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Locating a competitive facility in the plane with a robustness criterion*

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Abstract A new continuous location model is presented and embedded in the literature on robustness in facility location. The multimodality of the model is investigated, and a branch and bound method based on dc optimization is described. Numerical experience is reported, showing that the developed method allows one to solve in a few seconds problems with thousands of demand points.

Keywords: robustness, facility location, robust solutions, competitive location, Huff model, dc programming, branch and bound.

1 Introduction

The perception of the term robustness in the field of supply chain design, and, more specifically, in facility location, is wide. de Neufville (2004) defines robustness from the perspective of systems as "the ability of a system to maintain its operational capabilities under different circumstances", whereas Dong (2006) defines the robustness of a supply chain network as "the extent to which the network is able to carry out its functions despite some damage done to it, such as the removal of some of the nodes and/or links in a network."

The perception from the viewpoint of design from de Neufville (2004) and Dong (2006) comes close to the idea that robust means that a design

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performs under all circumstances. Robustness from this perspective is a measure of how robust a design is. What is generic here is that performance is seen from a YES/NO perspective. The design performs, works, fulfills specifications in a yes or no sense. If it always does, we perceive a design as robust. Formally, design x is robust if $(x, w) \in Q\ \forall w \in W$, where the set Q is a set of desired performance and W is the set of outcomes of the uncertain parameter w. In other words, if Q_x denotes the set of parameter values w such that $(x, w) \in Q$, x is seen as robust if $w \in Q_x$ for all $w \in W$.

In the literature on facility location, see e.g. Owen and Daskin (1998) and Snyder (2006), the design variable x expresses usually locations of facilities, the uncertain parameters w represent demand, buying power, population etc., and the set Q is described with a threshold concept: a cost or a reward function $f(x, w)$ and a threshold value τ are given, and Q is defined as the set

$$Q = \{(x, w) | f(x, w) \# \tau\}, \tag{1}$$

where # can be $<, >, \leq, \geq$.

Within this framework we can consider two main concepts of robustness, referred to in what follows as *deviation robustness* and *probabilistic robustness.*

In deviation robustness models, a nominal value μ of the uncertain parameter w is given; for any feasible x, one can pose the question of how far deviations from the nominal value μ may go such that the design still performs as intended:

$$R(x) = \min \{\|v - \mu\| : v \notin Q_x\}, \tag{2}$$

where $\| \cdot \|$ is a norm used to measure deviations in the parameter space W. Observe that, for any x not performing properly for the nominal value μ, i.e., $\mu \notin Q_x$, its robustness $R(x)$ is zero. The most robust solution, i.e., the solution x with maximum value for $R(x)$, is sought. Such deviation robustness concept is called in Olieman (2008) the maximum inscribed sphere problem, as (2) means one wishes to find a maximum sized sphere of circumstances around the nominal value μ for which the design is still feasible. This concept has the advantage that no information is needed about the set of realisations, no probability distribution is required nor a range or worst-case outcome. It is thus well suited to problems with very high uncertainty, as happens, for instance, in long-term planning problems as those encountered in facility location.

Under particular forms of function f in (1), a more tractable expression for R can be derived. Indeed, as shown in Hendrix, Mecking and Hendriks (1996) and Carrizosa and Nickel (2003), if $f(x, w)$ is linear in w, i.e., if f has the form

$$f(x, w) = c(x)^\top w \tag{3}$$

for a vector-valued function c, and # is $>$, then $R(x)$ can be expressed as

$$R(x) = \max \left\{ \frac{c(x)^\top \mu - \tau}{\|c(x)\|^\circ}, 0 \right\}, \tag{4}$$

where $\|\cdot\|^\circ$ denotes the norm dual to $\|\cdot\|$. For instance, if $\|\cdot\|$ is the ℓ_p norm, then $\|\cdot\|^\circ$ is the ℓ_q norm, with $1/p + 1/q = 1$.

Observe that, as soon as some x^* exists with strictly positive robustness $R(x^*)$, maximizing R turns out to be equivalent to maximizing $\overline{R}$, defined as

$$\overline{R}(x) = \frac{c(x)^\top \mu - \tau}{\|c(x)\|^\circ}. \tag{5}$$

Deviation robustness has been investigated in Carrizosa and Nickel (2003) within the field of continuous location for Weber problems. In such a problem, $\{p_i\}_{i\in I}$ is the set of demand points, which represent the geographical location of the customers, $f(x,w)$ is defined as the total transportation cost if a facility is located at x, and transportation costs to demand point p_i are assumed to be proportional to the distance $d_i(x)$ between x and demand point p_i. In other words, f is assumed to have the form (3), with $c(x) = (c_i(x))_{i\in I}$, $c_i(x) = d_i(x)$ and each w_i is an uncertain parameter representing the demand of a demand point p_i, for which just a nominal value μ_i is given. Robustness, as defined in (4) is maximized via a finite-time convergent algorithm for particular models of distance functions d_i and choices of $\|\cdot\|$. The reader is also referred to Hendrix et al. (1996) and Casado, Hendrix and García (2007) for applications of deviation robustness to other related problems.

Whereas deviation robustness can be seen as a worst-case measure, the concept of probabilistic robustness takes a probabilistic view, since it considers the circumstances w as a random variable $\mathbf{w}$ and defines robustness as

$$R(x) = P\{(x, \mathbf{w}) \in Q\}. \tag{6}$$

In the literature on location, we see this robustness back under the terminology of "threshold" (without being called robustness) in Drezner, Drezner and Shioge (2002), who address a threshold model which maximizes the probability of reaching a minimum market share τ

$$R(x) = P\{c(x)^\top \mathbf{w} \geq \tau\}, \tag{7}$$

where the functions c_i measure market share according to the Huff model, (Blanquero and Carrizosa 2009a, Drezner et al. 2002),

$$c_i(x) = \frac{1}{1 + h_i d_i^\lambda(x)}, \tag{8}$$

$d_i(x)$ is again the distance from demand point p_i to a facility located at x, $\lambda \geq 1$ (typically $\lambda = 2$) and h_i is typically a positive constant that represents the relative attractiveness of competing facilities.

