Bregman Strongly Nonexpansive Operators in Reflexive Banach Spaces

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Abstract

We present a detailed study of right and left Bregman strongly nonexpansive operators in reflexive Banach spaces. We analyze, in particular, compositions and convex combinations of such operators, and prove the convergence of the Picard iterative method for operators of these types. Finally, we use our results to approximate common zeroes of maximal monotone mappings and solutions to convex feasibility problems.

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1 Introduction

The theory and applications of nonexpansive operators in Banach spaces have been intensively studied for almost fifty years now [7, 23, 24, 25]. There are several important classes of nonexpansive operators which have remarkable properties not shared by all such operators. We refer, for example, to strongly nonexpansive operators which were introduced in [16]. This class of operators is of particular significance in fixed point, iteration and convex optimization theories mainly because it is closed under composition. It encompasses other noteworthy classes of nonexpansive operators. For example, in uniformly convex Banach spaces all firmly nonexpansive operators as well as all averaged operators are strongly nonexpansive [16]. A related class of operators comprises the quasi-nonexpansive operators. These operators still enjoy relevant fixed point properties although nonexpansivity is only required for each fixed point [22].

In this paper, we are concerned with certain analogous classes of operators which are, in some sense, strongly nonexpansive not with respect to the norm, but with respect to Bregman distances [3, 14, 18, 21]. Since these distances are not symmetric in general, it seems natural to distinguish between left and right Bregman strongly nonexpansive operators. The left variant has already been

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studied and applied in [31, 35]. We have recently introduced and studied several classes of right Bregman nonexpansive operators in reflexive Banach spaces [28, 29]. The present paper is devoted to a detailed study of right and left Bregman strongly nonexpansive operators.

Our paper is organized as follows. Section 2 contains certain essential preliminary results regarding properties of Bregman distances. In the next section we establish in detail two fundamental properties of left Bregman strongly nonexpansive operators. These properties concern the compositions of finitely many such operators. In Section 4 we establish analogous results for right Bregman strongly nonexpansive operators. The next section is devoted to convex combinations of a given finite number of right Bregman strongly nonexpansive operators. In Section 6 we bring out the connections between left and right Bregman strongly nonexpansive operators. Finally, in the last two sections we prove the convergence of the Picard iteration for right Bregman strongly nonexpansive operators and then use our results to find common zeroes of maximal monotone mappings and to solve convex feasibility problems.

2 Preliminaries

In this section we collect several definitions and results, which are pertinent to our study. Let X be a reflexive Banach space and let $f : X \to (-\infty, +\infty]$ be a function with effective domain $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. The Fenchel conjugate function $f^* : X^* \to (-\infty, +\infty]$, defined by $f^*(u) = \sup\{\langle u, x\rangle - f(x) : x \in \operatorname{dom} f\}$. We say that f is *admissible* if it is proper, convex, lower semicontinuous and Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. Under these assumptions we know that f is continuous in $\operatorname{int}\operatorname{dom} f$ (see [4, Fact 2.3, page 619]). Recall that f is called *cofinite* if $\operatorname{dom} f^* = X^*$. The subdifferential of f is the set-valued mapping $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) := \{u \in X^* : f(y) \geq f(x) + \langle u, y - x\rangle \ \forall y \in X\}, \quad x \in X.$$

The boundedness of f is inherited by the subdifferential and vice versa as the following result shows.

Proposition 2.1 (*cf.* [18, Proposition 1.1.11, page 16])**.** *If $f : X \to \mathbb{R}$ is continuous and convex, then $\partial f : 2^{X^*} \to X$ is bounded on bounded subsets if and only if f itself is bounded on bounded subsets.*

The function f is said to be *Legendre* if it satisfies the following two conditions.

(L1) $\operatorname{int}\operatorname{dom} f \neq \emptyset$ and ∂f is single-valued on its domain ($\operatorname{dom}\partial f = \{x \in X : \partial f(x) \neq \emptyset\}$).

(L2) $\operatorname{int}\operatorname{dom} f^* \neq \emptyset$ and ∂f^* is single-valued on its domain.

When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$ [30, Definition 1.3, page 3] (this is the case when f is an admissible function, not necessarily Legendre). The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [4]. Their definition is equivalent to conditions (L1) and (L2) because the space X is assumed to be reflexive (see [4, Theorems 5.4 and 5.6, page 634]). It is well known that in reflexive spaces $\nabla f = (\nabla f^*)^{-1}$ (see [4, Theorem 5.10, page 636]). When this fact is combined with conditions (L1) and (L2), we obtain

$$\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^* \quad \text{and} \quad \operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f.$$

It also follows that f is Legendre if and only if f^* is Legendre (see [4, Corollary 5.5, page 634]) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. When the Banach space X is smooth and strictly convex, in particular, a Hilbert space, the function $(1/p)\left\|\cdot\right\|^p$ with $p \in (1,\infty)$ is Legendre (*cf.* [4, Lemma 6.2, page 639]). For examples and more information regarding Legendre functions, see, for instance, [2, 4].

The bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0,+\infty)$ given by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{1}$$

is called the *Bregman distance with respect to* f (*cf.* [20]). With the function f we associate the bifunction $W^f : \operatorname{dom} f^* \times \operatorname{dom} f \to [0,+\infty)$ defined by

$$W^f(\xi,x) := f(x) - \langle \xi, x\rangle + f^*(\xi). \tag{2}$$

This function satisfies

$$W^f(\nabla f(x), y) = D_f(y,x)$$

for all $x \in \operatorname{int}\operatorname{dom} f$ and $y \in \operatorname{dom} f$ (*cf.* [27]).

We now recall the definition of a totally convex function which was introduced in [17, 18].

Definition 2.2 (Total convexity)**.** The function f is called *totally convex at a point* $x \in \operatorname{int}\operatorname{dom} f$ if its *modulus of total convexity at* x, $\nu_f(x,\cdot) : [0,+\infty) \to [0,+\infty]$, defined by

$$\nu_f(x,t) := \inf\left\{D_f(y,x) : y \in \operatorname{dom} f, \quad \|y-x\| = t\right\},$$

is positive whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point of $\operatorname{int}\operatorname{dom} f$.

Definition 2.3 (Total convexity on bounded subsets)**.** The function f is called *totally convex on bounded sets* if, for any nonempty and bounded set $E \subset X$, the *modulus of total convexity of* f on E, $\nu_f(E,t)$, is positive for any $t > 0$, where $\nu_f(E,\cdot) : [0,+\infty) \to [0,+\infty]$ is defined by

$$\nu_f(E,t) := \inf\left\{\nu_f(x,t) : x \in E \cap \operatorname{int}\operatorname{dom} f\right\}.$$

Relevant examples of functions f satisfying the above properties can be found in [10, 12, 13, 28]. The following result will be crucial for our analysis.

The following result will play a crucial rule in our results (*cf.* [18, Lemma 2.1.2, page 67]).

Proposition 2.4 (Property of total convexity on bounded subsets)**.** *The function* $f : X \to (-\infty,+\infty]$ *is totally convex on bounded subsets of* X *if and only if for any two sequences* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *in* $\operatorname{int}\operatorname{dom} f$ *and* $\operatorname{dom} f$, *respectively, such that the first one is bounded,*

$$\lim_{n\to\infty} D_f(y_n,x_n) = 0 \;\Rightarrow\; \lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Next we recall two boundedness properties (*cf.* [26, 34]).

Proposition 2.5 (Boundedness property - left variable)**.** *Let the function* $f : X \to (-\infty,+\infty]$ *be admissible and totally convex at a point* $x \in \operatorname{int}\operatorname{dom} f$. *Let* $\{x_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} f$. *If* $\{D_f(x_n,x)\}_{n\in\mathbb{N}}$ *is bounded, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded too.*

Proof. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not bounded. Then it contains a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\|x_{n_k}\| = +\infty.$$

Consequently, $\lim_{k\to\infty}\|x_{n_k}-x\| = +\infty$ and there exists some $k_0 > 0$ such that $\|x_{n_k}-x\| > 1$ for all $k > k_0$. Since f is totally convex at x, the modulus of total convexity $\nu_f(x,\cdot)$ is strictly increasing and positive on $(0,+\infty)$ (see [18, Proposition 1.2.2, page 18]), in particular, $\nu_f(x,1) > 0$ for all $x \in X$. So using [18, Proposition 1.2.2, page 18] again, we get that, for all $k > k_0$,

$$\nu_f(x, \|x_{n_k}-x\|) \geq \|x_{n_k}-x\|\, \nu_f(x,1) \to \infty.$$

Thus $\{\nu_f(x,\|x_n-x\|)\}_{n\in\mathbb{N}}$ is not bounded. Since, by definition,

$$\nu_f(x,\|x_n-x\|) \leq D_f(x_n,x), \tag{3}$$

for all $n \in \mathbb{N}$, this implies that the sequence $\{D_f(x_n,x)\}_{n\in\mathbb{N}}$ cannot be bounded. □

Proposition 2.6 (Boundedness property - right variable)**.** *Let $f : X \to \mathbb{R}$ be an admissible function such that ∇f^* is bounded on bounded subsets of* $\operatorname{dom} f^* = X^*$. *Let $x \in X$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$. If $\{D_f(x,x_n)\}_{n\in\mathbb{N}}$ is bounded, so is the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. Let β be an upper bound of the sequence $\{D_f(x,x_n)\}_{n\in\mathbb{N}}$. Then from the definition of W^f (see (2)) we obtain

$$f(x) - \langle \nabla f(x_n), x\rangle + f^*(\nabla f(x_n)) = W^f(\nabla f(x_n), x) = D_f(x,x_n) \leq \beta.$$

This implies that $\{\nabla f(x_n)\}_{n\in\mathbb{N}}$ is contained in the sub-level set, $\operatorname{lev}^{\psi}_{\leq}(\beta - f(x))$, of the function $\psi = f^* - \langle\cdot,x\rangle$. Since the function f^* is proper and lower semicontinuous, an application of the Moreau-Rockafellar Theorem (see [4, Fact 3.1, page 623]) shows that ψ is coercive, that is, $\lim_{\|x\|\to\infty}\psi(x) = +\infty$. Consequently, all sub-level sets of ψ are bounded. Hence the sequence $\{\nabla f(x_n)\}_{n\in\mathbb{N}}$ is bounded. By hypothesis, ∇f^* is bounded on bounded subsets of X^*. Therefore the sequence $\{x_n\}_{n\in\mathbb{N}} = \{\nabla f^*(\nabla f(x_n))\}_{n\in\mathbb{N}}$ is bounded too, as claimed. □

3 Composition of left Bregman strongly nonexpansive operators

This section is devoted to the properties of left Bregman strongly nonexpansive operators. These properties originate in [31]. We include here complete proofs for the convenience of the reader.

