tau \to -\infty$ we obtain that $U(t,s,A(s)) \subset A(t)$. □

Remark 1. This lemma was proved in [7] using the stronger condition that $\cup_{t\leq\tau} A(t)$ is bounded for all $\tau \in \mathbb{R}$.

We observe that since a pullback attractor $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ attracts every bounded set, for any backwards bounded family $\{K(t)\}_{t\in\mathbb{R}}$ we have

$$dist\,(U(t,s,K(s)), \mathscr{A}(t)) \leq dist\,(U(t,s,\cup_{r\leq\tau} K(r)), \mathscr{A}(t)) \to 0 \text{ as } s \to -\infty.$$

Hence, we highlight that the pullback attractor attracts in fact not only bounded sets but some families of sets as well. In particular, we obtain the following result.

Lemma 2.6. *If the pullback attractor $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ is backwards bounded, then*

$$dist\,(U(t,s,\mathscr{A}(s)), \mathscr{A}(t)) \to 0 \text{ as } s \to -\infty,$$

that is, it pullback attracts itself.

We would like to give a more detailed characterization of the dynamics inside the pullback attractor by using bounded complete trajectories in a similar way to the single-valued case [5, p.37]. For this aim we need to consider the particular case of generalized processes, which were introduced at first in [1], whereas in [2] the properties of ω-limit sets for such processes were studied. It is worth pointing out that in the autonomous case a comparison between the two approaches to the multivalued case using either multivalued semiflows or multivalued processes is given in [6].

Let us denote $W_\tau = C([\tau,\infty);X)$ and let $\mathscr{R} = \{\mathscr{R}(\tau)\}_{\tau\in\mathbb{R}}$ consists of maps $\varphi \in W_\tau$ satisfying:

$(H1)$ For any $\tau \in \mathbb{R}$ and $x \in X$ there exists $\varphi \in \mathscr{R}(\tau)$ such that $\varphi(\tau) = x$.
$(H2)$ $\varphi_s = \varphi\,|_{[\tau+s,\infty)} \in \mathscr{R}(\tau+s)$ for any $s \geq 0$, $\varphi \in \mathscr{R}(\tau)$ (translation property).

Consider also some additional assumptions, which will be needed in order to obtain good properties. Namely:

$(H3)$ Let $\varphi,\psi\in\mathscr{R}$ be such that $\varphi\in\mathscr{R}(\tau)$, $\psi\in\mathscr{R}(r)$ and $\varphi(s)=\psi(s)$ for some $s\geq r\geq\tau$. Then the function θ defined by

$$\theta(t):=\begin{cases}\varphi(t),\ t\in[\tau,s],\\ \psi(t),\ t\in[s,\infty),\end{cases}$$

belongs to $\mathscr{R}(\tau)$ (concatenation property).

$(H4)$ For any sequence $\varphi^n\in\mathscr{R}(\tau)$ such that $\varphi^n(\tau)\to\varphi_0$ in X, there exists a subsequence φ^{n_k} and $\varphi\in\mathscr{R}(\tau)$ such that

$$\varphi^{n_k}(t)\to\varphi(t),\,\forall t\geq\tau.$$

We define the multivalued map $U:\mathbb{R}_d\times X\to P(X)$ associated with the family $\mathscr{R}$ in the following way:

$$y\in U(t,s,x)\text{ if there is }\varphi\in\mathscr{R}(s)\text{ such that }y=\varphi(t),\ \varphi(s)=x.$$

The following lemma is straightforward to prove.

Lemma 2.7. *If $(H1)-(H2)$ hold, then U is a multivalued process. If, moreover, $(H3)$ holds, then U is a strict multivalued process.*

If $(H4)$ is satisfied, then $x\mapsto U(t,\tau,x)$ has closed graph for all $t\geq\tau$.

Definition 2.8. A map $\gamma:\mathbb{R}\to X$ is called a complete trajectory of $\mathscr{R}$ if

$$\varphi=\gamma|_{[\tau,+\infty)}\in\mathscr{R}(\tau),\text{ for all }\tau\in\mathbb{R}. \tag{2}$$

It is obvious that

$$\gamma(t)\in U(t,s,\gamma(s))\text{ for all }s\leq t. \tag{3}$$

The complete trajectory γ is said to be backwards (forwards) bounded if there exists $\tau\in\mathbb{R}$ such that $\cup_{r\leq t}\gamma(r)$ $(\cup_{r\geq t}\gamma(r))$ is bounded. It is bounded if $\cup_{r\in\mathbb{R}}\gamma(r)$ is a bounded set. Clearly, γ is bounded if and only if it is backwards and forwards bounded.

By the pullback attracting property and (3) it is easy to see that if $\gamma(\cdot)$ is a backwards bounded complete trajectory, then $\gamma(t)\subset\mathscr{A}(t)$ for any $t\in\mathbb{R}$, where $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$ is the pullback attractor. We shall show that if the pullback attractor is backwards bounded, then it consists exactly of all backwards bounded complete trajectories.

Theorem 2.9. *Assume that $(H1)-(H2)$, $(H4)$ hold and that U possesses the backwards bounded pullback attractor $\mathscr{A}(t)$. Then*

$$\mathscr{A}(t)=\{\gamma(t):\gamma\textit{ is a backwards bounded complete trajectory}\}. \tag{4}$$

Proof. We have already seen that $\gamma(t)\in\mathscr{A}(t)$ for any backwards bounded complete trajectory γ.