Even inspecting R in (7) is, in general, very hard, since it involves multivariate calculus. In Drezner et al. (2002) it is assumed that $\mathbf{w}$ has a normal distribution with mean μ and covariance matrix V, and thus R takes the simpler form

$$R(x) = \Phi\left(\frac{c(x)^\top \mu - \tau}{\sqrt{c(x)^\top V c(x)}}\right), \tag{9}$$

where Φ is the cumulative distribution function of the standard normal distribution. Hence, maximizing $R(x)$ is equivalent to maximizing the nonlinear fractional function $\frac{c(x)^\top \mu - \tau}{\sqrt{c(x)^\top V c(x)}}$.

The challenge in general with Huff-like models is that the market share functions c_i in (8) are neither convex nor concave. Hence, optimizing $f(x) = w^\top c(x)$ is a global optimization problem even for w fixed. This is inherited by the threshold model (7), even under the assumption that $\mathbf{w}$ follows a normal distribution. In Drezner et al. (2002), a multistart strategy is used, thus no guarantee of having found the true global optimum is provided. As will be seen later, the probabilistic model (9) can be seen as a particular case of the deviation robustness model, for which a global optimization approach is proposed here.

The remainder of this paper is organized as follows. In Section 2 we introduce the problem of locating a competitive facility in the plane, where competition is described by a Huff-like model, and deviation robustness is to be maximized. It is shown in particular that the model of Drezner et al. (2002) appears as a particular case for a given choice of the norm $\|\cdot\|$.

The multimodal character of the optimization problem is investigated. Deterministic solution approaches that guarantee a global optimum solution are discussed in Section 3. Numerical results are reported in Section 4. Finally we conclude in Section 5.

2 A competitive robustness location model

We address the problem of locating a competitive facility with uncertain demand optimizing a deviation robustness criterion. Users are identified by an index $i \in I := \{1, 2, \ldots, N\}$, a demand location p_i, and a nominal value μ_i for the demand. Market is captured following a Huff model: The market captured by the facility at x given demand w is $f(x, w) = c(x)^\top w$, where c is defined by (8).

The question which is answered is how far demand can fluctuate in a distance sense from its nominal value μ without capturing less than a given threshold value τ. The robustness $R(x)$ of a facility located at x, to be maximized, is given by (4), or (5) if some x^* exists with $R(x^*) > 0$. No assumptions are made on the norm $\|\cdot\|$, and different choices of the norm lead to different models. We may take, for instance, the ℓ_1 or ℓ_∞ norm to measure deviations with respect to the nominal value of the demand. Hence, for a given location x, $R(x)$ measures the maximum deviation (in the dual norm, ℓ_∞ or ℓ_1 respectively) in the demand w with respect to its nominal value μ such that the market captured remains above the threshold value τ.

In particular, the stochastic programming formulation of Drezner et al. (2002) in (9) is a particular case of our model (5) by defining the norm $\|\cdot\|$ as

$$\|w\| = \sqrt{w^\top V^{-1} w}, \tag{10}$$

which means that deviations with respect to the nominal vector of demands μ are measured by the dual $\|\cdot\|^\circ$ of $\|\cdot\|$,

$$\|w\|^\circ = \sqrt{w^\top V w}.$$

As the Huff-like continuous location problem is a global optimisation problem, one may expect that this structure is inherited by $R(x)$. The next research question is how multimodal is such model. We explore how parameters affect the number of local optima and we check the feasibility of nonlinear optimization local search to solve the optimization problem. The following repeatable experiment is carried out. A total of m demand points is randomly generated on $[0,1]^2$ with generated demand μ from $[0,1]$ and k competing facilities. The competing facilities give values to h_i in (8) according to

$$h_i = \sum_{j}^{k} \frac{1}{\delta_{ij}^2}, \tag{11}$$

where δ_{ij} is the Euclidean distance between demand point p_i and existing facility j. The threshold was kept on $\tau = 1$ and the norm $\|\cdot\|$ was the Euclidean norm for all experiments. The resulting objective function of a generated instance is depicted in Figure 1 having $k = 3$ competitors.

To get a feeling for the multimodality of the problem we also generated the same instance, but then having $k = 50$ competitors. As one can observe from Figure 2, the number of local maxima increases substantially. To investigate the trend for increasing number of demand points and competitors, we generated 50 intances for each setting varying the number of demand points as $m = 40, 200, 1000$ and the number of competing facilities as $k = 2, 4, 8, 16, 32$. Multistart using $10 \times k$ random starting points was applied for each generated instance to count the number of local optima found. FMINUNC was used to generate local optima. The average number of detected optima using this multistart strategy is given in Table 1. As can be observed, the number of optima depends mainly on the number of existing competing facilities.