Definition 3.1 (Left Bregman strongly nonexpansive operators)**.** We say that an operator $T : K \subset \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ is *left Bregman strongly nonexpansive* (L-BSNE) with respect to $S \subset \operatorname{dom} f$ if

$$D_f(p,Tx) \leq D_f(p,x) \tag{4}$$

for all $p \in S$ and $x \in K$, and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded, $p \in S$, and

$$\lim_{n\to\infty}(D_f(p,x_n) - D_f(p,Tx_n)) = 0, \tag{5}$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n,x_n) = 0. \tag{6}$$

Let $f : X \to (-\infty,+\infty]$ be admissible and let K be a nonempty subset of X. The *fixed point set* of an operator $T : K \to X$ is the set $\{x \in K : Tx = x\}$. It is denoted by $\operatorname{Fix}(T)$. Recall that a point $u \in K$ is said to be an *asymptotic fixed point* [31] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in K such that $x_n \rightharpoonup u$ as $n \to \infty$ (that is, $\{x_n\}_{n\in\mathbb{N}}$ is weakly convergent to u) and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. We denote the asymptotic fixed point set of T by $\widehat{\operatorname{Fix}}(T)$.

Next we list three types of left Bregman strong nonexpansivity.

Remark 3.2 (Types of left Bregman strong nonexpansivity)**.** We will use the following particular cases.

(i) An operator which satisfies (4)-(6) with respect to $S := \mathrm{Fix}\,(T)$ is called *properly L-BSNE.*

(ii) An operator which satisfies (4)-(6) with respect to $S := \widehat{\mathrm{Fix}}\,(T)$ is called *strictly L-BSNE* (this class of operators was first defined in [31]).

(iii) An operator which satisfies (4)-(6) with respect to $S := \mathrm{Fix}\,(T) = \widehat{\mathrm{Fix}}\,(T)$ is called *fully L-BSNE.* $\diamondsuit$

The next result shows that the composition of a finite family of strictly L-BSNE operators is also strictly L-BSNE.

Proposition 3.3 (Composition of strictly L-BSNE operators)**.** *Let $f : X \to \mathbb{R}$ be an admissible function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that ∇f^* is bounded on bounded subsets of* $\mathrm{dom}\, f^* = X^*$ *and let K be a nonempty subset of X. Let $\{T_i : i = 1, \ldots, N\}$ be N strictly L-BSNE operators from K into itself and consider the composition $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$. Assume that the set*

$$\widehat{F} = \bigcap_{i=1}^{N} \widehat{\mathrm{Fix}}\,(T_i)$$

is not empty. Then

(i) $\widehat{\mathrm{Fix}}\,(T) \subset \widehat{F}$*;*

(ii) *moreover if* $\widehat{\mathrm{Fix}}\,(T) \neq \emptyset$*, T is also strictly L-BSNE.*

Proof. Let $u \in \widehat{F}$. First we claim that if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded and

$$\lim_{n\to\infty} \left(D_f\left(u, x_n\right) - D_f\left(u, Tx_n\right)\right) = 0, \tag{7}$$

then, for all $i = 1, \ldots, N$, we have

$$\lim_{n\to\infty} \left\| y_n^i - y_n^{i-1} \right\| = 0,$$

where $y_n^i := T_i \circ T_{i-1} \circ \cdots \circ T_1 x_n$, $i = 1, \ldots, N$, under the conventions that $y_n^0 = x_n$ and $T_0 = I$, the identity operator.

Now we prove this claim. Since each T_i, $i = 1, \ldots, N$, is strictly L-BSNE operator we get from (4)

$$D_f\left(u, Tx_n\right) = D_f\left(u, y_n^N\right) \le D_f\left(u, y_n^{N-1}\right) \le \cdots \le D_f\left(u, T_1 x_n\right) \le D_f\left(u, x_n\right). \tag{8}$$

Hence, from (7), we get for all $i = 2, \ldots, N$, that

$$\lim_{n\to\infty} \left(D_f\left(u, y_n^{i-1}\right) - D_f\left(u, y_n^i\right)\right) \le \lim_{n\to\infty} \left(D_f\left(u, x_n\right) - D_f\left(u, Tx_n\right)\right) = 0.$$

Thus

$$\lim_{n\to\infty} \left(D_f\left(u, y_n^{i-1}\right) - D_f\left(u, y_n^i\right)\right) = 0. \tag{9}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, and both f and ∇f are bounded on bounded subsets, the sequence $\{D_f\left(u, x_n\right)\}_{n\in\mathbb{N}}$ is bounded too. Therefore if follows from (8) that $\left\{D_f\left(u, y_n^i\right)\right\}_{n\in\mathbb{N}}$ is bounded for each $i = 1, \ldots, N$. Since ∇f^* is bounded on bounded subsets of X^*, it follows from Proposition 2.6 that $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded. This together with (9) implies that

$$\lim_{n\to\infty} D_f\left(y_n^i, y_n^{i-1}\right) = 0$$

because T_i, $i = 1, \ldots, N$, is strictly L-BSNE. Since $\left\{y_n^{i-1}\right\}_{n\in\mathbb{N}}$ is bounded, we have from Proposition 2.4 that

$$\lim_{n\to\infty} \left\|y_n^i - y_n^{i-1}\right\| = 0,$$

as claimed.

Now we will prove our assertions.

(i) Let $u \in \widehat{F}$. Given $x \in \widehat{\operatorname{Fix}}(T)$, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$ converging weakly to x such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{10}$$

Since f is uniformly Fréchet differentiable on bounded subsets, it is also uniformly continuous on bounded subsets of X (see [1, Theorem 1.8, page 13]) and therefore we obtain from (10)

$$\lim_{n\to\infty} \left(f\left(Tx_n\right) - f\left(x_n\right)\right) = 0. \tag{11}$$

In addition, from [33, Proposition 2.1, page 474] we also obtain that ∇f is uniformly continuous on bounded subsets of X and thus again from (10) we get

$$\lim_{n\to\infty} \left\|\nabla f\left(Tx_n\right) - \nabla f\left(x_n\right)\right\|_* = 0. \tag{12}$$

From the definition of the Bregman distance (see (1)) we obtain that

$$\begin{aligned}
D_f\left(u, x_n\right) - D_f\left(u, Tx_n\right) &= \left[f\left(u\right) - f\left(x_n\right) - \left\langle \nabla f\left(x_n\right), u - x_n\right\rangle\right] \\
&\quad - \left[f\left(u\right) - f\left(Tx_n\right) - \left\langle \nabla f\left(Tx_n\right), u - Tx_n\right\rangle\right] \\
&= f\left(Tx_n\right) - f\left(x_n\right) - \left\langle \nabla f\left(x_n\right), u - x_n\right\rangle \\
&\quad + \left\langle \nabla f\left(Tx_n\right), u - Tx_n\right\rangle \\
&= f\left(Tx_n\right) - f\left(x_n\right) - \left\langle \nabla f\left(x_n\right) - \nabla f\left(Tx_n\right), u - x_n\right\rangle \\
&\quad + \left\langle \nabla f\left(Tx_n\right), x_n - Tx_n\right\rangle.
\end{aligned}$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [18, Propostion 1.1.11, page 16]). Thus both sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\nabla f\left(Tx_n\right)\}_{n\in\mathbb{N}}$ are bounded. Hence from these facts along with (10), (11) and (12), we deduce that

$$\lim_{n\to\infty} \left(D_f\left(u, x_n\right) - D_f\left(u, Tx_n\right)\right) = 0. \tag{13}$$

As we have already proved, from this property we obtain that

$$\lim_{n\to\infty} \left\|y_n^i - y_n^{i-1}\right\| = 0$$

for all $i = 1, \ldots, N$. Now from (10) we obtain that $\left\{y_n^N\right\}_{n\in\mathbb{N}}$ (note that $y_n^N = Tx_n$) converges weakly to x and since

$$\lim_{n\to\infty} \left\|y_n^N - y_n^{N-1}\right\| = 0$$

we get that $\left\{y_n^{N-1}\right\}_{n\in\mathbb{N}}$ also converges weakly to x. Repeating this argument for all $i = 1, \ldots N-2$, yields that the all sequences $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ converge weakly to x. Now, since

$$\lim_{n\to\infty}\left\|y_n^{i-1} - T_i y_n^{i-1}\right\| = 0$$

we get that $x \in \widehat{\operatorname{Fix}}(T_i)$ for each $i = 1, 2, \ldots, N$, so that $x \in \widehat{F}$, as asserted.