Let $z\in\mathscr{A}(t)$, $t\in\mathbb{R}$ be arbitrary. For any sequence $s_n\to-\infty$ we have that $z\in\mathscr{A}(t)\subset U(t,s_n,\mathscr{A}(s_n))$. Then there exists $u_n\in\mathscr{R}(s_n)$ such that $z=u_n(t)$ and $u_n(s_n)\in\mathscr{A}(s_n)$. By $(H2)$ we have $v_n^0=u_n|_{[t,\infty)}\in\mathscr{R}(t)$. Thus, $(H4)$ implies that up to a subsequence $v_n^0(r)\to v^0(r)$, for all $r\geq t$, where $v^0\in\mathscr{R}(t)$, $v^0(t)=z$. As $v^0(r)=\lim_{n\to\infty}u_n(r)$ and $u_n(r)\in U(r,s_n,\mathscr{A}(s_n))\subset U(r,s_n,B_\tau)$ for $s_n\leq\tau$, where $B_\tau=\cup_{s\leq\tau}\mathscr{A}(s)$ and τ is chosen in such a way that $B_\tau\in\mathscr{B}(X)$, we obtain that $v^0(r)\in\omega(r,B_\tau)\subset\mathscr{A}(r)$ for any $r\geq t$.

Let now $v_n^1=u_n|_{[t-1,\infty)}\in\mathscr{R}(t-1)$. Since $v_n^1(t-1)=u_n(t-1)\in U(t-1,s_n,\mathscr{A}(s_n))\subset U(t-1,s_n,B_\tau)$, passing to a subsequence $v_n^1(t-1)\to z_{-1}$. Therefore, arguing as before we obtain the existence of $v^1\in\mathscr{R}(t-1)$ and a subsequence of v_n^1 (denoted again by v_n^1) such that $v_n^1(r)\to v^1(r)$ for all $r\geq t-1$. Also, it is clear that $v^1(r)\in\mathscr{A}(r)$, for any $r\geq t-1$, and that $v^1(r)=v^0(r)$ if $r\geq t$. In particular, $v^1(t)=z$.

Proceeding in the same way we can define a sequence of functions $v^j \in \mathscr{R}(t-j)$, $j \in \mathbb{Z}^+$, such that $v^j(r) \in \mathscr{A}(r)$, for any $r \geq t-j$, $v^j(r) = v^{j-1}(r)$, for $r \geq t-j+1$, and $v^j(t) = z$.

We define $v(\cdot)$ by taking the common value of the functions $v^j(\cdot)$ for all $r \in \mathbb{R}$. Thus, $v(\cdot)$ is a complete trajectory of $\mathscr{R}$, $v(t) = z$ and it is backwards bounded since $v(r) \in \mathscr{A}(r)$, for any $r \in \mathbb{R}$, and the pullback attractor is backwards bounded. □

If the pullback attractor is bounded, that is, $\cup_{t\in\mathbb{R}}\mathscr{A}(t)$ is bounded, then it can be characterized by the union of all bounded complete trajectories.

Corollary 2.10. *Assume that $(H1)-(H2)$, $(H4)$ hold and that U possesses the bounded pullback attractor $\mathscr{A}(t)$. Then*

$$\mathscr{A}(t) = \{\gamma(t) : \gamma \textit{ is a bounded complete trajectory}\}. \tag{5}$$

Proof. If the pullback attractor $\mathscr{A}(t)$ is bounded, it follows from (4) that every backwards bounded complete trajectory γ is bounded as well. Hence, the result follows using again (4). □

We can obtain the same results but using condition $(H3)$ instead of $(H4)$.

Theorem 2.11. *Assume that $(H1)-(H3)$ hold and that U possesses the backwards bounded pullback attractor $\mathscr{A}(t)$. Then*

$$\mathscr{A}(t) = \{\gamma(t) : \gamma \textit{ is a backwards bounded complete trajectory}\}. \tag{6}$$

Proof. We know that $\gamma(t) \in \mathscr{A}(t)$ for any backwards bounded complete trajectory γ.

Let $z \in \mathscr{A}(t)$. By $(H1)$ there exists $\gamma^0 \in \mathscr{R}(t)$ such that $\gamma^0(t) = z$. Lemmas 2.5, 2.7 imply that $\gamma^0(r) \in \mathscr{A}(r)$ for any $r \geq t$. Further, since $z \in \mathscr{A}(t) \subset U(t, t-1, \mathscr{A}(t-1))$, there is $v^1 \in \mathscr{R}(t-1)$ satisfying $v^1(r) \in \mathscr{A}(r)$, for all $r \geq t-1$, and $v^1(t) = z$. Concatenating v^1 and γ^0 we obtain using $(H3)$ a function $\gamma^1 \in \mathscr{R}(t-1)$ such that $\gamma^1(r) \in \mathscr{A}(r)$, for all $r \geq t-1$, $\gamma^1(t) = z$ and $\gamma^1(r) = \gamma^0(r)$ if $r \geq t$. In the same way, we can define inductively a sequence of functions $\gamma^j \in \mathscr{R}(t-j)$, $j \in \mathbb{Z}^+$, such that $\gamma^j(r) \in \mathscr{A}(r)$, for all $r \geq t-j$, $\gamma^j(t) = z$ and $\gamma^j(r) = \gamma^{j-1}(r)$ if $r \geq t-j+1$. Taking γ as the common value of γ^j at any point $t \in \mathbb{R}$ we obtain a complete trajectory satisfying $\gamma(t) = z$ and $\gamma(r) \in \mathscr{A}(r)$ for any $r \in \mathbb{R}$. Since the pullback attractor is backwards bounded, γ is bacwards bounded as well. □

Corollary 2.12. *Assume that $(H1)-(H3)$ hold and that U possesses the bounded pullback attractor $\mathscr{A}(t)$. Then*

$$\mathscr{A}(t) = \{\gamma(t) : \gamma \textit{ is a bounded complete trajectory}\}. \tag{7}$$

3. **On the structure of the attractor for a nonautonomous multivalued scalar equation.**

