Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem

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In this paper, for both the sharp front surface quasi-geostrophic equation and the Muskat problem, we rule out the "splash singularity" blow-up scenario; in other words, we prove that the contours evolving from either of these systems cannot intersect at a single point while the free boundary remains smooth. Splash singularities have been shown to hold for the free boundary incompressible Euler equation in the form of the water waves contour evolution problem. Our result confirms the numerical simulations in earlier work, in which it was shown that the curvature blows up because the contours collapse at a point. Here, we prove that maintaining control of the curvature will remove the possibility of pointwise interphase collapse. Another conclusion that we provide is a better understanding of earlier work in which squirt singularities are ruled out; in this case, a positive volume of fluid between the contours cannot be ejected in finite time.

incompressible flow | porous media

We consider the following general transport evolution equation:

$$\begin{aligned}&\zeta_t(x,t)+v(x,t)\cdot\nabla\zeta(x,t)=0,\quad x\in\mathbb{R}^2,\quad t\in[0,\infty),\\&\zeta(x,0)=\zeta_0(x),\end{aligned}\qquad [1]$$

where ζ is an active scalar driven by the incompressible velocity $v(x,t)$:

$$\nabla\cdot v(x,t)=0.\qquad [2]$$

Depending upon our choice of the relation between the velocity and the scalar, we will obtain from this system both the surface quasi-geostrophic (SQG) equation for sharp fronts and the Muskat problem. In this paper, we present a unified method to establish the absence of splash singularities for both of these systems in different scenarios. Specifically, we show that the dynamics of a smooth contour cannot cause an intersection at a single point.

We obtain the SQG equations from systems **1** and **2** by expressing the velocity v in terms of a stream function

$$v=\nabla^{\perp}\psi=(-\partial_{x_2}\psi,\partial_{x_1}\psi),$$

where the function ψ satisfies $\zeta=-(-\Delta)^{1/2}\psi$. Here, $(-\Delta)^{1/2}$ is the Zygmund operator defined on the Fourier side by

$$\widehat{(-\Delta)^{1/2}}=|\xi|.$$

This may be shown to be equivalent to the condition

$$v(x,t)=(-R_2\zeta(x,t),R_1\zeta(x,t)),\qquad [3]$$

which relates the temperature to the velocity by means of the Riesz transforms R_1 and R_2.

The SQG system is physically important as a model of atmospheric turbulence and oceanic flows (see, e.g., refs. 1–3 and the references therein). This equation is derived in the situation of small Rossby and Ekman numbers and constant potential vorticity (4), where the scalar ζ is the evolution over time of the temperature of the fluid. SQG has been the subject of many studies from different points of view. Underlying its mathematical interest are its strong analogies to the 3D Euler equations (see refs. 1 and 5 for these discussions). A very actively studied question for this system has been the formation of singularities in finite time for smooth initial data (see, e.g., refs. 6–11 and references therein).

The SQG system furthermore has been used as a mathematical model in the meteorological process of frontogenesis. Here, the dynamics of hot and cold fluids are studied in the context of the formation and time evolution of weather sharp fronts in which the temperature exhibits discontinuity jumps (further information may be found in ref. 1 and the references therein). In light of this interest, Rodrigo (12) studied the case in which the initial temperature takes two different constant values on complementary domains:

$$\zeta_0(x)=\begin{cases}\zeta^1, & x\in\Omega_0,\\ \zeta^2, & x\in\mathbb{R}^2\backslash\Omega_0,\end{cases}\qquad [4]$$

where $\zeta^1\neq\zeta^2$. The initial data represent sharp fronts, and the interest is in their dynamics, which evolve by SQG. The transport character of Eq. **1** shows that the temperature as it evolves in time should have the form

$$\zeta(x,t)=\begin{cases}\zeta^1, & x\in\Omega(t),\\ \zeta^2, & x\in\mathbb{R}^2\backslash\Omega(t)\end{cases}.\qquad [5]$$

In this formulation, $\Omega(t)$ is a moving domain. Then, a contour dynamics problem is obtained by considering the time evolution

Significance

The formation of singularities for the evolution of the interphase between fluids with different characteristics is a fundamental problem in mathematical fluid mechanics. These contour dynamics problems are given by fundamental fluid laws such as Euler's equation, Darcy's law, and surface quasi-geostrophic (SQG) equations. This work proves that contours cannot intersect at a single point while the free boundary remains smooth—a "splash singularity"—for either the sharp front SQG equation or the Muskat problem. Splash singularities have been shown for water waves. The SQG equation has seen numerical evidence of single pointwise collapse with curvature blow-up. We prove that maintaining control of the curvature will remove the possibility of pointwise interphase collapse, confirming the numerical experiments.