A multistart strategy, as suggested by Drezner et al. (2002), can give us some confidence on the local optimum found. Indeed, if we knew we have about m optima, and we can assume the region of attraction of the global optimum to occupy $\frac{100}{m}$% of the search space, then the probability to detect a global optimum after r independent local searches is

$$P = 1 - \left(\frac{m-1}{m}\right)^r, \tag{12}$$

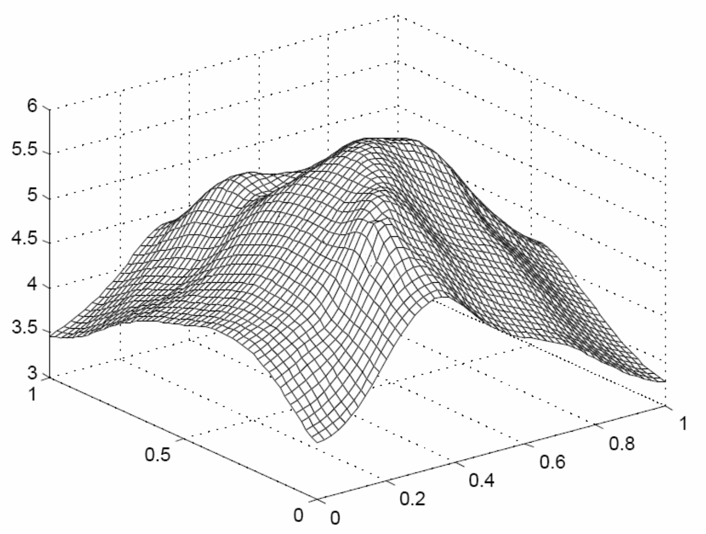

Fig. 1 Function $\overline{R}$ in (5). Randomly generated instance, $m = 200$, $k = 3$

Table 1 Average number of optima over 50 instances, k competitors m demand points

$m \setminus k$	2	4	8	16	32
40	2.76	3.74	6.76	13.88	26.12
200	3.06	5.82	13.78	31.72	54.40
1000	2.08	4.28	10.12	28.88	68.72

see for instance Hendrix and Toth (2010). Under these assumptions, if for example the number of local optima is $m = 10$, we need $r = 44$ trials to have a probability of $P = 99\%$ to reach the global optimum.

In other words, stochastic algorithms can reach, under some assumptions, a probabilistic target on effectiveness. Deterministic methods can be used to reach a guarantee on an accuracy of the reached optimum, Hendrix and Toth (2010). A specific method is elaborated in the next section.

3 A deterministic solution method

The basic idea in branch and bound methods consists of a recursive decomposition of the original problem into smaller disjoint subproblems until the solution is found. The method avoids visiting those subproblems which are known not to contain a solution. The initial set T_1 is subsequently partitioned in more and more refined subsets (branching) over which upper and lower bounds of an objective function value can be determined

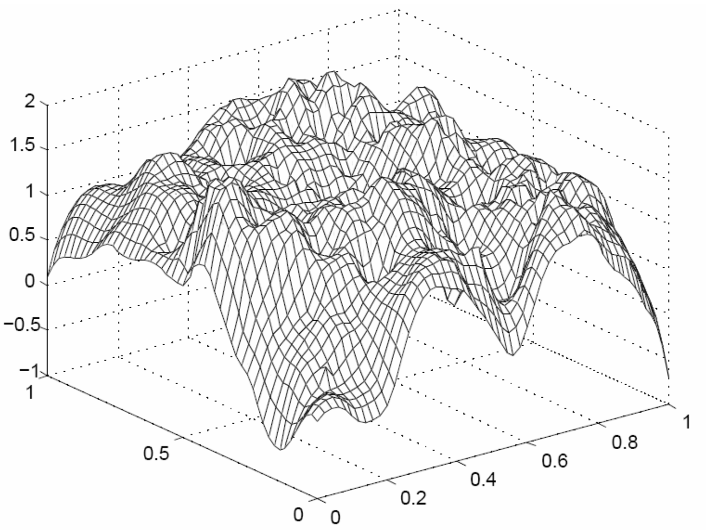

Fig. 2 Function $\overline{R}$ in (5). Randomly generated instance, $m = 200$, $k = 50$

(bounding). In continuous location optimization, the most common branching procedures use rectangles and simplices (triangles). They are known in the literature as Big Square Small Square (BSSS), Hansen, Peeters, Richard and Thisse (1985), and Big Triangle Small Triangle (BTST), Drezner and Suzuki (2004).

To construct bounds, the classical approach in continuous location, already advocated in the seminal paper Hansen et al. (1985), exploits monotonicity and bounds derived with interval arithmetic. In recent years, alternative bounding schemes have been proposed in the literature of continuous location based on expressing the objective as a difference of two convex functions, see Drezner (2007), Blanquero and Carrizosa (2009a).

In this paper we observe that the objective is not only dc (it can be written as a difference of two convex functions), but it can be written in terms of compositions of convex and convex monotonic functions. In the terminology of Blanquero and Carrizosa (2009a), the functions involved are *dcm functions* (difference of convex monotonic), and thus the bounding strategies developed in Blanquero and Carrizosa (2009a) can be used here.

The following key result, stated in Bello, Blanquero and Carrizosa (2009) and with straightforward proof, enables one to express the function in (8) as difference of convex monotonic functions.