3.1. **Setting of the problem.** Let us consider the differential inclusion

$$\begin{cases} \dfrac{du}{dt} + \lambda u \in b(t)H(u),\ t \geq s, \\ u(s) = u_s, \end{cases} \tag{8}$$

where $\lambda > 0$, H is the Heaviside function given by

$$H(u) = \begin{cases} 1 \text{ if } u > 0, \\ [-1, 1] \text{ if } u = 0, \\ -1 \text{ if } u < 0, \end{cases}$$

and $b : \mathbb{R} \to \mathbb{R}^+$ is a continuous function satisfying

$$0 < b_0 \leq b(t) \leq b_1.$$

The function $u : [s, +\infty) \to \mathbb{R}$ is called a solution of (8) if $u \in C([s, +\infty), \mathbb{R})$, $\dfrac{du}{dt} \in L^\infty_{loc}([s, +\infty), \mathbb{R})$ and there exists $h \in L^\infty_{loc}([s, +\infty), \mathbb{R})$ such that $h(t) \in H(u(t))$, for a.a. $t > s$, and

$$\frac{du}{dt} + \lambda u = b(t)h(t) \text{ for a.a. } t > s. \tag{9}$$

It is obvious that problem (8) possesses at least one solution for every initial data. If $u_s > 0$, the unique solution of the linear problem

$$\begin{cases} \dfrac{du}{dt} + \lambda u = b(t), \\ u(s) = u_s, \end{cases} \tag{10}$$

is clearly a solution to (8). When $u_s < 0$ the same is true replacing $b(t)$ by $-b(t)$ in the right-hand side of (10). Finally, if $u_s = 0$, then $u(t) \equiv 0$ is a solution, though not the only one, as we will see later on.

We will denote by $\mathscr{R}^b_s$ the set of all solutions of (8) starting at s. In the autonomous case, that is, when $b(t) \equiv b > 0$, the set of all solutions starting at $s = 0$ will be denoted by $\mathscr{R}_b$.

We will check first that the concatenation of two solutions gives us a new solution.

Lemma 3.1. *Let $u \in \mathscr{R}^b_\tau$, $v \in \mathscr{R}^b_r$ with $\tau \leq r \leq s$ and $u(s) = v(s)$. Then*

$$w(t) = \begin{cases} u(t) \text{ if } \tau \leq t \leq s, \\ v(t) \text{ if } t \geq s, \end{cases}$$

belongs to $\mathscr{R}^b_\tau$.

Proof. It is clear that w satisfies $w \in C([\tau, \infty), \mathbb{R})$, $\dfrac{dw}{dt} \in L^\infty_{loc}([\tau, +\infty), \mathbb{R})$ and there exists $h \in L^\infty_{loc}([\tau, +\infty), \mathbb{R})$ such that $h(t) \in H(w(t))$, for a.a. $t > \tau$, and

$$\frac{dw}{dt} + \lambda w = b(t)h(t) \text{ for a.a. } t > \tau.$$

The function h is defined by

$$h(t) = \begin{cases} h_1(t) \text{ if } t \in (\tau, s), \\ h_2(t) \text{ if } t > s, \end{cases}$$

where h_1, h_2 are the selections corresponding to u and v respectively in equality (9). □

The goal of this paper is to show that the solutions to (8) generate a multivalued process having a pullback attractor and to give a complete description of its structure.

3.2. **Strong comparison of solutions.** We shall firstly prove some strong comparison results for the solutions of (8).

To start with let us consider the autonomous case $b(t) \equiv b$.

Lemma 3.2. *Let $b(t) \equiv b$ and $x_0 \leq y_0$. Then there exist $\overline{y}, \underline{x} \in \mathscr{R}_b$ with $\overline{y}(0) = y_0$, $\underline{x}(0) = x_0$ such that*

$$x(t) \leq \overline{y}(t) \text{ for all } t \geq 0,\ x(\cdot) \in \mathscr{R}_b,\ x(0) = x_0, \tag{11}$$

$$\underline{x}(t) \leq y(t) \text{ for all } t \geq 0,\ y(\cdot) \in \mathscr{R}_b,\ y(0) = y_0. \tag{12}$$

Proof. Let us prove (11). We consider two cases.

Case 1. $y_0 \geq 0$.

Let $\overline{y}(\cdot)$ be the unique solution to the problem

$$\begin{cases} \dfrac{d\overline{y}}{dt} + \lambda \overline{y} = b, \\ \overline{y}(0) = y_0. \end{cases}$$

It is clear that $\overline{y}(t) \geq 0$ for all $t \geq 0$, and then $\overline{y} \in \mathscr{R}_b$. On the other hand, for any solution $x(t)$ with $x(0) = x_0$ it holds

$$\frac{dx}{dt} + \lambda x(t) = g(t),$$

where $g(t) \in bH(x(t))$ for a.a. t and $g \in L^{\infty}_{loc}(\mathbb{R}^+, \mathbb{R})$. Since $g(t) \leq b$, standard comparison between these two problems gives

$$x(t) \leq \overline{y}(t) \text{ for all } t \geq 0.$$

Case 2. $y_0 < 0$.

In this case it is easy to see that the solutions $x(t)$, $\overline{y}(t)$ to (8) corresponding to x_0 and y_0 are unique and solve the problem

$$\begin{cases} \dfrac{dz}{dt} + \lambda z = -b, \\ z(0) = z_0. \end{cases}$$

Again, standard comparison implies $x(t) \leq \overline{y}(t)$ for all $t \geq 0$.

The proof of (12) is rather similar. □

Corollary 3.3. *If $y_0 \geq 0$, then there exists $y \in \mathscr{R}_b$ such that $y(t) \geq 0$ for all $t \geq 0$.*

Proof. Since $x(t) \equiv 0$ is a solution, it follows directly from Lemma 3.2. □

This is a strong comparison principle. It implies in particular that for every initial data there exists a maximal and a minimal solution. Further, let us consider the nonautonomous case.

Lemma 3.4. *Let $u_s \geq 0$. Then there exists $\overline{y} \in \mathscr{R}_{b_1}$ with $\overline{y}(0) = u_s$ such that*

$$u(t) \leq \overline{y}(t-s) \text{ for all } t \geq s, \tag{13}$$

if $u \in \mathscr{R}_s^b$, $u(s) = u_s$.