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of the free boundary $\partial\Omega(t)$. For Eqs. **1–3**, the SQG equation for the evolution of a sharp front is then given by

$$x_t(\alpha) = \frac{\zeta^2 - \zeta^1}{2\pi} \int_{\mathbb{T}} \frac{x'(\alpha) - x'(\beta)}{|x(\alpha) - x(\beta)|} d\beta. \quad [6]$$

Here, the boundary is parameterized by the closed one-to-one curve $x(\alpha, t)$:

$$\partial\Omega(t) = \{x(\alpha,t) = (x_1(\alpha,t), x_2(\alpha,t)) : \alpha \in [-\pi, \pi] = \mathbb{T}\},$$

which satisfies the chord-arc condition (see ref. 13 for a detailed derivation of the contour equation in this form). Above the subscript t and the prime notation denote the partial derivatives in time and α (or β), respectively. In **[6]**, the time dependence is disregarded for notational simplicity.

Then, fundamental questions to study are the existence of front-type solutions and the possible singularity formation in the evolution of $\partial\Omega(t)$. These issues are comparable to the vortex-patch problem for the 2D Euler equations (see refs. 5 and 14), but the SQG front system is more singular (see ref. 12 for more details on this discussion).

Local-in-time existence and uniqueness in this situation were proven in ref. 12 for C^∞ contours using Nash–Moser arguments. This tool was used because the operator involved in the contour equation is considerably singular; it loses more than one derivative (Eq. **6**). In ref. 13, the result was extended within the chain of Sobolev spaces because of several cancellations. See also ref. 15 for a proof of local existence for analytic contours. In ref. 16, numerical simulations indicate the possibility of singularity formation on the free boundary. More specifically, initial data were shown in which the curvature blows up numerically because two branches of the fluid interphase collapse in a single point in a self-similar way. Therefore, this work provides an interesting stable scenario for a possible singularity formation. Recently, there has been an active interest in the study of almost sharp front-type weak solutions of the SQG equation (see ref. 17 and the references therein for more details).

We next discuss the Muskat problem; this system models the physical scenario of multiple fluids with different characteristics in porous media. Specifically, we will study the dynamics of interphases between fluids that are immiscible and incompressible. To derive the equations of the Muskat problem, system **1–2** is used, and we choose the velocity $v(x,t)$ to satisfy Darcy's law:

$$v(x,t) = -\nabla p(x,t) - (0, \zeta(x,t)). \quad [7]$$

Above the scalar $p(x,t)$ is the pressure, and in this situation, ζ is the scalar density. Also, the acceleration due to gravity and the viscosity of the fluid are set to unity to simplify the notation. Then, system **1–2** turns out to be the conservation of mass, which together with **[7]** yields the incompressible porous media (IPM) equation (18–20). By considering a solution of the form **5**, the interphase $\partial\Omega(t)$ is a free boundary, and it describes the density jump between each fluid. The evolution equation is given by

$$x_t(\alpha) = \frac{\zeta^2 - \zeta^1}{2\pi} \int \frac{(x_1(\alpha) - x_1(\beta))(x'(\alpha) - x'(\beta))}{|x(\alpha) - x(\beta)|^2} d\beta \quad [8]$$

(see ref. 21 for the whole derivation). Above, $\alpha, \beta \in \mathbb{T}$ for closed contours and $\alpha, \beta \in \mathbb{R}$ for the asymptotically flat case when $x(\alpha,t) - (\alpha, 0) \to 0$ as $\alpha \to \infty$. Further, $\alpha, \beta \in \mathbb{R}$ for periodic curves in the x_1 direction when $x(\alpha + (2\pi, 0), t) = x(\alpha, t)$. The integral in **[8]** is understood as a principal value when that is necessary.

The Muskat problem is a classical well-established problem (22). It has been highly studied, particularly because of strong similarities to the contour dynamics of fluids in Hele-Shaw cells (23). For both these completely different physical scenarios, it therefore is possible to reach similar conclusions. These problems may be studied in the situation of fluids with different viscosities (24) and with surface tension effects (25). Notice that formulation **8** above describes the case in which the viscosities and pressures are equal across the interphase (and **[8]** is in the situation with no surface tension).