(ii) Let $u \in \widehat{\operatorname{Fix}}(T)$ and $x \in K$. From assertion (i) we already know that $u \in \widehat{F}$. Since each T_i, $i = 1, 2, \ldots, N$, satisfies (4) we obtain that

$$\begin{aligned}
D_f(u, Tx) &= D_f(u, T_N \circ T_{N-1} \circ \cdots \circ T_1 x)\\
&\le D_f(u, T_{N-1} \circ \cdots \circ T_1 x)\\
&\le\\
&\vdots\\
&\le D_f(u, T_1 x)\\
&\le D_f(u, x).
\end{aligned}$$

Hence T also satisfies (4). Given $u \in \widehat{\operatorname{Fix}}(T)$ and a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\left(D_f(u, x_n) - D_f(u, Tx_n)\right) = 0, \tag{14}$$

we now prove that (6) is valid. We have already proved that (14) yields

$$\lim_{n\to\infty}\left\|y_n^i - y_n^{i-1}\right\| = 0$$

for all $i = 1, \ldots, N$. Since

$$\|x_n - Tx_n\| \le \left\|x_n - y_n^1\right\| + \left\|y_n^1 - y_n^2\right\| + \cdots + \left\|y_n^{N-1} - y_n^N\right\|$$

we get that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [18, Propostion 1.1.11, page 16]). Thus both sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\nabla f(Tx_n)\}_{n\in\mathbb{N}}$ are bounded. Since f is uniformly continuous on bounded subsets of X (see [1, Theorem 1.8, page 13]), we have

$$\lim_{n\to\infty}\left(f(Tx_n) - f(x_n)\right) = 0.$$

So from the definition of the Bregman distance (see (1)) we obtain that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

Hence T is strictly L-BSNE, as asserted. □

Proposition 3.4 (Composition of fully L-BSNE operators)**.** *Let $f : X \to \mathbb{R}$ be an admissible function which is bounded, uniformly Frchet differentiable and totally convex on bounded subsets of X. Assume that ∇f^* is bounded on bounded subsets of* $\operatorname{dom} f^* = X^*$ *and let K be a nonempty* *subset of X. For each $i = 1, \ldots, N$, let $T_i : K \subset X \to K$ be a fully L-BSNE operator, and let $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$. If*

$$F := \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$$

is nonempty, then T is also fully L-BSNE and $\operatorname{Fix}(T) = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$.

Proof. Indeed, from Proposition 3.3 it follows that

$$\operatorname{Fix}(T) \subset \widehat{\operatorname{Fix}}(T) \subset \bigcap_{i=1}^{N} \widehat{\operatorname{Fix}}(T_i) = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = F \subset \operatorname{Fix}(T),$$

which implies that all inclusions are equalities, as claimed. □

4 Composition of right Bregman strongly nonexpansive operators

This section is devoted to a detailed study of the compositions of right Bregman strongly nonexpansive operators.

Definition 4.1 (Right Bregman Strongly Nonexpansive operators)**.** We say that an operator $T : K \subset \operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ is *right Bregman strongly nonexpansive* (R-BSNE) with respect to $S \subset \operatorname{int}\operatorname{dom} f$ if

$$D_f(Tx, p) \le D_f(x, p) \tag{15}$$

for all $p \in S$ and $x \in K$, and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded, $p \in S$, and

$$\lim_{n\to\infty} \left(D_f(x_n, p) - D_f(Tx_n, p)\right) = 0, \tag{16}$$

it follows that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0. \tag{17}$$

Remark 4.2 (Types of right Bregman strong nonexpansivity)**.** We will use the following particular cases.

(i) An operator which satisfies (15)-(17) with respect to $S := \operatorname{Fix}(T)$ is called *properly R-BSNE.*

(ii) An operator which satisfies (15)-(17) with respect to $S := \widehat{\operatorname{Fix}}(T)$ is called *strictly R-BSNE.*

(iii) An operator which satisfies (15)-(17) with respect to $S := \operatorname{Fix}(T) = \widehat{\operatorname{Fix}}(T)$ is called *fully R-BSNE.* ◊

Remark 4.3. When condition (15) is satisfied with S being the set $\operatorname{Fix}(T)$, $\widehat{\operatorname{Fix}}(T)$ or $\operatorname{Fix}(T) = \widehat{\operatorname{Fix}}(T)$, the operator T is said to be properly R-QBNE, strictly R-QBNE or fully R-QBNE, respectively. The acronym QBNE stands for quasi-Bregman nonexpansive. ◊

Proposition 4.4 (Composition of strictly R-BSNE operators)**.** *Let $f : X \to \mathbb{R}$ be an admissible function which is bounded, uniformly continuous and totally convex on bounded subsets of X. Let K be a nonempty subset of X. Let $\{T_i : i = 1, \ldots, N\}$ be N strictly R-BSNE operators from K into itself, and $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$. Assume that the set*

$$\widehat{F} = \bigcap_{i=1}^{N} \widehat{\mathrm{Fix}}\,(T_i)$$

is not empty. Then

(i) $\widehat{\mathrm{Fix}}\,(T) \subset \widehat{F}$*;*

(ii) *moreover, if* $\widehat{\mathrm{Fix}}\,(T) \neq \emptyset$*,* T *is strictly R-BSNE.*

Proof. Let $u \in \widehat{F}$. First we claim that if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded and

$$\lim_{n\to\infty} \left(D_f\left(x_n, u\right) - D_f\left(Tx_n, u\right)\right) = 0, \tag{18}$$

then, for all $i = 1, \ldots, N$, we have

$$\lim_{n\to\infty} \left\|y_n^i - y_n^{i-1}\right\| = 0,$$

where $y_n^i := T_i \circ T_{i-1} \circ \cdots \circ T_1 x_n$, $i = 1, \ldots, N$, under the conventions that $y_n^0 = x_n$ and $T_0 = I$, the identity operator.

Now we prove this claim. Since each T_i, $i = 1, \ldots, N$, is strictly R-BSNE operator we get from (15)

$$D_f\left(Tx_n, u\right) = D_f\left(y_n^N, u\right) \leq D_f\left(y_n^i, u\right) \leq D_f\left(y_n^{i-1}, u\right) \leq \cdots \leq D_f\left(T_1 x_n, u\right) \leq D_f\left(x_n, u\right). \tag{19}$$

Hence, from (18), we get for all $i = 2, \ldots, N$, that

$$\lim_{n\to\infty} \left(D_f\left(y_n^{i-1}, u\right) - D_f\left(y_n^i, u\right)\right) \leq \lim_{n\to\infty} \left(D_f\left(x_n, u\right) - D_f\left(Tx_n, u\right)\right) = 0.$$

Thus

$$\lim_{n\to\infty} \left(D_f\left(y_n^{i-1}, u\right) - D_f\left(y_n^i, u\right)\right) = 0. \tag{20}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded and f is bounded on bounded subsets, the sequence $\{D_f\left(x_n, u\right)\}_{n\in\mathbb{N}}$ is bounded too. Therefore if follows from (19) that $\left\{D_f\left(y_n^i, u\right)\right\}_{n\in\mathbb{N}}$ is bounded for each $i = 1, \ldots, N$. Now it follows from Proposition 2.5 that $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded. This together with (20) implies that

$$\lim_{n\to\infty} D_f\left(y_n^{i-1}, y_n^i\right) = 0$$

because T_i, $i = 1, \ldots, N$, is strictly R-BSNE. Since $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded, Proposition 2.4 now implies that

$$\lim_{n\to\infty} \left\|y_n^i - y_n^{i-1}\right\| = 0,$$

as claimed.

Now we will prove our assertions.

(i) Let $u \in \widehat{F}$. Given $x \in \widehat{\operatorname{Fix}}(T)$, then there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset K$ converging weakly to x such that

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0. \tag{21}$$

Since f is uniformly Fréchet differentiable on bounded subsets, it is also uniformly continuous on bounded subsets of X (see [1, Theorem 1.8, page 13]) and therefore we obtain from (21) that

$$\lim_{n \to \infty} \left(f(Tx_n) - f(x_n)\right) = 0. \tag{22}$$

From the definition of the Bregman distance (see (1)) we obtain that

$$\begin{aligned} D_f(x_n, u) - D_f(Tx_n, u) &= \left[f(x_n) - f(u) - \langle \nabla f(u), x_n - u\rangle\right] \\ &\quad - \left[f(Tx_n) - f(u) - \langle \nabla f(u), Tx_n - u\rangle\right] \\ &= f(x_n) - f(Tx_n) + \langle \nabla f(u), Tx_n - x_n\rangle. \end{aligned}$$

Hence from (21) and (22) we deduce that

$$\lim_{n \to \infty} \left(D_f(x_n, u) - D_f(Tx_n, u)\right) = 0. \tag{23}$$

As we have already proved, from this property we obtain that

$$\lim_{n \to \infty} \left\|y_n^i - y_n^{i-1}\right\| = 0$$

for all $i = 1, \ldots, N$. Now from (21) we obtain that $\left\{y_n^N\right\}_{n \in \mathbb{N}}$ (note that $y_n^N = Tx_n$) converges weakly to x and since

$$\lim_{n \to \infty} \left\|y_n^N - y_n^{N-1}\right\| = 0$$

we get that $\left\{y_n^{N-1}\right\}_{n \in \mathbb{N}}$ also converges weakly to x. Repeating this argument for all $i = 1, \ldots N-2$, yields that the all sequences $\left\{y_n^i\right\}_{n \in \mathbb{N}}$ converge weakly to x. Now, since

$$\lim_{n \to \infty} \left\|y_n^{i-1} - T_i y_n^{i-1}\right\| = 0$$

we get that $x \in \widehat{\operatorname{Fix}}(T_i)$ for each $i = 1, 2, \ldots, N$, so that $x \in \widehat{F}$, as asserted.