Let $u_s > 0$. Then there exists $\underline{x} \in \mathscr{R}_{b_0}$ with $\underline{x}(0) = u_s$ such that

$$0 < \underline{x}(t-s) \leq u(t) \text{ for all } t \geq s, \tag{14}$$

if $u \in \mathscr{R}_s^b$, $u(s) = u_s$.

Proof. Let us prove the existence of $\overline{y}\in\mathscr{R}_{b_1}$. The solution $\overline{y}(\cdot)$ is chosen as the unique solution to the problem

$$\begin{cases} \dfrac{d\overline{y}}{dt}+\lambda\overline{y}=b_1, \\ \overline{y}(0)=u_s, \end{cases}$$

whereas $u(\cdot)$ is the solution of the problem

$$\begin{cases} \dfrac{du}{dt}+\lambda u(t)=g(t),\ t>s, \\ u(s)=u_s, \end{cases} \tag{15}$$

where $g(t)\in b(t)H(u(t))$ for a.a. $t>s$ and $g\in L^{\infty}_{loc}([s,\infty),\mathbb{R})$. Since $\overline{y}(t)\geq 0$ for any $t\geq 0$, we obtain that $\overline{y}\in\mathscr{R}_{b_1}$. The result follows from $b(t)\leq b_1$ and standard comparison theorems.

For the second case, if $u_s>0$, then $\underline{x}(\cdot)$ is chosen as the unique solution to the problem

$$\begin{cases} \dfrac{d\underline{x}}{dt}+\lambda\underline{x}=b_0, \\ \underline{x}(0)=u_s, \end{cases}$$

and it is easy to see that $\underline{x}(t)>0$ for all $t\geq 0$, so $\underline{x}\in\mathscr{R}_{b_0}$. On the other hand, $u(\cdot)$ is a solution of problem (15) and at least in some interval $[s,t_0]$ it is true that $u(t)>0$, so $g(t)=b(t)$ a.e. in (s,t). Again, from $b(t)\geq b_0$ and standard comparison theorems it follows that $u(t)\geq\underline{x}(t-s)$ for all $t\in[s,t_0]$. In fact, this inequality remains true whenever $u(t)>0$. Let $[s,T_{\max})$ be the maximal interval in which $u(t)>0$. Thus, either $T_{\max}=+\infty$ or $u(T_{\max})=0$. The last is not possible as in such a case we would have by continuity that $\underline{x}(T_{\max}-s)\leq 0$, but we have seen that $\underline{x}(t)>0$ for all $t\geq 0$. Therefore, $u(t)\geq\underline{x}(t-s)$ for all $t\geq s$ and we are done. □

Remark 2. If $u_s=0$, it is not possible to obtain (14) even without the assumption $0<\underline{x}(t-s)$, but for any solution $u\in\mathscr{R}^b_s$ satisfying $u(t)\geq 0$ for all $t\geq s$, by taking $\underline{x}(t)\equiv 0$ it is obvious that

$$\underline{x}(t-s)\leq u(t)\ \text{ for all } t\geq s.$$

Corollary 3.5. *If $u_s>0$, then $u(t)>0$ for all $t\geq s$ and all $u\in\mathscr{R}^b_s$ with $u(s)=u_s$.*

If $u_s<0$, then $u(t)<0$ for all $t\geq s$ and all $u\in\mathscr{R}^b_s$ with $u(s)=u_s$.

Hence, for any $u_s\neq 0$, the solution of problem (8) is unique and is given by

$$u(t)=e^{-\lambda(t-s)}u_s+\int_s^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau. \tag{16}$$

If $u_s=0$, there are infinitely many solutions given by

$$u_\infty\equiv 0, \tag{17}$$

$$u_r^+(t)=\begin{cases} 0 \text{ if } s\leq t\leq r, \\ \int_r^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau, \end{cases} \tag{18}$$

$$u_r^-(t)=\begin{cases} 0 \text{ if } s\leq t\leq r, \\ -\int_r^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau, \end{cases} \tag{19}$$

where $r\geq 0$ is arbitrary, and these are the only possible solutions.

Proof. Let $u_s>0$. The first statement follows from (14). This implies that any $u\in\mathscr{R}^b_s$ with $u(s)=u_s>0$ is a solution of the problem

$$\begin{cases} \dfrac{du}{dt}+\lambda u=b(t), \\ u(s)=u_s, \end{cases} \tag{20}$$

which possesses a unique solution defined by (16). We note that if $u(\cdot)$ is a solution of (8), then $-u(\cdot)$ is also a solution. Thus, the result follows for $u_s < 0$.

Let $u_s = 0$. It is obvious that $u_\infty \equiv 0$ is a solution. Using Lemma 3.1, the solution u_r^+ is obtained concatenating u_∞ with the unique solution to the problem

$$\begin{cases} \dfrac{du^+}{dt} + \lambda u^+ = b(t), \\ \quad u^+(r) = 0, \end{cases} \tag{21}$$

given by the expression

$$u^+(t) = \int_r^t e^{-\lambda(t-\tau)} b(\tau)\, d\tau.$$

Since $u^+(t) \geq 0$ for all $t \geq r$ we can see that $u^+(\cdot)$ is a solution to (8). In a similar way we prove that u_r^- is a solution.

There cannot be any other solutions. Indeed, let $u(\cdot)$ be an arbitrary solution such that $u(s) = 0$ and $u \neq u_\infty$. In view of the first two statements, there exists $r \geq s$ such that $u(t) = 0$, if $s \leq t \leq r$, and $u(t) \neq 0$ for any $t > r$. In the last situation, suppose for example that $u(t) > 0$ for any $t > r$. Then $u(\cdot)$ has to be the unique solution of problem (21), so $u(t) \equiv u_r^+(t)$. The case where $u(t) < 0$ is treated similarly. □

As the proof of the following lemma is rather similar to the proof of Lemma 3.2, we omit its proof.