The Muskat problem has been shown to exhibit instabilities and ill-posedness in several situations (see, for instance, refs. 18, 20, 24, and 25). For the situation we are studying in this paper, e.g., the contour evolution system (**[8]**), the instabilities in the system will appear when the heavy fluid lies on top of the light one (26).

When the light fluid lies above the heavy fluid, this is called the stable scenario, and in this case, the system has been shown to be well-posed (26). More generally, for the Muskat problem, the well-posedness condition amounts to the positivity of the difference of the gradient of the pressure jump at the interphase in Darcy's law (**[7]**) (27). This condition must hold for the initial data in order for the system to be well-posed (28). It is known in the literature as the Rayleigh–Taylor sign condition (23, 29). The stable framework gives rise to global-existence results for initial data with small norms (24, 25, 30, 31). On the other hand, global existence may be false for certain scenarios with large initial data. In ref. 21, it was proven that initial data exist in the stable regime for Eq. **8** such that the solution turns to the unstable regime in finite time. This interface initially is a smooth stable graph (with the heavier fluid below), but later it enters into an unstable regime. In other words, the interphase is transformed into a non-graph in finite time: when this happens, we say the interphase "turns over." The particular significance of a turnover is that the Rayleigh–Taylor condition breaks down. At some branch in the interphase, it is possible to localize the heavy fluid on top of the lighter one. Then later, the regularity of the contour breaks down (32), i.e., the Muskat problem develops a singularity in finite time starting from regular stable initial data.

We briefly discuss the 2D water waves problem, which is another incompressible fluid interphase dynamics equation. This system can be given by **[1]–[2]** together with the 2D density variable Euler equations:

$$\zeta(x,t)(v_t + v \cdot \nabla v)(x,t) = -\nabla p(x,t) - (0, \zeta(x,t)). \quad [9]$$

We have solutions to this system in the form of **[5]**, which establishes the evolution of a free boundary given by air, $\zeta^1 = 0$, and water, $\zeta^2 = 1$, governed by the gravity force. The velocity is assumed to be rotationally free on each side of $\partial\Omega(t)$ but concentrated on the moving interphase as a delta distribution:

$$\nabla^\perp \cdot v(x,t) = \omega(\alpha,t)\delta(x = x(\alpha,t)). \quad [10]$$

There is a large body of mathematical literature on the 2D water waves problem (see ref. 29 and the references therein). This system has been shown to be well-posed if the Rayleigh–Taylor condition is satisfied initially (33). Recent global-in-time results exist for small initial data (see refs. 34–36 and the references therein). On the other hand, for large initial data with an "overturning shape," the system develops finite time splash singularities (37–39). More precisely, there is a family of initial data satisfying the chord-arc condition such that the interface $x(\alpha,t)$ from the solution of the system **[1]**, **[2]**, **[5]**, **[9]**, and **[10]** satisfying $\zeta^1 = 0$ touches itself at a single point at time $t_s > 0$ while $x(\alpha, t_s)$ is smooth. In particular, the curvature is finite. We also would like to mention recent developments by C. Fefferman, A. D. Ionescu, and V. Lie (40) on the absence of splash singularities for two incompressible fluids.

With the results below, we prove that to have a pointwise collapse, the second derivative and therefore the curvature, must blow up. Splash singularities turn out to be false for the SQG sharp fronts and the Muskat problem. This phenomenon was observed numerically in ref. 16, in which computer solutions of the SQG sharp front system exhibit a pointwise collapse and the curvature blows up at the same finite time.

We also improve the result in ref. 41, in which it is shown that a positive volume of fluid between the contours cannot be ejected in finite time. That result is proved by showing that the velocity is bounded (8) for the Muskat problem for smooth contours. The velocity may be related to the density using singular integral operators with even kernels (**[7]**). Then, the fact that ζ is given by a step function (**[5]**) allows one to show that v is in L^∞. A cancellation used to establish that v is bounded was obtained previously by Bertozzi and Constantin (14). They applied it to the 2D vortex-patch problem to prove global regularity. The present work contributes the information that the level set cannot collapse even pointwise.

The pointwise collapse of smooth level sets, and therefore splash singularities, for regular solutions of SQG, IPM, and general active scalar equations has been studied extensively (see e.g., refs. 8 and 10). Although for initial data that are not necessarily a sharp front (**[4]**), the situation might be a priori less singular, and the problem is still open.