Proposition 1 *Given* $h > 0, \lambda \geq 1,$ *define* d_0 *as*

$$d_0 = \left(\frac{\lambda - 1}{h(\lambda + 1)} \right)^{\frac{1}{\lambda}}$$

and the functions $\Phi, \Phi^1, \Phi^2 : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *as*

$$\Phi(d) = \frac{1}{1+hd^\lambda} \tag{13}$$

$$\Phi^1(d) = \begin{cases} \Phi(d_0) + \Phi'(d_0)(d-d_0) & \text{if } d \leq d_0 \\ \Phi(d) & \text{if } d > d_0 \end{cases} \tag{14}$$

$$\Phi^2(d) = \begin{cases} \Phi(d_0) + \Phi'(d_0)(d-d_0) - \Phi(d) & \text{if } d \leq d_0 \\ 0 & \text{if } d > d_0 \end{cases} \tag{15}$$

One has:

1. $\Phi = \Phi^1 - \Phi^2$
2. Φ^1, Φ^2 *are smooth convex nonincreasing functions in* $\mathbb{R}_+$

Define

$$\Phi_i(d) = \frac{1}{1+h_i d^\lambda}, \qquad i = 1, 2, \ldots, N$$
$$\Psi(d_1, \ldots, d_N) = (\Phi_1(d_1), \ldots, \Phi_N(d_N)).$$

Observe that

$$c(x) = \Psi(d_1(x), \ldots, d_N(x)).$$

By Proposition 1, we can express the function $\Psi(d_1, \ldots, d_N)^\top \mu - \tau$ as a difference of convex nonincreasing functions in $\mathbb{R}_+$. We follow Blanquero and Carrizosa (2009a) to obtain bounds by respectively majorizing by a convex function the numerator and minorizing by a concave function the denominator in (4) on a given polytope T in $\mathbb{R}^2$.

First, one easily obtains a convex function U such that

$$c(x)^\top \mu - \tau \leq U(x) \qquad \forall x \in T. \tag{16}$$

Indeed, since, by assumption, each d_i is convex, if x_i^* is an arbitrary point of T and ξ_i is a subgradient of d_i at x_i^*, one has

$$d_i(x) \geq d_i(x_i^*) + \xi_i^\top (x - x_i^*) \qquad \forall x \in T. \tag{17}$$

Since Φ_i^1 is nonincreasing, one has by (17) that

$$\Phi_i^1(d_i(x)) \leq \Phi_i^1(d_i(x_i^*) + \xi_i^\top (x - x_i^*)) \qquad \forall x \in T. \tag{18}$$

Observe also that the right-hand side function in (18) is the composition of a convex and an affine function, and it is thus convex.

On the other hand, Φ_i^2 is a convex smooth function. Hence thus

$$\Phi_i^2(d) \geq \Phi_i^2(d_i(x_i^*)) + \left(\Phi_i^2(d_i(x_i^*))\right)' (d - d_i(x_i^*)) \qquad \forall d \geq 0, \tag{19}$$

thus

$$\Phi_i^2(d_i(x)) \geq \Phi_i^2(d_i(x_i^*)) + \left(\Phi_i^2(d_i(x_i^*))\right)' (d_i(x) - d_i(x_i^*)) \qquad \forall x \in T. \tag{20}$$

Moreover, Φ_i^2 is nonincreasing, $\left(\Phi_i^2(d_i(x_i^*))\right)' \leq 0$, thus the right-hand side function in (19) is concave.

Joining (18) and (20) we have that

$$\Phi_i^1(d_i(x) - \Phi_i^2(d_i(x)) \leq U(x) := \Phi_i^1(d_i(x_i^*) + \xi_i^\top (x - x_i^*)) - \Phi_i^2(d_i(x_i^*)) - \left(\Phi_i^2(d_i(x_i^*)\right)' (d_i(x) - d_i(x_i^*)), \tag{21}$$

and U is convex in T.

Let us consider now the denominator. The following proposition is a consequence of the results in Blanquero and Carrizosa (2000) and Blanquero and Carrizosa (2009b)

Proposition 2 *Given a norm $\|\cdot\|$, the functions $x \in \mathbb{R}^2 \longmapsto \|c_(x)\|$ and $d \longmapsto \|\Psi(d)\|$ can be expressed as the difference of convex functions. Moreover, if the norm $\|\cdot\|$ is monotonic in the positive orthant, then the function $d \longmapsto \|\Psi(d)\|$ can be expressed as the difference of two convex nonincreasing functions.*

Hence, given a polytope T in the plane, we can find a concave function L such that

$$\|c(x)\| \geq L(x) \qquad \forall x \in T. \tag{22}$$

Note that ℓ_p norms satisfy the monotonicity assumptions on $\|\cdot\|$. This assumption also holds for the norm (10) as soon as V is a matrix of non-negative elements, i.e., that the demand at different users are positively correlated. However, the monotonicity assumption does not hold for arbitrary norms, and thus for arbitrary norms, or, for instance, for norm (10) with negative correlations, the weaker result should be used. Nevertheless Drezner et al. (2002) claim that “it is likely that the distributions of buying power at two demand points are positively correlated. This might be due to good economic conditions or other factors resulting in either higher or lower than expected buying power in any community.” Hence, the strongest assumptions seem to be applicable also in real world problems.

If we can guarantee that $L(x) > 0$ for all $x \in T$, then U/L is the ratio of a convex over a positive concave function, thus it is quasiconvex. This implies it attains its maximum at extreme points on T, i.e.,

$$\frac{c(x)^\top \mu - \tau}{\|c(x)\|} \leq \frac{U(x)}{L(x)} \leq \max_{v \in ext(T)} \frac{U(v)}{L(v)}, \tag{23}$$

and thus

$$R(x) \leq \max\left\{0, \max_{v \in ext(T)} \frac{U(v)}{L(v)}\right\}. \tag{24}$$

Hence, we have an upper bound for the objective as soon as we can asssert that $L(x) > 0$ for all $x \in T$. Since L is concave on T, such a condition is equivalent to

$$\min_{v \in ext(T)} L(v) > 0. \tag{25}$$

4 Computational experiments

In this section we show how the bounding scheme outlined in Section 3 can be used to solve the problem

$$\max_{x \in S} R(x) \tag{26}$$

where $R(x)$ is given by (4) under the assumptions described in Section 2. The bounding strategy based on dcm functions here applied has been successfully used to solve other nonconvex location problems, see Blanquero and Carrizosa (2009a) and Bello et al. (2009).