Lemma 3.6. *Let $x_s \leq y_s$ and $s \in \mathbb{R}$. Then there exist $\overline{y}, \underline{x} \in \mathscr{R}_s^b$ with $\overline{y}(s) = y_s$, $\underline{x}(s) = x_s$ such that*

$$x(t) \leq \overline{y}(t) \text{ for all } t \geq s,\ x(\cdot) \in \mathscr{R}_s^b,\ x(x) = x_0, \tag{22}$$

$$\underline{x}(t) \leq y(t) \text{ for all } t \geq s,\ y(\cdot) \in \mathscr{R}_s^b,\ y(x) = y_0. \tag{23}$$

Let us come back to problem (8). We will show first that (10) generates a strict multivalued process.

We recall that $\mathscr{R}_\tau^b$ is the set of all solutions of (8) starting at τ. Then we take $\mathscr{R}(\tau) = \mathscr{R}_\tau^b$. It is quite obvious that $(H1)-(H2)$ hold. Also, $(H3)$ follows from Lemma 3.1. Therefore, (8) generates the strict multivalued semiflow U.

In order to prove the existence of a pullback attractor we need to obtain a family of compact pullback attracting sets.

Lemma 3.7. *There exists a bounded pullback attracting family of compact sets $\{K(t)\}_{t\in\mathbb{R}}$.*

Proof. Multiplying (9) by any solution u of (8) we obtain

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \lambda u^2 = b(t)\, h(t)\, u(t) \leq \frac{\lambda}{2} u^2 + \frac{1}{2\lambda} b_1.$$

Hence,

$$|u(t)|^2 \leq e^{-\lambda(t-s)} |u(s)|^2 + \frac{b_1}{\lambda^2}, \tag{24}$$

and the required family is given by $K(t) = \{y \in \mathbb{R} : |y| \leq \frac{\sqrt{b_1}}{\lambda}\}$. □

Lemma 3.8. $(H4)$ *is satisfied.*

Proof. Let $x_n \to x$ and $u_n \in \mathscr{R}_s^b$ be such that $u_n(s) = x_n$.

First, assume that $x > 0$. Then $x_n > 0$ for $n \geq n_0$. In view of Corollary 3.5 the function $u_n(\cdot)$ is the unique solution to problem (20) with initial data x_n. Hence, it converges in

$C([s,\infty),\mathbb{R})$ to the unique solution u to problem (20) with initial data x, which belongs to $\mathscr{R}_s^b$.

The case $x<0$ is proved in a similar way.

Let $x=0$. Passing to a subsequence if necessary, one can consider without loss of generality only the following three cases.

Case 1. $x_n>0$ for all n.

As before, the function $u_n(\cdot)$ is the unique solution to problem (20) with initial data x_n, which converges in $C([s,\infty),\mathbb{R})$ to the function $u_s^+\in\mathscr{R}_s^b$ defined in (18).

Case 2. $x_n<0$ for all n.

The proof is similar to the previous one

Case 3. $x_n=0$ for all n.

Using Corollary 3.5 one can suppose without loss of generality that one of the following options is satisfied for any n: $u_n=u_\infty$, $u_n=u_{r_n}^+$ or $u_n=u_{r_n}^-$, where $r_n\geq s$.

If $u_n=u_\infty$, it is obvious that $u_n\to 0\in\mathscr{R}_s^b$ in $C([s,\infty),\mathbb{R})$.

If $u_n=u_{r_n}^+$, we consider two cases. First, assume that r_n is bounded, so that up to a subsequence $r_n\to\bar{r}$. It follows easily that $u_n\to u_{\bar{r}}^+\in\mathscr{R}_s^b$ in $C([s,\infty),\mathbb{R})$. Secondly, if $r_n\to+\infty$, then $u_n\to 0\in\mathscr{R}_s^b$ in $C([s,\infty),\mathbb{R})$.

The case $u_n=u_{r_n}^-$ is similar to the previous one. □

Theorem 3.9. *The multivalued process U possesses the strictly invariant pulback attractor $\{\mathscr{A}(t)\}_{t\in\mathbb{R}}$. Moreover, it is bounded and*

$$\mathscr{A}(t)=\{\gamma(t):\gamma \text{ is a bounded complete trajectory}\}. \tag{25}$$

Proof. The multivalued process U is strict and the map $x\mapsto U(t,\tau,x)$ has closed graph for all $t\geq\tau$ in view of Lemma 3.8. On the other hand, from Lemma 3.7 there exists a bounded pullback attracting family of compact sets $\{K(t)\}_{t\in\mathbb{R}}$. The existence of the strictly invariant pullback attractor follows from Theorem 2.3 and Lemma 2.5. The characterization (25) is a consequence of Corollary 2.12. Since $\mathscr{A}(t)\subset K(t)$, the pullback attractor is bounded. □

Sometimes it is interesting due to physical motivations to consider only non-negative solutions. We cannot state that the positive cone $\mathbb{R}^+$ is positively invariant, as negative solutions can appear for initial data equal to zero. However, we can define in $\mathbb{R}^+$ a process U_b^+. Namely,

$$\mathscr{R}_s^{b,+}=\{u\in\mathscr{R}_s^b:u(t)\geq 0\ \forall t\geq s\}.$$

It is clear that any $u\in\mathscr{R}_s^b$ with $u(s)>0$ belongs to $\mathscr{R}_s^{b,+}$, and similary to the previous arguments one can check that $\mathscr{R}_s^{b,+}$ satisfies properties $(H1)-(H4)$. Hence, the map $U_b^+:\mathbb{R}_d\times\mathbb{R}^+\to P(\mathbb{R}^+)$ given by

$$U_b^+(t,s,x)=\{u(t):u\in\mathscr{R}_s^{b,+},\ u(s)=x\}$$

is a strict multivalued process and $U_b^+(t,s,x)=U_b(t,s,x)$ if $x>0$. The existence of a strictly invariant pulback attractor $\{\mathscr{A}^+(t)\}_{t\in\mathbb{R}}$ satisfying (25) is proved in a similar way.