We will explain the proof of our results first for the multiphase Muskat problem. Our reasoning is twofold. First, the Muskat scenario we present is well posed (41), and there are no Rayleigh–Taylor instabilities (42). Second, the proof in this case will appear more clearly. We will consider fluids that have three different constant values for the density:

$$\zeta(x_1,x_2,t)=\begin{cases}\zeta^1 & \text{in} \quad \{x_2>f(x_1,t)\},\\ \zeta^2 & \text{in} \quad \{f(x_1,t)>x_2>g(x_1,t)\},\\ \zeta^3 & \text{in} \quad \{g(x_1,t)>x_2\},\end{cases}$$

where we suppose that $f(x_1,t)>g(x_1,t)$ and that the two dynamic surfaces, which are defined by $x_2=f(x_1,t)$ and $x_2=g(x_1,t)$, can be parameterized as a graph at time $t=0$. The constant densities satisfy

$$\zeta^1<\zeta^2<\zeta^3. \qquad [11]$$

This keeps us in the stable situation. Furthermore, we work in the situation in which

$$\lim_{x_1\to\infty} f(x_1,t)=f_\infty>g_\infty=\lim_{x_1\to\infty} g(x_1,t).$$

Then, our result may be stated as follows:

Theorem 1. *Suppose the free boundaries* $f(\alpha,t)$ *and* $g(\alpha,t)$ *are smooth for* $\alpha\in\mathbb{R}$ and $t\in[0,T)$ *with* $T>0$ *arbitrary. Define the distance*:

$$0<S(t)=\min_{\alpha\in\mathbb{R}}(f(\alpha,t)-g(\alpha,t))\ll\min\{f_\infty-g_\infty,1\}. \qquad [12]$$

Then, the following uniform lower bound for $t\in[0,T)$ *holds*:

$$S(t)\geq\exp\left(\ln(S(0))\exp\left(\int_0^t C(f,g)(s)ds\right)\right). \qquad [13]$$

Here, $C(f,g)$ *is a smooth function of* $||f''||_{L^\infty}+||g''||_{L^\infty}$ *and* $||f||_{L^\infty}+||g||_{L^\infty}$, *which is defined in* **[17]** *below.*

After proving Theorem 1, we will extend these results to the SQG sharp front system based on the previous approach used for the Muskat problem.

Theorem 2. *Consider a smooth curve* $x(\alpha,t)$ *that is a solution to the sharp front SQG system for* $t\in[0,T)$ *with* $T>0$ *arbitrary. Let* $S(t)>0$ *be defined as the minimum distance between two different branches of the interphase that are approaching each other as* $t\to T^+$. *Then,* $S(t)$ *is bounded below by an explicitly computable positive function that goes to zero double-exponentially fast for* t *traveling to infinity*.

Finally, at the end of this paper, we will show additional scenarios in which our result will hold, such as the multiphase SQG system. For Muskat, we also consider the cases of closed contours and overturning shaped interphases, although in those situations, Rayleigh–Taylor instabilities appear and the interphases have to be analytic for there to be bona fide solutions (32).

The Multiphase Muskat Problem

The contour equation for the multiphase Muskat problem may be written as

$$\begin{aligned} f_t(\alpha)&=\int_{\mathbb{R}}\left(\zeta^{21}K(f,f)+\zeta^{32}K(f,g)\right)(\alpha,\beta)d\beta,\\ g_t(\alpha)&=\int_{\mathbb{R}}\left(\zeta^{32}K(g,g)+\zeta^{21}K(g,f)\right)(\alpha,\beta)d\beta,\end{aligned} \qquad [14]$$

where $\zeta^{21}\stackrel{\text{def}}{=}(\zeta^2-\zeta^1)/(2\pi)$, $\zeta^{32}\stackrel{\text{def}}{=}(\zeta^3-\zeta^2)/(2\pi)$,

$$K(f,g)(\alpha,\beta)=\frac{\beta\delta_\beta(f',g')(\alpha)}{\beta^2+\left(\delta_\beta(f,g)(\alpha)\right)^2},$$

$$\delta_\beta(f,g)(\alpha)=f(\alpha)-g(\alpha-\beta),$$

and for simplicity we denote

$$\delta_\beta f(\alpha)=f(\alpha)-f(\alpha-\beta).$$

We remark that it is possible to recover **[8]** by taking $\zeta^{32}=g(\alpha)=0$ and $x(\alpha,t)=(\alpha,f(\alpha,t))$ (see ref. 41 for a detailed derivation of this equation).