(ii) Let $u \in \widehat{\operatorname{Fix}}(T)$ and $x \in K$. From assertion (i) we already know that $u \in \widehat{F}$. Since each T_i, $i = 1, 2, \ldots, N$, satisfies (15) we obtain that

$$\begin{aligned} D_f(Tx, u) &= D_f(T_N \circ T_{N-1} \circ \cdots \circ T_1 x, u) \\ &\leq D_f(T_{N-1} \circ \cdots \circ T_1 x, u) \\ &\leq \\ &\vdots \\ &\leq D_f(T_1 x, u) \\ &\leq D_f(x, u). \end{aligned}$$

Hence T also satisfies (4). Given $u \in \widehat{\operatorname{Fix}}(T)$ and a bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \left(D_f(x_n, u) - D_f(Tx_n, u)\right) = 0, \tag{24}$$

we now prove that (6) is valid. We have already proved that (24) yields

$$\lim_{n\to\infty} \left\| y_n^i - y_n^{i-1} \right\| = 0$$

for all $i = 1, \ldots, N$. Since

$$\|x_n - Tx_n\| \leq \left\| x_n - y_n^1 \right\| + \left\| y_n^1 - y_n^2 \right\| + \cdots + \left\| y_n^{N-1} - y_n^N \right\|$$

we get that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [18, Propostion 1.1.11, page 16]). Thus both sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\nabla f(x_n)\}_{n\in\mathbb{N}}$ are bounded. Since f is uniformly continuous on bounded subsets of X (see [1, Theorem 1.8, page 13]),

$$\lim_{n\to\infty} \left(f(Tx_n) - f(x_n) \right) = 0.$$

So from the definition of the Bregman distance (see (1)) we obtain that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

Hence T is strictly R-BSNE, as asserted. □

Using Proposition 4.4 we get an analogous result of Proposition 3.4 for fully R-BSNE operators.

Proposition 4.5 (Composition of fully R-BSNE operators)**.** *Let $f : X \to \mathbb{R}$ be an admissible function which is bounded, uniformly continuous and totally convex on bounded subsets of X. For each $i = 1, \ldots, N$, let $T_i : K \subset X \to K$ be a fully R-BSNE operator , and let $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$. If*

$$F := \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$$

is nonempty, then T is also fully R-BSNE and $\operatorname{Fix}(T) = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$.

5 Convex combinations of right Bregman strongly nonexpansive operators

The goal of this section is to prove that, under certain assumptions regarding the function f, the set of R-BSNE operators is closed under convex combinations. We start by defining the convex combination operator of finitely many operators.

Definition 5.1. Given a finite family $\{T_i : i = 1, \ldots, N\}$ of operators from $K \subset \operatorname{int} \operatorname{dom} f$ into $\operatorname{int} \operatorname{dom} f$, and given weights $\{w_i\}_{i=1}^N \subset (0, 1)$ satisfying $\sum_{i=1}^N w_i = 1$, the convex combination operator $T_C : K \to \operatorname{int} \operatorname{dom} f$ is defined by

$$T_C = \sum_{i=1}^{N} w_i T_i.$$

Remark 5.2. Note that, for any finite family $\{T_i : i = 1, \ldots, N\}$ of operators from $K \subset \operatorname{int}\operatorname{dom} f$ into $\operatorname{int}\operatorname{dom} f$, and any $x, p \in K$, the convexity of f implies that

$$\begin{aligned}
D_f(T_C x, p) &= f\left(\sum_{i=1}^{N} w_i T_i x\right) - f(p) - \left\langle \nabla f(p), \sum_{i=1}^{N} w_i T_i x - p \right\rangle \\
&\leq \sum_{i=1}^{N} w_i f(T_i x) - \sum_{i=1}^{N} w_i f(p) - \sum_{i=1}^{N} w_i \langle \nabla f(p), T_i x - p\rangle \\
&= \sum_{i=1}^{N} w_i D_f(T_i x, p). \qquad (25)
\end{aligned}$$

$\diamondsuit$

We start studying the convex combination operator by considering a family of finitely many strictly R-BSNE operators.

Proposition 5.3 (Convex combination of strictly R-BSNE operators)**.** *Let $f : X \to (-\infty, +\infty]$ be an admissible function which is bounded, uniformly continuous and totally convex on bounded subsets of* $\operatorname{int}\operatorname{dom} f$. *Let $\{T_i : 1 \leq i \leq N\}$ be a finite family of strictly R-BSNE operators from $K \subset \operatorname{int}\operatorname{dom} f$ into $\operatorname{int}\operatorname{dom} f$. Assume that the set*

$$\widehat{F} := \bigcap_{i=1}^{N} \widehat{\operatorname{Fix}}(T_i) \neq \emptyset.$$

Then, given weights $\{w_i\}_{i=1}^{N} \subset (0,1)$ satisfying $\sum_{i=1}^{N} w_i = 1$,

(i) $\widehat{\operatorname{Fix}}(T_C) \subset \widehat{F}$*;*

(ii) *moreover if $\widehat{\operatorname{Fix}}(T_C) \neq \emptyset$, T_C is strictly R-BSNE with respect to $\widehat{\operatorname{Fix}}(T_C) \subset \widehat{F}$.*

Proof. (i) Let $x \in \widehat{\operatorname{Fix}}(T_C)$, that is, there exists a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ converging weakly to x such that $\lim_{n\to\infty} \|x_n - T_C x_n\| = 0$. We claim that

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0 \qquad (26)$$

for each $k \in \{1, \ldots, N\}$. Indeed, since $\lim_{n\to\infty} \|x_n - T_C x_n\| = 0$, the sequence $\{T_C x_n\}_{n\in\mathbb{N}}$ is bounded and therefore the uniform continuity of f on bounded subsets of X implies that

$$\lim_{n\to\infty} (f(x_n) - f(T_C x_n)) = 0.$$

Given $p \in \widehat{F}$, by the definition of the Bregman distance (see (1)) we have

$$\begin{aligned}
D_f(x_n, p) - D_f(T_C x_n, p) &= [f(x_n) - f(p) - \langle \nabla f(p), x_n - p\rangle] \\
&\quad - [f(T_C x_n) - f(p) - \langle \nabla f(p), T_C x_n - p\rangle] \\
&= f(x_n) - f(T_C x_n) + \langle \nabla f(p), T_C x_n - x_n\rangle.
\end{aligned}$$

Hence we obtain that

$$\lim_{n\to\infty} (D_f(x_n, p) - D_f(T_C x_n, p)) = 0. \qquad (27)$$

Fix $k \in \{1, \ldots, N\}$. Since every T_i is strictly R-BSNE, using inequality (25), we get

$$\begin{aligned} D_f(T_C x_n, p) &\leq \sum_{i=1}^{N} w_i D_f(T_i x, p) \\ &= w_k D_f(T_k x_n, p) + \sum_{i \neq k} w_i D_f(T_i x_n, p) \\ &\leq w_k D_f(T_k x_n, p) + (1 - w_k) D_f(x_n, p) \\ &= w_k (D_f(T_k x_n, p) - D_f(x_n, p)) + D_f(x_n, p), \end{aligned} \tag{28}$$

which implies that

$$\lim_{n\to\infty} (D_f(x_n, p) - D_f(T_k x_n, p)) \leq \frac{1}{w_k} \lim_{n\to\infty} (D_f(x_n, p) - D_f(T_C x_n, p)) = 0.$$

Since T_k is strictly R-BSNE and $p \in \widehat{F} \subset \widehat{\mathrm{Fix}}(T_k)$, it follows that

$$\lim_{n\to\infty} D_f(x_n, T_k x_n) = 0. \tag{29}$$

Since T_k is strictly R-QBNE and $p \in \widehat{F} \subset \widehat{\mathrm{Fix}}(T_k)$, we have $D_f(T_k x_n, p) \leq D_f(x_n, p)$. In addition, since the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded and f is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f$, it follows that the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ is bounded too. The boundedness of these two sequences implies that the sequence $\{D_f(x_n, p)\}_{n\in\mathbb{N}}$ and therefore the sequence $\{D_f(T_k x_n, p)\}_{n\in\mathbb{N}}$ are also bounded. Now we can apply Proposition 2.5 to assure that $\{T_k x_n\}_{n\in\mathbb{N}}$ is bounded. Thus the total convexity of f on bounded subsets of $\operatorname{int}\operatorname{dom} f$ and Proposition 2.4 imply (26), as claimed. That is, $x \in \widehat{\mathrm{Fix}}(T_k)$ for all k and so $\widehat{\mathrm{Fix}}(T_C) \subset \widehat{F}$, as asserted.

(ii) Since we already proved that $\widehat{\mathrm{Fix}}(T_C) \subset \widehat{F}$, the fact that T_C is strictly R-QBNE is a consequence of each T_k being strictly R-QBNE. Indeed, for any $p \in \widehat{\mathrm{Fix}}(T_C)$, $x \in K$, using inequality (25), we get

$$D_f(T_C x, p) \leq \sum_{i=1}^{N} w_i D_f(T_i x, p) \leq D_f(x, p).$$

It remains to prove that given a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ and $p \in \widehat{\mathrm{Fix}}(T_C) \subset \widehat{F}$ satisfying $\lim_{n\to\infty} (D_f(x_n, p) - D_f(T_C x_n, p)) = 0$, it follows that

$$\lim_{n\to\infty} D_f(x_n, T_C x_n) = 0.$$

So fixing $k \in \{1, \ldots, N\}$, since $p \in \widehat{F}$, inequality (28) holds. Therefore inequality (29) remains true and likewise we have (26), that is

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0. \tag{30}$$

Since $k \in \{1, \ldots, N\}$ is arbitrary, it follows that

$$\lim_{n\to\infty} \|x_n - T_C x_n\| \leq \lim_{n\to\infty} \sum_{i=1}^{N} w_i \|x_n - T_i x_n\| = 0$$

and therefore $\{T_C x_n\}_{n\in\mathbb{N}}$ is bounded. Thus the sequence $\{\nabla f(T_C x_n)\}_{n\in\mathbb{N}}$ is bounded too, because ∇f is bounded on bounded sets (see Proposition 2.1), and

$$\lim_{n\to\infty}\left(f(x_n)-f(T_C x_n)\right)=0.$$

Thus from the definition of the Bregman distance (see (1)), we see that

$$\lim_{n\to\infty} D_f(x_n, T_C x_n)=0.$$

Hence T_C is indeed strictly R-BSNE. □

The convex combination of properly R-BSNE operators $\{T_i\}_{i=1}^N$ turns out to be also properly R-BSNE, and its fixed point set coincides with the intersection of all the fixed point sets of the operators T_i. Before proving this, we first show that this also holds for properly R-QBNE operators. To this end, we use a lemma the proof of which, given in [27], is included here for the sake of completeness.