3.3. **Structure of the pullback attractor.** In this section we will provide a full description of the dynamics inside the pullback attractor.

First, we shall obtain an upper bound of any bounded complete trajectory of problem (8). Hence, we obtain at once a lower bound.

Theorem 3.10. *There exists a maximal bounded complete trajectory $\xi_M(\cdot)$ of $\mathscr{R}_b$, which means that for any bounded complete trajectory $\psi(\cdot)$ we have*

$$-\xi_M(t) \le \psi(t) \le \xi_M(t) \text{ for all } t \in \mathbb{R}.$$

It is defined by

$$\xi_M(t) = \int_{-\infty}^{t} e^{-\lambda(t-\tau)} b(\tau)\, d\tau.$$

Moreover, for any $u_0 \neq 0$ we have

$$\lim_{s\to-\infty} U_b(t,s,u_0) = \xi_M(t) \text{ if } u_0 > 0, \tag{26}$$

$$\lim_{s\to-\infty} U_b(t,s,u_0) = -\xi_M(t) \text{ if } u_0 < 0, \tag{27}$$

and the interval $[-\xi_M(t), \xi_M(t)]$ is positively invariant.

Proof. For any $x > 0$ we have

$$U_b(t,s,x) = e^{-\lambda(t-s)}x + \int_s^t e^{-\lambda(t-\tau)} b(\tau)\, d\tau \to \int_{-\infty}^{t} e^{-\lambda(t-\tau)} b(\tau)\, d\tau \overset{def}{=} \xi_M(t)$$

as $s \to -\infty$. It is obvious from $b_0 \le b(t) \le b_1$ that

$$\frac{b_0}{\lambda} \le \xi_M(t) \le \frac{b_1}{\lambda}.$$

Also, $\xi_M(\cdot)$ is a complete trajectory, which follows from

$$U_b(t,\tau,\psi(\tau)) = \lim_{s\to-\infty} U_b(t,\tau,U_b(\tau,s,x)) = \lim_{s\to-\infty} U_b(t,s,x) = \xi_M(t).$$

Here, we have used $(H4)$ and the uniqueness of solutions for positive initial data.

Hence, (26) is proved and (27) follows from the fact that if $u(\cdot)$ is a solution of (8), then $-u(\cdot)$ is also a solution.

Let now $\psi(\cdot)$ be an arbitrary bounded global trajectory. Let $\phi > 0$ be such that $\psi(s) \le \phi$ for all n and s. Then, $\psi(t) \in U_b(t,s,\psi(s))$ and Lemma 3.6 imply that

$$\psi(t) \le U_b(t,s,\phi) \to \xi_M(t) \text{ as } s \to -\infty.$$

The inequality $-\xi_M(t) \le \psi(t)$ is obtained in the same way.

Finally, using again Lemma 3.6 and the uniqueness of solutions for positive initial data for any $-\xi_M(s) \le u_s \le \xi_M(s)$, $u \in \mathscr{R}_s^b$, $u(s) = u_s$, $t \ge s$, we have

$$-\xi_M(t) = U_b(t,s,-\xi_M(s)) \le u(t) \le U_b(t,s,\xi_M(s)) = \xi_M(t),$$

so the interval $[-\xi_M(t), \xi_M(t)]$ is positively invariant. □

We shall show further that $\xi_M(\cdot)$ $(-\xi_M(\cdot))$ is the only bounded strictly positive (negative) complete trajectory.

Lemma 3.11. *If $\psi(\cdot)$ is a bounded complete trajectory of $\mathscr{R}_b$ such that $\psi(t) > 0$ ($\psi(t) < 0$) for all $t \in \mathbb{R}$, then $\psi(t) \equiv \xi_M(t)$ ($\psi(t) \equiv -\xi_M(t)$)*

Proof. Let us consider the case where $\psi(t) > 0$. In view of Corollary 3.5 $\psi(\cdot)$ is a solution of problem (20) on any interval $[s,t]$, so

$$\psi(t) = e^{-\lambda(t-s)}\psi(s) + \int_s^t e^{-\lambda(t-\tau)} b(\tau)\, d\tau.$$

As $s < t$ is arbitrary, passing to the limit as $s \to -\infty$ we obtain that

$$\psi(t) = \int_{-\infty}^{t} e^{-\lambda(t-\tau)} b(\tau)\, d\tau = \xi_M(t) \text{ for any } t \in \mathbb{R}.$$

The second case is proved in a similar way. □

Moreover, one can prove that every solution with positive (negative) initial data approaches to the complete solution $\xi_M(t)$ $(-\xi_M(t))$ asymptotically as $t \to +\infty$.

Lemma 3.12. *Let $u_s > 0$, $v_s < 0$. Then*

$$|U_b(t,s,u_s) - \xi_M(t)| \to 0,$$
$$|U_b(t,s,v_s) + \xi_M(t)| \to 0, \text{ as } t \to +\infty.$$

Proof. By Corollary 3.5 we have

$$U_b(t,s,u_s) = e^{-\lambda(t-s)}u_s + \int_s^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau.$$

Hence,

$$|U_b(t,s,u_s) - \xi_M(t)| = \left| e^{-\lambda(t-s)}u_s + \int_s^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau - \int_{-\infty}^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau \right|$$
$$\leq e^{-\lambda(t-s)}u_s + e^{-\lambda t}\int_{-\infty}^s e^{\lambda\tau}b(\tau)\,d\tau \to 0 \text{ as } t \to +\infty.$$

The second case is checked similarly. □

When the initial data is equal to 0, uniqueness of solutions fails. In this particular case, apart from the zero solution, there exist solutions which approach asymptotically to the complete bounded trajectories $\pm\xi_M(t)$ as time increases.