We next check the evolution of **[12]** and denote $\alpha_t\in\mathbb{R}$ such that $S(t)=f(\alpha_t,t)-g(\alpha_t,t)$. We use the Rademacher theorem to obtain that $S(t)$ is differentiable almost everywhere and that $S_t(t)=f_t(\alpha_t,t)-g_t(\alpha_t,t)$ (see refs. 43 and 44 for the whole argument). We plug this identity into **[14]** to split the integration regions as

$$\begin{aligned} S_t(t)&=\int_{|\beta|<S(t)}d\beta+\int_{S(t)<|\beta|<1}d\beta+\int_{|\beta|>1}d\beta\\ &=I+II+III.\end{aligned}$$

For the first integral, we bound the kernels K in absolute value using the crucial identity

$$f'(\alpha_t,t)=g'(\alpha_t,t)$$

to find

$$I\leq2\left(\zeta^{21}||f''||_{L^\infty}+\zeta^{32}||g''||_{L^\infty}\right)\int_{|\beta|<S(t)}1d\beta,$$

and therefore we obtain

$$I\leq C(||f''||_{L^\infty}+||g''||_{L^\infty})S(t).$$

For the second integral, we further split $II=\zeta^{21}II_1+\zeta^{32}II_2$. We will show how to deal with II_1 and observe that II_2 is analogous. Notice that we have

$$II_1=\int_{S(t)<|\beta|<1}\frac{\beta\delta_\beta f'(\alpha_t)\left[\left(\delta_\beta(g,f)(\alpha_t)\right)^2-\left(\delta_\beta f(\alpha_t)\right)^2\right]}{D(g,f,\beta)}d\beta, \qquad [15]$$

where the denominator is given by

$$D(g,f,\beta)\stackrel{\text{def}}{=}\left[\beta^2+\left(\delta_\beta f(\alpha_t)\right)^2\right]\left[\beta^2+\left(\delta_\beta(g,f)(\alpha_t)\right)^2\right].$$

We further split

$$II_1=-\int_{S(t)<|\beta|<1}\frac{\beta\delta_\beta f'(\alpha_t)S(t)\delta_\beta(g,f)(\alpha_t)}{D(g,f,\beta)}d\beta-\int_{S(t)<|\beta|<1}\frac{\beta\delta_\beta f'(\alpha_t)S(t)\delta_\beta f(\alpha_t)}{D(g,f,\beta)}d\beta \qquad [16]$$

to obtain

$$II_1\le 2\|f''\|_{L^\infty}S(t)\int_{S(t)<|\beta|<1}|\beta|^{-1}d\beta.$$

The last calculation yields

$$II_1\le -2\|f''\|_{L^\infty}S(t)\ln S(t).$$

The term II_2 can be estimated similarly, and we obtain

$$II\le -C\left(\|f''\|_{L^\infty}+\|g''\|_{L^\infty}\right)S(t)\ln S(t).$$

For the last term III, we arrange the terms as in **[16]** to find

$$III\le C(f,g)S(t)\int_{|\beta|>1}|\beta|^{-2}d\beta,$$

with

$$C(f,g)=C\left(\|f''\|_{L^\infty}+\|g''\|_{L^\infty}\right)\left(\|f\|_{L^\infty}+\|g\|_{L^\infty}+1\right). \qquad [17]$$

Collecting all the previous estimates, we obtain that

$$S_t(t)\ge C(f,g)S(t)\ln S(t).$$

A further time integration yields **[13]**. Notice that $\ln(S(0))<0$. Thus, $S(t)$ cannot go to zero in finite time.

SQG Sharp Front

For the SQG sharp front equation, we choose the parameterization for the contour equation that yields the equation

$$x_t(\alpha)=\int_{\mathbb{T}}\frac{\delta_\beta x'(\alpha)}{|\delta_\beta x(\alpha)|}d\beta, \qquad [18]$$

where we take $\zeta^2-\zeta^1=2\pi$ for the sake of simplicity. We now assume without loss of generality that the pointwise approaching "splash" is going to take place in a small ball B of radius $\epsilon_0/2$ and center $(0,0)$. The two branches of the interfaces will be approaching horizontally so that they are represented by $(\alpha,f(\alpha,t))$ and $(\alpha,g(\alpha,t))$ inside $2B$ with $f>g$. We then find for the chart $x(\alpha)=(\alpha,f(\alpha))$ for $\alpha\in(-\epsilon_0,\epsilon_0)$ the equation