Lemma 5.4. *Let $f : X \to (-\infty,+\infty]$ be a Legendre function and let the weights $\{w_i\}_{i=1}^N \subset (0,1)$ be such that $\sum_{i=1}^N w_i = 1$. Let $\{x_i\}_{i=1}^N$ be a family of points in $\operatorname{int}\operatorname{dom} f$ and assume that*

$$f\left(\sum_{i=1}^N w_i x_i\right)=\sum_{i=1}^N w_i f(x_i). \tag{31}$$

Then $x_1 = x_2 = \ldots = x_N$.

Proof. If $x_k \neq x_l$ for some $k,l \in \{1,2,\ldots,N\}$, then from the strict convexity of f in $\operatorname{int}\operatorname{dom} f$ we get

$$f\left(\frac{t_k}{t_k+t_l}x_k+\frac{t_l}{t_k+t_l}x_l\right)<\frac{t_k}{t_k+t_l}f(x_k)+\frac{t_l}{t_k+t_l}f(x_l).$$

Using this inequality, we obtain

$$\begin{aligned}
f\left(\sum_{i=1}^N w_i x_i\right) &= f\left((t_k+t_l)\left(\frac{t_k}{t_k+t_l}x_k+\frac{t_l}{t_k+t_l}x_l\right)+\sum_{i\neq k,l} w_i x_i\right)\\
&\leq (t_k+t_l) f\left(\frac{t_k}{t_k+t_l}x_k+\frac{t_l}{t_k+t_l}x_l\right)+\sum_{i\neq k,l} w_i f(x_i)\\
&< (t_k+t_l)\left(\frac{t_k}{t_k+t_l}f(x_k)+\frac{t_l}{t_k+t_l}f(x_l)\right)+\sum_{i\neq k,l} w_i f(x_i)\\
&= \sum_{i=1}^N w_i f(x_i).
\end{aligned}$$

This contradicts assumption (31). □

Proposition 5.5 (Convex combination of properly R-QBNE operators)**.** *Let $f : X \to (-\infty,+\infty]$ be a Legendre function and let $\{T_i : 1 \leq i \leq N\}$ be a family of properly R-QBNE operators from $K \subset \operatorname{int}\operatorname{dom} f$ into $\operatorname{int}\operatorname{dom} f$. Assume that the set*

$$F=\bigcap_{i=1}^N \operatorname{Fix}(T_i)\neq\emptyset. \tag{32}$$

Then, given weights $\{w_i\}_{i=1}^N \subset (0,1)$ satisfying $\sum_{i=1}^N w_i = 1$, the convex combination operator T_C is properly R-QBNE with respect to $\operatorname{Fix}(T_C) = F$.

Proof. Let $p \in F$ and $x \in K$. Since every T_i is properly R-QBNE, from inequality (25) we get

$$D_f(T_C x, p) \leq \sum_{i=1}^{N} w_i D_f(T_i x, p) \leq D_f(x, p),$$

that is, T_C is R-QBNE with respect to F. We now show that $F = \operatorname{Fix}(T_C)$, so that we could conclude that T_C is properly R-QBNE. The fact that $F \subset \operatorname{Fix}(T_C)$ is clear. To prove the other inclusion, let $u \in \operatorname{Fix}(T_C)$, $p \in F$ and $k \in \{1, 2 \ldots, N\}$. Then, by inequality (25),

$$D_f(u, p) = D_f(T_C u, p) \leq \sum_{i=1}^{N} w_i D_f(T_i u, p) \leq \sum_{i \neq k}^{N} w_i D_f(T_i u, p) + w_k D_f(T_k u, p).$$

Therefore

$$w_k D_f(u, p) = \left(1 - \sum_{i \neq k}^{N} w_i\right) D_f(u, p) \leq w_k D_f(T_k u, p).$$

Since $w_k > 0$, we may conclude that $D_f(u, p) \leq D_f(T_k u, p)$. This, when combined with the facts that T_k is R-QBNE and $p \in F \subset \operatorname{Fix}(T_k)$, implies that $D_f(u, p) = D_f(T_k u, p)$. Thus,

$$D_f\left(\sum_{i=1}^{N} w_i T_i u, p\right) = D_f(u, p) = \sum_{i=1}^{N} w_i D_f(u, p) = \sum_{i=1}^{N} w_i D_f(T_i u, p).$$

By the definition of the Bregman distance (see (1)), from the previous equality we deduce that

$$f\left(\sum_{i=1}^{N} w_i T_i u\right) = \sum_{i=1}^{N} w_i f(T_i u)$$

for all $u \in \operatorname{Fix}(T_C)$. So Lemma 5.4 applies to yield that $T_i u = T_j u$ for all $i, j = 1, \ldots, N$. Hence $u = \sum_{i=1}^{N} w_i T_i u = T_j u$ for all $j = 1, \ldots, N$, which means that $\operatorname{Fix}(T_C) \subset F = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$, as claimed. □

Proposition 5.6 (Convex combination of properly R-BSNE operators)**.** *Let $f : X \to (-\infty, +\infty]$ be an admissible function which is bounded, uniformly continuous and totally convex on bounded subsets of* $\operatorname{int}\operatorname{dom} f$*. Let $\{T_i : 1 \leq i \leq N\}$ be a family of properly R-BSNE operators from $K \subset$* $\operatorname{int}\operatorname{dom} f$ *into* $\operatorname{int}\operatorname{dom} f$*. Assume that the set*

$$F = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i) \neq \emptyset.$$

Then, given weights $\{w_i\}_{i=1}^{N} \subset (0, 1)$ satisfying $\sum_{i=1}^{N} w_i = 1$, the convex combination operator T_C is properly R-BSNE with respect to $\operatorname{Fix}(T_C) = F$*.*

Proof. Since every T_i is properly R-QBNE, Proposition 5.5 assures us that T_C is properly R-QBNE with respect to $F = \operatorname{Fix}(T_C)$. Now, given a bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ and $p \in \operatorname{Fix}(T_C)$ satisfying $\lim_{n \to \infty} (D_f(x_n, p) - D_f(T_C x_n, p)) = 0$, exactly the same argument as in Proposition 5.3 applies to prove that

$$\lim_{n \to \infty} D_f(x_n, T_C x_n) = 0.$$

Thus T_C is indeed properly R-BSNE. □

For the norm analog of this result, see [32, Lemmata 1.3 and 1.4, pages 282-283].

6 Connections between L-BSNE and R-BSNE operators

Let $f : X \to (-\infty, +\infty]$ be a Legendre function, $T : K \subset \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ an operator and let $S \subset \operatorname{int}\operatorname{dom} f$ be a nonempty set. The *conjugate operator* associated with T is defined by

$$T^* := \nabla f \circ T \circ \nabla f^* : \nabla f(K) \to \operatorname{int}\operatorname{dom} f^*. \tag{33}$$

This operator was first studied in [28], where its basic properties are collected in Proposition 2.7 there.

Fact 6.1. If ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$ and T is R-BSNE with respect to S, then T^* is L-BSNE with respect to $\nabla f(S)$. To see this, we first recall that

$$D_{f^*}(\nabla f(y), \nabla f(x)) = D_f(x, y) \quad \forall x, y \in \operatorname{int}\operatorname{dom} f.$$

Since T is R-QBNE with respect to S, this implies that T^* is L-QBNE with respect to $\nabla f(S)$. Now let $\{\xi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\nabla f(K)$ such that

$$\lim_{n\to\infty}(D_{f^*}(\eta, \xi_n) - D_{f^*}(\eta, T^*\xi_n)) = 0$$

for a point $\eta \in \nabla f(S)$. Then the sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$, defined by $x_n = \nabla f^*(\xi_n)$, is bounded and

$$\lim_{n\to\infty}(D_f(x_n, p) - D_f(Tx_n, p)) = 0$$

for $p = \nabla f^*(\eta) \in S$. Since T is R-BSNE, it follows that

$$\lim_{n\to\infty} D_{f^*}(T^*\xi_n, \xi_n) = \lim_{n\to\infty} D_f(x_n, Tx_n) = 0,$$

so T^* is indeed L-BSNE with respect to $\nabla f(S)$, as claimed.

Analogously, it is possible to prove that if ∇f is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f$, then the converse implication holds. Therefore we arrive at the following result.

Fact 6.2. If ∇f and ∇f^* are bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f$ and $\operatorname{int}\operatorname{dom} f^*$, respectively, then T is R-BSNE with respect to S if and only if T^* is L-BSNE with respect to $\nabla f(S)$.

Thus, if $S = \operatorname{Fix}(T)$, since $\nabla f(\operatorname{Fix}(T)) = \operatorname{Fix}(T^*)$ (see [28, Proposition 2.7 (iii)]), we have the following relationship between properly right and left BSNE operators.

Fact 6.3. If ∇f and ∇f^* are bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f$ and $\operatorname{int}\operatorname{dom} f^*$, respectively, then T is properly R-BSNE if and only if T^* is properly L-BSNE.

Furthermore, under certain continuity assumptions on the gradient mappings we obtain relations between the asymptotic fixed point sets of T and T^*. If ∇f^* is uniformly continuous on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$, by the definition of asymptotic fixed points it is easy to check that $\widehat{\operatorname{Fix}}(T^*) \subset \nabla f\left(\widehat{\operatorname{Fix}}(T)\right)$. In an analogous way, we can show that when ∇f is uniformly continuous on bounded subsets of $\operatorname{int}\operatorname{dom} f$, then the reverse inclusion holds, resulting in the following statement (see [28, Proposition 2.7 (vi)]).

Fact 6.4. If ∇f and ∇f^* are uniformly continuous on bounded subsets of $\operatorname{int}\operatorname{dom} f$ and $\operatorname{int}\operatorname{dom} f^*$, respectively, then $\widehat{\operatorname{Fix}}(T^*) = \nabla f\left(\widehat{\operatorname{Fix}}(T)\right)$.

So in this case, if $S = \widehat{\mathrm{Fix}}\,(T)$, Facts 6.2 and 6.4, yield the following equivalence between strictly right and left BSNE operators.