Lemma 3.13. *For any $s \in \mathbb{R}$ there exist at least two bounded complete trajectories ϕ_s^+, ϕ_s^- such that $\phi_s^+(t) = \phi_s^-(t) = 0$, for all $t \leq s$, and*

$$\left|\phi_s^+(t) - \xi_M(t)\right| \to 0 \text{ as } t \to +\infty,$$
$$\left|\phi_s^-(t) + \xi_M(t)\right| \to 0 \text{ as } t \to +\infty.$$

Proof. For $t \geq s$ we consider the function u_s^+ defined in (18), which is a solution to (8). Further,

$$\left|u_s^+(t) - \xi_M(t)\right| = \left|\int_s^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau - \int_{-\infty}^t e^{-\lambda(t-\tau)}b(\tau)\,d\tau\right|$$
$$= e^{-\lambda t}\int_{-\infty}^s e^{\lambda\tau}b(\tau)\,d\tau \to 0 \text{ as } t \to +\infty.$$

Then, we concatenate this function with $v(t) = 0$ in $(-\infty, s]$ and obtain the desired bounded complete trajectory ϕ_s^+.

The proof for the other one is rather similar by using u_0^- from (19). □

The functions $\pm\xi_M(t)$, $\phi_s^\pm$ as defined in Theorem 3.10 and Lemma 3.13 and $\psi(t) \equiv 0$ are the only possible bounded complete trajectories of $\mathscr{R}_b$.

Corollary 3.14. *Let $\psi(\cdot)$ be a bounded complete trajectories of $\mathscr{R}_b$. Then either $\psi(t) \equiv \pm\xi_M(t)$, $\psi(t) \equiv 0$ or there exists $s \in \mathbb{R}$ such that $\psi(t) = \phi_s^\pm(t)$.*

Proof. In view of Lemma 3.11 if $\psi(t) > 0$ ($\psi(t) < 0$) for all $t \in \mathbb{R}$, then $\psi(t) \equiv \xi_M(t)$ ($\psi(t) \equiv -\xi_M(t)$). Therefore, assume that for some $s \in \mathbb{R}$ it holds that $\psi(s) = 0$. It follows from Corollary 3.5 that $\psi(t) = 0$, for all $t \leq s$, and also that either $\psi(t) = 0$, for any $t \geq s$, or there exists $r \geq s$ for which either $\psi(t) = u_s^+(t)$ or $\psi(t) = u_s^-(t)$ for all $t \geq s$. Hence, either $\psi(t) \equiv \phi_s^+(t)$ or $\psi(t) \equiv \phi_s^-(t)$. □

At light of the previous results we are able to provide a full description of the structure of the pullback attractor for the multivalued process generated by (8).

In view of (25) the pullback attractor $\mathscr{A}(t)$ consists of all bounded complete trajectories. Three of these trajectories are the functions $\psi_1(t) \equiv \xi_M(t)$, $\psi_2(t) \equiv -\xi_M(t)$, $\psi_3(t) \equiv 0$, which can be considered as nonautonomous equilibria. Corollary 3.14 implies that the other possible complete trajectories are the functions $\phi_s^{\pm}(t)$.

We note that $|\phi_s^+(t) - \xi_M(t)| \underset{t\to+\infty}{\to} 0$ and $\phi_s^+(t) = 0$ for all $t \leq s$ (see Lemma 3.13). Thus, ϕ_s^+ is a heteroclinic connection from 0 to the nonautonomous equilibria ξ_M. In the same way, ϕ_s^- is a heteroclinic connection from 0 to the nonautonomous equilibria $-\xi_M$.

Therefore, the pullback attractor consists of three nonautonomous equilibria and the heteroclininc connections which go from 0 to the non-zero equilibria.

We observe that, in our application, we have defined three nonautonomous equilibria. One of them is a classical equilibrium and the other ones are defined as the unique bounded complete trajectories which are not equal to 0 at any point. For a more general concept of nonautonomous equilibria see [5, Chapter 13].

If we consider the multivalued process U_b^+, then it can be proved in a similar way the existence of a pullback attractor, which is characterized by the two equilibria $\psi_1(t) \equiv \xi_M(t)$, $\psi_3(t) \equiv 0$ and the heteroclinic connections from 0 to the nonautonomous equilibria ξ_M.

Remark 3. Using the methods developed in [8], [11] and [15] we might study other properties for the solutions of problem (8) such as regularity in stronger spaces or the existence of trajectory attractors. However, in this paper we focus mainly on the structure of the pullback attractor.

3.4. **Structure of the global attractor in the autonomous case.** Let us consider now the autonomous inclusion, that is, the case where $b(t) \equiv b > 0$. We recall that the set of all solutions starting at $s = 0$ is denoted by $\mathscr{R}_b$, which is a subset of the space $C([0,\infty);\mathbb{R})$.

In a similar way to the nonautonomous case one can check that this set satisfies the following properties:

$(K1)$ For any $x \in \mathbb{R}$ there exists $\varphi \in \mathscr{R}_b$ such that $\varphi(0) = x$.

$(K2)$ $\varphi_\tau(\cdot) = \varphi(\cdot + \tau) \in \mathscr{R}_b$ for any $\tau \geq 0$, $\varphi(\cdot) \in \mathscr{R}_b$ (translation property).

$(K3)$ Let $\varphi_1, \varphi_2 \in \mathscr{R}_b$ be such that $\varphi_2(0) = \varphi_1(s)$, where $s > 0$. Then the function $\varphi(\cdot)$, defined by

$$\varphi(t) = \begin{cases} \varphi_1(t) \text{ if } 0 \leq t \leq s, \\ \varphi_2(t-s) \text{ if } s \leq t, \end{cases}$$

belongs to $\mathscr{R}_b$ (concatenation property).