Instances of (26) are generated in the following way:

- The feasible region is assumed to be the unit square $S = [0, 1] \times [0, 1]$.
- There are 10 existing facilities, with locations randomly and uniformly distributed in S.
- Demand points are also randomly and uniformly generated in S. The number N of demand points ranges from very small ($N = 10$) to large ($N = 10000$).

For each number N of demand points, a nominal vector μ was randomly generated in $[0, 1]^N$, and the optimal objective value z_{nom} of the problem $\max_{x \in S} c(x)^\top \mu$ was computed. Observe that in order to compute z_{nom}, a global optimization problem is solved. The optimization problem (26) was solved using different values of the threshold, ranging from $0.3z_{nom}$ to $1.5z_{nom}$ with a step of $0.05z_{nom}$. The norm $\| \cdot \|$ considered in (26) was the Euclidean.

For each choice of N and τ, ten instances were generated and solved using the BSSS method. The program code was written in Fortran, compiled by Intel Fortran 10.1 and ran on a 2.4GHz computer under Windows XP. The solutions were found to an accuracy of 10^{-8}.

The bounds for the numerator of $R(x)$ were computed according to the procedure proposed in Blanquero and Carrizosa (2009a), as has been detailed in Section 3. Regarding the denominator, a dcm decomposition for it was obtained combining the same procedure with Theorem 1 in Blanquero and Carrizosa (2009b), since the norm considered is monotonic in $\mathbb{R}^N_+$. The bounds obtained using this result are better than those provided by Proposition 1.1 in Blanquero and Carrizosa (2000), which could have also been used.

Tables 2 to 11 report, for the different threshold values τ and number N of demand points, statistics on the number of iterations, the memory usage, measured via the maximum size of the list of squares to be inspected in the branch and bound, and the CPU time.

The numerical results for different threshold values τ are shown in Tables where one can observe that the computational effort needed to solve the problem decreases as soon as the threshold value grows, especially when the threshold exceeds z_{nom}, since in that case $R(x) = 0 \;\; \forall x \in S$, and the algorithm quickly closes the gap.

Average CPU time and number of iterations for different number of demand points is shown in Figures 3 and 4 for different threshold values τ. For the instances with positive robustness, i.e., with $\tau < z_{nom}$, the average running times and number of iterations increase at most linearly in the number N of demand points. Finally, Figure 5 illustrates the evolution of the average CPU time as a function of the threshold value, for N equal to 100, 1000, 5000 and 10000.

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	157	393	268,50	22	64	45,10	0,000	0,016	0,008
20	198	291	240,90	31	53	40,80	0,000	0,016	0,013
50	242	472	328,00	44	78	58,80	0,016	0,047	0,030
100	316	1100	435,70	71	176	92,90	0,047	0,203	0,081
200	339	550	412,30	78	133	103,70	0,125	0,219	0,153
500	367	1347	635,20	108	195	150,00	0,344	1,250	0,588
1000	397	1116	562,60	93	203	160,90	0,719	2,063	1,036
2000	434	1273	662,40	145	258	184,20	1,594	4,719	2,445
5000	526	2136	847,10	192	357	283,60	4,906	19,891	7,834
10000	713	3854	1724,90	305	570	396,70	13,188	71,328	31,888

Table 2 Computational results for $\tau = 0.3z_{nom}$

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	160	542	276,20	21	101	45,60	0,000	0,016	0,003
20	182	338	228,40	25	56	39,70	0,000	0,016	0,009
50	240	626	341,80	38	101	55,70	0,016	0,063	0,030
100	294	551	387,10	57	103	77,30	0,047	0,094	0,069
200	338	724	459,30	63	142	95,50	0,125	0,266	0,170
500	353	1237	670,60	96	187	138,20	0,328	1,156	0,619
1000	339	1168	661,40	79	216	159,40	0,625	2,172	1,223
2000	455	1293	728,30	136	283	184,20	1,688	4,781	2,686
5000	544	3198	1303,50	182	483	293,30	5,016	29,672	12,075
10000	800	3204	1556,90	252	438	352,10	14,766	59,344	28,794

Table 3 Computational results for $\tau = 0.4z_{nom}$

5 Conclusion

Two different robustness concepts, deviation robustness and probabilistic, are described. The deviation concept has been elaborated in a new generic

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	155	484	239,20	22	88	38,80	0,000	0,016	0,008
20	160	1147	306,70	23	255	56,80	0,000	0,031	0,013
50	223	722	373,30	31	137	64,90	0,016	0,063	0,036
100	271	448	321,10	43	88	65,10	0,047	0,078	0,059
200	305	810	474,90	47	146	89,60	0,125	0,297	0,178
500	362	1339	662,10	70	258	134,60	0,328	1,250	0,616
1000	355	1355	838,30	75	206	150,40	0,656	2,500	1,550
2000	486	1111	757,90	96	238	164,20	1,797	4,125	2,795
5000	627	2022	1254,80	197	333	258,20	5,781	18,750	11,605
10000	751	3185	1670,50	193	509	331,30	13,859	58,844	30,883

Table 4 Computational results for $\tau = 0.5z_{nom}$

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	136	430	210,40	20	84	34,40	0,000	0,016	0,005
20	130	723	247,30	19	158	42,90	0,000	0,031	0,013
50	177	528	294,50	25	93	48,50	0,016	0,047	0,028
100	215	450	303,20	30	87	54,90	0,047	0,094	0,058
200	260	492	386,90	34	93	62,60	0,094	0,172	0,142
500	391	4166	923,70	51	987	180,00	0,359	3,844	0,856
1000	544	2427	907,10	81	570	168,90	1,000	4,500	1,681
2000	596	1458	1011,70	75	215	160,40	2,203	5,375	3,744
5000	991	2446	1324,40	127	434	239,40	9,188	22,594	12,241
10000	916	3034	1916,40	143	447	280,90	16,891	56,047	35,456