Fact 6.5. If ∇f and ∇f^* are bounded and uniformly continuous on bounded subsets of $\operatorname{int}\operatorname{dom} f$ and $\operatorname{int}\operatorname{dom} f^*$, respectively, then T is strictly R-BSNE if and only if T^* is strictly L-BSNE.

It is natural to wonder whether results regarding R-BSNE operators can be obtained from known results concerning L-BSNE operators by conjugation (see (33)) and *vice versa*. In this direction, bearing in mind, for instance, the connection we have just presented, we are able to show that Propositions 4.4 could be deduced from Propositions 3.3 under different suitable conditions, most of them imposed on the conjugate function.

Proposition 6.6. *Let $f : X \to (-\infty, +\infty]$ be a cofinite Legendre function such that f^* is totally convex on bounded subsets of X^*. Assume that ∇f and ∇f^* are bounded and uniformly continuous on bounded subsets of $\operatorname{int}\operatorname{dom} f$ and X^*, respectively. Let $\{T_i : i = 1, \ldots, N\}$ be N strictly R-BSNE operators from K into itself, where $K \subset X$, and let $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$. If the set*

$$\widehat{F} = \bigcap_{i=1}^{N} \widehat{\mathrm{Fix}}\,(T_i) \neq \emptyset,$$

then $\widehat{\mathrm{Fix}}\,(T) \subset \widehat{F}$. Furthermore, if $\widehat{\mathrm{Fix}}\,(T)$ is not empty, then T is strictly R-BSNE.

Proof. We consider, for each $i = 1, \ldots, N$, the conjugate operator

$$T_i^* := \nabla f \circ T_i \circ \nabla f^* : \nabla f\,(K) \to \nabla f\,(K)\,.$$

Since ∇f^* is bounded and uniformly continuous on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$, we know from Fact 6.5 that, for any $i = 1, \ldots, N$, T_i^* is strictly L-BSNE with respect to $\widehat{\mathrm{Fix}}\,(T^*) = \nabla f\left(\widehat{\mathrm{Fix}}\,(T)\right)$. Therefore, if

$$\widehat{F} := \bigcap_{i=1}^{N} \widehat{\mathrm{Fix}}\,(T_i)$$

is nonempty, we can show that so is

$$\widehat{F^*} := \bigcap_{i=1}^{N} \widehat{\mathrm{Fix}}\,(T_i^*)\,.$$

Indeed, $x \in \widehat{F}$ if and only if $x \in \widehat{\mathrm{Fix}}\,(T_i) = \nabla f^*\left(\widehat{\mathrm{Fix}}\,(T_i^*)\right)$ for all $i = 1, \ldots, N$. In other words, $x \in \widehat{F}$ if and only if $\nabla f\,(x) \in \widehat{\mathrm{Fix}}\,(T_i^*)$ for any $i = 1, \ldots, N$, which is equivalent to $\nabla f\,(x) \in \widehat{F^*}$. This means that

$$\nabla f\left(\widehat{F}\right) = \widehat{F^*}. \tag{34}$$

Thus the family $\{T_i^* : i = 1, \ldots, N\}$ satisfies the hypotheses of Proposition 3.3, and consequently, if we denote $T^* = T_n^* \circ \cdots \circ T_1^*$, we see that

$$\widehat{\mathrm{Fix}}\,(T^*) \subset \widehat{F^*}. \tag{35}$$

Note that T^* is the conjugate operator of $T = T_n \circ \cdots \circ T_1$, that is,

$$T^* = (\nabla f \circ T_n \circ \nabla f^*) \circ \cdots \circ (\nabla f \circ T_1 \circ \nabla f^*) = \nabla f \circ T_n \circ \cdots \circ T_1 \circ \nabla f^* = \nabla f \circ T \circ \nabla f^*$$

and then $\widehat{\mathrm{Fix}}\,(T) = \nabla f^* \left(\widehat{\mathrm{Fix}}\,(T^*)\right)$. Hence one deduces from (34) and (35) that

$$\widehat{\mathrm{Fix}}\,(T) = \nabla f^* \left(\widehat{\mathrm{Fix}}\,(T^*)\right) \subset \nabla f^* \left(\widehat{F^*}\right) = \widehat{F}.$$

If we assume that $\widehat{\mathrm{Fix}}\,(T)$ is nonempty, then so is $\widehat{\mathrm{Fix}}\,(T^*)$ and thus Proposition 3.3 assures us that T^* is strictly L-BSNE. It follows that T is strictly R-BSNE, as asserted. □

Taking into account that the hypotheses of Proposition 6.6 seem to be stronger than those of Propositions 4.4, we see that the conjugation technique does not seem to lead to the best possible results.

Regarding the convex combination operator, the results proved in Section 5 can also be recovered from analogous results associated with the so-called block operator defined and analyzed in [27]. We recall here its definition and main properties [27].

Definition 6.7 (Block operator)**.** Let $\{T_i : 1 \leq i \leq N\}$ be N operators from X into X and let the weights $\{w_i\}_{i=1}^{N} \subset (0,1)$ satisfy $\sum_{i=1}^{N} w_i = 1$. Then the *block operator* corresponding to $\{T_i : 1 \leq i \leq N\}$ and $\{w_i : 1 \leq i \leq N\}$ is defined by

$$T_B := \nabla f^* \circ \left(\sum_{i=1}^{N} w_i \nabla f \circ T_i\right). \tag{36}$$

Proposition 6.8 (Block operator of strictly L-BSNE operators)**.** *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$*. If each T_i, $i = 1, \ldots, N$, is a strictly L-BSNE operator from $K \subset X$ into X and the set*

$$\widehat{F} := \bigcap \left\{\widehat{\mathrm{Fix}}\,(T_i) : 1 \leq i \leq N\right\} \neq \emptyset,$$

then $\widehat{\mathrm{Fix}}\,(T_B) \subset \widehat{F}$. Furthermore, if $\widehat{\mathrm{Fix}}\,(T_B) \neq \emptyset$, then T_B is also strictly L-BSNE.

Proposition 6.9 (Block operator of properly L-BSNE operators)**.** *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that ∇f^* and is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$*. If each T_i, $i = 1, \ldots, N$, is a properly L-BSNE operator from $K \subset X$ into X and the set*

$$F := \bigcap \{\mathrm{Fix}\,(T_i) : 1 \leq i \leq N\} \neq \emptyset,$$

then T_B is also properly L-BSNE and $F = \mathrm{Fix}\,(T_B)$.

These propositions can be applied to deduce the theses in Propositions 5.3, and 5.6 under different conditions on the function f as we show in the following two propositions.

Proposition 6.10. *Let $f : X \to (-\infty, +\infty]$ be a cofinite Legendre function such that f^* is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X^*. Assume that ∇f and ∇f^* are bounded and uniformly continuous on bounded subsets of* $\operatorname{int}\operatorname{dom} f$ *and X^*, respectively. Let $\{T_i : i = 1, \ldots, N\}$ be N strictly R-BSNE operators from $K \subset X$ into* $\operatorname{int}\operatorname{dom} f$*. If the set*

$$\widehat{F} = \bigcap_{i=1}^{N} \widehat{\mathrm{Fix}}\,(T_i) \neq \emptyset,$$

then, given weights $\{w_i\}_{i=1}^{N} \subset (0,1)$ satisfying $\sum_{i=1}^{N} w_i = 1$, every asymptotic fixed point of the convex combination operator T_C belongs to $\widehat{F}$, that is, $\widehat{\mathrm{Fix}}\,(T_C) \subset \widehat{F}$. Furthermore, if $\widehat{\mathrm{Fix}}\,(T_C)$ is not empty, then T_C is strictly R-BSNE.

Proof. We consider the conjugate operators $T_i^* = \nabla f \circ T_i \circ \nabla f^*$ from $\nabla f(K)$ into X^*, $i = 1, \ldots, N$, which are strictly L-BSNE with respect to $\widehat{\mathrm{Fix}}(T_i^*) = \nabla f\left(\widehat{\mathrm{Fix}}(T_i)\right)$ (see Facts 6.4 and 6.5). In Proposition 6.6 we proved that $\nabla f\left(\widehat{F}\right) = \widehat{F^*} := \bigcap_{i=1}^N \widehat{\mathrm{Fix}}(T_i^*)$. So $\widehat{F^*} \neq \emptyset$ and Proposition 6.8 ensures that the asymptotic fixed points of the block operator $T_B^* := \nabla f \circ \left(\sum_{i=1}^N w_i \nabla f^* \circ T_i^*\right)$ belong to $\widehat{F^*}$, that is, $\widehat{\mathrm{Fix}}(T_B^*) \subset \widehat{F^*}$. Note that T_B^* is the conjugate operator of T_C:

$$
\begin{aligned}
T_B^* &= \nabla f \circ \left(\sum_{i=1}^N w_i \nabla f^* \circ (\nabla f \circ T_i \circ \nabla f^*)\right) \\
&= \nabla f \circ \left(\sum_{i=1}^N w_i T_i \circ \nabla f^*\right) \\
&= \nabla f \circ \left(\sum_{i=1}^N w_i T_i\right) \circ \nabla f^* \\
&= \nabla f \circ T_C \circ \nabla f^*.
\end{aligned}
$$

Then, by Proposition 6.8,

$$
\widehat{\mathrm{Fix}}(T_C) = \nabla f^*\left(\widehat{\mathrm{Fix}}(T_B^*)\right) \subset \nabla f^*\left(\widehat{F^*}\right) = \widehat{F}.
$$

If $\widehat{\mathrm{Fix}}(T_C) \neq \emptyset$, then so is $\widehat{\mathrm{Fix}}(T_B^*)$ and thus Proposition 6.8 assures us that T_B^* is strictly L-BSNE. So from Fact 6.5 it follows that T_C is strictly R-BSNE. □

Following the same arguments used in the previous proposition, it is readily proved that Proposition 5.6 is a consequence of Proposition 6.9 under suitable conditions on f and f^*.