$(K4)$ For any sequence $\varphi^n(\cdot) \in \mathscr{R}_b$ such that $\varphi^n(0) \to \varphi_0$ in $\mathbb{R}$, there exists a subsequence φ^{n_k} and $\varphi \in \mathscr{R}_b$ such that

$$\varphi^{n_k}(t) \to \varphi(t), \forall t \geq 0.$$

We can define the multivalued map $G_b : \mathbb{R}^+ \to P(\mathbb{R})$ by

$$G_b(t,x) = \{y : y = u(t),\ u \in \mathscr{R}_b,\ u(0) = x\}.$$

Since $(K1)-(K3)$ hold, G_b is a strict multivalued semiflow [10, Lemma 5]. This means that $G_b(0,\cdot)$ is the identity map and that

$$G_b(t+s,x) = G_b(t, G_b(s,x)) \text{ for all } x \in \mathbb{R},\ t \geq s \geq 0.$$

Lemma 3.2 implies that the multivalued semiflow G_b is order-preserving in the following sense [3]: if $x_0 \leq y_0$, then

1. there exists $\underline{x}(t) \in G_b(t,x_0)$ such that
$$\underline{x}(t) \le y(t), \quad \text{for all } y(t) \in G_b(t,y_0);$$
2. there exists $\overline{y}(t) \in G_b(t,y_0)$ such that
$$x(t) \le \overline{y}(t), \quad \text{for all } x(t) \in G_b(t,x_0).$$

For short, we denote this property by $G_b(t,x_0) \le G_b(t,y_0)$ whenever $x_0 \le y_0$.

The compact set $\mathscr{A}$ is said to be a global attractor for G_b if:

1. $\mathscr{A} \subset G_b(t,\mathscr{A}), \forall t \ge 0$ (negatively semi-invariance);
2. For any bounded set $B \subset \mathbb{R}$,
$$dist(G_b(t,B),\mathscr{A}) \to 0, \text{ as } t \to +\infty. \tag{28}$$
3. It is minimal, that is, for any closed set C satisfying (28) it holds $\mathscr{A} \subset C$.

It is called invariant if, moreover, $\mathscr{A} = G_b(t,\mathscr{A})$, for all $t \ge 0$.

Property $(K4)$ implies that the map $x \mapsto G_b(t,x)$ has closed graph for any $t \ge 0$. Also, in view of Lemma 3.7 the compact set $K = \{y \in \mathbb{R} : |y| \le \frac{\sqrt{b_1}}{\lambda}\}$ is attracting, that is,
$$dist(G_b(t,B),K) \to 0, \text{ as } t \to +\infty,$$
for every bounded set B.

Hence, G_b possesses the global invariant attractor $\mathscr{A} = \omega(K)$, where $\omega(K)$ is the ω-limit set given by $\omega(K) = \bigcap_{s\ge 0} \overline{\bigcup_{t\ge s} G_b(t,B)}$ (see [16, p.11-12] or [13, Theorem 4 and Remark 7]).

In order to study the structure of the attractor we need to recall the concepts of equilibria (or fixed points) and of complete trajectories of $\mathscr{R}_b$.

We say that $z \in \mathbb{R}$ is a fixed point of $\mathscr{R}_b$ if $\varphi(t) \equiv z$ belongs to $\mathscr{R}_b$. In view of $(K1)-(K4)$ this is equivalent to the property
$$z \in G_b(t,z) \text{ for any } t \ge 0.$$
See [10, Lemma 7]. It is also easy to see that z is a fixed point if and only if
$$\lambda z \in bH(z).$$
Hence, the autonomous problem (8) possesses three fixed points given by
$$z_1^+ = \frac{b}{\lambda},\ z_1^- = -\frac{b}{\lambda},\ z_0 = 0.$$
We observe that $z_1^+ = \xi_M(t)$, $z_1^- = -\xi_M(t)$, so $z_1^{\pm}$ coincide with the nonautonomous nonzero equilibria. This fact reinforce the choice of $\pm\xi_M(t)$ as generalized equilibria in the nonautonomous case.

A map $\gamma : \mathbb{R} \to \mathbb{R}$ is called a complete trajectory of $\mathscr{R}_b$ if
$$\gamma(\cdot + h)|_{[0,+\infty)} \in \mathscr{R}_b,\ \forall h \in \mathbb{R},$$
that is, if $\gamma|_{[\tau,+\infty)}$ is a solution of (8) on $(\tau,+\infty)$ for any $\tau \in \mathbb{R}$.

Every complete trajectory $\gamma(\cdot)$ of $\mathscr{R}_b$ satisfies
$$\gamma(t+s) \in G_b(t,\gamma(s)),\ \forall t \ge 0, s \in \mathbb{R}. \tag{29}$$
Conversely, $(K1)-(K4)$ imply that every continuous map $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying (29) is a complete trajectory of $\mathscr{R}_b$ (see [6, Lemma 5] or [10, Lemma 8]).

Let $\mathbb{K}$ be the set of all bounded complete trajectories of $\mathscr{R}_b$. In view of [10, Theorems 9 or 10] the global attractor can be described as the union of all bounded complete trajectories:
$$\mathscr{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma \in \mathbb{K}\}.$$

Corollary 3.14 implies that if γ is a bounded complete trajectory, then either it is a fixed point or

$$\gamma(t) = \phi_s^+(t) = \begin{cases} 0 \text{ if } t \leq s, \\ \frac{b}{\lambda}\left(1 - e^{-\lambda(t-s)}\right) \text{ if } t \geq s, \end{cases}$$

$$\gamma(t) = \phi_s^-(t) = \begin{cases} 0 \text{ if } t \leq s, \\ -\frac{b}{\lambda}\left(1 - e^{-\lambda(t-s)}\right) \text{ if } t \geq s, \end{cases}$$

where $s \in \mathbb{R}$. Therefore, the attractor is characterized by the fixed points and the bounded complete trajectories connecting them. The trajectories which are not equilibria tend to $z_0 = 0$ when $t \to -\infty$ (in fact they reach 0 at some s and remain there for all $t \leq s$) and converg to z_1^+ or z_1^- as $t \to +\infty$. In other words, it consists of the three fixed points and the heteroclinic connections which go from 0 to either z_1^+ or z_1^-.

In conclusion, we can see that using the generalized concept of nonautonomous equilibria the structure of the global attractor in the autonomous case and of the pullback attractor in the nonautonomous case share the same dynamical features.

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