Table 5 Computational results for $\tau = 0.6z_{nom}$

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	123	371	193,70	19	71	32,40	0,000	0,016	0,005
20	117	369	201,10	16	70	32,30	0,000	0,016	0,009
50	156	440	232,80	23	74	37,60	0,016	0,047	0,022
100	192	380	244,60	29	57	41,60	0,031	0,078	0,045
200	229	488	314,80	32	71	48,20	0,078	0,172	0,117
500	306	779	448,30	41	130	69,60	0,281	0,719	0,416
1000	465	655	571,80	68	157	93,80	0,859	1,203	1,058
2000	482	1055	795,40	63	219	125,20	1,797	3,906	2,947
5000	647	1400	1031,40	89	248	167,60	6,000	12,891	9,542
10000	834	2169	1361,30	112	337	183,60	15,563	40,016	25,181

Table 6 Computational results for $\tau = 0.7z_{nom}$

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	117	403	191,00	18	84	33,00	0,000	0,016	0,005
20	103	243	165,60	15	45	27,50	0,000	0,016	0,009
50	126	531	221,10	19	103	37,70	0,000	0,047	0,023
100	150	514	231,30	21	100	38,00	0,031	0,094	0,044
200	179	301	231,50	24	46	35,60	0,063	0,109	0,086
500	243	706	359,20	33	101	54,60	0,234	0,656	0,339
1000	335	721	472,60	45	130	73,10	0,609	1,391	0,881
2000	376	795	525,90	49	149	81,20	1,406	2,938	1,948
5000	445	1766	812,40	58	265	118,10	4,141	16,359	7,509
10000	555	1352	856,60	72	179	113,00	10,344	25,016	15,869

Table 7 Computational results for $\tau = 0.8z_{nom}$

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	106	612	196,10	16	156	38,60	0,000	0,000	0,000
20	93	206	139,60	15	32	21,50	0,000	0,016	0,008
50	108	288	156,10	15	47	24,40	0,016	0,016	0,016
100	119	580	201,80	17	137	36,00	0,016	0,109	0,039
200	144	239	183,10	21	39	27,60	0,047	0,094	0,069
500	190	443	260,60	28	65	39,00	0,172	0,406	0,242
1000	245	500	332,80	32	87	51,80	0,453	0,922	0,616
2000	253	523	359,60	31	101	52,80	0,938	1,938	1,331
5000	318	1081	529,00	45	165	74,10	3,000	9,984	4,895
10000	372	852	531,80	50	109	69,10	6,922	15,734	9,853

Table 8 Computational results for $\tau = 0.9z_{nom}$

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	90	234	142,20	12	46	23,60	0,000	0,016	0,002
20	83	145	115,40	12	26	17,00	0,000	0,016	0,003
50	94	226	130,80	14	37	20,20	0,000	0,016	0,006
100	99	210	138,40	14	38	20,00	0,016	0,031	0,023
200	115	190	140,10	15	29	19,50	0,031	0,078	0,052
500	149	239	179,70	16	37	25,00	0,141	0,219	0,167
1000	174	295	208,20	22	41	29,10	0,328	0,547	0,386
2000	163	393	230,10	19	59	32,00	0,594	1,453	0,853
5000	181	440	273,50	25	73	37,70	1,672	4,078	2,533
10000	164	388	244,40	24	50	32,10	3,063	7,156	4,533