Proposition 6.11. *Let $f : X \to \mathbb{R}$ be a cofinite Legendre function such that f and f^* are bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X and X^*, respectively. Assume that ∇f and ∇f^* are bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f$ *and X^*, respectively. For each $i = 1, \ldots, N$, let $T_i : K \subset X \to X$ be a properly R-BSNE operator. If the set*

$$
F = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset,
$$

then, given weights $\{w_i\}_{i=1}^N \subset (0,1)$ satisfying $\sum_{i=1}^N w_i = 1$, the convex combination operator T_C is properly R-BSNE with respect to $\mathrm{Fix}(T_C) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

7 Picard iteration for R-BSNE operators

In this section we are concerned with the iterates of R-BSNE operators, their compositions and their convex combinations.

Definition 7.1 (Weakly sequentially continuous mapping)**.** A mapping $B : X \to X^*$ is called *weakly sequentially continuous* if for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$, $x_n \rightharpoonup x$ implies that $Bx_n \rightharpoonup Bx$ as $n \to \infty$.

Proposition 7.2 (Weak convergence)**.** *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \operatorname{int}\operatorname{dom} f$ is bounded and that*

$$\lim_{n\to\infty} D_f(x_n, u) \tag{37}$$

exists for any weak subsequential limit u of $\{x_n\}_{n\in\mathbb{N}}$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly.

Proof. It suffices to prove the uniqueness of weak subsequential limits of $\{x_n\}_{n\in\mathbb{N}}$ because, since $\{x_n\}_{n\in\mathbb{N}}$ is bounded and X is reflexive, we know that there is at least one. Assume that u and v are any two weak subsequential limits of $\{x_n\}_{n\in\mathbb{N}}$. From (37) we know that

$$\lim_{n\to\infty} \left(D_f(x_n, u) - D_f(x_n, v)\right)$$

exists. From the definition of the Bregman distance (see (1)) we get

$$\begin{aligned} D_f(x_n, u) - D_f(x_n, v) &= [f(x_n) - f(u) - \langle \nabla f(u), x_n - u\rangle] \\ &- [f(x_n) - f(v) - \langle \nabla f(v), x_n - v\rangle] \\ &= f(v) - f(u) + \langle \nabla f(v) - \nabla f(u), x_n\rangle \\ &+ \langle \nabla f(u), u\rangle - \langle \nabla f(v), v\rangle \end{aligned}$$

and therefore

$$\lim_{n\to\infty} \langle \nabla f(v) - \nabla f(u), x_n\rangle$$

exists. Since u and v are weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$, there are subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_{m_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup u$ and $x_{m_k} \rightharpoonup v$ as $k \to \infty$. Thus

$$\begin{aligned} \langle \nabla f(v) - \nabla f(u), u\rangle &= \lim_{k\to+\infty} \langle \nabla f(v) - \nabla f(u), x_{n_k}\rangle \\ &= \lim_{n\to+\infty} \langle \nabla f(v) - \nabla f(u), x_n\rangle \\ &= \lim_{k\to+\infty} \langle \nabla f(v) - \nabla f(u), x_{m_k}\rangle \\ &= \langle \nabla f(v) - \nabla f(u), v\rangle . \end{aligned}$$

Hence $\langle \nabla f(v) - \nabla f(u), v - u\rangle = 0$, which implies that $u = v$ because f is strictly convex in $\operatorname{int}\operatorname{dom} f$ which implies the strict monotonicity of ∇f in $\operatorname{dom} \nabla f$. ☐

Definition 7.3 (Asymptotic regularity)**.** An operator $T : K \to K$ is called *asymptotically regular* if, for any $x \in K$, we have

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| = 0. \tag{38}$$

In the following result we prove that any R-BSNE operator is asymptotically regular.

Proposition 7.4 (R-BSNE operators are asymptotically regular)**.** *Let $f : X \to (-\infty, +\infty]$ be an admissible function which is totally convex on bounded subsets of* $\operatorname{int}\operatorname{dom} f$*. Let K be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$*. Let T be a strictly (properly) R-BSNE operator from K into itself such that* $\widehat{\operatorname{Fix}}(T) \neq \emptyset$ $(\operatorname{Fix}(T) \neq \emptyset)$*. Then T is asymptotically regular.*

Proof. Assume that T is strictly R-BSNE (see Remark 4.2(ii)). Let $u \in \widehat{\operatorname{Fix}}(T)$ and $x \in K$. From (15) we get that

$$D_f\left(T^{n+1}x, u\right) \le D_f(T^n x, u) \le \cdots \le D_f(Tx, u).$$

Thus $\lim_{n\to\infty} D_f(T^n x, u)$ exists and the sequence $\{D_f(T^n x, u)\}_{n\in\mathbb{N}}$ is bounded. Now Proposition 2.5 implies that $\{T^n x\}_{n\in\mathbb{N}}$ is also bounded for any $x \in K$. Since the limit $\lim_{n\to\infty} D_f(T^n x, u)$ exists, we have

$$\lim_{n\to\infty} \left[D_f(T^n x, u) - D_f\left(T^{n+1}x, u\right)\right] = 0.$$

From (16) and (17) we get

$$\lim_{n\to\infty} D_f\left(T^n x, T^{n+1}x\right) = 0.$$

Since $\{T^n x\}_{n\in\mathbb{N}}$ is bounded, we now obtain from Proposition 2.4 that

$$\lim_{n\to\infty} \left\|T^n x - T^{n+1}x\right\| = 0.$$

In other words, T is asymptotically regular. The proof in the case where T is properly R-BSNE is identical when we take $u \in \operatorname{Fix}(T)$ (see Remark 4.2(i)). ☐

We now state and prove the main result of this section.

Theorem 7.5 (Picard iteration)**.** *Let $f : X \to (-\infty, +\infty]$ be an admissible and totally convex function. Let K be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$ and let $T : K \to K$ be a strictly R-QBNE operator. Then the following assertions hold.*

- (i) *If $\widehat{\operatorname{Fix}}(T)$ is nonempty, then $\{T^n x\}_{n\in\mathbb{N}}$ is bounded for each $x \in K$.*
- (ii) *If, furthermore, T is asymptotically regular, then, for each $x \in K$, $\{T^n x\}_{n\in\mathbb{N}}$ converges weakly to an element of $\widehat{\operatorname{Fix}}(T)$.*

Proof. (i) See [29, Proposition 3.2].

- (ii) We know that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded (by assertion (i)). Let a subsequence $\{T^{n_k} x\}_{k\in\mathbb{N}}$ of $\{T^n x\}_{n\in\mathbb{N}}$ converge weakly to some u. Define $x_n = T^n x$ for any $n \in \mathbb{N}$. Since T is asymptotically regular, it follows from (38) that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Therefore we have $x_{n_k} \rightharpoonup u$ and $\|x_{n_k} - Tx_{n_k}\| \to 0$ as $k \to \infty$, which means that $u \in \widehat{\operatorname{Fix}}(T)$. Thus we have proved that any weak subsequential limit of $\{T^n x\}_{n\in\mathbb{N}}$ belongs to $\widehat{\operatorname{Fix}}(T)$. Since T is strictly R-QBNE, it follows that the limit $\lim_{n\to\infty} D_f(T^n x, u)$ exists for any weak subsequential limit u of the sequence $\{T^n x\}_{n\in\mathbb{N}}$. The result now follows immediately from Proposition 7.2.

☐

Corollary 7.6 (Picard iteration for fully R-BSNE operators)**.** *Let the function $f : X \to (-\infty, +\infty]$ be an admissible function which is totally convex on bounded subsets of $\operatorname{int}\operatorname{dom} f$. Let K be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$. Let $T : K \to K$ be a fully R-BSNE operator with $\operatorname{Fix}(T) \neq \emptyset$. Then $\{T^n x\}_{n\in\mathbb{N}}$ converges weakly to an element in $\operatorname{Fix}(T)$ for each $x \in K$.*

Proof. This result follows immediately from Theorem 7.5 and Proposition 7.4. ☐

Remark 7.7. For the norm analog of this result, see [32, Corollary 2.4, page 286]. If $\operatorname{Fix}(T) \neq \widehat{\operatorname{Fix}}(T)$, but $\widehat{\operatorname{Fix}}(T) \neq \emptyset$, then we only know that, for a strictly R-BSNE operator T, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges weakly to an element in $\widehat{\operatorname{Fix}}(T)$ for each $x \in K$. ◊

Remark 7.8 (Common fixed point - composition case)**.** Let $f : X \to \mathbb{R}$ be an admissible function which is bounded, uniformly continuous and totally convex on bounded subsets of X. Let K be a nonempty, closed and convex subset of X.

Let $\{T_i : 1 \le i \le N\}$ be N operators from K into itself which are R-BSNE with respect to $\widehat{\mathrm{Fix}}\,(T_i) = \mathrm{Fix}\,(T_i) \neq \emptyset$ for each $1 \le i \le N$ and let $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$. From Proposition 4.5 we obtain that if $\bigcap \{\mathrm{Fix}\,(T_i) : 1 \le i \le N\} \neq \emptyset$, then T is also R-BSNE with respect to $\widehat{\mathrm{Fix}}\,(T) = \mathrm{Fix}\,(T) = \bigcap \{\mathrm{Fix}\,(T_i) : 1 \le i \le N\}$.

From Theorem 7.5 we now get that $\{T^n x\}_{n\in\mathbb{N}}$ converges weakly to a common fixed point of the given family of R-BSNE operators. Similarly, if we just assume that each T_i is strictly R-BSNE, $1 \le i \le N$, with $\widehat{\mathrm{Fix}}\,(T_i) \neq \emptyset$, then we get weak convergence of the sequence $\{T^n x\}_{n\in\mathbb{N}}$ to a common asymptotic fixed point. $\Diamond$

Remark 7.9 (Common fixed point - convex combination case)**.** Let $f : X \to (-\infty, +\infty]$ be an admissible function which is bounded, uniformly continuous and totally convex on bounded subsets of $\mathrm{int}\,\mathrm{dom}\, f$. Let K be a nonempty, closed and convex subset of X.