$$f_t(\alpha)=\int_{-\epsilon_0}^{\epsilon_0}\frac{\delta_\beta f'(\alpha)}{\sqrt{\beta^2+\left(\delta_\beta f(\alpha)\right)^2}}d\beta+\int_{\epsilon_0}^{-\epsilon_0}\frac{\delta_\beta(f',g')(\alpha)}{\sqrt{\beta^2+\left(\delta_\beta(f,g)(\alpha)\right)^2}}d\beta+R(f),$$

where $R(f)$ is the remainder in the integral equation in the second component in **[18]**. For the chart $x(\alpha)=(\alpha,g(\alpha))$, we similarly have

$$g_t(\alpha)=\int_{-\epsilon_0}^{\epsilon_0}\frac{\delta_\beta(g',f')(\alpha)}{\sqrt{\beta^2+\left(\delta_\beta(g,f)(\alpha)\right)^2}}d\beta+\int_{\epsilon_0}^{-\epsilon_0}\frac{\delta_\beta g'(\alpha)}{\sqrt{\beta^2+\left(\delta_\beta g(\alpha)\right)^2}}d\beta+R(g),$$

and $R(g)$ is again the remainder given by **[18]**. We now define as before

$$S(t)\stackrel{\text{def}}{=}\min_{[-\epsilon_0,\epsilon_0]}(f(\alpha,t)-g(\alpha,t))=f(\alpha_t,t)-g(\alpha_t,t),$$

with $\alpha_t\in(-\epsilon_0/2,\epsilon_0/2)$. We proceed as in the previous section to follow $S_t(t)$ for almost every t. We find that the integrals above may be handled similarly, except for $R(f)$ and $R(g)$. For these remainder terms, we can choose ϵ_0 small enough so that $R(f)-R(g)=O(S(t))$. In fact,

$$|R(f)|\le\frac{\|x''\|_{L^\infty}}{c_{CA}}\int_{\mathbb{T}\setminus[-\epsilon_0,\epsilon_0]}d\beta,$$

where $c_{CA}>0$ is the chord-arc constant of the curve outside the ball B:

$$|\delta_\beta x(\alpha)|\ge c_{CA}|\beta|,\ \alpha\in[-\epsilon_0/2,\epsilon_0/2],\ \beta\in\mathbb{T}\setminus[-\epsilon_0,\epsilon_0].$$

The analogous estimate for $R(g)$ follows similarly. We thus can obtain

$$S_t(t)\ge C(x)S(t)\ln S(t),$$

where $C(x)=C\left(\|x''\|_{L^\infty},c_{CA},\epsilon_0\right)$. We therefore again control the size of $S(t)$ from below by double-exponential time decay.

Additional Scenarios for Muskat and SQG

This analysis also works for the multiphase SQG sharp front system. In that case, the equations for the 2π -periodic contours $f(\alpha,t)$ and $g(\alpha,t)$ are given by

$$f_t(\alpha)=\int_{\mathbb{T}}\left(\zeta^{21}\Sigma(f,f)+\zeta^{32}\Sigma(f,g)\right)(\alpha,\beta)d\beta,$$
$$g_t(\alpha)=\int_{\mathbb{T}}\left(\zeta^{32}\Sigma(g,g)+\zeta^{21}\Sigma(g,f)\right)(\alpha,\beta)d\beta.$$

Above, $\zeta^{21}=(\zeta^2-\zeta^1)/(2\pi)$ and $\zeta^{32}=(\zeta^3-\zeta^2)/(2\pi)$ with no order needed in the size of ζ^1, ζ^2, and ζ^3 as in **[11]**, because there are no instabilities for SQG. The kernel $\Sigma(f,g)(\alpha,\beta)$ behaves like

$$\Sigma(f,g)(\alpha,\beta)=\frac{\delta_\beta(f',g')(\alpha)}{\sqrt{\beta^2+\left(\delta_\beta(f,g)(\alpha)\right)^2}}$$

for β close to 0 and $f(\alpha)$ close to $g(\alpha-\beta)$. Hence, the same approach as described previously for SQG follows.

We end by proposing two additional scenarios. These are closed and overturning shaped contours for the Muskat equation (**[8]**). In those cases, the same results can be shown as for the

SQG sharp fronts. However, because of the Rayleigh–Taylor instabilities, the solutions to the interphase equations have to be analytic to make rigorous mathematical sense (21).

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