Table 9 Computational results for $\tau = 1.0z_{nom}$

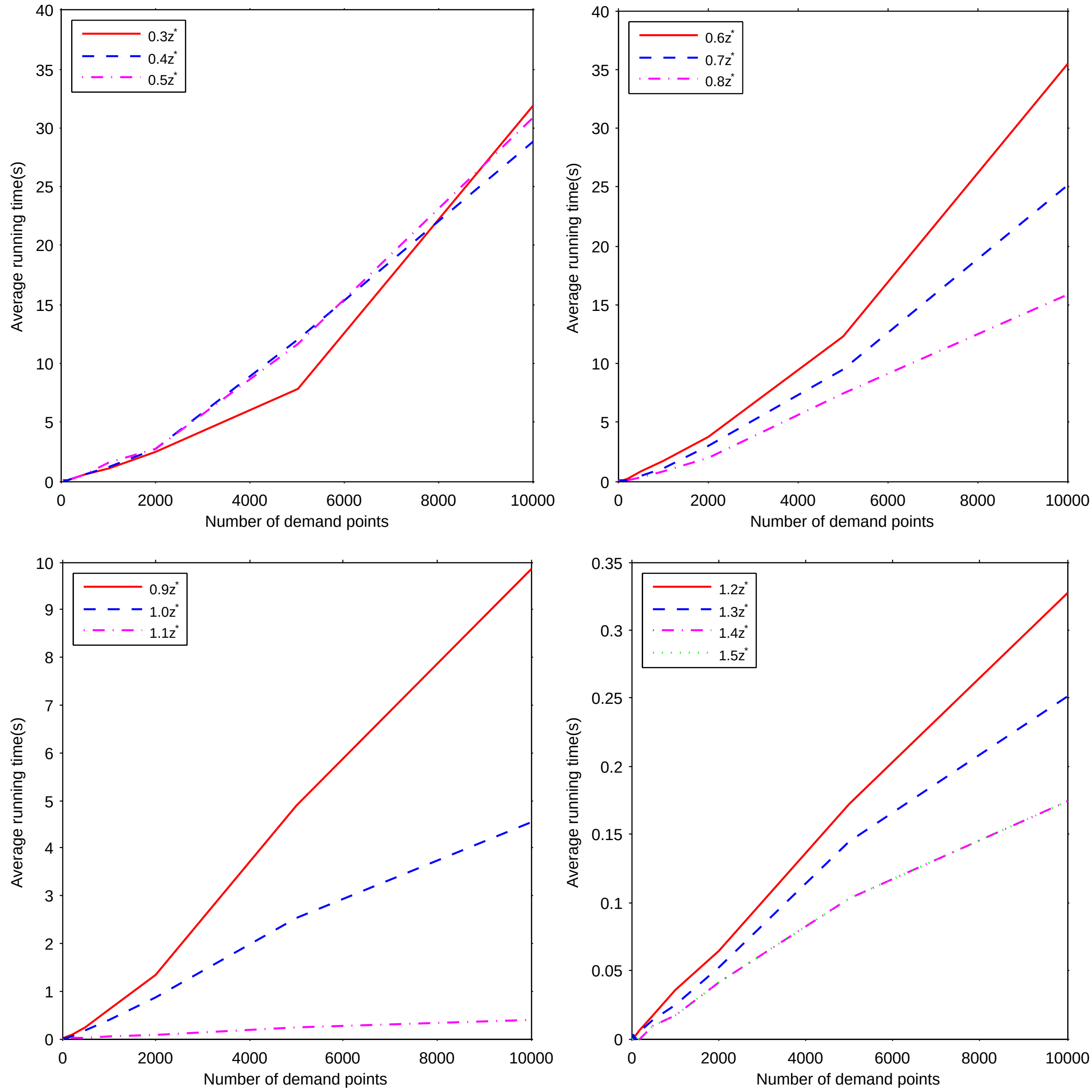

Fig. 3 Average running times versus number of demand points for different thresholds

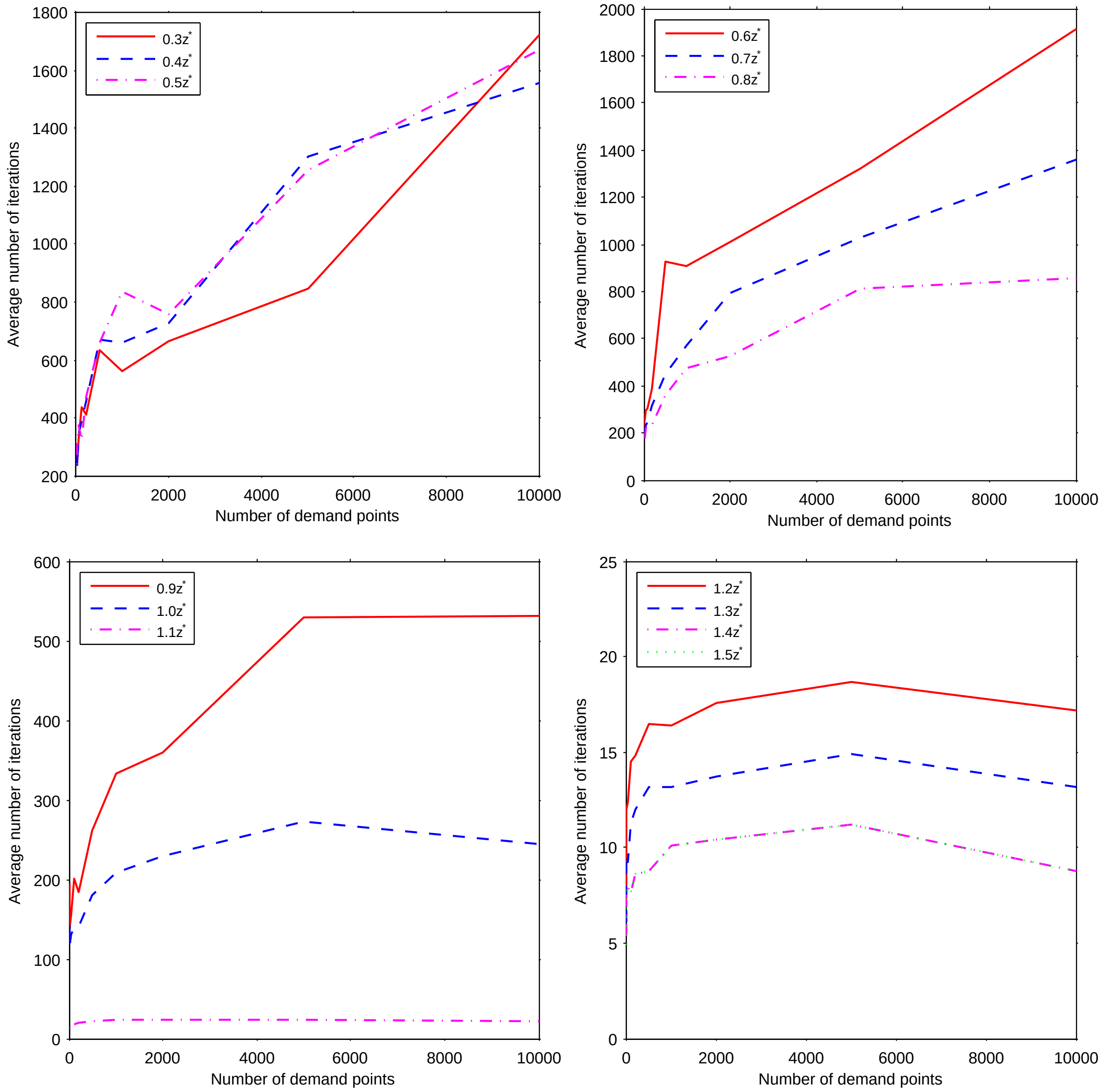

Fig. 4 Average number of iterations versus number of demand points for different thresholds