Let $\{T_i : 1 \le i \le N\}$ be N operators from K to $\mathrm{int}\,\mathrm{dom}\, f$ which are R-BSNE with respect to $\widehat{\mathrm{Fix}}\,(T_i) = \mathrm{Fix}\,(T_i) \neq \emptyset$ for each $1 \le i \le N$. From Proposition 5.6 we know that if $\bigcap \{\mathrm{Fix}\,(T_i) : 1 \le i \le N\} \neq \emptyset$, then the convex combination operator T_C is also R-BSNE with respect to $\widehat{\mathrm{Fix}}\,(T) = \mathrm{Fix}\,(T) = \bigcap \{\mathrm{Fix}\,(T_i) : 1 \le i \le N\}$. Then Theorem 7.5 guarantees that $\{T_C^n x\}_{n\in\mathbb{N}}$ converges weakly to a common fixed point of the given family of R-BSNE operators. Similarly, if we just assume that each T_i is strictly R-BSNE, $1 \le i \le N$, with $\widehat{\mathrm{Fix}}\,(T_i) \neq \emptyset$, then we get weak convergence of the sequence $\{T_C^n x\}_{n\in\mathbb{N}}$ to a common asymptotic fixed point. $\Diamond$

8 Applications

In this section we present two applications of the Picard iteration proposed in the previous section. The first application concerns common zeroes of maximal monotone mappings and the second is an algorithm for solving convex feasibility problems.

8.1 Zeroes of Maximal Monotone Operators

Let $A : X \to 2^{X^*}$ be a maximal monotone set-valued mapping. The problem of finding an element $x \in X$ such that $0^* \in Ax$ is very important in Optimization Theory and Nonlinear Analysis. In this section we use the Picard iteration to find common zeroes of N maximal monotone set-valued mappings.

Definition 8.1 (Conjugate ∇f-resolvent)**.** Let $A : X \to 2^{X^*}$ be a set-valued mapping. The *conjugate resolvent* of A with respect to f, or the conjugate ∇f-resolvent, is the operator $\mathrm{CRes}_A^f : X^* \to 2^{X^*}$ defined by

$$\mathrm{CRes}_A^f := (I + A \circ \nabla f^*)^{-1}. \tag{39}$$

In the following proposition we collect several properties of conjugate resolvents (*cf.* [28]).

Proposition 8.2 (Properties of conjugate ∇f-resolvents)**.** *Let $f : X \to (-\infty, +\infty]$ be an admissible function and let $A : X \to 2^{X^*}$ be a set-valued mapping such that* $\mathrm{int}\,\mathrm{dom}\, f \cap \mathrm{dom}\, A \neq \emptyset$. *The following statements hold.*

(i) $\mathrm{dom}\,\mathrm{CRes}_A^f \subset \mathrm{int}\,\mathrm{dom}\, f^*$.

(ii) $\operatorname{ran} \operatorname{CRes}_A^f \subset \operatorname{int} \operatorname{dom} f^*$.

(iii) $\nabla f^* \left(\operatorname{Fix}\left(\operatorname{CRes}_A^f\right)\right) = \operatorname{int} \operatorname{dom} f \cap A^{-1}(0^*)$.

(iv) *Suppose, in addition, that A is a monotone mapping. Then the following assertions also hold.*

(a) *If $f|_{\operatorname{int} \operatorname{dom} f}$ is strictly convex, then the operator CRes_A^f is single-valued on its domain and properly R-BSNE (if $\operatorname{Fix}\left(\operatorname{CRes}_A^f\right) \neq \emptyset$).*

(b) *If $f : X \to \mathbb{R}$ is such that $\operatorname{ran} \nabla f \subset \operatorname{ran}(\nabla f + A)$, then $\operatorname{dom} \operatorname{CRes}_A^f = \operatorname{int} \operatorname{dom} f^*$.*

Remark 8.3. In [28, Proposition 5.5] it was actually proved that, under the assumptions in (iv)(a), the operator CRes_A^f is right Bregman firmly nonexpansive, which is more restrictive than being properly R-BSNE. We know that, if f is Legendre, and bounded and uniformly continuous on bounded subsets of X, then for every right Bregman firmly nonexpansive operator T, $\widehat{\operatorname{Fix}}(T) = \operatorname{Fix}(T)$ (see [35]). So under these assumptions on f, the operator CRes_A^f is R-BSNE.

The following proposition [28, Theorem 5.7] is essential for our convergence result.

Proposition 8.4 (Surjectivity result). *Let $f : X \to \mathbb{R}$ be a strictly convex, cofinite and admissible function, and let $A : X \to 2^{X^*}$ be a set-valued monotone mapping. Then A is maximal monotone if and only if $\operatorname{dom} \operatorname{CRes}_A^f = X^*$.*

Now we present a variant of the Picard iterative method.

Proposition 8.5 (Common zeroes). *Let $f : X \to \mathbb{R}$ be a cofinite Legendre function such that f^* is uniformly continuous and totally convex on bounded subsets of X^*. Assume that ∇f^* is weakly sequentially continuous. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, be N maximal monotone set-valued mappings such that $Z := \bigcap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. For each $\xi_1 \in X^*$, consider the sequence $\{\xi_n\}_{n \in \mathbb{N}}$ generated iteratively by*

$$\xi_{n+1} = \operatorname{CRes}_{A_N}^f \circ \operatorname{CRes}_{A_{N-1}}^f \circ \cdots \circ \operatorname{CRes}_{A_1}^f (\xi_1), \quad n \geq 1. \tag{40}$$

Then the sequence $\{\nabla f^*(\xi_n)\}_{n \in \mathbb{N}}$ converges weakly to an element in Z.

Proof. From Proposition 8.4 we know that each $T_i = \operatorname{CRes}_{A_i}^f$, $i = 1, 2, \ldots, N$, is an operator from X^* into itself. Since

$$\bigcap_{i=1}^N \operatorname{Fix}\left(\operatorname{CRes}_{A_i}^f\right) = \bigcap_{i=1}^N \nabla f\left(A_i^{-1}(0^*)\right) = \nabla f(Z) \neq \emptyset, \tag{41}$$

Proposition 8.2(iii), (iv)(a) and Remark 8.3 guarantee that each T_i, $i = 1, 2, \ldots, N$, is R-BSNE with respect to $\operatorname{Fix}(T_i) = \widehat{\operatorname{Fix}}(T_i)$. Now the result follows immediately from Remark 7.8 applied to X^*. □

Remark 8.6. Analogously to the previous case, using convex combinations of the conjugate resolvents instead of their composition, we can consider a different Picard iterative method defined by means of the scheme

$$\xi_{n+1} = \sum_{i=1}^N w_i \operatorname{CRes}_{A_i}^f (\xi_n), \quad n \geq 1.$$

The sequence $\{\nabla f^*(\xi_n)\}_{n \in \mathbb{N}}$ generated by this scheme converges weakly to a common zero of the given maximal monotone set-valued mappings. ◊

8.2 Convex feasibility problems

Let K_i, $i = 1, \ldots, N$, be N nonempty, closed and convex subsets of X. The convex feasibility problem (CFP) is to find an element in the assumed nonempty intersection $\bigcap_{i=1}^{N} K_i$.

Given a set $K \subset \operatorname{int}\operatorname{dom} f$, the *right Bregman projection* [28] onto K is the operator $\overrightarrow{\operatorname{proj}}_K^f : \operatorname{int}\operatorname{dom} f \to K$ defined by

$$\overrightarrow{\operatorname{proj}}_K^f(x) := \operatorname{argmin}_{y \in K} \{D_f(x, y)\} = \{z \in K : D_f(x, z) \leq D_f(x, y)\ \forall y \in K\} \tag{42}$$

for each $x \in \operatorname{int}\operatorname{dom} f$.

It is clear that $\operatorname{Fix}\left(\overrightarrow{\operatorname{proj}}_{K_i}^f\right) = K_i$ for any $i = 1, \ldots, N$. We proved in [28] that the right Bregman projection $\overrightarrow{\operatorname{proj}}_{K_i}^f$ is strictly R-BSNE. If, in addition, $f : X \to \mathbb{R}$ is Legendre and uniformly continuous on bounded subsets of X, and ∇f is weakly sequentially continuous, then $\operatorname{Fix}\left(\overrightarrow{\operatorname{proj}}_{K_i}^f\right) = \widehat{\operatorname{Fix}}\left(\overrightarrow{\operatorname{proj}}_{K_i}^f\right)$ (*cf.* [28]). Therefore, if we take $T_i = \overrightarrow{\operatorname{proj}}_{K_i}^f$ in Remark 7.8, then we get an algorithm for solving convex feasibility problems. More precisely, we arrive at the following result (for relevant related results see [9, Theorem 4.1], [6, Theorem 4.1 and Section 5] and [8, Section 5]).

Proposition 8.7 (Picard's iteration for solving the CFP)**.** *Let $f : X \to \mathbb{R}$ be a cofinite Legendre function which is bounded, uniformly continuous and totally convex on bounded subsets of X. Assume that ∇f is weakly sequentially continuous. Let K_i, $i = 1, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i \neq \emptyset$. Then, for each $x_1 \in X$, the sequence $\{x_n\}_{n \in \mathbb{N}}$ generated iteratively by*

$$x_{n+1} = \overrightarrow{\operatorname{proj}}_{K_N}^f \circ \overrightarrow{\operatorname{proj}}_{K_{N-1}}^f \circ \cdots \circ \overrightarrow{\operatorname{proj}}_{K_1}^f(x_n), \quad n \geq 1,$$

converges weakly to an element in $\bigcap_{i=1}^{N} K_i$.

Remark 8.8. Analogously, using convex combinations of the right Bregman projections instead of their composition, we can consider a different Picard iterative method defined by means of the scheme

$$x_{n+1} = \sum_{i=1}^{N} w_i \overrightarrow{\operatorname{proj}}_{K_i}^f(x_n), \quad n \geq 1.$$

The sequence generated by this scheme also converges weakly to a solution of the convex feasibility problem. $\diamondsuit$

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