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	8	19	11,50	3	7	4,50	0,000	0,000	0,000
20	10	25	18,00	4	10	6,70	0,000	0,000	0,000
50	11	24	17,90	4	10	7,30	0,000	0,016	0,002
100	13	33	18,80	5	12	7,50	0,000	0,016	0,005
200	12	26	20,00	5	10	8,50	0,000	0,016	0,008
500	15	28	22,30	5	12	9,00	0,016	0,031	0,023
1000	16	35	23,70	8	11	9,10	0,031	0,063	0,045
2000	17	34	24,40	9	14	10,20	0,063	0,141	0,092
5000	15	36	24,50	6	11	9,30	0,141	0,328	0,228
10000	14	28	21,40	5	10	8,80	0,266	0,516	0,400

Table 10 Computational results for $\tau = 1.1z_{nom}$

N	Iterations			Max squares			Time (s)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	5	14	8,00	2	5	3,80	0,000	0,000	0,000
20	7	16	12,00	3	7	5,10	0,000	0,016	0,002
50	7	17	12,40	4	8	5,60	0,000	0,000	0,000
100	7	20	14,50	4	9	6,40	0,000	0,016	0,002
200	10	18	14,80	5	9	7,40	0,000	0,016	0,006
500	10	20	16,50	5	10	7,60	0,016	0,031	0,017
1000	10	21	16,40	5	10	7,70	0,016	0,047	0,036
2000	12	21	17,60	7	10	8,50	0,047	0,078	0,064
5000	14	21	18,70	5	10	8,50	0,125	0,203	0,172
10000	10	21	17,20	4	10	8,10	0,188	0,406	0,327

Table 11 Computational results for $\tau = 1.2z_{nom}$

robust competitive continuous location model. We show it also captures stochastic programming models that follow the probabilistic approach.

The model inherits the multimodal character of the underlying Huff-model. We found that the number of optima for such a model mainly depends on the number of existing competitive facilities and it does not increase substantially with the number of demand points.

A branch and bound approach gives a guaranteed global optimum of a competitive location model. The computational experiments reported support the idea that using dc-programming techniques enables one to solve problems with thousands of demand points in a few seconds.

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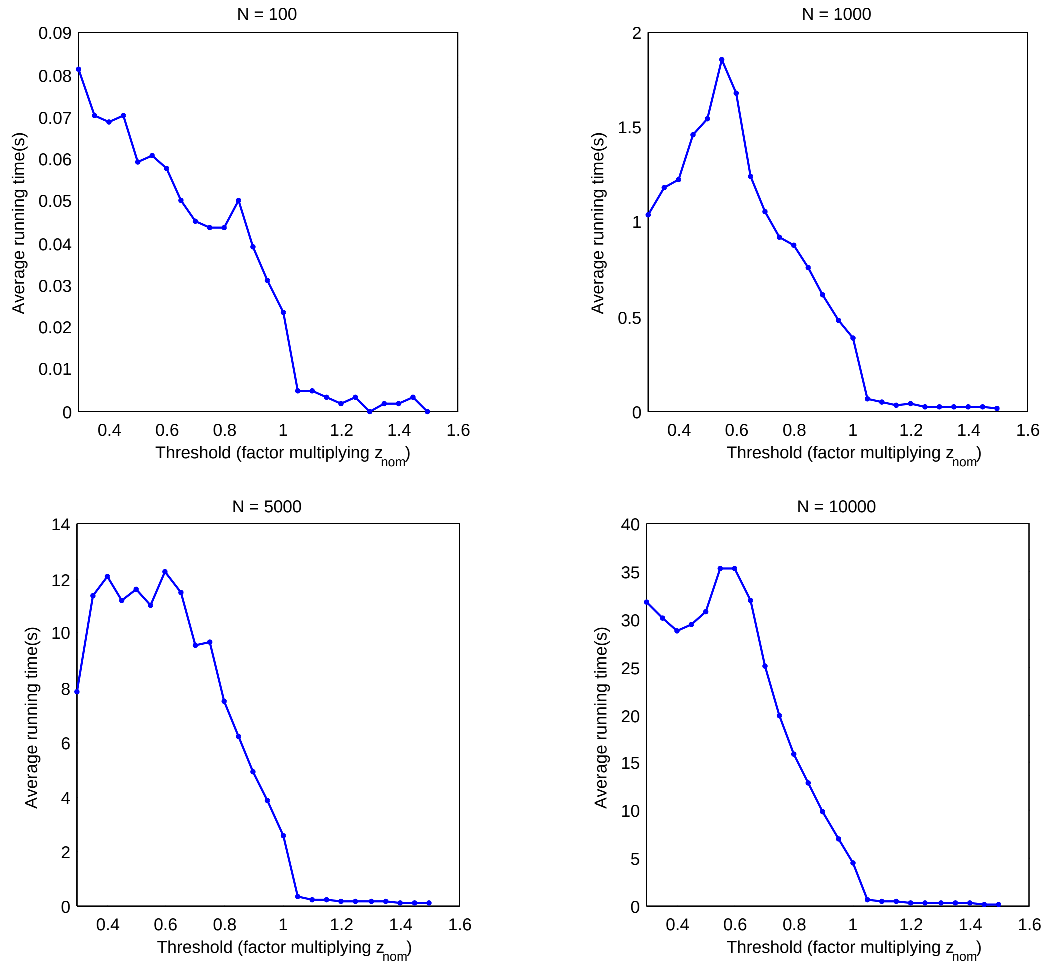

Fig. 5 Average running times versus thresholds for different number of demand points

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